The Annals of Applied Probability
2012, Vol. 22, No. 4, 1576–1610
DOI: 10.1214/11-AAP802

ROBUST MAXIMIZATION OF ASYMPTOTIC GROWTH

BY CONSTANTINOS KARDARAS[1] AND SCOTT ROBERTSON

Boston University and Carnegie Mellon University

This paper addresses the question of how to invest in a robust growth-optimal way in a market where the instantaneous expected return of the underlying process is unknown. The optimal investment strategy is identified using a generalized version of the principal eigenfunction for an elliptic second-order differential operator, which depends on the covariance structure of the underlying process used for investing. The robust growth-optimal strategy can also be seen as a limit, as the terminal date goes to infinity, of optimal arbitrages in the terminology of Fernholz and Karatzas [*Ann. Appl. Probab.* **20** (2010) 1179–1204].

Discussion. This paper addresses the question of how to invest optimally in a market when the financial planning horizon is long, and the dynamics of the underlying assets are uncertain. For long time-horizons, it is reasonable to question whether fixed parameter estimation, especially for drift rates, remain valid. Therefore, determining a robust way to invest across potential model misidentifications is desirable, if not indispensable.

On the canonical space of continuous functions from $[0, \infty)$ to $\mathbb{R}^d$, let X denote the coordinate mapping, which should be thought as representing the (relative) price of certain underlying assets, discounted by some baseline wealth process. It is assumed that there exists a probability $\mathbb{Q}$ under which X has dynamics of the form $dX_t = \sigma(X_t)\, dW_t^{\mathbb{Q}}$, where $c := \sigma\sigma'$ represents the instantaneous covariance matrix, and $W^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$. The significance of the local martingale probability $\mathbb{Q}$ lies in that it acts as a "dominating" measure used to form a class of probabilities Π, out of which an unknown representative is supposed to capture the true dynamics of the process. The class Π is built by exactly all probabilities satisfying the following two conditions:

- First, under $\mathbb{P} \in \Pi$ the coordinate mapping X stays in an open and connected subset $E \subseteq \mathbb{R}^d$. Qualitatively, if X represents either asset prices or relative capitalizations, this condition asserts that assets should not cease to exist over the time horizon.

Received June 2010; revised July 2011.

[1]Supported in part by the NSF Grant DMS-09-08461.

MSC2010 subject classifications. 60G44, 60G46, 60H05.

Key words and phrases. Asymptotic growth rate, robustness, generalized martingale problem, optimal arbitrage.

- Second, for $t \geq 0$, each $\mathbb{P} \in \Pi$ is absolutely continuous with respect to $\mathbb{Q}$ on $\sigma(X_s, 0 \leq s \leq t)$. This last fact implies that the volatility process of X under each $\mathbb{P} \in \Pi$ is the same; even though model misidentification is possible, the allowable models are not permitted to be wildly inconsistent with one another.

Note that the family Π as described above does not necessarily induce any ergodic or stability property of the assets, although it certainly contains all such models; in particular, models where the assets display transient behavior are allowable. Furthermore, it is *not* assumed that $\mathbb{Q} \in \Pi$. Indeed, it is often the case that X "explodes" under $\mathbb{Q}$; more precisely, with ζ denoting the first exit time of X from E, $\mathbb{Q}[\zeta < \infty] > 0$ is allowed.

There are good reasons to let the class of models be defined in the above way. While the covariance structure given by the function c is easy to assess, the returns process of X under the "true" probability is statistically impossible to estimate in practice.[2]

Given that the underlying dynamics are only specified within a range of models $\mathbb{P} \in \Pi$, a natural question is to find a reasonable criterion for "optimal investment in X." Here, optimal investment is defined as a wealth process which ensures the largest possible worst-case (with respect to the whole class of models) asymptotic growth rate. Given the set $\mathcal{V}$ of all possible positive stochastic integrals against X starting from unit initial capital, the asymptotic growth rate of $V \in \mathcal{V}$ under $\mathbb{P} \in \Pi$ is defined as the largest $\gamma \in \mathbb{R}_+$ such that $\lim_{t \uparrow \infty} \mathbb{P}[(1/t) \log V_t \geq \gamma] = 1$ holds. (An alternative definition of asymptotic growth rate via almost-sure limits is also considered in the paper.) With this definition, the investor seeks to find a wealth process in $\mathcal{V}$ that achieves maximal growth rate uniformly over all possible models in Π, or at least in a large enough suitable subclass of Π that covers all "nonpathological" cases.

The solution to the above problem is given in terms of a generalized version of the principal eigenvalue-eigenvector pair (λ^*, η^*) of the eigenvalue equation

$$\frac{1}{2} \sum_{i,j=1}^{d} c_{i,j}(x) \frac{\partial^2 \eta}{\partial x_i \, \partial x_j}(x) = -\lambda \eta(x), \qquad x \in E. \tag{0.1}$$

More precisely, the main result of Section 2 states that, when restricted to a large sub-class Π^* of Π, λ^* is the maximal growth rate, and the process $V \in \mathcal{V}$ defined via $V_t = e^{\lambda^* t} \eta^*(X_t)$ achieves this maximal growth rate. There are, of course,

[2]Actually, under continuous-time observations, perfect estimation of c is possible. More realistically, high-frequency data give good estimators for c. In contrast, consider a one-dimensional model for an asset-price of the form $dX_t / X_t = b\,dt + 0.2\,dW_t$, where $b \in \mathbb{R}$—note that $\sigma = 0.2$ is considered a "typical" value for annualized volatility. Given observations $(X_t)_{t \in [0,T]}$, where $T > 0$, the best linear unbiased estimator for b is $\hat{b}_T := (1/T) \log(X_T / X_0)$. Easy calculations show that in order for $|\hat{b}_T - b| \leq 0.01$ to happen with probability at least 95%, one needs $T \approx 1600$ (in years). This simple exercise demonstrates the futility of attempting to estimate drifts.

technicalities on an analytical level arising from the use of the eigenvalue equation (0.1), since it is unreasonable in the present setting to assume either that c is uniformly positive definite on E or that E is bounded with smooth boundary. [Consider, e.g., the case where X represents the prices of d assets. In this instance $E = (0, \infty)^d$, which is unbounded with corners. Furthermore, once the stock price goes to zero, it remains stuck there. Thus, the covariance matrix c degenerates along the boundary of E and hence cannot be both continuous and uniformly elliptic.] In order to allow for degenerate c and unbounded E with nonsmooth boundary, but still retain some tractability in the problem, it is assumed that E can be "filled up" by bounded subregions with smooth boundary and that c is continuous and pointwise strictly positive definite. Under this assumption, [25], Chapter 4, gives a detailed account of eigenvalue equations of the form (0.1).

Growth-optimal trading in the face of model uncertainty has been investigated by other authors. One strand of research considers the case where asset returns are assumed stationary and ergodic. In [2], asymptotically growth-optimal trading strategies based upon historical data are constructed. There have been a number of follow-up papers on this topic; see [1], [14] and the references cited within. In contrast to the aforementioned approach, knowledge of the entire past is not required in this paper. In fact, the optimal strategy is only based on the current level of X and is, therefore, closely-related to the idea of functionally-generated portfolios studied in [9]. Furthermore, it is also not assumed here that X represents asset returns; in fact, the primary example is when X are relative capitalizations, and not asset returns. In this setting, stationarity of the relative capitalizations does not automatically transfer to stationarity of returns.

The concept of robust growth optimality is also related to that of robust utility optimization, the idea of which dates back to [11] and is considered in detail in [10, 13, 26, 28] and [29], amongst others. (There is also recent literature on optimal stopping under model ambiguity—see, e.g., [3].) Though this paper differs from those mentioned in not considering penalty functions and by focusing on growth rather than general utility functions, the growth-optimal strategy provides a "good" long-term robust optimal strategy for general utility functions due to the exponential increase in terminal wealth as time progresses. Two recent papers which are close in spirit to the present paper are [18] and [17]. Reference [18] considers long-run robust utility maximization in the case of model uncertainty for power and logarithmic utility, and [17] addresses the problem of finding wealth processes that minimize long-term downside risk. The precise manner in which the class of models is defined in these papers can only be identified up to a (stochastic) affine perturbation away from a fixed model. This paper differs from the above two in that, to the extent that underlying economic factors affect the asset dynamics, it is only through the drift of X. Furthermore, there is no a priori fixed model from which all other models are recovered via perturbations. This enables the class of models to be determined by qualitative properties, without additional technical restrictions. However, here, as well as in [17], there is a fundamental

PDE, playing the role of an ergodic Bellman equation, which governs the robust trading strategies.

The problem of constructing robust growth-optimal strategies can be extended to the case where even the covariance matrix c is not known precisely, but rather assumed to belong to a class of admissible matrices $\mathcal{C}$. Such a situation has been studied in [7], in the setting of optimal arbitrage mentioned below. In such a setting, one does not even assume the existence of a dominating probability $\mathbb{Q}$, and the probabilities in $\mathbb{P}$ can be mutually singular. It is left for future research to establish a natural definition of an "extremely" robust growth-optimal trading strategy in terms of sub-solutions of (0.1) which are uniform over $\mathcal{C}$.

A second goal of the present paper is to relate robust growth-optimal trading strategies to optimal arbitrages, as considered in [6]. Optimal arbitrages are trading strategies designed to outperform the benchmark process used for discounting almost surely over a given time horizon. In [6], it was shown that, under certain assumptions, the existence of an optimal arbitrage on a finite time horizon $[0, T]$, $T \in \mathbb{R}_+$, is equivalent to $\mathbb{Q}[\zeta \leq T] > 0$ (positive probability of explosion of the coordinate process under $\mathbb{Q}$ before T), when E is the simplex in $\mathbb{R}^d$. In fact, optimal arbitrages are naturally expressed in terms of (conditional) tails of the distribution of ζ under $\mathbb{Q}$.

For a fixed $T > 0$, denote by $(V_t^T)_{t\in[0,T]}$ the optimal arbitrage in the interval $[0, T]$. The robust growth-optimal wealth processes $(V_t)_{t\in\mathbb{R}+}$ considered here can be regarded as a long-term limit of the optimal arbitrages; this is a topic taken up in Section 4. A better understanding of this connection requires exploring a particular probability $\mathbb{P}^*$, under which X has dynamics of the form $dX_t = (c(X_t)\nabla \log \eta^*(X_t))\, dt + \sigma(X_t)\, dW_t^{\mathbb{P}^*}$ for $t \in \mathbb{R}_+$, where $W^{\mathbb{P}^*}$ is a standard Brownian motion under $\mathbb{P}^*$. Loosely speaking, ergodicity of X under $\mathbb{P}^*$ implies that on any compact time interval $[0, \tau]$ the collection of processes $((V_t^T)_{t\in[0,\tau]})_{T\in\mathbb{R}+}$ converges to the robust growth-optimal wealth process $(V_t)_{t\in[0,\tau]}$ as the horizon T becomes large. This is part of the reason why Section 3 is devoted to investigating the properties of X under $\mathbb{P}^*$. An application of ergodic results for unbounded functions from [22], coupled with powerful probabilistic arguments, allows us to show the aforementioned convergence of optimal arbitrages to the robust growth-optimal one. Furthermore, convergence of the probabilities $\mathbb{Q}[\cdot \mid \zeta > T]$ to $\mathbb{P}^*$ on $\mathcal{F}_\tau$ as $T \uparrow \infty$ in the total-variation norm is established. This extends results on diffusions conditioned to remain in a bounded region, first obtained in [24], to regions with nonsmooth boundaries where the matrix c need not be uniformly positive definite, and where the process X under $\mathbb{Q}$ need not be m-reversing for any measure m.

In the special one-dimensional case, considered in Section 5, simple tests for transience and recurrence of diffusions are readily available. This allows us to provide tight conditions upon c in the case of a bounded interval, in which $\lambda^* = 0$ or $\lambda^* > 0$, and characterize both the nature of η^* and of $\mathbb{P}^*$. The main message is essentially the following: if X can explode to both endpoints under $\mathbb{Q}$, then everything works out nicely, in the sense that $\lambda^* > 0$ and X is positive recurrent under $\mathbb{P}^*$. The technical proof of this result relies heavily on singular Sturm–Liouville theory and is given in Section 7.

Finally, Section 6 provides examples that illustrate the results obtained in previous sections. In contrast to the case where c is uniformly positive definite on E, multi-dimensional examples where the function η^* does not vanish on the boundary of E, even if E is bounded, are given.

1. The set-up.

Consider an open and connected set $E \subseteq \mathbb{R}^d$ and a function c mapping E to the space of $d \times d$ matrices. For $\alpha \in (0, 1]$, recall that a function $f : E \mapsto \mathbb{R}$ is called *locally $C^{2,\alpha}$ on E* if for all bounded, open, connected $D \subset E$ such that $\bar{D} \subset E$ it follows that $f \in C^{2,\alpha}(\bar{D})$. For a definition of the Hölder space $C^{2,\alpha}$, see [5], Chapter 5.1. The following assumptions will be in force throughout.

ASSUMPTION 1.1. For each $x \in E$, $c(x)$ is a symmetric and strictly positive definite $d \times d$ matrix. For $1 \le i, j \le d$, $c_{ij}(x)$ is locally $C^{2,\alpha}$ on E for some $\alpha \in (0, 1]$. Furthermore, there exists a sequence $(E_n)_{n \in \mathbb{N}}$ of bounded open connected subsets of E such that each boundary ∂E_n is $C^{2,\alpha}$, $\bar{E}_n \subset E_{n+1}$ for $n \in \mathbb{N}$ and $E = \bigcup_{n=1}^{\infty} E_n$.

1.1. *The generalized martingale problem on E.*

It will now be discussed how Assumption 1.1 implies the existence of a unique solution to the generalized martingale problem on E for the operator L which acts on $f \in C^2(E)$ via

$$(1.1) \qquad (Lf)(x) = \frac{1}{2} \sum_{i,j=1}^{d} c_{ij}(x) \frac{\partial^2 f}{\partial x_i \, \partial x_j}(x), \qquad x \in E.$$

Let $\hat{E} = E \cup \triangle$ be the one-point compactification of E; the point $\triangle$ is identified with ∂E if E is bounded and with ∂E plus the point at ∞ if E is unbounded. Let $C(\mathbb{R}_+, \hat{E})$ be the space of continuous functions from $[0, \infty)$ to $\hat{E}$. For $\omega \in C(\mathbb{R}_+, \hat{E})$, define the exit times

$$\zeta_n(\omega) := \inf\{t \in \mathbb{R}_+ \mid \omega_t \notin E_n\},$$
$$\zeta(\omega) := \lim_{n \uparrow \infty} \zeta_n(\omega).$$

Then define

$$\Omega = \{\omega \in C(\mathbb{R}_+, \hat{E}) \mid \omega_{\zeta+t} = \triangle \text{ for all } t \in \mathbb{R}_+ \text{ if } \zeta(\omega) < \infty\}.$$

Let $X = (X_t)_{t \in \mathbb{R}_+}$ be the coordinate mapping process for $\omega \in \Omega$. Set $\mathcal{B} = (\mathcal{B}_t)_{t \in \mathbb{R}_+}$ to be the natural filtration of X. It follows that the smallest σ-algebra that is generated by $\bigcup_{t \in \mathbb{R}_+} \mathcal{B}_t$, denoted by $\mathcal{B}_\infty$, is actually the Borel σ-algebra on Ω.

Furthermore, $\mathcal{B}_\infty$ is also the smallest σ-algebra that is generated by $\bigcup_{n\in\mathbb{N}}\mathcal{B}_{\zeta_n}$, since paths in Ω stay in $\triangle$ upon arrival.

A solution to the generalized martingale problem on E is a family of probability measures $(\mathbb{Q}_x)_{x\in\hat{E}}$ such that $\mathbb{Q}_x[X_0 = x] = 1$ and

$$f(X_{t\wedge\zeta_n}) - \int_0^{t\wedge\zeta_n} (Lf)(X_s)\,ds$$

is a $(\Omega, (\mathcal{B}_t)_{t\in\mathbb{R}_+}, \mathbb{Q}_x)$-martingale for all $n \in \mathbb{N}$ and all $f \in C^2(E)$ with Lf given as in (1.1).

Assumption 1.1 ensures a solution to the generalized martingale problem, as the following proposition, taken from [25], Theorem 1.13.1, shows.

PROPOSITION 1.2. *Under Assumption 1.1, there is a unique solution $(\mathbb{Q}_x)_{x\in\hat{E}}$ to the generalized martingale problem on E. The family $(\mathbb{Q}_x)_{x\in\hat{E}}$ possesses the strong Markov property.*

Set $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ to be the right-continuous enlargement of $(\mathcal{B}_t)_{t\in\mathbb{R}_+}$. Furthermore, with $\mathcal{F}$ denoting the smallest σ-algebra that contains $\bigcup_{t\in\mathbb{R}_+}\mathcal{F}_t$, we have $\mathcal{F} = \mathcal{B}_\infty$. Assumption 1.1 implies that

$$f(X_{t\wedge\zeta_n}) - \int_0^{t\wedge\zeta_n} (Lf)(X_s)\,ds$$

is a $(\Omega, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{Q}_x)$-martingale for all $n = 1, 2, 3, \ldots$ and $f \in C^2(E)$ since f and Lf are bounded on each E_n. By setting $f(x) = x^i, i = 1, \ldots, d$, and $f(x) = x^i x^j, i, j = 1, \ldots, d$, it follows that, for each n and each $x \in \hat{E}$, $X_{t\wedge\zeta_n}$ is a $(\Omega, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{Q}_x)$-martingale with quadratic covariation process $\int_0^\cdot \mathbb{I}_{\{t\le\zeta_n\}} c(X_t)\,dt$.

1.2. *Asymptotic growth rate.* For a fixed $x_0 \in E$, set $\mathbb{Q} = \mathbb{Q}_{x_0}$. In the sequel, whenever there is no subscript associated to the probabilities, it will be tacitly assumed that they only charge the event $\{X_0 = x_0\}$.

Denote by Π the class of probabilities on $(\Omega, \mathcal{F})$ which are *locally absolutely continuous* with respect to $\mathbb{Q}$ (written $\mathbb{P} \ll_{\text{loc}} \mathbb{Q}$) and for which the coordinate process X does not explode, that is,

$$\Pi = \big[P \in M_1(\Omega, \mathcal{F}) : \mathbb{P}|_{\mathcal{F}_t} \ll \mathbb{Q}|_{\mathcal{F}_t} \text{ for all } t \ge 0 \text{ and } \mathbb{P}[\zeta < \infty] = 0\big]. \tag{1.2}$$

For each $\mathbb{P} \in \Pi$, X is a $(\Omega, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$-semimartingale such that $\mathbb{P}[X \in C(\mathbb{R}_+, E)] = 1$. Therefore, X admits the representation

$$X = x_0 + \int_0^\cdot b_t^{\mathbb{P}}\,dt + \int_0^\cdot \sigma(X_t)\,dW_t^{\mathbb{P}},$$

where $W^{\mathbb{P}}$ is a standard d-dimensional Brownian motion on $(\Omega, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$, σ is the unique symmetric strictly positive definite square root of c and $b^{\mathbb{P}}$ is a d-dimensional $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-progressively measurable process.

Let $(\xi_t)_{t\in\mathbb{R}_+}$ be an adapted process. For $\mathbb{P}\in\Pi$, define

$$\mathbb{P}\text{-}\liminf_{t\to\infty}\xi_t := \operatorname*{ess\,sup}_{\mathbb{P}}\Big\{\chi \text{ is } \mathcal{F}\text{-measurable} \;\Big|\; \lim_{t\to\infty}\mathbb{P}[\xi_t \geq \chi] = 1\Big\}.$$

If, in addition, $\mathbb{P}[\xi_t > 0] = 1$ for each $t\in\mathbb{R}_+$, let

$$g(\xi;\mathbb{P}) := \sup\Big\{\gamma\in\mathbb{R} \;\Big|\; \mathbb{P}\text{-}\liminf_{t\to\infty}(t^{-1}\log\xi_t) \geq \gamma, \mathbb{P}\text{-a.s.}\Big\}$$

be the *asymptotic growth rate of* ξ *under* $\mathbb{P}$. Since $\mathbb{P}\in\Pi$ and $\mathbb{Q}$ are not necessarily equivalent on $\mathcal{F}$, $g(\xi;\mathbb{P})$ indeed depends on $\mathbb{P}\in\Pi$. The following result, the proof of which is straightforward and hence omitted, provides an alternative representation for $g(\xi;\mathbb{P})$.

LEMMA 1.3. *For a given* $\mathbb{P}\in\Pi$ *and an adapted real-valued process* $(\xi_t)_{t\in\mathbb{R}_+}$ *such that* $\mathbb{P}[\xi_t > 0] = 1$ *for all* $t\in\mathbb{R}_+$,

$$g(\xi;\mathbb{P}) = \sup\Big\{\gamma\in\mathbb{R} \;\Big|\; \lim_{t\to\infty}\mathbb{P}[t^{-1}\log\xi_t \geq \gamma] = 1\Big\}.$$

1.3. *The problem.* The basic object in our study will be the class of all possible nonnegative wealth processes that one can achieve by investing in the d assets whose price processes are modeled via X. Whenever ϑ is a d-dimensional predictable process, that is, X-integrable under $\mathbb{Q}$ (and, as a consequence, X-integrable under any $\mathbb{P}\in\Pi$, as $\mathbb{P}\ll_{\mathrm{loc}}\mathbb{Q}$), define the process $V^{\vartheta} = 1 + \int_0^{\cdot}\vartheta_t'\,dX_t$, where the prime symbol ($'$) denotes transposition throughout the text. Then let $\mathcal{V}$ denote the class of all processes V^{ϑ} of the previous form, where we additionally have $V^{\vartheta}\geq 0$ up to $\mathbb{Q}$-evanescent sets. (Of course, $V^{\vartheta}\geq 0$ also holds up to $\mathbb{P}$-evanescent sets for all $\mathbb{P}\in\Pi$.) Naturally, ϑ represents the position that an investor takes on the assets whose discounted price-processes are given by X, and V^{ϑ} represents the resulting wealth from trading starting from unit capital, constrained not to go negative at any time.

The problem considered is to calculate

$$\sup_{V\in\mathcal{V}}\inf_{\mathbb{P}\in\Pi} g(V;\mathbb{P}) \tag{1.3}$$

and to find $V^*\in\mathcal{V}$ that attains this value, at least for all $\mathbb{P}$ in a large sub-class of Π that will be soon defined. To this end, for a given $\lambda\in\mathbb{R}$ and L as in (1.1), define the cone of positive harmonic functions with respect to $L+\lambda$ as

$$H_\lambda := \{\eta\in C^2(E) \mid L\eta = -\lambda\eta \text{ and } \eta > 0\}. \tag{1.4}$$

Set

$$\lambda^* := \sup\{\lambda\in\mathbb{R} \mid H_\lambda \neq \varnothing\}. \tag{1.5}$$

Since $H_0\neq\varnothing$ (take $\eta\equiv 1$), it follows that $\lambda^*\geq 0$. If $H_{\lambda^*}\neq\varnothing$, then, by construction, there is an $\eta^*\in C^2(E)$ satisfying

$$L\eta^* = -\lambda^*\eta^*, \tag{1.6}$$

and λ^* is the largest real for which such an η^* exists. Thus λ^* is a generalized version of the principal eigenvalue for L on E. The following result, taken from [25], Theorem 4.3.2, states that, indeed, $H_{\lambda^*} \neq \varnothing$.

PROPOSITION 1.4. *Let Assumption 1.1 hold. Then* $0 \leq \lambda^* < \infty$ *and* $H_{\lambda^*} \neq \varnothing$.

REMARK 1.5. To connect Proposition 1.4 with [25], Theorem 4.3.2, note that $\lambda_c(D)$ therein is equal to $-\lambda^*$. Note also that, by its construction, $\Pi = \varnothing$ if there exists a $t > 0$ such that $\mathbb{Q}[\zeta > t] = 0$. However, by [25], Theorem 4.4.4, it follows that if such a $t > 0$ exists, then $\lambda^* = \infty$. Proposition 1.4 thus implies that $\mathbb{Q}[\zeta > t] > 0$ for all $t > 0$. It is also directly shown in the proof of Theorem 2.1 below that under Assumption 1.1, $\Pi \neq \varnothing$.

REMARK 1.6. Proposition 1.4 makes no claim regarding the uniqueness of η^* corresponding to λ^*. For example, when $E = (0, \infty)$ and $c \equiv 1$, it holds that $\lambda^* = 0$; hence η^* could be either x or 1. For this E and c, Example 4.7 in Section 4 shows that even when uniqueness fails, a particular choice of η^* may be advantageous.

The following result, taken from [25], Theorems 4.3.3 and 4.3.4, provides a way of checking if a particular pair (η, λ) such that $\eta \in H_\lambda$ corresponds to an optimal pair (η^*, λ^*) and if the optimal pair is unique.

PROPOSITION 1.7. *Let Assumption 1.1 hold. Let (η, λ) be such that $\eta \in H_\lambda$. Then there exists a unique solution $(\mathbb{P}_x^\eta)_{x\in\hat{E}}$ to the generalized martingale problem on $\hat{E}$ for the operator*

$$L^\eta = L + c\nabla \log \eta \cdot \nabla, \tag{1.7}$$

and $(\mathbb{P}_x^\eta)_{x\in\hat{E}}$ possesses the strong Markov property. Furthermore, if the coordinate mapping process X is recurrent under $(\mathbb{P}_x^\eta)_{x\in E}$, then η is unique up to multiplication by a positive constant, $\eta^* = \eta$ and $\lambda^* = \lambda$.

REMARK 1.8. Proposition 1.7 only covers the case where the coordinate mapping process X is recurrent under $(\mathbb{P}_x^\eta)_{x\in E}$. It should be noted, however, that even when the coordinate mapping process X under $(\mathbb{P}_x^\eta)_{x\in E}$ is transient, $\eta = \eta^*$ and $\lambda = \lambda^*$ is still possible. Indeed, in Example 4.7 from Section 4, $\lambda^* = 0$ even though $\mathbb{Q}_x[\zeta < \infty] > 0$ for all $x \in E$, and thus $\eta^* = 1$ does not yield a recurrent process.

2. The min–max result.

2.1. *The result.* For future reference, let η^* be a solution of (1.6) corresponding to λ^* with $\eta^*(x_0) = 1$, and define the function $\ell^* : E \mapsto \mathbb{R}$ via

$$\ell^*(x) = \log \eta^*(x) \qquad \text{for } x \in E. \tag{2.1}$$

The following result identifies λ^* with the value in (1.3).

THEOREM 2.1. *Let Assumption 1.1 hold. Let η^* be a solution of (1.6) corresponding to λ^* with $\eta^*(x_0) = 1$, and define V^* via $V_t^* = e^{\lambda^* t}\eta^*(X_t)$ for all $t \in \mathbb{R}_+$. Define also*

$$\Pi^* := \Big\{\mathbb{P} \in \Pi \mid \mathbb{P}\text{-}\liminf_{t\to\infty}(t^{-1}\log\eta^*(X_t)) \geq 0, \mathbb{P}\text{-}a.s.\Big\}.$$

Then $V^* \in \mathcal{V}$ and $g(V^*;\mathbb{P}) \geq \lambda^*$ for all $\mathbb{P} \in \Pi^*$. Furthermore,

$$\lambda^* = \sup_{V\in\mathcal{V}}\inf_{\mathbb{P}\in\Pi^*} g(V;\mathbb{P}) = \inf_{\mathbb{P}\in\Pi^*}\sup_{V\in\mathcal{V}} g(V;\mathbb{P}). \tag{2.2}$$

REMARK 2.2. The normalized eigenfunction η^* in the statement of Theorem 2.1 may not be unique. Since the class of measures Π^* depends upon η^*, the variational problems in (2.2) also change with η^*. However, the value λ^* is the same no matter which η^* is chosen.

For a given η^*, it may seem artificial to restrict attention to Π^*. However, no matter which $\eta^* \in H_{\lambda^*}$ is chosen, Π^* contains all the probabilities $\mathbb{P}$ such that X is tight in E, and hence naturally corresponds to those $\mathbb{P}$ for which X is stable. To see this, assume that X is tight, and let $\epsilon > 0$ and $K^\epsilon \subseteq E$ be compact such that $\sup_{t\geq 0}\mathbb{P}[X_t \notin K^\epsilon] \leq \epsilon$. Set $\beta^\epsilon = \max_{x\in K^\epsilon}|\log\eta^*(x)|$, and note that for any $\delta > 0$ and $t > \beta^\epsilon/\delta$,

$$\mathbb{P}[t^{-1}\log\eta^*(X_t) < -\delta] \leq \mathbb{P}[|t^{-1}\log\eta^*(X_t)| > \delta; X_t \notin K^\epsilon] \leq \epsilon.$$

Thus, $\lim_{t\to\infty}\mathbb{P}[t^{-1}\log\eta^*(X_t) \geq -\delta] = 1$ for all $\delta > 0$; hence, $\mathbb{P} \in \Pi^*$.

PROOF OF THEOREM 2.1. To see why $V^* \in \mathcal{V}$, note that Itô's formula gives, for each $n \in \mathbb{N}$, each $t \in \mathbb{R}_+$ and each $\mathbb{P} \in \Pi$,

$$\begin{aligned} V^*_{t\wedge\zeta_n} &= 1 + \int_0^{t\wedge\zeta_n} e^{\lambda^* s}\nabla\eta^*(X_s)'\,dX_s \\ &= 1 + \int_0^{t\wedge\zeta_n} V_s^*\nabla\ell^*(X_s)'\,dX_s. \end{aligned} \tag{2.3}$$

Since $\mathbb{P}[\zeta < \infty] = 0$ for all $\mathbb{P} \in \Pi$, it follows that the equalities in (2.3) hold under $\mathbb{P}$ when we replace $t \wedge \zeta_n$ with t for all $t \in \mathbb{R}_+$. By the construction of Π^*, $\mathbb{P}[\lim_{t\to\infty} t^{-1}\log(V_t^*) \geq \gamma] = 1$ holds for all $\gamma < \lambda^*$ and all $\mathbb{P} \in \Pi^*$. Therefore, Lemma 1.3 implies $g(V^*;\mathbb{P}) \geq \lambda^*$ for all $\mathbb{P} \in \Pi^*$. In particular, $\lambda^* \leq \sup_{V\in\mathcal{V}}\inf_{\mathbb{P}\in\Pi^*} g(V;\mathbb{P})$.

Now, let λ_n^*, η_n^* and ℓ_n^* be the equivalents of λ^*, η^* and ℓ^* when E is replaced by E_n in (1.4), (1.5), (1.6) and (2.1). Assumption 1.1 gives that c is uniformly elliptic on E_n and hence $\eta_n^* \in C^{2,\alpha}(\bar{E}_n)$ and vanishes on ∂E_n [25], Theorem 3.5.5. Furthermore, there exists a solution to the generalized martingale problem $(\mathbb{P}^*_{x,n})_{x\in E_n}$

for the operator $L^{\eta_n^*}$ in (1.7) and the coordinate process X under $(\mathbb{P}_{x,n}^*)_{x\in E_n}$ is recurrent in E_n ([25], proof of Theorem 4.2.4). This latter fact gives the uniqueness (up to multiplication by a positive constant) of η_n^*.

Set $\mathbb{P}_n^* = \mathbb{P}_{x_0,n}^*$. It follows that $\mathbb{P}_n^*[\zeta < \infty] = 0$ and $\lim_{t\to\infty} \mathbb{P}_n^*[t^{-1}\log\eta^*(X_t) = 0] = 1$ since there exists a $K_n > 0$ such that $1/K_n < \eta^* < K_n$ on E_n. Thus, $\mathbb{P}_n^* \in \Pi^*$ if $\mathbb{P}_n^* \ll_{\mathrm{loc}} \mathbb{Q}$. To show the latter, let $(\mathbb{Q}_{x,n})_{x\in\hat{E}_n}$ be the solution to the generalized martingale problem for L on $\hat{E}_n$. Let $\mathbb{Q}_n = \mathbb{Q}_{x_0,n}$. It follows from [25], Corollary 4.1.2, and the recurrence of X under $\mathbb{P}_n^*$ that for $t > 0$,

$$\left.\frac{d\mathbb{P}_n^*}{d\mathbb{Q}_n}\right|_{\mathcal{B}_t} = e^{\lambda_n^* t}\frac{\eta_n^*(X_t)}{\eta_n^*(x_0)}\mathbb{I}_{\{\zeta_n > t\}}, \tag{2.4}$$

and thus $\mathbb{P}_n^*|_{\mathcal{B}_t} \ll \mathbb{Q}_n|_{\mathcal{B}_t}$. This immediately gives $\mathbb{P}_n^*|_{\mathcal{B}_{t\wedge\zeta_n}} \ll \mathbb{Q}_n|_{\mathcal{B}_{t\wedge\zeta_n}}$ for each n. But, $\mathbb{Q}_n|_{\mathcal{B}_{t\wedge\zeta_n}} = \mathbb{Q}|_{\mathcal{B}_{t\wedge\zeta_n}}$. If $B \in \mathcal{B}_t$ is such that $\mathbb{Q}[B] = 0$, then $\mathbb{Q}[B \cap \{\zeta_n > t\}] = 0$. Since $B \cap \{\zeta_n > t\} \in \mathcal{B}_{t\wedge\zeta_n}$, it follows that $\mathbb{P}_n^*[B \cap \{\zeta_n > t\}] = 0$. But, $\mathbb{P}_n^*[\zeta_n > t] = 1$ for each t so $\mathbb{P}_n^*[B \cap \{\zeta_n > t\}] = 0$ implies $\mathbb{P}_n^*[B] = 0$. Therefore, $\mathbb{P}_n^*|_{\mathcal{B}_t} \ll \mathbb{Q}|_{\mathcal{B}_t}$ and hence $\mathbb{P}_n^*|_{\mathcal{F}_t} \ll \mathbb{Q}|_{\mathcal{F}_t}$ as well, proving $\mathbb{P}_n^* \in \Pi^*$.

Let V_n^* be defined via $V_n^*(t) = e^{\lambda_n^*(t\wedge\zeta_n)}\eta_n^*(X_{t\wedge\zeta_n})$ for $t \in \mathbb{R}_+$ [in order to avoid the cumbersome notation $V_{n,t}^*$ for $t \in \mathbb{R}_+$, we simply use $V_n^*(t)$ here]. The same computations as in (2.3) show that, for all $\mathbb{P} \in \Pi$,

$$V_n^* = 1 + \int_0^\cdot \mathbb{I}_{\{t\le\zeta_n\}} e^{\lambda_n t}\nabla\eta_n^*(X_t)'\,dX_t$$

and hence $V_n^* \in \mathcal{V}$. Note that V_n^* stays strictly positive under $\mathbb{P}_n^*$ since $\mathbb{P}_n^*[\zeta_n < \infty] = 0$. Now, $g(V_n^*;\mathbb{P}_n^*) \le \lambda_n^*$ is immediate since E_n is bounded, and hence η_n^* is bounded above on E_n. Furthermore, V_n^* is the *numéraire portfolio in $\mathcal{V}$ under $\mathbb{P}_n^*$*, which means that V/V_n^* is a (nonnegative) $\mathbb{P}_n^*$-supermartingale for all $V \in \mathcal{V}$. To wit, consider any other $V \in \mathcal{V}$, and write $V = 1 + \int_0^\cdot \vartheta_t'\,dX_t$. A straightforward use of Itô's formula using the fact that $L\eta_n^*(x) = -\lambda_n^*(x)\eta_n^*(x)$ holds for all $x \in E_n$ gives that, under $\mathbb{P}_n^*$,

$$\frac{V}{V_n^*} = \int_0^\cdot \left(\frac{\vartheta_t - V_t\nabla\ell_n^*(X_t)}{V_n^*(t)}\right)' d\big(X_t - c(X_t)\nabla\ell_n^*(X_t)\,dt\big);$$

since the process $X - \int_0^\cdot c(X_t)\nabla\ell_n^*(X_t)\,dt$ is a local $\mathbb{P}_n^*$-martingale, the numéraire property of V_n^* in $\mathcal{V}$ under $\mathbb{P}_n^*$ follows. In view of the nonnegative supermartingale convergence theorem, the nonnegative supermartingale property of V/V_n^* under $\mathbb{P}_n^*$ gives that $\limsup_{t\to\infty}\log(V_t/V_n^*(t)) \le 0$ in the $\mathbb{P}_n^*$-a.s. sense. Therefore, $g(V;\mathbb{P}_n^*) \le g(V_n^*;\mathbb{P}_n^*)$ holds for all $V \in \mathcal{V}$. Since $g(V_n^*;\mathbb{P}_n^*) \le \lambda_n^*$, $\sup_{V\in\mathcal{V}} g(V;\mathbb{P}_n^*) \le \lambda_n^*$ holds, and $\inf_{\mathbb{P}\in\Pi^*}\sup_{V\in\mathcal{V}} g(V;\mathbb{P}) \le \inf_{n\in\mathbb{N}}\lambda_n^*$. However, $\downarrow\lim_{n\to\infty}\lambda_n^* = \lambda^*$ holds in view of Assumption 1.1 ([25], Theorem 4.4.1). This gives $\inf_{\mathbb{P}\in\Pi^*}\sup_{V\in\mathcal{V}} g(V;\mathbb{P}) \le \lambda^*$ and completes the argument. $\square$

2.2. *An "almost sure" class of measures.* For a fixed $\eta^* \in H_{\lambda^*}$, define the following class of probability measures:

$$\Pi^*_{\text{a.s.}} := \Big\{\mathbb{P} \in \Pi \mid \liminf_{t\to\infty}(t^{-1}\log \eta^*(X_t)) \geq 0, \mathbb{P}\text{-a.s.}\Big\}.$$

It is straightforward to check that $\Pi^*_{\text{a.s.}} \subseteq \Pi^*$. Furthermore, as will be seen in Section 3, it can be easier to verify inclusion in $\Pi^*_{\text{a.s.}}$ than Π^*. For $\mathbb{P} \in \Pi$ and $V \in \mathcal{V}$ define

$$g_{\text{a.s.}}(V;\mathbb{P}) := \sup\Big\{\gamma \in \mathbb{R} \mid \liminf_{t\to\infty}(t^{-1}\log V_t) \geq \gamma, \mathbb{P}\text{-a.s.}\Big\}$$

as the "almost sure" growth of the wealth V. The following result is the analog of Theorem 2.1 for the class of measures $\Pi^*_{\text{a.s.}}$ and for the growth rate $g_{\text{a.s.}}(V;\mathbb{P})$.

PROPOSITION 2.3. *Let Assumption 1.1 hold. Let $\eta^* \in H_{\lambda^*}$ be such that $\eta^*(x_0) = 1$, and define $\Pi^*_{\text{a.s.}}$ as above. Define $V^* \in \mathcal{V}$ by $V^*_t = e^{\lambda^* t}\eta^*(X_t), t \geq 0$, as in Theorem 2.1. Then $g_{\text{a.s.}}(V^*;\mathbb{P}) \geq \lambda^*$ for all $\mathbb{P} \in \Pi^*_{\text{a.s.}}$ and*

$$\lambda^* = \sup_{V\in\mathcal{V}} \inf_{\mathbb{P}\in\Pi^*_{\text{a.s.}}} g_{\text{a.s.}}(V;\mathbb{P}) = \inf_{\mathbb{P}\in\Pi^*_{\text{a.s.}}} \sup_{V\in\mathcal{V}} g_{\text{a.s.}}(V;\mathbb{P}).$$

REMARK 2.4. Concerning the class Π^*, in Remark 2.2 it was discussed that when the coordinate process X is $\mathbb{P}$-tight, then $\mathbb{P} \in \Pi^*$. In contrast, a useful characterization of even a subset of $\Pi^*_{\text{a.s.}}$ independent of η^* is difficult. On the positive side, if $\mathbb{P}$ is such that X never exits E_n for some n, then $\mathbb{P} \in \Pi^*_{\text{a.s.}}$. However, even if X is positive recurrent under $\mathbb{P}$, it cannot immediately be said that $\mathbb{P} \in \Pi^*_{\text{a.s.}}$

PROOF OF PROPOSITION 2.3. By construction of the class $\Pi^*_{\text{a.s.}}$ it follows that $g_{\text{a.s.}}(V^*;\mathbb{P}) \geq \lambda^*$ for all $\mathbb{P} \in \Pi^*_{\text{a.s.}}$. Thus $\lambda^* \leq \sup_{V\in\mathcal{V}} \inf_{\mathbb{P}\in\Pi^*_{\text{a.s.}}} g_{\text{a.s.}}(V;\mathbb{P})$. The inequality $\lambda^* \geq \inf_{\mathbb{P}\in\Pi^*_{\text{a.s.}}} \sup_{V\in\mathcal{V}} g_{\text{a.s.}}(V;\mathbb{P})$ follows by essentially the same argument as in Theorem 2.1. Specifically, let $\lambda^*_n, \eta^*_n, \ell^*_n, V^*_n$ and $\mathbb{P}^*_n$ be as in the proof of Theorem 2.1. It was shown therein that $\mathbb{P}^*_n \in \Pi$ for each n and that the coordinate process is recurrent in E_n under $(\mathbb{P}^*_{x,n})_{x\in E_n}$. In fact, $\mathbb{P}^*_n \in \Pi^*_{\text{a.s.}}$ because there is a $K_n > 0$ such that $1/K_n < \eta^* < K_n$ on E_n and hence, $\mathbb{P}^*_n$-a.s., $\lim_{t\to\infty} t^{-1}\log\eta^*(X_t) = 0$. Furthermore, since η^*_n is bounded from above on E_n it holds that $g_{\text{a.s.}}(V^*_n;\mathbb{P}^*_n) \leq \lambda^*_n$. Using the numéraire property of V^*_n under $\mathbb{P}^*_n$ and the supermartingale convergence theorem, it follows that $g_{\text{a.s.}}(V;\mathbb{P}^*_n) \leq g_{\text{a.s.}}(V^*;\mathbb{P}^*_n)$ holds for all $V \in \mathcal{V}$. Therefore, $\sup_{V\in\mathcal{V}} g_{\text{a.s.}}(V;\mathbb{P}^*_n) \leq \lambda^*_n$ and

$$\inf_{\mathbb{P}\in\Pi^*_{\text{a.s.}}} \sup_{V\in\mathcal{V}} g_{\text{a.s.}}(V;\mathbb{P}^*_n) \leq \inf_{n\in\mathbb{N}} \lambda^*_n = \lambda^*$$

since $\downarrow \lim_{n\to\infty} \lambda^*_n = \lambda^*$ as seen in the proof of Theorem 2.1. This completes the argument. $\square$

3. An interesting probability measure. Let $\eta^* \in H_{\lambda^*}$, and let $(\mathbb{P}^*_x)_{x\in\hat{E}}$ be the solution to the generalized martingale problem on $\hat{E}$ for the operator L^{η^*} given in (1.7). Set $\mathbb{P}^* \equiv \mathbb{P}^*_{x_0}$.

It is of great interest to know whether $\mathbb{P}^* \in \Pi^*$. To begin with, if this is indeed true and $g(V^*, \mathbb{P}^*) = \lambda^*$, the pair $(V^*, \mathbb{P}^*)$ constitutes a saddle point for the minimax problem described in (2.2). Indeed, using the numéraire property of V^* under $\mathbb{P}^*$ and the definition of Π^*, it follows that, in this case (see the proof of Theorem 2.1)

$$g(V; \mathbb{P}^*) \leq g(V^*; \mathbb{P}^*) \leq g(V^*; \mathbb{P}) \qquad \text{for all } V \in \mathcal{V} \text{ and } \mathbb{P} \in \Pi^*.$$

Furthermore, in Section 4 where connections between robust growth-optimal portfolios and optimal arbitrages are studied, the behavior of the coordinate process X under $\mathbb{P}^*$ becomes important. To this end, presented in the sequel are some results that explore the behavior of X under $\mathbb{P}^*$. In particular, Propositions 3.4 and 3.6 give sufficient conditions to ensure that $\mathbb{P}^* \in \Pi^*$.

REMARK 3.1. By construction, if $\mathbb{P}^*[\zeta < \infty] > 0$ then $\mathbb{P}^* \notin \Pi^*$. Example 4.7 provides a case when explosion of X under $\mathbb{P}^*$ occurs *for some* $\eta^* \in H_{\lambda^*}$. Furthermore, [23] contains an example showing that for *for all* $\eta^* \in H_{\lambda^*}$, the probability $\mathbb{P}^*$ that is constructed from η^* leads to explosive behavior of X under $\mathbb{P}^*$; hence, none of the candidate $\mathbb{P}^*$ is in Π^*. Now, consider the case when $\eta^* \in H_{\lambda^*}$ is such that X is nonexplosive under $\mathbb{P}^*$. In this instance, Corollary 3.7 shows that if $\lambda^* = 0$, then $\mathbb{P}^* \in \Pi^*$. As for when $\lambda^* > 0$, although only *sufficient* conditions ensuring that $\mathbb{P}^* \in \Pi^*$ are presented in this section, examples where $\mathbb{P}^* \notin \Pi^*$ have not been found. It is an open question whether, under Assumption 1.1, $\mathbb{P}^* \in \Pi^*$ holds whenever $\lambda^* > 0$ and $\eta^* \in H_{\lambda^*}$ is such that X is nonexplosive under $\mathbb{P}^*$. See Example 6.5 in Section 6 for a potential counterexample.

The first result gives conditions under which $\mathbb{P}^* \in \Pi$ and relates the tail probabilities of ζ under $\mathbb{Q}$ and robust growth-optimal strategies.

PROPOSITION 3.2. *Let Assumption 1.1 hold, and let $\eta^* \in H_{\lambda^*}$ be such that $\mathbb{P}^*_x[\zeta < \infty] = 0$ holds for all $x \in E$. Then $\mathbb{P}^* \in \Pi$ and*

$$\mathbb{Q}_x[\zeta > T] = \eta^*(x)\mathbb{E}^{\mathbb{P}^*}_x\left[\frac{1}{V^*_T}\right] \qquad \textit{holds for all } T \in \mathbb{R}_+ \textit{ and } x \in E. \tag{3.1}$$

PROOF. In a similar manner to (2.4), if $\mathbb{P}^*[\zeta < \infty] = 0$, then it follows from [25], Corollary 4.1.2, that

$$\left.\frac{d\mathbb{P}^*}{d\mathbb{Q}}\right|_{\mathcal{B}_t} = e^{\lambda^* t}\frac{\eta^*(X_t)}{\eta^*(x_0)}\mathbb{I}_{\{\zeta > t\}}$$

from which it immediately holds that $\mathbb{P}^* \ll_{\text{loc}} \mathbb{Q}$, and hence $\mathbb{P}^* \in \Pi$. Given $V_T^* = \exp(\lambda^* T)\eta^*(X_T)$, the equality in (3.1) follows immediately from [25], Theorem 4.1.1. $\square$

Recall from Remark 2.2 that $\mathbb{P}^*$-tightness of $(X_t)_{t\in\mathbb{R}_+}$ implies that $\mathbb{P}^* \in \Pi^*$. The following result is useful because it shows that, under Assumption 1.1, positive recurrence and tightness of $(X_t)_{t\in\mathbb{R}_+}$ under $\mathbb{P}^*$ are equivalent notions. Note that, in general, even in the one-dimensional bounded case, the behavior of $(X_t)_{t\in\mathbb{R}_+}$ under $\mathbb{P}^*$ can vary from positive recurrence to transience as is shown in the examples in Section 6.1.

PROPOSITION 3.3. *Let Assumption 1.1 hold. Then the following are equivalent*:

(1) *The coordinate mapping process X is positive recurrent under $(\mathbb{P}_x^*)_{x\in E}$.*
(2) *For some $x \in E$ the family of random variables $(X_t)_{t\geq 0}$ is $\mathbb{P}_x^*$-tight in E.*

PROOF. Under Assumption 1.1, X is recurrent under $(\mathbb{P}_x^*)_{x\in E}$ if for any $x, y \in E$ and $\varepsilon > 0$, if $\tau_{B(y,\varepsilon)}$ is the first time the coordinate process enters into the closed ball of radius ε around y, then $\mathbb{P}_x^*[\tau_{B(y,\varepsilon)} < \infty] = 1$. Note that if X is recurrent under $(\mathbb{P}_x^*)_{x\in E}$, then for all $x \in E$, $\mathbb{P}_x^*[\zeta < \infty] = 0$ ([25], Theorem 2.8.1). Furthermore, given that X is recurrent under $(\mathbb{P}_x^*)_{x\in E}$, then X is further positive recurrent under $(\mathbb{P}_x^*)_{x\in E}$ if there exists a function $\tilde{\eta}^* > 0$ such that $\tilde{L}^*\tilde{\eta}^* = 0$ and $\tilde{\eta}^* \in \mathbb{L}^1(E, \text{Leb})$ where $\tilde{L}^*$ is the formal adjoint to L^* ([25], Section 4.9). Under Assumption 1.1, and recalling the definition of ℓ^* from (2.1), $\tilde{L}^*$ is the differential operator acting on $f \in C^2(E)$ by

$$\tilde{L}^* f(x) = \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2}{\partial x_i\, \partial x_j}(c_{ij}(x) f(x)) - \sum_{i=1}^d \frac{\partial}{\partial x_i}((c(x)\nabla\ell^*(x))_i f(x)).$$

Assume that X is positive recurrent under $(\mathbb{P}_x^*)_{x\in E}$ and normalize $\tilde{\eta}^*$ so that $\int_E \tilde{\eta}^*(y)\,dy = 1$. By the ergodic theorem ([25], Theorem 4.9.9) it follows that for any bounded measurable function $f : E \mapsto \mathbb{R}$,

$$\lim_{t\uparrow\infty} \mathbb{E}_x^{\mathbb{P}^*}[f(X_t)] = \int_E f(y)\tilde{\eta}^*(y)\,dy. \tag{3.2}$$

Since $\tilde{\eta}^*$ is a probability density, for any $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset E$ such that

$$\int_{K_\varepsilon^c} \tilde{\eta}^*(y)\,dy \leq \varepsilon.$$

Thus, taking $f_\varepsilon(x) = \mathbb{I}_{K_\varepsilon^c}(x)$ in (3.2), the continuity of X and $\mathbb{P}^*[\zeta < \infty] = 0$ imply that $(X_t)_{t\geq 0}$ is $\mathbb{P}_x^*$-tight for any $x \in E$.

As for the reverse implication, assume for some $x \in E$ that $(X_t)_{t\geq 0}$ is $\mathbb{P}_x^*$-tight in E, and for each ε let $K_\varepsilon \subset E$ be a compact set such that

$$\inf_{t\geq 0} \mathbb{P}_x^*[X_t \in K_\varepsilon] \geq 1-\varepsilon. \tag{3.3}$$

Under Assumption 1.1 there are only three possibilities for the coordinate process X under $(\mathbb{P}_x^*)_{x\in\hat{E}}$ ([25], Section 2.2.8):

(1) X is transient: for all $x \in E$ and $n \in \mathbb{N}$, $\mathbb{P}_x^*[X \text{ is eventually in } E_n^c] = 1$;
(2) X is null recurrent: X is recurrent and for any $\phi \in C^2(E)$, $\phi > 0$ such that $\tilde{L}^*\phi = 0$, $\int_E \phi(y)\,dy = \infty$;
(3) X is positive recurrent: X is recurrent but not null recurrent.

Clearly, if $(X_t)_{t\geq t_0}$ is $\mathbb{P}_x^*$-tight in E for some $x \in E$, then X cannot be transient. Furthermore, if X were null recurrent, then for each $x \in E$ and any compact set $K \subset E$ it would follow that ([25], Theorem 4.9.5)

$$\lim_{t\uparrow\infty} \frac{1}{t}\int_0^t \mathbb{P}_x^*[X_s \in K]\,ds = 0.$$

But, by the assumption of tightness, for the compact set $K_\varepsilon \subset E$ appearing in (3.3),

$$\liminf_{t\uparrow\infty} \frac{1}{t}\int_0^t \mathbb{P}_x^*[X_s \in K_\varepsilon]\,ds \geq (1-\varepsilon).$$

Therefore, X cannot be null-recurrent. Thus X is positive recurrent under $(\mathbb{P}_x^*)_{x\in E}$. □

The following result is useful when point-wise estimates for η^* are available.

PROPOSITION 3.4. *Let Assumption 1.1 hold, and let ℓ^* be as in (2.1). If $\lambda^* > 0$, $\mathbb{P}^*[\zeta < \infty] = 0$ and*

$$\lim_{n\uparrow\infty} \inf_{x\in E_n^c} \frac{1}{2}\nabla\ell^*(x)'c(x)\nabla\ell^*(x) \geq \lambda^*, \tag{3.4}$$

then $\mathbb{P}^* \in \Pi^*$.

REMARK 3.5. If c is uniformly elliptic on E, and E is bounded with a smooth boundary, λ^* corresponds to the principal eigenvalue for L acting on functions η which vanish on ∂E. Since $(e^{\lambda^* t}\eta^*(X_t))^{-1}$ is a $\mathbb{P}^*$-supermartingale, it follows that $\mathbb{P}^*[\zeta < \infty] = 0$. Furthermore, Hopf's lemma asserts that $\nabla\eta^*$ does not vanish on ∂E, so (3.4) holds as well; indeed, the quantity on the left-hand side is unbounded from above.

PROOF OF PROPOSITION 3.4. That $\mathbb{P}^* \in \Pi$ follows by Proposition 3.2. Recall that $\eta^*(x_0) = 1$. Now,

$$\begin{aligned}\frac{1}{t}\ell^*(X_t) &= \frac{1}{t}\int_0^t \left(\frac{1}{2}\nabla \ell^*(X_s)' c(X_s) \nabla \ell^*(X_s) - \lambda^*\right) ds \\ &\quad + \frac{1}{t}\int_0^t \nabla \ell^*(X_s)'\sigma(X_s)\, dW_s^{\mathbb{P}^*},\end{aligned} \tag{3.5}$$

where $W^{\mathbb{P}^*}$ is a Brownian motion under $\mathbb{P}^*$. By (3.4), there is a $\tilde{\lambda} > 0$ such that, for n large enough,

$$\begin{aligned}&\int_0^t \nabla \ell^*(X_s)' c(X_s) \nabla \ell^*(X_s)\, ds \\ &\quad \geq \tilde{\lambda}\int_0^t \mathbb{I}_{\{X_s \in E_n^c\}}\, ds + \int_0^t \nabla \ell^*(X_s)' c(X_s) \nabla \ell^*(X_s) \mathbb{I}_{\{X_s \in E_n\}}\, ds.\end{aligned} \tag{3.6}$$

Under Assumption 1.1, X is either positive recurrent, null recurrent or transient under $(\mathbb{P}_x^*)_{x\in E}$. If X is positive recurrent, then, since $\lambda^* > 0$ implies that η^* is not identically constant, it follows that ([25], Theorem 4.9.5) for n large enough

$$\lim_{t\uparrow\infty} \int_0^t \nabla \ell^*(X_s)' c(X_s) \nabla \ell^*(X_s) \mathbb{I}_{\{X_s \in E_n\}}\, ds = \infty, \qquad \mathbb{P}^*\text{-a.s.}$$

Similarly, if X is either null recurrent or transient it follows that (again, by [25], Theorem 4.9.5)

$$\lim_{t\uparrow\infty} \tilde{\lambda}\int_0^t \mathbb{I}_{\{X_s \in E_n^c\}}\, ds = \infty, \qquad \mathbb{P}^*\text{-a.s.}$$

Using (3.6) it thus holds in each case

$$\lim_{t\uparrow\infty} \int_0^t \nabla \ell^*(X_s)' c(X_s) \nabla \ell^*(X_s)\, ds = \infty, \qquad \mathbb{P}^*\text{-a.s.}$$

Let $M = \int_0^\cdot \nabla \ell^*(X_s)'\sigma(X_s)\, dW_s^{\mathbb{P}^*}$, so that

$$[M, M] = \int_0^\cdot \nabla \ell^*(X_s)' c(X_s) \nabla \ell^*(X_s)\, ds.$$

By the Dambins, Dubins and Schwarz theorem ([15], Theorem 3.4.6), there exists a standard Brownian motion (under $\mathbb{P}^*$) B such that $M = B_{[M,M]_\cdot}$. Therefore, one can write (3.5) as

$$\frac{1}{t}\ell^*(X_t) = -\lambda^* + \frac{[M, M]_t}{2t}\left(1 + 2\frac{B_{[M,M]_t}}{[M, M]_t}\right).$$

By the strong law of large numbers,

$$\lim_{t\uparrow\infty} \frac{B_{[M,M]_t}}{[M, M]_t} = 0, \qquad \mathbb{P}^*\text{-a.s.,}$$

which means that

$$\liminf_{t\uparrow\infty} \frac{1}{t}\ell^*(X_t) \geq -\lambda^* + \liminf_{t\uparrow\infty} \frac{[M,M]_t}{2t}, \qquad \mathbb{P}^*\text{-a.s.} \tag{3.7}$$

If X is positive recurrent under $\mathbb{P}^*$, then $\mathbb{P}^* \in \Pi^*$ as shown in Proposition 3.3 and Remark 2.2. Otherwise, note that because of (3.4), for any $\delta > 0$ and $n \in \mathbb{N}$ large enough,

$$\begin{aligned} -\lambda^* + \frac{[M,M]_t}{2t} &\geq -\delta \frac{1}{t}\int_0^t \mathbb{I}_{\{X_s \in E_n^c\}}\,ds - \lambda^* \frac{1}{t}\int_0^t \mathbb{I}_{\{X_s\in E_n\}}\,ds \\ &\geq -\delta - \lambda^* \frac{1}{t}\int_0^t \mathbb{I}_{\{X_s \in E_n\}}\,ds. \end{aligned}$$

Now, if X is null-recurrent under $\mathbb{P}^*$, then from [25], Theorem 4.9.5, it follows that

$$\lim_{t\uparrow\infty} \frac{1}{t}\int_0^t \mathbb{I}_{\{X_s\in E_n\}}\,ds = 0, \qquad \mathbb{P}^*\text{-a.s.}$$

proving, in view of (3.7), that $\mathbb{P}^* \in \Pi^*_{\text{a.s.}}$, and hence $\mathbb{P}^* \in \Pi^*$. Clearly,

$$\{X \text{ eventually in } E_n^c\} \subseteq \left\{\lim_{t\uparrow\infty} \frac{1}{t}\int_0^t \mathbb{I}_{\{X_s\in E_n\}}\,ds = 0\right\}.$$

Therefore, if X is transient it follows that $\mathbb{P}^* \in \Pi^*$. $\square$

Another result giving a condition on whether $\mathbb{P}^* \in \Pi^*$ based on the tail-decay of the distribution of ζ under $\mathbb{Q}$ will be established.

PROPOSITION 3.6. *Let Assumption 1.1 hold. If* $\mathbb{P}^*[\zeta < \infty] = 0$ *and*

$$\liminf_{t\uparrow\infty}\left(-\frac{1}{t}\log \mathbb{Q}[\zeta > t]\right) \geq \lambda^*, \tag{3.8}$$

then $\mathbb{P}^* \in \Pi^*$.

When $\lambda^* = 0$ the fact that $\mathbb{Q}[\zeta > t] \leq 1$ immediately yields that $\mathbb{P}^* \in \Pi^*$.

COROLLARY 3.7. *Let Assumption 1.1 hold. If* $\lambda^* = 0$ *and* $\mathbb{P}^*[\zeta < \infty] = 0$ *then* $\mathbb{P}^* \in \Pi^*$.

PROOF OF PROPOSITION 3.6. That $\mathbb{P}^* \in \Pi$ follows by Proposition 3.2. Also, by Proposition 3.2, using the fact that $V_t^* = \exp(\lambda^* t)\eta^*(X_t)$ for $t \in \mathbb{R}_+$,

$$\log\left(\mathbb{E}^{\mathbb{P}^*}\left[\frac{1}{\eta^*(X_t)}\right]\right) = \lambda^* t + \log(\mathbb{Q}[\zeta > t]) - \log \eta^*(x_0).$$

Thus, (3.8) implies

$$\limsup_{t\uparrow\infty}\left(\frac{1}{t}\log\left(\mathbb{E}^{\mathbb{P}^*}\left[\frac{1}{\eta^*(X_t)}\right]\right)\right) \leq 0. \tag{3.9}$$

Now, by Chebyshev's inequality, for each $\epsilon > 0$,

$$\begin{aligned}\frac{1}{t}\log\left(\mathbb{P}^*\left[\frac{1}{t}\log\eta^*(X_t) \le -\epsilon\right]\right) &= \frac{1}{t}\log\left(\mathbb{P}^*\left[\frac{1}{\eta^*(X_t)} \ge \exp(\epsilon t)\right]\right) \\ &\le \frac{1}{t}\log\left(\exp(-\epsilon t)\mathbb{E}^{\mathbb{P}^*}\left[\frac{1}{\eta^*(X_t)}\right]\right) \\ &= -\epsilon + \frac{1}{t}\log\left(\mathbb{E}^{\mathbb{P}^*}\left[\frac{1}{\eta^*(X_t)}\right]\right).\end{aligned}$$

In conjunction with (3.9), this gives

$$\limsup_{t\uparrow\infty}\left(\frac{1}{t}\log\left(\mathbb{P}^*\left[\frac{1}{t}\log\eta^*(X_t) \le -\epsilon\right]\right)\right) \le -\epsilon,$$

which implies, in particular, that

$$\lim_{t\uparrow\infty}\mathbb{P}^*\left[\frac{1}{t}\log\eta^*(X_t) \le -\epsilon\right] = 0.$$

Since this is true for all $\epsilon > 0$, it follows that $\mathbb{P}^* \in \Pi^*$. □

REMARK 3.8. From [25], Theorem 4.4.4 (note that there, λ_c is used in place of $-\lambda^*$),

$$-\lambda^* = \lim_{n\uparrow\infty}\lim_{t\uparrow\infty}\frac{1}{t}\log\mathbb{Q}[\zeta_n > t].$$

Since $\mathbb{Q}[\zeta_n > t] \le \mathbb{Q}[\zeta > t]$ it holds that

$$\lambda^* + \liminf_{t\uparrow\infty}\frac{1}{t}\log\mathbb{Q}[\zeta > t] \ge 0.$$

In particular, (3.8) is really equivalent to

$$\lim_{t\uparrow\infty}\left(\frac{1}{t}\log\mathbb{Q}[\zeta > t]\right) = -\lambda^*.$$

4. Connections with optimal arbitrages. In [6], and quite close to the setting considered here, the authors treat the problem of *optimal arbitrage* on a given finite time horizon. We briefly mention the main points below, sending the interested reader to [6] for a more in-depth treatment.

Consider a class of probabilities $(\mathbb{P}_x)_{x\in E}$ on $(\Omega, \mathcal{F}_\infty)$ under which the coordinate process X has Markovian structure, and with the property that $\mathbb{P}_x \ll_{\text{loc}} \mathbb{Q}_x$ holds for all $x \in E$. Define a function $U : \mathbb{R}_+ \times E \mapsto [0, 1]$ via the following recipe: for $(T, x) \in \mathbb{R}_+ \times E$, set

$$1/U(T, x) = \sup\{v \in \mathbb{R}_+ \mid \exists V \in \mathcal{V} \text{ such that } \mathbb{P}_x[V_T \ge v] = 1\}.$$

In words, $1/U(T,x)$ is the maximal capital that one can realize at time T starting from unit initial capital when the market configuration at the initial time is $x \in E$. Equivalently, $U(T,x)$ is the minimal capital required in order to ensure at least one unit of wealth at time T when the market configuration at the initial time is $x \in E$. Arbitrage on the finite time interval $[0,T]$ exists if and only if $U(T,x) < 1$. Using the notation of the present paper and recalling that for $x_0 \in E$ the subscripts in the probability measures are dropped, it is shown in [6] that arbitrage over a time horizon $[0,T]$ exists if and only if $\mathbb{Q}[\zeta > T] < 1$. Furthermore, it is established that $U(T,x) = \mathbb{Q}_x[\zeta > T]$ for all $(T,x) \in \mathbb{R}_+ \times E$, and that the optimal arbitrage exists and is given by $V^T = (V_t^T)_{t\in[0,T]}$, where

$$V_t^T = \frac{\mathbb{Q}[\zeta > T \mid \mathcal{F}_t]}{\mathbb{Q}[\zeta > T]} = \frac{U(T-t, X_t)}{U(T,x_0)} \qquad \text{for } t \in [0,T]. \tag{4.1}$$

Observe that the optimal arbitrage V^T in (4.1) is normalized so that $V_0^T = 1$. In [6], the normalization is such that the *terminal* value of the optimal relative arbitrage is unit; as already mentioned, in that case $U(T,x_0)$ is the minimal capital required at time zero to ensure a unit of capital at time T.

REMARK 4.1. In [6], Sections 10–12, the problem of optimal arbitrage is specified to when E is the interior of the simplex on $\mathbb{R}^{d-1}$, that is,

$$E = \left\{ x \in \mathbb{R}^{d-1} \,\Big|\, \min_{i=1,\dots,d-1} x_i > 0, \text{ and } \sum_{i=1}^{d-1} x_i < 1 \right\}.$$

(In fact, in [6] the simplex $\Delta_+^d := \{x \in \mathbb{R}^d \mid \min_{i=1,\dots,d} x_i > 0, \text{ and } \sum_{i=1}^d x_i = 1\}$ is used. Since $x = (x_i)_{i=1,\dots,d-1} \in E \iff (x, 1 - \sum_{i=1}^{d-1} x_i) \in \Delta_+^d$, E is in trivial one-to-one correspondence with Δ_+^d. For the purposes of this paper, the state space has to be an open set; for this reason, E as defined above will be used throughout.) The interpretation is that the coordinate process X represents the *relative* capitalizations of stocks, and the corresponding optimal arbitrages are in fact *relative* arbitrages with respect to the market portfolio. In principle, the treatment of [6] does not really utilize the special structure of the simplex; therefore, the general case is considered.

It is natural to study the asymptotic behavior of these optimal arbitrages as the time-horizon becomes arbitrarily large. It is shown below that, under suitable assumptions, the sequence of wealth processes $(V^T)_{T\in\mathbb{R}_+}$ (parameterized via their maturity) converges to the robust asymptotically growth-optimal wealth process.

A tool in proving this convergence will be Proposition 3.2. In view of that result, it follows that if $\lambda^* > 0$ and $\mathbb{P}_x^*[\zeta < \infty] = 0$ for each $x \in E$, arbitrage occurs if and only if the local $\mathbb{P}_x^*$-martingale $1/V^*$ is a strict local $\mathbb{P}_x^*$-martingale in the terminology of [4]. If $1/V^*$ is a $\mathbb{P}_x^*$-martingale, then, even though arbitrage does not exist, it is still possible to construct robust growth-optimal trading strategies, as seen in Example 6.7.

REMARK 4.2. Equation (3.1) holds when ζ, η^*, V^* and $\mathbb{P}_x^*$ are replaced by ζ_n, η_n^*, V_n^* and $\mathbb{P}_{x,n}^*$, where these quantities appear in the proof of Theorem 2.1. In this case, and when $E = (0,\infty)^d$, conditioning upon $\zeta_n > T$ can be interpreted as forcing a diversity condition in the market since $X \in E_n$ implies there exists some $\delta > 0$ such that no one asset's relative capitalization is above $1 - \delta$. Conditioned upon never exiting E_n for $n \in \mathbb{N}$, the robust growth optimal wealth process V_n^* is thus identified with the long-run version of the arbitrage constructed in [21].

Equation (3.1) may be re-written as

$$e^{\lambda^* T}\mathbb{Q}_x[\zeta > T] = \eta^*(x)\mathbb{E}_x^{\mathbb{P}^*}\left[\frac{1}{\eta^*(X_T)}\right]. \tag{4.2}$$

Thus, to study the asymptotic behavior of V_t^T as $T \uparrow \infty$ in (4.1), it is necessary to study the long-time (as $T \uparrow \infty$) behavior of $\mathbb{E}_x^{\mathbb{P}^*}[(\eta^*(X_T))^{-1}]$. Assume that X is positive recurrent (or, equivalently, tight) under $(\mathbb{P}_x^*)_{x\in E}$ with invariant probability measure μ. Under Assumption 1.1, [22], Theorem 1.2 (iii), equations (3.29), (3.30) extends the ergodic result in (3.2) to functions f which are integrable with respect to μ. Thus, for all positive measurable functions $f : E \mapsto \mathbb{R}$,

$$\lim_{T\uparrow\infty} \mathbb{E}_x^{\mathbb{P}^*}[f(X_T)] = \int_E f\,d\mu, \tag{4.3}$$

and this limit is the same for all $x \in E$. This yields the following proposition:

PROPOSITION 4.3. *Let Assumption 1.1 hold. Suppose that $\eta^* \in H_{\lambda^*}$ is such that*

$$\lim_{n\uparrow\infty} \sup_{x\in E_n^c} \eta^*(x) = 0. \tag{4.4}$$

Then $\mathbb{P}_x^*[\zeta < \infty] = 0$ for all $x \in E$, and the following are equivalent:

(1) $\lim_{T\uparrow\infty} e^{\lambda^* T}\mathbb{Q}_x[\zeta > T] = \kappa\eta^*(x)$ *for all $x \in E$ where $\kappa > 0$ does not depend upon x;*

(2) $\limsup_{T\uparrow\infty} e^{\lambda^* T}\mathbb{Q}_x[\zeta > T] < \infty$ *for some $x \in E$;*

(3) *X is positive recurrent under $(\mathbb{P}_x^*)_{x\in E}$ and $\int_E (\eta^*)^{-1}\,d\mu < \infty$ where μ is the invariant measure for X.*

REMARK 4.4. Note that (3) implies (1) even if (4.4) does not hold. Note also that, by Example 4.7 below, some condition like (4.4) is necessary for (1), (2) and (3) to be equivalent.

PROOF OF PROPOSITION 4.3. Let $x \in E$. Note that $(e^{\lambda^* t}\eta^*(X_t))^{-1}$ is a $\mathbb{P}_x^*$ super-martingale. By (4.4), if $\mathbb{P}_x^*[\zeta < \infty] > 0$, then the super-martingale property would be violated. Thus an explosion cannot occur.

Regarding the equivalences, (1) ⇒ (2) is trivial. As for (2) ⇒ (3), if (2) holds, then by (4.2) it follows that there is some $T_0 \geq 0$ such that

$$\sup_{T \geq T_0} \mathbb{E}_x^{\mathbb{P}^*}\left[\frac{1}{\eta^*(X_T)}\right] < \infty.$$

Therefore, (4.4) yields that $(X_T)_{T \geq T_0}$ form a $\mathbb{P}_x^*$ tight family of random variables for each $x \in E$. By Proposition 3.3 it follows that X is positive recurrent under $(\mathbb{P}_x^*)_{x \in E}$; hence, (4.3) gives

$$\int_E \frac{1}{\eta^*} \, d\mu = \lim_{T \uparrow \infty} \mathbb{E}_x^{\mathbb{P}^*}\left[\frac{1}{\eta^*(X_T)}\right] \leq \limsup_{T \uparrow \infty} \mathbb{E}_x^{\mathbb{P}^*}\left[\frac{1}{\eta^*(X_T)}\right] < \infty$$

proving (3). Implication (3) ⇒ (1) follows by applying (4.3) to $1/\eta^*$ and using (4.2). □

The following is the main result of the section.

THEOREM 4.5. *Suppose that $\eta^* \in H_{\lambda^*}$ is such that $\mathbb{P}^*[\zeta < \infty] = 0$ and that condition* (1) *in Proposition* 4.3 *holds. Fix $\mathbb{P} \in \Pi$. Then, for any fixed $t \in \mathbb{R}_+$,*

$$\mathbb{P}\text{-}\lim_{T \to \infty} \sup_{\tau \in [0,t]} |V_\tau^T - V_\tau^*| = 0. \tag{4.5}$$

Additionally, for each $T \in \mathbb{R}_+$, let $(\vartheta_t^T)_{t \in [0,T]}$ be a predictable process such that

$$V^T = 1 + \int_0^\cdot V_t^T (\vartheta_t^T)' \, dX_t. \tag{4.6}$$

With ℓ^* as in (2.1) *and $\vartheta^* = \nabla \ell^*(X)$, it follows that, for any fixed $t \in \mathbb{R}_+$,*

$$\mathbb{P}\text{-}\lim_{T \to \infty} \int_0^t (\vartheta_\tau^T - \vartheta_\tau^*)' c(X_\tau)(\vartheta_\tau^T - \vartheta_\tau^*) \, d\tau = 0. \tag{4.7}$$

PROOF. Fix $t \in \mathbb{R}_+$. Equation (4.1), coupled with condition (1) in Proposition 4.3, implies that $\mathbb{P}\text{-}\lim_{T \to \infty} V_t^T = V_t^*$. Let $Z^T = (Z_\tau^T)_{\tau \in [0,t]}$ be defined via $Z_\tau^T := V^T / V^*$. The arguments used in the proof of Theorem 2.1 show that V^* is the numéraire portfolio in $\mathcal{V}$ under $\mathbb{P}^*$, that is, that Z^T is a nonnegative $\mathbb{P}^*$-supermartingale on $[0, t]$ for all $T \in (t, \infty)$. Then [16], Theorem 2.5, implies that $\mathbb{P}^*\text{-}\lim_{T \to \infty} \sup_{\tau \in [0,t]} |Z_\tau^T - 1| = 0$. Using the fact that $\mathbb{P}^*[\inf_{\tau \in [0,t]} V_\tau^* > 0] = 1$, it follows that $\mathbb{P}^*\text{-}\lim_{T \to \infty} \sup_{\tau \in [0,t]} |V_\tau^T - V_\tau^*| = 0$. Now, with $R^T = (R_\tau^T)_{\tau \in [0,t]}$ defined via

$$R^T = \int_0^\cdot (\vartheta_s^T - \vartheta_s^*)' \big(dX_s - c(X_s) \nabla \ell^*(X_s) \, ds\big),$$

it holds that $Z^T = 1 + \int_0^\cdot Z_s^T \, dR_s$. Invoking [16], Theorem 2.5, again yields $\mathbb{P}^*$-$\lim_{T \to \infty} [R^T, R^T]_t = 0$ for all $t \in \mathbb{R}_+$. As

$$[R^T, R^T]_t = \int_0^t (\vartheta_\tau^T - \vartheta_\tau^*)' c(X_\tau)(\vartheta_\tau^T - \vartheta_\tau^*) \, d\tau,$$

(4.7) follows, with $\mathbb{P}^*$ replacing $\mathbb{P}$ there.

Up to now, the validity of both (4.5) and (4.7), for the special case $\mathbb{P} = \mathbb{P}^* \in \Pi$ has been shown. For a general $\mathbb{P} \in \Pi$, the result follows by noting that $\mathbb{P}^*$ and $\mathbb{P}$ are equivalent on each $\mathcal{F}_{\zeta_n}$, $n \in \mathbb{N}$, and that $\lim_{n\to\infty} \mathbb{P}[\zeta_n > t] = 1$. Indeed, for any $\epsilon > 0$ pick $n_\epsilon \in \mathbb{N}$ large enough so that $\mathbb{P}[\zeta_{n_\epsilon} \le t] \le \epsilon/2$. Then pick $\delta_\epsilon > 0$ so that $\mathbb{P}[A] \le \epsilon/2$ holds whenever $A \in \mathcal{F}_{\zeta_{n_\epsilon}}$ and $\mathbb{P}^*[A] \le \delta_\epsilon$. Finally, pick $T_\epsilon \in \mathbb{R}_+$ large enough so that

$$\mathbb{P}^*\Big[\sup_{\tau\in[0,t]} |V_\tau^T - V_\tau^*| \ge \epsilon\Big] \le \delta_\epsilon$$

as well as

$$\mathbb{P}^*\bigg[\int_0^t (\vartheta_\tau^T - \vartheta_\tau^*)' c(X_\tau)(\vartheta_\tau^T - \vartheta_\tau^*)\, d\tau \ge \epsilon\bigg] \le \delta_\epsilon$$

holds whenever $T \ge T_\epsilon$. Therefore, for all $T \ge T_\epsilon$,

$$\begin{aligned}\mathbb{P}\Big[\sup_{\tau\in[0,t]} |V_\tau^T - V_\tau^*| \ge \epsilon\Big] &\le \mathbb{P}\Big[\sup_{\tau\in[0,\zeta_n\wedge t]} |V_\tau^T - V_\tau^*| \ge \epsilon\Big] + \mathbb{P}[\zeta_{n_\epsilon} \le t] \\ &\le \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

This establishes (4.5). Similarly, we establish (4.7). $\square$

REMARK 4.6. The result of Theorem 4.5 is expected to hold in greater generality than its assumptions suggest. It is conjectured that the results hold under Assumption 1.1, but it is an open question. See Example 6.4 in Section 6 for a potential counterexample. The next example shows that it can even hold when $\lambda^* = 0$.

EXAMPLE 4.7. Let $E = (0, \infty)$ and $c(x) = 1$ for $x \in E$. It is straightforward to check that

$$U(T, x) = \mathbb{Q}_x[\zeta > T] = 2\Phi\big(x/\sqrt{T}\big) - 1 \qquad \text{for } (T, x) \in \mathbb{R}_+ \times E,$$

where Φ is the cumulative distribution function of the standard normal law. With $x_0 = 1$, it follows that

$$V_t^T = \frac{2\Phi(X_t/\sqrt{T-t}) - 1}{2\Phi(1/\sqrt{T}) - 1} \qquad \text{for } t \in [0, T].$$

From this explicit formula it is straightforward that $\mathbb{P}\text{-}\lim_{T\to\infty} \sup_{\tau\in[0,t]} |V_\tau^T - X_\tau| = 0$ holds whenever $t \in \mathbb{R}_+$. Observe that $V^* = X$ exactly for the choice $\eta^*(x) = x$ corresponding to $\lambda^* = 0$, and $\mathbb{P}^*$ being the probability that makes X behave as a three-dimensional Bessel process. Remember that in this example the dimensionality of the set of principal eigenfunctions is two—the other one is $\eta \equiv 1$. It is interesting to note that the sequence (V^T) "chooses" to converge to the optimal strategy of the optimal probability $\mathbb{P}^*$ that satisfies $\mathbb{P}^* \in \Pi$.

As in [8], Section 5.1, for $T \in \mathbb{R}_+$ and $x \in E$, define the measure $\mathbb{P}_x^{\star,T}$ on $\mathcal{F}_T$ via

$$\mathbb{P}_x^{\star,T}[A] = \mathbb{Q}_x[A \mid \zeta > T] \qquad \text{for } A \in \mathcal{F}_T.$$

It is shown therein that, for each $t \in [0, T]$ and $x \in E$,

$$\frac{d\mathbb{P}_x^{\star,T}}{d\mathbb{Q}_x}\bigg|_{\mathcal{F}_t} = \frac{U(T-t, X_t)}{U(T, x)} \mathbb{I}_{\{\zeta > t\}}.$$

Furthermore, under the assumption $U \in C^{1,2}((0, T) \times E)$, the coordinate process X under $(\mathbb{P}_x^{\star,T})_{x \in E}$ has dynamics on $[0, T]$ of

$$\begin{aligned} dX_\tau &= c(X_\tau) \frac{\nabla_x U(T-\tau, X_\tau)}{U(T-\tau, X_\tau)} d\tau + \sigma(X_\tau)\, dW_\tau^{\mathbb{P}^{\star,T}} \\ &= c(X_\tau) \vartheta_\tau^T\, d\tau + \sigma(X_\tau)\, dW_\tau^{\mathbb{P}^{\star,T}} \end{aligned}$$

using the notation of (4.6) in Theorem 4.5. Assuming $\mathbb{P}_x^*[\zeta < \infty] = 0$, it follows that $\mathbb{P}_x^{\star,T}$ and $\mathbb{P}_x^*$ are equivalent on $\mathcal{F}_t$ for $t \in [0, T]$ with

$$\begin{aligned} \frac{d\mathbb{P}_x^{\star,T}}{d\mathbb{P}_x^*}\bigg|_{\mathcal{F}_t} = \exp\biggl(&-\frac{1}{2}\int_0^t (\vartheta_\tau^T - \vartheta_\tau^*)' c(X_\tau)(\vartheta_\tau^T - \vartheta_\tau^*)\, d\tau \\ &+ \int_0^t (\vartheta_\tau^T - \vartheta_\tau^*)' \sigma(X_\tau)\, dW_\tau^{\mathbb{P}^*}\biggr). \end{aligned} \tag{4.8}$$

Thus the results of Theorem 4.5 immediately imply the following:

PROPOSITION 4.8. *Suppose the hypotheses of Theorem 4.5 hold. Then, for any $t \in \mathbb{R}_+$, $\mathbb{P}_x^{\star,T}$ converges in variation norm to $\mathbb{P}_x^*$ on $\mathcal{F}_t$ as $T \uparrow \infty$.*

PROOF. The process on the right-hand side of (4.8) is the process $Z^T = V^T / V^*$ in the proof of Theorem 4.5. Since, for each $A \in \mathcal{F}_t$,

$$|\mathbb{P}_x^{\star,T}(A) - \mathbb{P}_x^*(A)| \le \mathbb{E}_x^{\mathbb{P}^*}[|Z_t^T - 1|],$$

the result follows from [16], Theorem 2.5(i). □

REMARK 4.9. In [24], a similar result to Proposition 4.8 is obtained, though not in the setting of convergence of relative arbitrages. Namely, it is assumed that

$$\lim_{T \uparrow \infty} \frac{\nabla_x U(T, x)}{U(T, x)} = \nabla \ell^*(x) \qquad \text{for } x \in E, \tag{4.9}$$

where the convergence takes place exponentially fast with rate λ^* and is uniform on compact subsets of E. Under this assumption, the measures $\mathbb{P}_x^{\star,T}$ are shown to weakly converge as $T \uparrow \infty$ to $\mathbb{P}_x^*$ on $\mathcal{F}_t$ for each $t \in \mathbb{R}_+$.

In the case where E is bounded with smooth boundary, and c is uniformly elliptic over E, (4.9) holds if there exists a function $H : E \mapsto \mathbb{R}$ such that, for each $i = 1, \ldots, d$,

$$\sum_{j=1}^{d} c_{ij}(x) \frac{\partial}{\partial_{x_j}} H(x) = f_i(x); \qquad f_i(x) := -\frac{1}{2} \sum_{j=1}^{d} \frac{\partial}{\partial_{x_j}} c_{ij}(x), \qquad i = 1, \ldots, d.$$

In vector notation, this gradient condition takes the form $\nabla H = c^{-1} f$, and f is the *Fichera drift* associated to $\mathbb{Q}$. Under this hypothesis, the measure $m(dx) = \exp(2H(x))\, dx$ is reversing for the transition probability function $\mathbb{Q}(t, x, \cdot)$, and the convergence result in (4.9) follows by representing $U(T, x) = \mathbb{Q}_x[\zeta > T]$ as an eigenfunction expansion where the underlying space is $L^2(E, m)$; see [24].

5. A thorough treatment of the one-dimensional case. This section considers the case $d = 1$, where $E = (\alpha, \beta)$ is a bounded interval. If $E = \mathbb{R}$, then $\lambda^* = 0$ holds by Proposition 1.7, because the coordinate process under $\mathbb{Q}$ is recurrent. If E is a half-bounded interval, it is possible for:

- $\lambda^* = 0$, even though there is explosion under $\mathbb{Q}$; see Example 4.7.
- $\lambda^* > 0$, even though there is no explosion under $\mathbb{Q}$; see Example 6.6 with $d = 1$.

Hence making a general statement connecting $\lambda^* > 0$ with explosion or nonexplosion under $\mathbb{Q}$ is difficult. Thus to enlighten the connections with relative arbitrages, the following will assumed throughout the section:

ASSUMPTION 5.1. Assumption 1.1 holds for $E = (\alpha, \beta)$ with $-\infty < \alpha < \beta < \infty$.

Under the validity of Assumption 5.1, results are provided that almost completely cover all the cases that can occur.

The first proposition establishes point-wise tests for c which yield $\lambda^* > 0$ or $\lambda^* = 0$. The second proposition gives integral tests which yield $\lambda^* > 0$ or $\lambda^* = 0$. Condition (5.11) is equivalent to the coordinate process X under $(\mathbb{Q}_x)_{x \in [\alpha, \beta]}$, exploding to both α, β with positive probability. Additionally, condition (5.11) not only yields $\lambda^* > 0$ but also that $\mathbb{P}^* \in \Pi^*_{\text{a.s.}}$ (and hence $\mathbb{P}^* \in \Pi^*$).

Recall the following facts regarding explosion, transience, recurrence and positive recurrence in the one-dimensional case under Assumption 5.1; see [25], Chapter 5.1:

- Since E is bounded the coordinate process X under $(\mathbb{Q}_x)_{x \in [\alpha, \beta]}$ is transient. Furthermore it explodes to α and/or β with positive probability if, for some $x_0 \in (\alpha, \beta)$,

$$\int_{\alpha}^{x_0} \frac{x - \alpha}{c(x)}\, dx < \infty \quad \text{and/or} \quad \int_{x_0}^{\beta} \frac{\beta - x}{c(x)}\, dx < \infty.$$

- The coordinate process X under $(\mathbb{P}_x^*)_{x\in(\alpha,\beta)}$ is recurrent if

$$\int_\alpha^{x_0} \frac{1}{(\eta^*(x))^2}\,dx = \infty \quad \text{and} \quad \int_{x_0}^{\beta} \frac{1}{(\eta^*(x))^2}\,dx = \infty. \tag{5.1}$$

  If either of the integrals in (5.1) are finite, then the coordinate process X is transient towards the endpoint with finite integral.
- The coordinate process X under $(\mathbb{P}_x^*)_{x\in(\alpha,\beta)}$ is positive recurrent if (5.1) holds and if

$$\int_\alpha^{\beta} \frac{(\eta^*(x))^2}{c(x)}\,dx < \infty. \tag{5.2}$$

PROPOSITION 5.2 (Pointwise result). *Let Assumption 5.1 hold. If*

$$\sup_{x\in(\alpha,\beta)} \frac{(x-\alpha)^2(\beta-x)^2}{c(x)} < \infty, \tag{5.3}$$

then $\lambda^* > 0$. *If*

$$\lim_{x\downarrow\alpha} \frac{(x-\alpha)^2}{c(x)} = \infty \quad or \quad \lim_{x\uparrow\beta} \frac{(\beta-x)^2}{c(x)} = \infty, \tag{5.4}$$

then $\lambda^* = 0$.

REMARK 5.3. We thank an anonymous referee for suggesting the short, self-contained proof to Proposition 5.2 below.

PROOF OF PROPOSITION 5.2. By [25], Theorem 4.4.5 (note that λ_c from [25], Theorem 4.4.5, is equal to $-\lambda^*$ here), λ^* admits the following variational representation:

$$\lambda^* = \sup_{\substack{\eta\in C^2(\alpha,\beta)\\ \eta>0}} \inf_{x\in(\alpha,\beta)} \frac{-c(x)\eta''(x)}{2\eta(x)}, \tag{5.5}$$

where the $'$ symbol is used to signify a derivative with respect to x (and not to denote matrix transposition as it was used in previous sections).

Let $\eta(x) = \sqrt{(x-\alpha)(\beta-x)}$. If (5.3) holds, then

$$\inf_{x\in(\alpha,\beta)} \frac{-c(x)\eta''(x)}{2\eta(x)} = \inf_{x\in(\alpha,\beta)} \frac{(\beta-\alpha)^2 c(x)}{8(x-\alpha)^2(\beta-x)^2} > 0$$

and hence $\lambda^* > 0$.

Now, assume (5.4) holds for $x\downarrow\alpha$. The proof for $x\uparrow\beta$ is the same. Let $a>\alpha$, and consider the case when $c\equiv 1$ and $E=(\alpha,a)$. Since Assumption 1.1 clearly holds in this setting, let λ_a^* represent the generalized principle eigenvalue. Set $\lambda_a = \frac{\pi^2}{2(a-\alpha)^2}$ and consider the function $\phi(x) = \sin(\sqrt{2\lambda_a}(x-\alpha))$. It can be directly verified that $-\frac{1}{2}\phi''(x) = \lambda_a \phi(x)$ and that both (5.1) and (5.2) hold [with $c \equiv 1$, β replaced by a and $x_0 \in (\alpha, a)$]. Thus, Proposition 1.7 implies that $\lambda_a^* = \lambda_a = \frac{\pi^2}{2(a-\alpha)^2}$. Plugging this into (5.5) (again, for $c \equiv 1$ and β replaced by a) gives for all $\eta \in C^2(\alpha, a)$, $\eta > 0$

$$\inf_{x \in (\alpha, a)} \frac{-\eta''(x)}{2\eta(x)} \leq \frac{\pi^2}{2(a-\alpha)^2}. \tag{5.6}$$

Now, for the general case, it is clearly true that $\lambda^* \geq 0$. Assume by way of contradiction that $\lambda^* > 0$. By (5.5) it follows that there exists a $\tilde{\lambda} > 0$ and $\eta \in C^2(\alpha, \beta)$, $\eta > 0$ such that

$$\tilde{\lambda} \leq \inf_{x \in (\alpha, \beta)} \frac{-c(x)\eta''(x)}{2\eta(x)}. \tag{5.7}$$

Let $M > 0$. Since (5.4) holds, there is an α_M such that for $x \in (\alpha, \alpha_M)$,

$$M \leq \frac{(x-\alpha)^2}{c(x)} \leq \frac{(\alpha_M - \alpha)^2}{c(x)}. \tag{5.8}$$

Together, (5.7) and (5.8) give

$$\frac{\tilde{\lambda} M}{(\alpha_M - \alpha)^2} \leq \inf_{x \in (\alpha, \alpha_M)} \frac{\tilde{\lambda}}{c(x)} \leq \inf_{x \in (\alpha, \alpha_M)} \frac{-\eta''(x)}{2\eta(x)}. \tag{5.9}$$

By (5.6) with $a = \alpha_M$, it follows that

$$\inf_{x \in (\alpha, \alpha_M)} \frac{-\eta''(x)}{2\eta(x)} \leq \frac{\pi^2}{2(\alpha_M - \alpha)^2}. \tag{5.10}$$

Combining (5.9) and (5.10) gives

$$\frac{\tilde{\lambda} M}{(\alpha_M - \alpha)^2} \leq \frac{\pi^2}{2(\alpha_M - \alpha)^2}$$

or that $M \leq \pi^2/(2\tilde{\lambda})$. This is a contradiction since M was arbitrary. Thus $\lambda^* = 0$. □

The proof of the following result is lengthy and technical; for this reason, it is delayed until Section 7.

PROPOSITION 5.4 (Integral result). *Let Assumption 5.1 hold. If*

$$\int_\alpha^\beta \frac{(x-\alpha)(\beta - x)}{c(x)}\,dx < \infty, \tag{5.11}$$

then:

(1) $\lambda^* > 0$.

(2) *For any* $\eta^* \in H_{\lambda^*}$, $\lim_{x\downarrow\alpha} \eta^*(x) = 0 = \lim_{x\uparrow\beta} \eta^*(x)$.

(3) *For any* $\eta^* \in H_{\lambda^*}$, *the coordinate process* X *under* $(\mathbb{P}^*_x)_{x\in(\alpha,\beta)}$ *is positive recurrent and so by Proposition* 1.7, η^* *is unique up to multiplication by a positive constant.*

(4) $\mathbb{P}^* \in \Pi^*_{\text{a.s.}}$ *and hence* $\mathbb{P}^* \in \Pi^*$.

If, for some $a \in (\alpha, \beta)$,

$$\int_\alpha^a \frac{(x-\alpha)^2}{c(x)}\,dx = \infty \quad \textit{or} \quad \int_a^\beta \frac{(\beta - x)^2}{c(x)}\,dx = \infty, \tag{5.12}$$

then $\lambda^* = 0$.

6. Examples.

6.1. *One-dimensional examples*. The following examples display a variety of outcomes regarding η^* and $\mathbb{P}^*$. Proofs of all the statements follow from Propositions 5.2, 5.4 and/or from the tests for recurrence, null recurrence or positive recurrence under $\mathbb{P}^*$ given in equations (5.1) and (5.2) in conjunction with Proposition 1.7.

The first three Examples 6.1–6.3, all consider $E = (0, 1)$ and display the different possible outcomes depending upon the rate of decay (to zero) of c at 0 and 1. The fourth Example 6.4 shows that it is possible that $\lambda^* > 0$, $\mathbb{P}^* \in \Pi^*_{\text{a.s.}}$, and the coordinate process is positive recurrent under $\mathbb{P}^*$, while $(\eta^*)^{-1}$ fails to be integrable with respect to the invariant measure under $\mathbb{P}^*$; thus, the results of Section 4, and in particular Theorem 4.5, are not applicable. Finally, Example 6.5 shows that even if $\lambda^* > 0$, there is no explosion of X under $\mathbb{P}^*$ and η^* is unique (up to multiplication), no conclusion can be made as to if $\mathbb{P}^* \in \Pi^*_{\text{a.s.}}$ or $\mathbb{P}^* \in \Pi^*$, based on results of this article.

EXAMPLE 6.1. Let $E = (0, 1)$ and $c(x) = x(1-x)$. Then:

- Equation (5.11) holds and so the results of Proposition 5.4 follow.
- $\eta^*(x) = x(1-x)$, $\lambda^* = 1$.
- Equation (4.4) holds as well as condition (3) in Proposition 4.3. Thus the results of Theorem 4.5 and Proposition 4.8 follow.

EXAMPLE 6.2. Let $E = (0, 1)$ and $c(x) = x^2(1-x)^2$. Then:

- $\mathbb{Q}[\zeta < \infty] = 0$.
- $\eta^*(x) = \sqrt{x(1-x)}$, $\lambda^* = 1/8$.
- The coordinate process X is null recurrent under $(\mathbb{P}^*_x)_{x\in E}$; however, $\mathbb{P}^* \in \Pi^*_{\text{a.s.}}$.

Note that there is a multidimensional generalization of this in Example 6.7.

EXAMPLE 6.3. Let $E = (0, 1)$ and $c(x) = x^3(1-x)^3$. Then:

- $\mathbb{Q}[\zeta < \infty] = 0$.
- $\lambda^* = 0$ by either Proposition 5.2 or 5.4.
- η^* can be any affine function $\alpha + \beta x$ such that $\eta^* > 0$ on $(0, 1)$. For any such η^*, $\mathbb{P}^* \in \Pi^*_{\text{a.s.}}$.

EXAMPLE 6.4. Let $E = (0, \hat{x})$, where

$$\hat{x} := \min\left\{x > 0 \,\middle|\, \int_0^x \log(-\log(y))\,dy = 0\right\} \approx 0.75.$$

Furthermore, let $c : E \mapsto \mathbb{R}_+$ be defined via

$$c(x) = -2x\log(x)\int_0^x \log(-\log(y))\,dy \qquad \text{for } x \in E.$$

Then:

- Equation (5.11) holds and so the results of Proposition 5.4 follow.
- $\eta^*(x) = \int_0^x \log(-\log(y))\,dy$, $\lambda^* = 1$.
- $(\eta^*)^{-1}$ is not integrable with respect to the invariant measure for $\mathbb{P}^*$.

EXAMPLE 6.5. Let $E = (0, \infty)$ and

$$c(x) = \frac{4(x^{3/2}\int_0^x \cos(y^{-1/2})\,dy + 4x^2 - x^{5/2})}{2 - \sin(x^{-1/2})} \qquad \text{for } x \in E.$$

Then:

- $\mathbb{Q}[\zeta < \infty] = 0$.
- $\eta^*(x) = \int_0^x \cos(y^{-1/2})\,dy + 4\sqrt{x} - x$, $\lambda^* = 1$.
- The coordinate process X under $(\mathbb{P}^*_x)_{x\in E}$ is null-recurrent. No conclusions as to whether or not $\mathbb{P}^* \in \Pi^*_{\text{a.s.}}$ or Π^* can be drawn based on the results of the paper (see Propositions 3.4 and 3.2) since

$$\limsup_{x\downarrow 0}\left(\frac{1}{2}\nabla\ell^*(x)'c(x)\nabla\ell^*(x) - \lambda^*\right) = 0,$$
$$\liminf_{x\downarrow 0}\left(\frac{1}{2}\nabla\ell^*(x)'c(x)\nabla\ell^*(x) - \lambda^*\right) = -\frac{2}{3}.$$

6.2. *Multi-dimensional examples*. The following examples show that the optimal η^* need not vanish on the boundary of E even when E is bounded, and that strictly positive asymptotic growth rate is possible even when $\mathbb{Q}[\zeta < \infty] = 0$.

EXAMPLE 6.6 (Correlated geometric Brownian motion). Let $E = (0, \infty)^d$, and define the matrix c via

$$c_{ij}(x) = x_i x_j A_{ij}, \qquad 1 \le i, j \le d,$$

where A is a symmetric, strictly positive definite $d \times d$ matrix. Define the vectors $\hat{A}, \hat{B} \in \mathbb{R}^d$ by

$$\hat{A}_i = A_{ii} \qquad (1 \le i \le d), \qquad \hat{B} = \tfrac{1}{2} A^{-1} \hat{A}.$$

Then

$$\eta^*(x) = \prod_{i=1}^{d} x_i^{\hat{B}_i}, \qquad \lambda^* = \frac{1}{8} \hat{A}' A^{-1} \hat{A}, \tag{6.1}$$

and $\mathbb{P}^* \in \Pi^*_{\text{a.s.}}$.

To see the validity of the above claims, set η, λ as the respective right-hand sides of (6.1). A straightforward calculation shows that $L\eta = -\lambda\eta$ and hence that $\lambda^* \ge \lambda$. Set $(\mathbb{P}^\eta_x)_{x \in \hat{E}}$ as the solution to the generalized martingale problem for L^η, as in (1.7) and $\mathbb{P}^\eta = \mathbb{P}^\eta_{x_0}$. The coordinate process X under $\mathbb{P}^\eta$ is given by $X = \exp(aW)$ where a is the unique positive definite square root of A and W a Brownian motion under $\mathbb{P}^\eta$. Thus, under $\mathbb{P}^\eta$,

$$\frac{1}{t} \log \eta(X_t) = \frac{1}{t} \hat{B}' a W_t.$$

The strong law of large numbers for Brownian motion gives that $\mathbb{P}^\eta \in \Pi^*_{\text{a.s.}}$. Theorem 2.1 then yields $\lambda^* \le \sup_{V \in \mathcal{V}} g(V; \mathbb{P}^\eta) \le \lambda$, and hence $\lambda^* = \lambda$, $\eta^* = \eta$ and $\mathbb{P}^* = \mathbb{P}^\eta$.

EXAMPLE 6.7 (Relative capitalizations of a correlated geometric Brownian motion). For $d \ge 2$, let

$$E = \left\{ x \in \mathbb{R}^{d-1} \mid \min_{i=1,\dots,d-1} x_i > 0; \sum_{i=1}^{d-1} x_i < 1 \right\}.$$

For the matrix A of Example 6.6, define the $(d-1)$-dimensional square matrix $\mathcal{A}$ by

$$\mathcal{A}_{ij} = A_{ij} - A_{id} - A_{jd} + A_{dd}, \qquad 1 \le i, j \le d-1,$$

and the matrix c via

$$c_{ij}(x) = x_i x_j \big(\mathcal{A}_{ij} - (\mathcal{A}x)_i - (\mathcal{A}x)_j + x'\mathcal{A}x\big), \qquad 1 \le i, j \le d-1.$$

Set the $(d-1)$-dimensional vectors

$$\hat{\mathcal{A}}_i = \mathcal{A}_{ii} \qquad (1 \le i \le d-1), \qquad \hat{\mathcal{B}} = \tfrac{1}{2} \mathcal{A}^{-1} \hat{\mathcal{A}}.$$

Then

$$\eta^*(x) = \left(\prod_{i=1}^{d-1} x_i^{\hat{\mathcal{B}}_i} \right) \left(1 - \sum_{i=1}^{d-1} x_i \right)^{1 - \sum_{i=1}^{d-1} \hat{\mathcal{B}}_i}, \qquad \lambda^* = \frac{1}{8} \hat{\mathcal{A}}' \mathcal{A}^{-1} \hat{\mathcal{A}}, \tag{6.2}$$

and $\mathbb{P}^* \in \Pi^*_{\text{a.s.}}$. Furthermore, the coordinate process under $\mathbb{P}^*$ on the simplex has the same dynamics as the coordinate process under $\mathbb{P}^*$ in Example 6.6 moved to the simplex.

To prove the validity of the claims, set η, λ as the right-hand sides of (6.2), that is,

$$\eta(x) = \left(\prod_{i=1}^{d-1} x_i^{\hat{\mathcal{B}}_i}\right)\left(1 - \sum_{i=1}^{d-1} x_i\right)^{1-\sum_{i=1}^{d-1}\hat{\mathcal{B}}_i} \qquad \text{for } x \in E, \lambda = \frac{1}{8}\hat{\mathcal{A}}'\mathcal{A}^{-1}\hat{\mathcal{A}}.$$

A long calculation shows that $L\eta = -\lambda\eta$. Let $(\mathbb{P}^\eta_x)_{x\in\hat{E}}$ be the solution to the generalized martingale problem for L^η as in (1.7), and set $\mathbb{P}^\eta = \mathbb{P}^\eta_{x_0}$.

Rewrite $\tilde{\mathbb{P}}^*$ for the probability measure $\mathbb{P}^*$ of Example 6.6, and let $\tilde{X}$ be the coordinate process taking values in $(0,\infty)^d$. As shown in Example 6.6, $X = \exp(aW^{\tilde{\mathbb{P}}^*})$, where a is the unique positive definite square root of A, and $W^{\tilde{\mathbb{P}}^*}$ is a standard Brownian motion under $\tilde{\mathbb{P}}^*$. Let $\tilde{Y} = \tilde{X}/(\mathbf{1}_d'\tilde{X})$, where $\mathbf{1}_d$ is the vector of all 1's in $\mathbb{R}^d$, and define $Y = (\tilde{Y}_1, \dots, \tilde{Y}_{d-1})$, which is an E-valued process. Note that $\tilde{Y}$ be recovered from Y since $\tilde{Y}_d = 1 - \sum_{i=1}^{d-1} Y_i$. Using Itô's formula it can be shown that Y has dynamics

$$dY_t = c(Y_t)\frac{\nabla\eta(Y_t)}{\eta(Y_t)}\,dt + \tilde{\sigma}(Y_t)\,dW_t^{\tilde{\mathbb{P}}^*},$$

where $\tilde{\sigma}$ is the $(d-1)\times d$ matrix given by

$$\tilde{\sigma}(x)_{ij} = x_i\left(a_{ij} - \sum_{l=1}^{d-1} x_l a_{lj} - \left(1 - \sum_{l=1}^{d-1} x_l\right)a_{dj}\right) \qquad \text{for } x \in E.$$

It can be verified that $\tilde{\sigma}\tilde{\sigma}' = c$—indeed, this is how c was constructed. Thus, using the one-to-one correspondence between weak solutions of SDEs and solutions to the Martingale problem ([27], Chapter 5.4) and the uniqueness of solutions to the Martingale problem under Assumption 1.1 ([25], Theorem 1.12.1), it follows that $\mathbb{P}^\eta[A] = \tilde{\mathbb{P}}^*[Y \in A]$ holds for all $A \in \mathcal{F}$. Since $\tilde{X} = \exp(aW^{\tilde{\mathbb{P}}^*})$, it follows that $\log\eta(Y) = \hat{\beta}(*)'aW^{\tilde{\mathbb{P}}^*} - \log(\mathbf{1}_d' e^{aW^{\tilde{\mathbb{P}}^*}})$, where

$$\hat{\beta}(*)_i = \hat{\beta}_i, \qquad 1 \le i \le d-1, \qquad \hat{\beta}(*)_d = 1 - \sum_{j=1}^{d-1}\hat{\beta}_j.$$

Thus it follows that $\tilde{\mathbb{P}}^*$-a.s., $\lim_{t\uparrow\infty}\frac{1}{t}\log\eta(Y_t) = 0$. Hence, with X denoting the coordinate process in E, $\lim_{t\uparrow\infty}\frac{1}{t}\log\eta(X_t) = 0$ holds $\mathbb{P}^\eta$-a.s., which implies that $\mathbb{P}^\eta \in \Pi^*_{\text{a.s.}}$. The same argument as in Example 6.6 yields the optimality of η, λ and $\mathbb{P}^\eta$.

An interesting numerical example. Using the same notation as in Examples 6.6 and 6.7, consider for $d = 3$ the matrix A and associated vectors $\hat{B}$, $\hat{\mathcal{B}}$ given by

$$A = \begin{pmatrix} 5/3 & 3 & 0 \\ 3 & 7 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} -7/4 \\ 5/4 \\ 1/2 \end{pmatrix}, \qquad \hat{\mathcal{B}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The eigenvalues of A are 1 and $13/3(1 \pm \sqrt{145/169})$, and hence A is positive definite. The η^* from (6.1) and (6.2), respectively, are

$$\eta^*(x, y, x) = \sqrt[4]{\frac{y^5 z^2}{x^7}} \qquad \text{for } (x, y, z) \in (0, \infty)^3,$$

$$\eta^*(x, y) = \frac{y(1 - x - y)}{x} \qquad \text{for } x > 0, y > 0, x + y < 1.$$

Therefore, η^* goes to ∞ along the boundary of E in each case, even when the region is bounded.

7. Proof of Proposition 5.4.

The proof of Proposition 5.4 relies upon the following two auxiliary results. As in the proof of Proposition 5.2, the symbol $'$ is used to identify derivatives.

LEMMA 7.1. *Let Assumption 5.1 hold. Let $\eta \in C^2(\alpha, \beta)$ be strictly positive and strictly concave. If* (5.12) *holds, then*

$$\inf_{x \in (\alpha, \beta)} \frac{-c(x)\eta''(x)}{2\eta(x)} = 0.$$

PROOF. The proof will be given for the integral near α in (5.12); the proof near β is the same. Let $\eta \in C^2(\alpha, \beta)$ be strictly positive and strictly concave. Set

$$\delta(\eta) = \inf_{x \in (\alpha, \beta)} \frac{-c(x)\eta''(x)}{2\eta(x)}.$$

Let $x_0 \in (\alpha, \beta)$ and normalize η so that $\eta(x_0) = 1$. Note that this will not change the value of $\delta(\eta)$. Using integration by parts, for $\alpha < x < x_0$,

$$\eta(x) = 1 - (x_0 - x)\eta'(x_0) - \int_x^{x_0} (y - x)(-\eta''(y))\, dy$$

and hence

$$\int_\alpha^{x_0} \mathbb{I}_{\{y \geq x\}}(y - x)(-\eta''(y))\, dy \leq 1 + (\beta - \alpha)|\eta'(x_0)|.$$

Fatou's lemma and the concavity of η yield

$$\int_\alpha^{x_0} (y - \alpha)(-\eta''(y))\, dy \leq 1 + (\beta - \alpha)|\eta'(x_0)|. \tag{7.1}$$

The positivity and concavity of η yield for $\alpha < \alpha_m < y < x_0$ that

$$\eta(y) = \eta\left(\frac{y-\alpha_m}{x_0-\alpha_m}x_0 + \frac{x_0-y}{x_0-\alpha_m}\alpha_m\right) \geq \frac{y-\alpha_m}{x_0-\alpha_m},$$

and so, letting $\alpha_m \downarrow \alpha$, it follows that $\eta(y) \geq (y-\alpha)/(x_0-\alpha)$. Thus, if $\delta(\eta) > 0$ and (5.12) holds, then

$$\int_\alpha^{x_0} (y-\alpha)(-\eta''(y))\,dy \geq 2\delta(\eta)\int_\alpha^{x_0} \frac{(y-\alpha)\eta(y)}{c(y)}\,dy$$
$$\geq \frac{2\delta(\eta)}{x_0-\alpha}\int_\alpha^{x_0}\frac{(y-\alpha)^2}{c(y)}\,dy = \infty,$$

which contradicts (7.1). Thus, $\delta(\eta) = 0$ proving the result. $\square$

LEMMA 7.2. *Let Assumption 5.1 hold. Let $\lambda > 0$ and $\eta \in H_\lambda$ be such that*

$$\lim_{x\downarrow\alpha}\eta(x) = 0 = \lim_{x\uparrow\beta}\eta(x) \tag{7.2}$$

and

$$\int_\alpha^\beta \frac{\eta^2(x)}{c(x)}\,dx < \infty. \tag{7.3}$$

Then, $\lambda^* = \lambda$ and $\eta^* = \eta$. The coordinate process X under $(\mathbb{P}^*_x)_{x\in(\alpha,\beta)}$ is positive recurrent, and hence, by Proposition 1.7, η^* is unique up to multiplication by a positive constant. Furthermore, $\mathbb{P}^* \in \Pi^*_{\text{a.s.}}$.

PROOF. If X is recurrent under $(\mathbb{P}^*_x)_{x\in E}$, then from Proposition 1.7, $\lambda^* = \lambda$ and $\eta^* = \eta$, and η^* is unique up to multiplication by a positive constant. Furthermore, by (7.3), positive recurrence will follow with the invariant measure $\tilde{\eta}$ that has density proportional to η^2/c with respect to Lebesgue measure, appropriately normalized so $\tilde{\eta}$ is a probability measure.

To check recurrence it will be shown that (5.1) holds near α; the proof near β is the same. Note that, since $\eta \in H_\lambda$ and (7.2) holds, there exists a unique $x_0 \in (\alpha,\beta)$ such that $\eta'(x_0) = 0$. For $\alpha < x < x_0$,

$$\int_x^{x_0}\frac{2\lambda\eta(y)^2}{c(y)}\,dy = -\int_x^{x_0}\eta(y)\eta''(y)\,dy = \eta(x)\eta'(x) + \int_x^{x_0}\eta'(y)^2\,dy.$$

Thus, as $x \downarrow \alpha$ since η is positive and concave, it must hold that $\eta(x)\eta'(x) > 0$, and hence, by (7.3), it follows that $\int_\alpha^{x_0}\eta'(y)^2\,dy < \infty$. Therefore, by the concavity of η and (7.2),

$$0 \leq \liminf_{x\downarrow\alpha}\eta(x)\eta'(x) \leq \lim_{x\downarrow\alpha}\int_0^x \eta'(y)^2\,dy = 0. \tag{7.4}$$

This implies that, for any $\varepsilon > 0$, there is an x_ε near α such that for $x \in (\alpha, x_\varepsilon)$, $\eta^2(x) \le 2\varepsilon(x-\alpha)$, or that

$$\int_\alpha^{x_\varepsilon} \frac{1}{\eta(y)^2}\,dy \ge \frac{1}{2\varepsilon}\int_\alpha^{x_\varepsilon} \frac{1}{y-\alpha}\,dy = \infty,$$

and recurrence follows. It remains to prove that $\mathbb{P}^* \in \Pi^*_{\text{a.s.}}$. To this end, it follows from equations (3.5) and (3.7) in the proof of Proposition 3.4 that $\mathbb{P}^* \in \Pi^*_{\text{a.s.}}$ if

$$\liminf_{t\uparrow\infty} \frac{1}{t}\int_0^t \left(\frac{1}{2}c(X_s)\left(\frac{\eta'(X_s)}{\eta(X_s)}\right)^2 - \lambda\right) ds \ge 0, \qquad \mathbb{P}^*\text{-a.s.}$$

By the ergodic theorem ([25], Theorem 4.9.5) and the monotone convergence theorem it follows that

$$\begin{aligned}\liminf_{t\uparrow\infty} \frac{1}{t}\int_0^t &\left(\frac{1}{2}c(X_s)\left(\frac{\eta'(X_s)}{\eta(X_s)}\right)^2 - \lambda\right) ds \\ &\ge \int_\alpha^\beta \left(\frac{1}{2}c(y)\left(\frac{\eta'(y)}{\eta(y)}\right)^2 - \lambda\right)\frac{\eta(y)^2}{c(y)}\,dy, \qquad \mathbb{P}^*\text{-a.s.}\end{aligned}$$

Continuing, $\eta \in H_\lambda$ implies

$$\int_\alpha^\beta \left(\frac{1}{2}c(y)\left(\frac{\eta'(y)}{\eta(y)}\right)^2 - \lambda\right)\frac{\eta(y)^2}{c(y)}\,dy = \lim_{x\downarrow\alpha}\eta(x)\eta'(x) - \lim_{x\uparrow\beta}\eta(x)\eta'(x) = 0,$$

where the last equality follows from (7.4) since the same equality holds near β. Thus, $\mathbb{P}^* \in \Pi^*_{\text{a.s.}}$. $\square$

In what follows, the proof of Proposition 5.4 will be given.

The proof of how (5.12) implies $\lambda^* = 0$ is handled first. By (5.5), it suffices to consider strictly concave functions η. However, since (5.12) holds, Lemma 7.1 applies and hence $\delta(\eta) = 0$ for all such η. Thus $\lambda^* = 0$.

Regarding the assertions when (5.11) holds, in light of Lemma 7.2 it suffices to show that (5.11) yields the existence of a $\lambda > 0$, $\eta \in H_\lambda$ such that conditions (7.2) and (7.3) are satisfied. To this end, define the σ-finite measure m via $m(dx) = c(x)^{-1}\,dx$. Note that condition (7.3) now reads $\eta \in L^2((\alpha,\beta), m)$. The desired pair (λ, η) are the principle eigenvalue and eigenfunction for the operator $(L, \mathcal{D}(L))$ where $(L\eta)(x) = -(1/2)c(x)\eta''(x)$ for $x \in (\alpha, \beta)$, and the domain $\mathcal{D}(L)$ consists of functions which vanish at α, β and is constructed so that $(L, \mathcal{D}(L))$ is self adjoint in $L^2((\alpha,\beta), m)$. $\mathcal{D}(L)$ is highly dependent upon the behavior of m near α and β. The study of the spectral properties of such operators falls under the name *Sturm–Liouville theory*. For a detailed exposition on the topics covered/results given below, see [20] and [30].

The case when $m((\alpha,\beta)) < \infty$ is called the *regular case*. Here $\mathcal{D}(L)$ is given by

$$\begin{aligned}\mathcal{D}(L) = \{\eta \in L^2((\alpha,\beta), m) \mid \eta' \in AC(\alpha,\beta), \eta(\alpha) = \eta(\beta) = 0,\\ c\eta'' \in L^2((\alpha,\beta), m)\},\end{aligned} \tag{7.5}$$

and the existence of a $\lambda > 0$, $\eta \in H_\lambda \cap \mathcal{D}(L)$ is given by [20], Theorem 2.7.4, and [30], Theorem 10.12.1.

Now, suppose that (5.11) holds, but for some $a \in (\alpha, \beta)$ either $m((\alpha, a)) = \infty$ or $m((a, \beta)) = \infty$, or both. These cases are called the *singular cases*. In each of these three cases there exists a domain $\mathcal{D}(L) \subset L^2((\alpha, \beta), m)$, similar to that in (7.5), such that $(L, \mathcal{D}(L))$ is self adjoint. For explicit formulas for the domains, see [30], Chapters 7 and 10.

According to [30], Theorem 10.12.1(8), if the spectrum of $(L, \mathcal{D}(L))$ is discrete and bounded from below, then in fact there exists a $\lambda > 0$ and $\eta \in H_\lambda \cap \mathcal{D}(L)$ such that (7.2) holds [this last fact follows by construction of $\mathcal{D}(L)$ but also because otherwise $\eta \notin L^2((\alpha, \beta), m)$].

To prove the spectrum is discrete and bounded from below, it suffices to treat the case of one regular and one singular endpoint. This follows using the *spectral decomposition method* on which a detailed description may be found in [12]. Without loss of generality, consider the case when α is regular and β is singular. Under the transformation $z = f(x) = \int_\alpha^x (1/c(y))\,dy$, (α, β) is taken to be $(0, \infty)$. Set $\varphi(z) = \eta(x)$ and $g(z) = f^{-1}(z)$. Note that $\eta \in L^2((\alpha, \beta), m)$ is equivalent to $\varphi \in L^2((0, \infty), \mathrm{Leb}) \equiv L^2(0, \infty)$. Furthermore, the operator $(M, \mathcal{D}(M))$ defined by

$$(M\varphi)(z) = -\frac{1}{2}\left(\frac{1}{g'(z)}\varphi'(z)\right)', \qquad \mathcal{D}(M) = \{\varphi \mid \varphi(z) = \eta(x), \eta \in \mathcal{D}(L)\}$$

is self-adjoint in $L^2(0, \infty)$. Let $N > 0$ and

$$Q_N = \{v \in C_0((N, \infty), \mathbb{C}) \mid v \in AC_{\mathrm{loc}}(0, \infty), v' \in L^2(0, \infty)\},$$

where C_0 means that v is continuous and compactly supported in (N, ∞). For $v \in Q_N$, set

$$I(v, N) = \frac{1}{2}\int_N^\infty \frac{|v'(z)|^2}{g'(z)}\,dz.$$

According to [19], Lemma 4.2, $(M, \mathcal{D}(M))$ has a discrete spectrum bounded from below if and only if for each $\theta > 0$ there exists an $N > 0$ such that

$$I(v, N) \geq \theta \int_N^\infty v(z)^2\,dz$$

for each real valued $v \in Q_N$. To show this, fix $\theta > 0$. For any $N > 0$ and $v \in Q_N$,

$$v(z) = -\int_z^\infty v'(\tau)\,d\tau.$$

Since $\tau = f(g(\tau))$, it follows that $g'(\tau) = c(g(\tau)) > 0$. By Hölder's inequality, for real valued $v \in Q_N$,

$$v(z)^2 \leq \left(\int_z^\infty \frac{v'(\tau)^2}{g'(\tau)}\,d\tau\right)\left(\int_z^\infty g'(\tau)\,d\tau\right) \leq 2I(v, N)\bigl(\beta - g(z)\bigr).$$

Therefore,

$$\theta \int_N^\infty v(z)^2\,dz \le 2\theta I(v,N) \int_N^\infty \big(\beta - g(z)\big)\,dz$$
$$= 2\theta I(v,N) \int_{g(N)}^\beta \frac{\beta - x}{c(x)}\,dx,$$

where the last equality follows from the substitution $x = g(z)$ or $z = f(x)$. Since $\lim_{z\uparrow\infty} g(x) = \beta$, by (5.11)

$$2\theta \int_{g(N)}^\beta \frac{\beta - x}{c(x)}\,dx \le 1$$

for N large enough, yielding the desired result. □

Acknowledgments. The authors would like to express their gratitude toward the Associate Editor and four anonymous referees for their very constructive remarks that helped improve the paper significantly.

REFERENCES

[1] ALGOET, P. (1992). Universal schemes for prediction, gambling and portfolio selection. *Ann. Probab.* **20** 901–941. MR1159579

[2] ALGOET, P. H. and COVER, T. M. (1988). Asymptotic optimality and asymptotic equipartition properties of log-optimum investment. *Ann. Probab.* **16** 876–898. MR0929084

[3] BAYRAKTAR, E., KARATZAS, I. and YAO, S. (2012). Optimal stopping for dynamic convex risk measures. *Illinois J. Math.* To appear.

[4] ELWORTHY, K. D., LI, X. M. and YOR, M. (1997). On the tails of the supremum and the quadratic variation of strictly local martingales. In *Séminaire de Probabilités*, *XXXI*. *Lecture Notes in Math.* **1655** 113–125. Springer, Berlin. MR1478722

[5] EVANS, L. C. (1998). *Partial Differential Equations*. *Graduate Studies in Mathematics* **19**. Amer. Math. Soc., Providence, RI. MR1625845

[6] FERNHOLZ, D. and KARATZAS, I. (2010). On optimal arbitrage. *Ann. Appl. Probab.* **20** 1179–1204. MR2676936

[7] FERNHOLZ, D. and KARATZAS, I. (2011). Optimal arbitrage under model uncertainty. *Ann. Appl. Probab.* **21** 2191–2225.

[8] FERNHOLZ, D. and KARATZAS, I. (2010). Probabilistic aspects of arbitrage. In *Contemporary Quantitative Finance* 1–17. Springer, Berlin. MR2732837

[9] FERNHOLZ, R. (2001). Equity portfolios generated by functions of ranked market weights. *Finance Stoch.* **5** 469–486. MR1861997

[10] FÖLLMER, H. and GUNDEL, A. (2006). Robust projections in the class of martingale measures. *Illinois J. Math.* **50** 439–472 (electronic). MR2247836

[11] GILBOA, I. and SCHMEIDLER, D. (1989). Maxmin expected utility with nonunique prior. *J. Math. Econom.* **18** 141–153. MR1000102

[12] GLAZMAN, I. M. (1966). *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators*. Israel Program for Scientific Translations, Jerusalem. Translated from the Russian by the IPST Staff. MR0190800

[13] GUNDEL, A. (2005). Robust utility maximization for complete and incomplete market models. *Finance Stoch.* **9** 151–176. MR2211122

[14] GYÖRFI, L., URBÁN, A. and VAJDA, I. (2007). Kernel-based semi-log-optimal empirical portfolio selection strategies. *Int. J. Theor. Appl. Finance* **10** 505–516. MR2319423

[15] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940

[16] KARDARAS, C. (2010). The continuous behavior of the numéraire portfolio under small changes in information structure, probabilistic views and investment constraints. *Stochastic Process. Appl.* **120** 331–347. MR2584897

[17] KNISPEL, T. (2010). Asymptotic minimization of robust "downside risk." Available at www.stochastik.uni-hannover.de/fileadmin/institut/pdf/DownsideRisk.pdf.

[18] KNISPEL, T. (2010). Asymptotics of robust utility maximization. Available at www.stochastik.uni-hannover.de/fileadmin/institut/pdf/PowerUtility.pdf.

[19] KWONG, M. K. and ZETTL, A. (1981). Discreteness conditions for the spectrum of ordinary differential operators. *J. Differential Equations* **40** 53–70. MR0614218

[20] MIKLAVČIČ, M. (1998). *Applied Functional Analysis and Partial Differential Equations*. World Scientific, River Edge, NJ. MR1784426

[21] OSTERRIEDER, J. and RHEINLÄNDER, T. (2006). Arbitrage opportunities in diverse markets via a non-equivalent measure change. *Annals of Finance* **2** 287–301.

[22] PINCHOVER, Y. (1992). Large time behavior of the heat kernel and the behavior of the Green function near criticality for nonsymmetric elliptic operators. *J. Funct. Anal.* **104** 54–70. MR1152459

[23] PINCHOVER, Y. (1995). On nonexistence of any λ_0-invariant positive harmonic function, a counter example to Stroock's conjecture. *Comm. Partial Differential Equations* **20** 1831–1846. MR1349233

[24] PINSKY, R. G. (1985). On the convergence of diffusion processes conditioned to remain in a bounded region for large time to limiting positive recurrent diffusion processes. *Ann. Probab.* **13** 363–378. MR0781410

[25] PINSKY, R. G. (1995). *Positive Harmonic Functions and Diffusion*. *Cambridge Studies in Advanced Mathematics* **45**. Cambridge Univ. Press, Cambridge. MR1326606

[26] QUENEZ, M.-C. (2004). Optimal portfolio in a multiple-priors model. In *Seminar on Stochastic Analysis, Random Fields and Applications IV*. *Progress in Probability* **58** 291–321. Birkhäuser, Basel. MR2096294

[27] ROGERS, L. C. G. and WILLIAMS, D. (2000). *Diffusions, Markov Processes, and Martingales*. *Cambridge Mathematical Library* **2**. Cambridge Univ. Press, Cambridge. MR1780932

[28] SCHIED, A. (2007). Optimal investments for risk- and ambiguity-averse preferences: A duality approach. *Finance Stoch.* **11** 107–129. MR2284014

[29] SCHIED, A. and WU, C.-T. (2005). Duality theory for optimal investments under model uncertainty. *Statist. Decisions* **23** 199–217. MR2236457

[30] ZETTL, A. (2005). *Sturm–Liouville Theory*. *Mathematical Surveys and Monographs* **121**. Amer. Math. Soc., Providence, RI. MR2170950

DEPARTMENT OF MATHEMATICS AND STATISTICS
BOSTON UNIVERSITY
111 CUMMINGTON STREET
BOSTON, MASSACHUSETTS 02215
USA
E-MAIL: kardaras@bu.edu

DEPARTMENT OF MATHEMATICAL SCIENCES
CARNEGIE MELLON UNIVERSITY
WEAN HALL 6113
PITTSBURGH, PENNSYLVANIA 15213
USA
E-MAIL: scottrob@andrew.cmu.edu