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ANOMALOUS DISSIPATION IN A STOCHASTIC INVISCID DYADIC MODEL

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A stochastic version of an inviscid dyadic model of turbulence, with multiplicative noise, is proved to exhibit energy dissipation in spite of the formal energy conservation. As a consequence, global regular solutions cannot exist. After some reductions, the main tool is the escape bahavior at infinity of a certain birth and death process.

1. Introduction. The dyadic model of turbulence has been introduced in, among others, [15, 16, 27] and [13], as a simplified model of fluid dynamics equations in order to investigate a number of properties which are out of reach at present for more realistic models. In this paper we study a suitable random perturbation of the classical dyadic model under which we are able to prove anomalous dissipation of energy.

On a complete filtered probability space (Ω, F_t, P), let $(W_n)_{n\geq 1}$ be a sequence of independent Brownian motions. Consider the infinite system of stochastic differential equations in Stratonovich form

$$dX_n = (k_{n-1}X_{n-1}^2 - k_n X_n X_{n+1})\,dt + k_{n-1}X_{n-1} \circ dW_{n-1} - k_n X_{n+1} \circ dW_n \tag{1}$$

for $n \geq 1$, with $X_0(t) = 0$. Denote by l^2 the Hilbert space of real square summable sequences $x = (x_n)_{n\geq 1}$ and set $\|x\|^2 = \sum_{n=1}^{\infty} x_n^2$. We call *energy* of $X(t) := (X_n(t))_{n\geq 1}$ the quantity $\mathcal{E}(t) := \frac{1}{2}\|X(t)\|^2$. Assume for simplicity to have a deterministic initial condition

$$X(0) = x, \qquad x = (x_n)_{n\geq 1} \in l^2.$$

The sequence of positive numbers $(k_n)_{n\geq 1}$ will be specified later on; the most natural case in analogy with fluid dynamics is $k_n = \lambda^n$ for some $\lambda > 1$.

System (1) is *formally* energy preserving. By the Stratonovich form of Itô's formula (see [19]), we have

$$\begin{aligned}\tfrac{1}{2}\,dX_n^2 &= X_n \circ dX_n \\ &= (k_{n-1}X_{n-1}^2 X_n - k_n X_n^2 X_{n+1})\,dt \\ &\qquad + k_{n-1}X_{n-1}X_n \circ dW_{n-1} - k_n X_n X_{n+1} \circ dW_n.\end{aligned}$$

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If we sum formally these identities and use the boundary condition $X_0(t) = 0$, we readily have $\frac{1}{2} d \sum_{n=1}^{\infty} X_n^2 = 0$, namely,

$$\mathcal{E}(t) = \mathcal{E}(0), \qquad P\text{-a.s.}$$

The aim of this paper is to prove rigorously an opposite statement, a property that we could call *anomalous dissipation*. We need the notion of energy controlled solution that will be given in the next section.

THEOREM 1. *Assume $k_n = \lambda^n$ for some $\lambda > 1$. Given $x \in l^2$, let $X(t)$ be the unique energy controlled solution of equation* (1). *Then, for all $t > 0$,*

$$P\big(\mathcal{E}(t) = \mathcal{E}(0)\big) < 1$$

and for all $\varepsilon > 0$ there exist t such that

$$P\big(\mathcal{E}(t) < \varepsilon\big) > 0.$$

Moreover, if $\mathcal{E}(0)$ is sufficiently small, then $\mathcal{E}(t)$ decays to zero at least exponentially fast both almost surely and in $\mathcal{L}^1$.

As a consequence of this theorem, in Section 7 we will also prove that *global regular solutions cannot exist*.

The proof of the theorem is built along Sections 3–6 and concluded in Section 6.1.

Being inspired by fluid dynamics, the results of Theorem 1 could be interpreted as a form of *turbulent dissipation*. Dynamically speaking, it is a dissipation due to a very fast cascade mechanism; energy moves faster and faster from low to high wave numbers n and escapes to infinity in finite time.

Results of anomalous dissipation for *linear* stochastic systems with additive noise have been proved by [22, 23]. The notion of anomalous dissipation of these papers is different and based on invariant measures, but conceptually the question is the same. The dyadic model of the present paper is, to our knowledge, the first nonlinear stochastic case where anomalous dissipation is proved to occur. Moreover, the proof is entirely different from those of the additive noise linear stochastic case.

Nonlinear models with anomalous dissipation have been discovered before in the deterministic case (see [3, 5, 7–9, 15–17, 27]). Our model is a multiplicative random perturbation of models of these forms. However, let us stress that the proof given here is totally different from the proofs of the deterministic literature, based on monotonicity and positivity properties of the deterministic part of system (1). These properties are lost in the stochastic case.

Theorem 1 remains true when $k_n \leq Cn^{\alpha}$ for some $\alpha > 1$. However, we make here the assumption $k_n = \lambda^n$ in analogy with the deterministic literature on dyadic models.

The proof is based on three main ingredients: (i) Girsanov's transform allows us to reduce the problem to a linear stochastic equation; (ii) second moments of components satisfy a closed system, without moments of products; (iii) this closed system is the forward equation of a birth and death process, the escape of which at infinity can be understood.

Some issues in this procedure are not trivial. One of them is the equivalence of laws on infinite time horizon (see Proposition 18). Its proof is nonstandard; moreover, it is restricted to a range of values of parameters, the generalization being open. Concerning the idea that square moments could satisfy a closed equation, related to jump process, we have been inspired by previous works ([1, 10, 11] and [21]), however, devoted to different models; the link here with the nonlinear model and the stochasticity is more transparent via Girsanov's transform.

1.1. *The multiplicative noise in Euler's equations.* The noise we introduced in equation (1), motivated by energy conservation, may appear peculiar from the physical point of view. Nevertheless, it is the natural choice if we compare the dyadic model with some other equations of fluid dynamics like Euler's equations or diffusions of passive scalars.

Let us give a picture of this analogy in the case of Euler's equations, which have the form

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = 0, \qquad \operatorname{div} u = 0$$

with appropriate initial and boundary conditions (u and p are the velocity and pressure field, resp.). Let us think of periodic boundary conditions for sake of simplicity. The Lagrangian motion of particles is given by the equation

$$\frac{dY(t)}{dt} = u(t, Y(t)).$$

A natural way to randomly perturb Euler dynamics (see [24]) is by adding a white noise to the Lagrangian motion,

$$dY(t) = u(t, Y(t))\,dt + \sum_j \sigma_j(t, Y(t))\,dW^j(t),$$

where $(W^j(t))_{t\geq 0}$ are independent Brownian motions and σ_j are given vector fields. By standard rules of stochastic calculus, applied formally, one can see that Euler equations take the stochastic form (with a new pressure $\widetilde{p}$)

$$du + [u \cdot \nabla u + \nabla \widetilde{p}]\,dt + \sum_j \sigma_j \cdot \nabla u \circ dW^j(t) = 0, \qquad \operatorname{div} u = 0, \tag{2}$$

where Stratonovich operation has to be used. Rigorous results and physical arguments in support of this kind of stochastic perturbation of the Lagrangian motion and the corresponding PDE with multiplicative Stratonovich noise (in the viscous case) can be found in [24, 25]. In addition, let us mention the wide literature on stochastic passive scalar equations (see, e.g., [20]) where multiplicative Stratonovich noise of the form above is used.

In abstract form, equation (2) takes the form

$$du + B(u,u)\,dt + B(\circ\, dW, u), \tag{3}$$

where $W(t) := \sum_j \sigma_j(x) W^j(t)$ and $B(u,v) := u \cdot \nabla v$. Notice that for sufficient regular u and v the following identity holds $\langle B(u,v), v\rangle = 0$.

In [8] the authors argued that after Fourier or wavelets transforms and certain simplifications, the deterministic system

$$\frac{dX_n}{dt} = k_{n-1} X_{n-1}^2 - k_n X_n X_{n+1}, \qquad n \geq 1, \tag{4}$$

describes some idealized features of the deterministic equation $\frac{du}{dt} + B(u,u) = 0$. Equation (4) has the form $\frac{d}{dt} X = \tilde{B}(X,X)$ where

$$\tilde{B}(X,Y)_{(n)} = k_{n-1} X_{n-1} Y_{n-1} - k_n X_n Y_{n+1}.$$

Formally we have $\langle \tilde{B}(X,Y), Y\rangle = 0$. Thanks to $\langle B(u,u), u\rangle = 0$ and $\langle \tilde{B}(X,X), X\rangle = 0$ the perturbation of u and X does not modify the energy balance (at a formal level).

Standing this idealized discretization $\tilde{B}(X,Y)$ of $B(u,v)$, the natural analog of equation (3) is

$$dX_n + \tilde{B}(X,X)_{(n)}\,dt + \tilde{B}(\circ\, dW, X)_{(n)}, \qquad n \geq 1,$$

which is precisely system (1).

2. Itô's formulation. For the rigorous formulation of equation (1) and a basic theorem of existence and uniqueness, we follow [4]. The Itô form of equation (1) is

$$\begin{aligned} dX_n = {} & (k_{n-1} X_{n-1}^2 - k_n X_n X_{n+1})\,dt + k_{n-1} X_{n-1}\,dW_{n-1} \\ & - k_n X_{n+1}\,dW_n - \tfrac{1}{2}(k_n^2 + k_{n-1}^2) X_n\,dt. \end{aligned} \tag{5}$$

Let us define the concept of weak solution for this equation. By a filtered probability space (Ω, F_t, P) we mean a probability space (Ω, F_∞, P) and a right-continuous filtration $(F_t)_{t\geq 0}$ such that F_∞ is the σ-algebra generated by $\bigcup_{t\geq 0} F_t$.

DEFINITION 2. Given $x \in l^2$, a weak solution of equation (1) in l^2 is a filtered probability space (Ω, F_t, P), a sequence of independent Brownian motions

$(W_n)_{n\geq 1}$ on (Ω, F_t, P) and an l^2-valued stochastic process $(X_n)_{n\geq 1}$ on (Ω, F_t, P) with continuous adapted components X_n, such that

$$\begin{aligned} X_n(t) = x_n &+ \int_0^t \bigl(k_{n-1}X_{n-1}^2(s) - k_n X_n(s) X_{n+1}(s)\bigr)\,ds \\ &+ \int_0^t k_{n-1}X_{n-1}(s)\,dW_{n-1}(s) - \int_0^t k_n X_{n+1}(s)\,dW_n(s) \\ &- \int_0^t \frac{1}{2}(k_n^2 + k_{n-1}^2)X_n(s)\,ds \end{aligned}$$

for each $n \geq 1$, with $X_0 = 0$. We denote this solution by

$$(\Omega, F_t, P, W, X)$$

or simply by X.

DEFINITION 3. We call energy controlled solutions the solutions of Definition 2 which satisfy

$$P\left(\sum_{n=1}^{\infty} X_n^2(t) \leq \sum_{n=1}^{\infty} x_n^2\right) = 1 \tag{6}$$

for all $t \geq 0$.

The following simple proposition (proved in [4]) clarifies that a process satisfying (5) rigorously satisfies also (1).

PROPOSITION 4. *If X is a weak solution, for every $n \geq 1$ the process $(X_n(t))_{t\geq 0}$ is a continuous semimartingale, hence, the two Stratonovich integrals*

$$\int_0^t k_{n-1}X_{n-1}(s) \circ dW_{n-1}(s) - \int_0^t k_n X_{n+1}(s) \circ dW_n(s)$$

are well defined and equal to

$$\begin{aligned} &\int_0^t k_{n-1}X_{n-1}(s)\,dW_{n-1}(s) - \frac{1}{2}\int_0^t k_{n-1}^2 X_n(s)\,ds \\ &\quad - \int_0^t k_n X_{n+1}(s)\,dW_n(s) - \frac{1}{2}\int_0^t k_n^2 X_n(s)\,ds. \end{aligned}$$

Hence, X satisfies the Stratonovich equations (1).

The main result proved in [4] is the well posedness in the weak probabilistic sense in the class of energy controlled solutions.

THEOREM 5. *Given $(x_n) \in l^2$, there exists one and only one energy controlled solution of equation* (1).

3. Girsanov's transformation. Formally, let us write equation (5) in the form

$$dX_n = k_{n-1}X_{n-1}(X_{n-1}\,dt + dW_{n-1}) - k_n X_{n+1}(X_n\,dt + dW_n) - \tfrac{1}{2}(k_n^2 + k_{n-1}^2)X_n\,dt.$$

The simple idea is that $X_n\,dt + dW_n$ is a Brownian motion with respect to a new measure Q on (Ω, F), simultaneously for every n, hence, the equations become linear SDEs under Q. We use details about Girsanov's theorem that can be found in [26], Chapter VIII, and an infinite dimensional version proved in [6, 12, 18].

Assume that $(X_n)_{n\geq 1}$ is an energy controlled solution. Due to the boundedness of $\sum_{n=1}^\infty X_n^2(t)$ [see (6)], the process $Y_t := -\sum_{n=1}^\infty \int_0^t X_n(s)\,dW_n(s)$ is well defined, is a martingale and its quadratic variation $[Y, Y]_t$ is $\int_0^t \sum_{n=1}^\infty X_n^2(s)\,ds$. For the same reason, Novikov criterium applies, so $\mathcal{N}(Y)_t := \exp(Y_t - [Y, Y]_t)$ is a strictly positive martingale. Define the set function Q on $\bigcup_{t\geq 0} F_t$ by setting

$$\frac{dQ}{dP}\bigg|_{F_t} = \mathcal{N}(Y)_t = \exp\left(-\sum_{n=1}^\infty \int_0^t X_n(s)\,dW_n(s) - \frac{1}{2}\int_0^t \sum_{n=1}^\infty X_n^2(s)\,ds\right) \tag{7}$$

for every $t \geq 0$. We also denote by Q its extension to the terminal σ-field F_∞. In general we cannot prove it is absolutely continuous with respect to P, but we shall see at least a case when this is true. Notice also that Q and P are equivalent on each F_t, by the strict positivity. Define

$$B_n(t) = W_n(t) + \int_0^t X_n(s)\,ds.$$

Under Q, $(B_n(t))_{n\geq 1, t\in[0,T]}$ is a sequence of independent Brownian motions. Since

$$\int_0^t k_{n-1}X_{n-1}(s)\,dB_{n-1}(s) = \int_0^t k_{n-1}X_{n-1}(s)\,dW_{n-1}(s) + \int_0^t k_{n-1}X_{n-1}(s)X_{n-1}(s)\,ds$$

and similarly for $\int_0^t k_n X_{n+1}(s)\,dB_n(s)$, we see that

$$X_n(t) = X_n(0) + \int_0^t k_{n-1}X_{n-1}(s)\,dB_{n-1}(s) - \int_0^t k_n X_{n+1}(s)\,dB_n(s) - \int_0^t \frac{1}{2}(k_n^2 + k_{n-1}^2)X_n(s)\,ds.$$

This is a linear stochastic equation. Girsanov's transformation has removed the nonlinearity. Let us collect the previous facts.

THEOREM 6. *If (Ω, F_t, P, W, X) is an energy controlled solution of the nonlinear equation (1), then it satisfies the linear equation*

$$dX_n = k_{n-1}X_{n-1}\,dB_{n-1} - k_n X_{n+1}\,dB_n - \tfrac{1}{2}(k_n^2 + k_{n-1}^2)X_n\,dt, \tag{8}$$

where the processes

$$B_n(t) = W_n(t) + \int_0^t X_n(s)\, ds$$

are a sequence of independent Brownian motions on (Ω, F_t, Q), Q *defined by* (7).

One may also check that

$$dX_n = k_{n-1}X_{n-1} \circ dB_{n-1} - k_n X_{n+1} \circ dB_n$$

so the previous computations could be described at the level of Stratonovich calculus.

4. Closed equation for $\mathbb{E}^Q[X_n^2(t)]$. Let (Ω, F_t, P, W, X) be an energy controlled solution of the nonlinear equation (1) with initial condition $x \in l^2$ and let Q be the measure given by Theorem 6. Denote by $\mathbb{E}^Q$ the mathematical expectation on (Ω, F_t, Q). We have

$$\begin{aligned}
\tfrac{1}{2}\, dX_n^2 &= X_n\, dX_n + \tfrac{1}{2}\, d[X_n]_t \\
&= -\tfrac{1}{2}(k_n^2 + k_{n-1}^2)X_n^2\, dt + dM_n + \tfrac{1}{2}(k_{n-1}^2 X_{n-1}^2 + k_n^2 X_{n+1}^2)\, dt, \\
\tfrac{1}{2}\, dX_n^4 &= 4X_n^3\, dX_n + \tfrac{12}{2} X_n^2\, d[X_n]_t \\
&= -\tfrac{1}{2}(k_n^2 + k_{n-1}^2)X_n^4\, dt + dM_n + \tfrac{12}{2} X_n^2 (k_{n-1}^2 X_{n-1}^2 + k_n^2 X_{n+1}^2)\, dt, \\
de^{X_n^2} &= e^{X_n^2}\, dX_n^2 + \tfrac{1}{2} e^{X_n^2}\, d[X_n^2, X_n^2], \\
de^{X_n} &= e^{X_n}\, dX_n + \tfrac{1}{2} e^{X_n}\, d[X_n, X_n] \\
&= \cdots e^{X_n}(k_n^2 + k_{n-1}^2) X_n\, dt + dM_n + \tfrac{1}{2} e^{X_n} \tfrac{1}{2}(k_{n-1}^2 X_{n-1}^2 + k_n^2 X_{n+1}^2)\, dt,
\end{aligned}$$

where

$$M_n(t) = \int_0^t k_{n-1} X_{n-1}(s) X_n(s)\, dB_{n-1}(s) - \int_0^t k_n X_n(s) X_{n+1}(s)\, dB_n(s).$$

Notice that

$$\mathbb{E}^Q \int_0^T X_n^4(t)\, dt < \infty \tag{9}$$

for each $n \geq 1$. Indeed, for an energy controlled solution, from (6) we have, with P-probability one,

$$\sum_{n=1}^{\infty} X_n^4(t) \leq \max_n X_n^2(t) \sum_{n=1}^{\infty} X_n^2(t) \leq \left(\sum_{n=1}^{\infty} x_n^2 \right)^2.$$

But P and Q are equivalent on F_t, hence,

$$Q\left(\sum_{n=1}^{\infty} X_n^4(t) \leq \left(\sum_{n=1}^{\infty} x_n^2 \right)^2 \right) = 1.$$

This implies (9).

From (9), $M_n(t)$ is a martingale for each $n \geq 1$. Moreover, $\mathbb{E}^Q[\sum_{n=1}^{\infty} X_n^2(t)] < \infty$ because $(X_n)_{n\geq 1}$ is an energy controlled solution [again, as above, condition (6) is invariant under the change of measure $P \leftrightarrow Q$ on F_t] and thus, in particular, $\mathbb{E}^Q[X_n^2(t)]$ is finite for each $n \geq 1$. From the previous equation we deduce the following.

PROPOSITION 7. *For every energy controlled solution X, $\mathbb{E}^Q[X_n^2(t)]$ is finite for each $n \geq 1$ and satisfies*

$$\frac{d}{dt}\mathbb{E}^Q[X_n^2] = -(k_n^2 + k_{n-1}^2)\mathbb{E}^Q[X_n^2] + k_{n-1}^2\mathbb{E}^Q[X_{n-1}^2] + k_n^2\mathbb{E}^Q[X_{n+1}^2]$$

for $t \geq 0$.

The first remarkable fact of this result is that $\mathbb{E}^Q[X_n^2]$ satisfies a closed equation. The second one is that this is the forward equation of a continuous-time Markov chain, as we shall discuss in the next section. See [1, 10] for different examples with the same structure.

5. Associated birth and death process. In this section we will make thorough use of birth and death processes. We do not suppose that all the readers are familiar with the field, so we will be more detailed.

Let us set

$$p_n(t) = \frac{1}{\|x\|^2}\mathbb{E}^Q[X_n^2(t)], \qquad p(t) = (p_n(t))_{n\geq 1}, \qquad t \geq 0,$$

and set also $p_0(t) \equiv 0$. Introduce the positive numbers $(\lambda_n)_{n\geq 1}$ and $(\mu_n)_{n\geq 1}$, defined as

$$\lambda_n = k_n^2, \qquad \mu_n = k_{n-1}^2.$$

By Proposition 7, we have

$$\begin{cases} \dfrac{d}{dt}p_n(t) = -(\lambda_n + \mu_n)p_n(t) + \lambda_{n-1}p_{n-1}(t) + \mu_{n+1}p_{n+1}(t), & t \geq 0, \\ p_n(0) = \dfrac{x_n^2}{\|x\|^2}. \end{cases} \tag{10}$$

We observe that $\sum_{n=1}^{\infty} p_n(t) = 1$ when $t = 0$ and, moreover,

$$\sum_{n=1}^{\infty} p_n(t) \leq 1 \tag{11}$$

for all $t > 0$ (since X is an energy controlled solution).

The system (10) can be conveniently put in matrix form, $\frac{d}{dt}p(t) = p(t)A$, where A is an infinite matrix with null row sums and nonnegative off-diagonal entries.

In the theory of continuous-time Markov chains this is usually referred to as a q-matrix. Since it has tridiagonal form, all the processes with q-matrix A will be *birth and death processes*.

By studying A we will be able to identify exactly one process ξ_t on some new probability space $(S, \mathcal{S}, \mathcal{P})$ such that $p_n(t) = \mathcal{P}(\xi_t = n)$. Since ξ will turn out to be *dishonest* (meaning that $\mathcal{P}$-a.s. ξ will escape to infinity in finite time), the conclusion will be that $\lim_{t\to\infty} \sum_{n=1}^{\infty} \mathbb{E}^{Q}[X_n^2(t)] = 0$.

5.1. *Minimal process.* In general, given a q-matrix A, there can be many processes χ with different laws $y_n(t) = \mathcal{P}(\chi(t) = n)$, all satisfying either the forward $y' = yA$ or the backward $y' = Ay$ equations associated with A. Whether the solutions of the two systems are unique depends on some well-studied properties of the q-matrix.

In the present case, A is stable (no $-\infty$ entries appear in the diagonal) and conservative (no mass disappears at zero because $\mu_1 = 0$). It is well known that to any stable q-matrix is associated a process, called *minimal*, whose law satisfies both systems of equations.

The latter is the naive process that anyone would construct from A, as follows. Given a probability space $(S, \mathcal{S}, \mathcal{P})$, let ξ_t be a continuous-time Markov chain on the positive integers, with initial distribution

$$\mathcal{P}(\xi_0 = n) = p_n(0), \qquad n = 1, 2, \dots,$$

and jump rates given by A entries, that is, the process waits in a state n for an exponential time with rate $\lambda_n + \mu_n$ and then jumps at $n+1$ or $n-1$ with probabilities π_n and $1 - \pi_n$, respectively, where

$$\pi_n := \frac{\lambda_n}{\lambda_n + \mu_n}.$$

Let $\tau \in [0, \infty]$ denote the first time such that in $[0, \tau)$ the process has undergone infinitely many jumps. We say that the process reaches the boundary at time τ and we give no special "return" rule if the process reaches the boundary in finite time. Hence, if for $\omega \in S$, $\tau(\omega) < \infty$, then $\xi_t(\omega)$ is not defined for $t \geq \tau(\omega)$. Notice that, given $s > 0$, $\mathcal{P}(\tau > s) = \sum_{n=1}^{\infty} \mathcal{P}(\xi_s = n)$ could be less than 1.

If the minimal solution of a q-matrix is honest, it is the unique solution for each one of the two systems and the q-matrix itself is called regular. As anticipated, the minimal solution, which is the law of the process described above, is not regular if the coefficients k_n grow too fast (Proposition 8 below), nevertheless it is the unique solution of the forward equations (Proposition 9 below), while the backward equations have infinite solutions.

This uniqueness is very important because it ensures that $\mathcal{P}(\xi_t = n) = p_n(t) := \|x\|^{-2}\mathbb{E}^Q[X_n^2(t)]$. If we denote by $\mathcal{E}$ the total energy of X,

$$\mathcal{E}(t) := \frac{1}{2}\sum_{n=1}^{\infty} X_n^2(t) \tag{12}$$

this means in particular that we can study $\mathbb{E}^Q[\mathcal{E}(t)]$ through $\mathcal{P}(\tau > t)$,

$$\mathbb{E}^Q[\mathcal{E}(t)] = \frac{1}{2}\sum_{n=1}^{\infty}\mathbb{E}^Q[X_n^2(t)] = \mathcal{E}(0)\sum_{n=1}^{\infty} p_n(t) = \mathcal{E}(0)\mathcal{P}(\tau > t), \tag{13}$$

which will be the aim of Section 5.2.

PROPOSITION 8. *The q-matrix A is not regular if and only if $\sum_n nk_n^{-2} < \infty$.*

For the proof we make use of results by Reuter and Anderson, which are efficiently exposed in the book by the latter [2]. It is, however, not too difficult an exercise to prove the "if" direction with elementary notions. Truly, Proposition 11 and Lemma 14 below provide such an argument and we refer the reader who wants some insight to them.

PROOF OF PROPOSITION 8. By Corollary 2.2.5 of [2], in the conservative case, the minimal solution is honest if and only if the backward equations have a unique solution.

By Theorem 3.2.2 of [2] the q-matrix of a birth and death process has a unique solution of the backward equations if and only if the following quantity is infinite:

$$R = \sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n} + \frac{\mu_n}{\lambda_n\lambda_{n-1}} + \frac{\mu_n\mu_{n-1}}{\lambda_n\lambda_{n-1}\lambda_{n-2}} + \cdots + \frac{\mu_n\cdots\mu_2}{\lambda_n\cdots\lambda_2\lambda_1}\right).$$

Since $\lambda_n = \mu_{n+1}$, we get $R = \sum_n n\lambda_n^{-1} = \sum_n nk_n^{-2}$. $\square$

PROPOSITION 9. *The forward system of equations* (10), *together with condition* (11) *admits a unique solution.*

Here again the proposition can be seen as a simple application of a result from the book by Anderson, specifically Theorem 3.2.3 of [2].

Uniqueness could also be proved with an analytic approach, based on the parabolic structure of the equation which is apparent if we remember $\lambda_n = \mu_{n+1}$ and we rewrite (10) as

$$\frac{d}{dt}p_n(t) = \lambda_n\big(p_{n+1}(t) - p_n(t)\big) - \lambda_{n-1}\big(p_n(t) - p_{n-1}(t)\big), \qquad t \geq 0. \tag{14}$$

Nevertheless, we believe it would be interesting to show a completely different and entirely elementary proof that maybe could also be used when the parabolic nature is lost and the associated process is no more a simple birth and death.

PROOF OF PROPOSITION 9. By linearity we can suppose $p_n(0) = 0$, with condition (11) still holding and $-1 \le p_n(t) \le 1$.

Suppose by contradiction that $(p_n)_{n\in\mathbb{N}}$ is a nonzero solution. Without loss of generality we can suppose $p_1(t_0) = \delta > 0$ for some $t_0 > 0$ [take the largest n_0 such that $p_n \equiv 0$ for all $n < n_0$, so we have $p_{n_0}(t_0) > 0$ for some $t_0 > 0$; then shift and rename the indexes of the sequence (p_n) in such a way that the new p_1 is the old p_{n_0}]. Define the partial sums

$$\phi_n(t) := \sum_{j=1}^{n} p_j(t).$$

We notice that a simple computation starting from (14) yields

$$\frac{d}{dt}\phi_n = \lambda_n(p_{n+1} - p_n). \tag{15}$$

Then define the times

$$t_n := \inf\{t | \phi_n(t) \ge n\delta\}, \qquad n \ge 1. \tag{16}$$

We claim that for all $n \ge 1$,

$$t_n \le t_{n-1} \quad \text{and} \quad p_n(t_n) \ge \delta$$

so that the sequence $(t_n)_{n\ge 0}$ is finite (in fact, decreasing), in contradiction with the position $\phi_n(t) \le 1$ for all n and for all t.

We shall prove the claim by induction.

For $n = 1$, by definition $t_1 \le t_0 < +\infty$ and $p_1(t_1) = \phi_1(t_1) \ge \delta$.

Let us suppose that the claim holds for n. By the definition of t_n, $\frac{d}{dt}\phi_n(t_n) \ge 0$. By (15), this implies $p_{n+1}(t_n) \ge p_n(t_n) \ge \delta$ and hence

$$\phi_{n+1}(t_n) = \phi_n(t_n) + p_{n+1}(t_n) \ge n\delta + \delta$$

thus, $t_{n+1} \le t_n$. This implies that $\phi_n(t_{n+1}) \le n\delta$, so that necessarily

$$p_{n+1}(t_{n+1}) \ge \delta.$$

The induction and the proof are complete. $\square$

We remark the fact that given the condition $\sum_{n=1}^{\infty} nk_n^{-2} < \infty$, the forward equations have a unique solution while the backward have infinitely many. This fact might appear a bit disconcerting if one notices that A is symmetric and hence, forward and backward equations are formally identical. The explanation is that any proper solution $p_n(t)$ of the forward system of equations must be summable in the sense that $\sum_n p_n(t) \le 1$. On the contrary, if $\{q_n(t; k)\}_n$ is a solution of the backward equations with initial condition $q_n(0; k) = \delta_{k,n}$, it must satisfy $\sum_k q_n(t; k) \le 1$ for all n.

5.2. *Time of escape.* In this section we study the law of τ, the time of escape to infinity of the minimal process. The main result is Proposition 11, which is generalized by Lemma 14.

LEMMA 10. *Suppose $\sum_{i=1}^{\infty} k_i^{-2} < \infty$ and that the minimal process starts from* 1. *For $n \geq 1$, the number of times the minimal process visits state n is a geometric r.v. with mean $(k_n^2 + k_{n-1}^2) \sum_{i=n}^{\infty} k_i^{-2}$.*

PROOF. We follow ideas from Feller [14]. Let $p_{i,j}$ denote the transition probabilities of the discrete time Markov chain embedded in continuous-time minimal process and let $\sigma^{(i)} = \{\sigma_n^{(i)}\}_{n>i}$ denote the probabilities that the chain starting from states n larger than i will never get to i. Then $\sigma^{(i)}$ is the maximal solution of

$$x_n = \sum_{j>i} p_{n,j} x_j, \qquad n > i, \tag{17}$$

satisfying $0 \leq x_n \leq 1$ for all n. (This solution can be zero.)

If we let $x_i = 0$ for sake of notation, in our case the system (17) reduces to

$$x_n = \frac{\mu_n}{\lambda_n + \mu_n} x_{n-1} + \frac{\lambda_n}{\lambda_n + \mu_n} x_{n+1}, \qquad n \geq i + 1,$$

yielding

$$x_{n+1} - x_n = \frac{\mu_n}{\lambda_n}(x_n - x_{n-1}), \qquad n \geq i + 1,$$

and then by induction, for $n \geq i$,

$$x_{n+1} - x_n = x_{i+1} \prod_{j=k+1}^{n} \frac{\mu_j}{\lambda_j} = x_{i+1} \frac{k_i^2}{k_n^2},$$

$$x_n = x_{i+1} k_i^2 \sum_{m=i}^{n-1} k_m^{-2}.$$

By hypothesis the sums are bounded and hence, the maximal solution is obtained by choosing x_{i+1} such that $\lim_n x_n = 1$, that is,

$$\sigma_{i+1}^{(i)} = \left(k_i^2 \sum_{m=i}^{\infty} k_m^{-2} \right)^{-1},$$

hence, the chain is transient.

Now suppose that the chain is starting from 1. It will visit i at least once. When it does, the probability that it is the last visit is $p_{i,i+1}\sigma_{i+1}^{(i)}$, so by strong Markov property, the total number of visits to i is a geometric random variable with mean

$$\left(p_{i,i+1}\sigma_{i+1}^{(i)}\right)^{-1} = (k_i^2 + k_{i-1}^2) \sum_{m=i}^{\infty} k_m^{-2}$$

as required. $\square$

PROPOSITION 11. *Suppose* $\nu_\infty := \sum_{n=1}^\infty nk_n^{-2} < \infty$ *and that the minimal process starts from* 1. *Let* T_n *be the total time the minimal process spends in the state* n,

$$T_n := \mathcal{L}\{t \geq 0 : \xi_t = n\},$$

so that the time of escape at infinity is $\tau = \sum_{n=0}^\infty T_n$.

Then for all $n \geq 1$, T_n *is an exponential r.v. with mean* $\nu_n := \sum_{i=n}^\infty k_i^{-2}$ *and in particular*

$$\mathbb{E}^{\mathcal{P}}(\tau) = \sum_{n=1}^\infty \nu_n = \sum_{n=1}^\infty nk_n^{-2} = \nu_\infty.$$

Moreover, there exists $h > 0$ *such that for all* t

$$e^{-t/\nu_1} \leq \mathcal{P}(\tau > t) \leq e^{-t/\nu_\infty + h}.$$

PROOF. The total time spent in a state n is the sum of many i.i.d. exponential waiting times of rate $k_n^2 + k_{n-1}^2$.

Since the sum of a geometric number of i.i.d. exponential r.v.'s is exponential, by Lemma 10, T_n is exponential and its mean is as required.

The lower bound comes easily from $\tau = \sum_n T_n$, since

$$\mathcal{P}(\tau > t) \geq \mathcal{P}(T_1 > t) = e^{-t/\nu_1}.$$

For the upper bound, let $H := -\sum_{n=1}^\infty \nu_n \log \nu_n$ and let

$$h := H/\nu_\infty + \log \nu_\infty = -\sum_{n=1}^\infty \frac{\nu_n}{\nu_\infty} \log \frac{\nu_n}{\nu_\infty}.$$

We need to prove that

$$\mathcal{P}(\tau > t) \leq \nu_\infty e^{(H-t)/\nu_\infty}.$$

If t is such that $\nu_\infty e^{(H-t)/\nu_\infty} \geq 1$, we are done. Otherwise, define the sequence of numbers $(\theta_n)_{n\geq 1}$ in such a way that for all n,

$$e^{-t\theta_n/\nu_n} = \nu_n e^{(H-t)/\nu_\infty}.$$

The numbers θ_n are positive since $\nu_n \leq \nu_\infty$, moreover,

$$\sum_{n=1}^\infty \theta_n = \sum_{n=1}^\infty \left[-\frac{1}{t}\nu_n \log \nu_n - \frac{1}{t}(H-t)\frac{\nu_n}{\nu_\infty} \right] = 1.$$

Now,

$$\begin{aligned} \mathcal{P}(\tau > t) &\leq \mathcal{P}\left(\bigcup_{n=1}^\infty \{T_n > \theta_n t\} \right) \leq \sum_{n=1}^\infty \mathcal{P}(T_n > \theta_n t) \\ &= \sum_{n=1}^\infty e^{-t\theta_n/\nu_n} = e^{(H-t)/\nu_\infty} \sum_{n=1}^\infty \nu_n \end{aligned} \tag{18}$$

and the proof is complete. $\square$

REMARK 12. Our argument did not make use of the joint law of the T_n's which is unknown. If the r.v.'s T_n were independent, a standard exponential bound would yield $\mathcal{P}(\tau > t) \le e^{-t/\nu_1 + h'}$, so we have the strong feeling that $1/\nu_\infty$ is not a sharp bound for the true rate, which could actually be $1/\nu_1$.

It should also be noted that one cannot get rid of h in the previous statement; one can prove that $\frac{d}{dt} \log \mathcal{P}(\tau > t)|_{t=0} = 0$.

The following lemma clarifies that either $\tau = \infty$ a.s. or τ belongs to any interval $[a, b]$ with positive probability.

LEMMA 13. *Suppose $\mathcal{P}(\tau > T) < 1$ for some T. Then $\mathcal{P}(\tau > a) > \mathcal{P}(\tau > b)$ whenever $a < b$ and in particular $\mathcal{P}(\tau > t) < 1$ for all $t > 0$.*

PROOF. For $i \ge 1$ and $t \ge 0$, let

$$q_i(t) := \mathcal{P}(\tau > t | \xi_0 = i) = \sum_{n=1}^{\infty} p_{i,n}(t).$$

By Chapman–Kolmogorov,

$$\mathcal{P}(\tau > t) = \sum_{m=1}^{\infty} p_m(t-s) q_m(s) \le \mathcal{P}(\tau > t - s). \tag{19}$$

Suppose by contradiction that $\mathcal{P}(\tau > a) = \mathcal{P}(\tau > b)$, then letting $s = b - a$, $t = b$, we have equality in (19). This implies that $q_m(s) = 1$ for all m so that we get equality for *any* choice of $t > 0$, meaning that $\mathcal{P}(\tau > \cdot)$ is periodic as well as nonincreasing. The only possibility is $\mathcal{P}(\tau > t) = 1$ for all t. $\square$

We will need the following lemma, that is, the probability of no explosion is larger if one starts from 1 than in any other case. This is a well-known fact whose proof is a standard exercise we do not repeat here.

LEMMA 14. $\mathcal{P}(\tau > t) \le \mathcal{P}(\tau > t | \xi_0 = 1)$.

6. Bounds on the energy. We are finally able to give statements on the decay of the energy $\mathcal{E}(t)$ as $t \to \infty$. First of all we obtain exponential estimates under Q, both in $\mathcal{L}^1$ and pathwise. Then we introduce a smallness condition on $\mathcal{E}(0)$ which is what we need to translate these results under P. The section concludes with the proof of Theorem 1.

PROPOSITION 15. *Suppose $\nu_\infty := \sum_{n=1}^\infty n k_n^{-2} < \infty$. Let X be an energy controlled solution and denote by $\mathcal{E}(t) := \frac{1}{2}\sum_{n=1}^\infty X_n^2(t)$ its total energy at time t. Then*

$$\lim_{t\to\infty} \mathbb{E}^Q[\mathcal{E}(t)] = 0.$$

In particular there exists a positive number h such that for all $t \geq 0$

$$\mathbb{E}^Q[\mathcal{E}(t)] \leq e^{-t/\nu_\infty + h}\mathcal{E}(0). \tag{20}$$

PROOF. By definition

$$\mathbb{E}^Q[\mathcal{E}(t)] = \frac{1}{2}\sum_n \mathbb{E}^Q[X_n^2(t)] = \mathcal{E}(0)\sum_n p_n(t),$$

whence, by Proposition 9,

$$= \mathcal{E}(0)\sum_n \mathcal{P}(\xi_t = n) = \mathcal{E}(0)\mathcal{P}(\xi_t < \infty) = \mathcal{E}(0)\mathcal{P}(\tau > t).$$

Finally, we apply Lemma 14 and Proposition 11 and we find (20). □

Using Borel–Cantelli arguments, one can deduce from (20) some Q-a.s. statements about the decrease to zero of the energy, at least on given sequences of times going to infinity. To extend this to all sequences we will need the following lemma.

LEMMA 16. *Let X be an energy controlled solution and denote by $\mathcal{E}(t) := \frac{1}{2}\sum_{n=1}^\infty X_n^2(t)$ its total energy at time t. Then for every $t \geq s \geq 0$ we have*

$$Q\big(\mathcal{E}(t) \leq \mathcal{E}(s)\big) = 1.$$

PROOF. Let $s \geq 0$ be given and set $\chi = X(s)$. Consider the linear equation on $[s, \infty)$ with initial condition χ. It is proved in [4] that it has a unique strong solution Y, with the property

$$Q\left(\sum_{n=1}^\infty Y_n^2(t) \leq \sum_{n=1}^\infty \chi_n^2\right) = 1$$

for every $t \in [s, \infty)$ (the result in [4] is for constant initial conditions, but the extension to nonanticipative square integrable random initial conditions is straightforward). But also X restricted to $[s, \infty)$ is a solution of the same equation, hence, equal to Y, on $[s, \infty)$. The previous identity is thus equal to the claim of the lemma. □

Now we can prove Q-a.s. exponential decay of energy.

PROPOSITION 17. *Under the same hypothesis of Proposition 15, the total energy of solutions goes to zero at least exponentially fast pathwise under Q,*

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathcal{E}(t) \le -\frac{1}{\nu_\infty}, \qquad Q\text{-}a.s.$$

PROOF. Let $\varepsilon > 0$ be given. Set $\alpha := 1/\nu_\infty + \varepsilon$. We have

$$Q\big(n^{-1} \log \mathcal{E}(n) > \alpha\big) \le e^{-\alpha n} \mathbb{E}^Q[\mathcal{E}(n)] \le C e^{-\varepsilon n},$$

where, by Proposition 15, $C = e^h \mathcal{E}(0)$ does not depend on n. Hence, the above probabilities are summable on n and by Borel–Cantelli lemma there exists a measurable set N with $Q(N) = 0$ and the following property: for every $\omega \in N^c$ there exists $n_0(\omega)$ such that, for all $n \ge n_0(\omega)$, $\mathcal{E}(n, \omega) \le e^{-\alpha n}$. Taking the supremum for $n < n_0(\omega)$, we obtain that there exists a constant $C(\omega) > 0$ such that

$$\mathcal{E}(n, \omega) \le C(\omega) e^{-\alpha n}$$

for all $\omega \in N^c$ and $n \ge 0$. From Lemma 16, there exists a measurable set $\widetilde{N}$ with $Q(\widetilde{N}) = 0$ such that $\mathcal{E}(r, \omega) \le \mathcal{E}(\lfloor r \rfloor, \omega)$ for all $\omega \in \widetilde{N}^c$ and all rational numbers $r \in [0, \infty)$. This implies, for $\omega \in N^c \cap \widetilde{N}^c$,

$$\mathcal{E}(r, \omega) \le \mathcal{E}(\lfloor r \rfloor, \omega) \le C(\omega) e^{-\alpha \lfloor r \rfloor} \le C'(\omega) e^{-\alpha r}$$

with $C'(\omega) = C(\omega) e^{\alpha'}$, for all $r \in [0, \infty) \cap \mathbb{Q}$.

With Q probability one, the function $\mathcal{E}(t)$ is lower semicontinuous, being the supremum in N of the functions $\sum_{n=1}^N X_n^2(t)$ which are continuous. Thus we get

$$Q\big(\mathcal{E}(t) \le C' e^{-\alpha t} \text{ for every } t \ge 0\big) = 1.$$

Letting ε go to zero on the rationals completes the proof. $\square$

The reader should be aware that this proposition does not automatically hold under P. Truly, P and Q are equivalent on all F_t, but C' is F_∞-measurable and not F_t-measurable for any t.

In general, we cannot prove that P and Q are equivalent on F_∞, so we cannot translate such claim into a similar statement on the original nonlinear equation (1).

When $\mathcal{E}(0)$ is small enough, however, we can prove the equivalence and hence, compute exponential upper bounds for $\mathcal{E}(t)$, both P-a.s. and in mean value.

PROPOSITION 18. *Let X be an energy controlled solution and denote by $\mathcal{E}(t) := \frac{1}{2} \sum_{n=1}^\infty X_n^2(t)$ its total energy at time t. Suppose $\nu_\infty := \sum_{n=1}^\infty n k_n^{-2} < \infty$ and $\nu_\infty \mathcal{E}(0) < 1$. Then*

$$\mathbb{E}^Q\big[e^{\int_0^\infty \mathcal{E}(t)\,dt}\big] < \infty,$$

so that in particular, P and Q are equivalent on F_∞.

PROOF. By Proposition 15 and the definition of energy controlled solution, we have

$$\mathbb{E}^Q[\mathcal{E}(t)] < \mathcal{E}(0)e^{-t/\nu_\infty+h} \qquad \forall t \geq 0,$$
$$0 \leq \mathcal{E}(t) \leq \mathcal{E}(0), \qquad Q\text{-a.s.}$$

Let $X := \int_0^\infty \mathcal{E}(t)\,dt \geq 0$ and $x \geq 0$. Then

$$\begin{aligned} xQ(X > x) &\leq \mathbb{E}^Q[X; X > x] = \int_0^\infty \mathbb{E}^Q[\mathcal{E}(t); X > x]\,dt \\ &\leq \int_0^\infty \min\bigl(\mathbb{E}^Q[\mathcal{E}(t)]; \mathcal{E}(0)Q(X > x)\bigr)\,dt \\ &\leq \mathcal{E}(0)\int_0^\infty \min\bigl(e^{-t/\nu_\infty+h}; Q(X > x)\bigr)\,dt \\ &= \mathcal{E}(0)\int_0^u Q(X > x)\,dt + \mathcal{E}(0)\int_u^\infty e^{-t/\nu_\infty+h}\,dt, \end{aligned}$$

where u is such that $e^{-u/\nu_\infty+h} = Q(X > x)$. Hence,

$$xQ(X > x) \leq \mathcal{E}(0)Q(X > x)u + \mathcal{E}(0)\nu_\infty e^{-u/\nu_\infty+h} = \mathcal{E}(0)Q(X > x)(u + \nu_\infty).$$

If $Q(X > x) = 0$ for some $x > 0$, then clearly X is bounded and we are done. Otherwise we get

$$u \geq \frac{x}{\mathcal{E}(0)} - \nu_\infty,$$

that is

$$Q(X > x) = e^{-u/\nu_\infty+h} \leq e^{-x/(\nu_\infty\mathcal{E}(0))+h+1},$$

yielding

$$Q(e^X > y) \leq y^{-1/(\nu_\infty\mathcal{E}(0))}e^{h+1}$$

and finally

$$\mathbb{E}^Q[e^X] = \int_0^\infty Q(e^X > y)\,dy \leq 1 + e^{h+1}\int_1^\infty y^{-1/(\nu_\infty\mathcal{E}(0))}\,dy < \infty,$$

where we used $\nu_\infty\mathcal{E}(0) < 1$. $\square$

The heuristic behind the proof, which is quite hidden, that is, given that $\mathbb{E}^Q[X]$ is bounded, $\mathbb{E}^Q[e^X]$ is maximum if X is spread as much as possible. Since $X = \int \mathcal{E}(t)\,dt$ and $\mathcal{E}(t) \in [0, \mathcal{E}(0)]$, this is done by choosing $\mathcal{E}(t, \omega) \in \{0, \mathcal{E}(0)\}$, and in particular $\mathcal{E}(t, \omega) = \mathcal{E}(0)\mathbb{I}_{[0,y(\omega)]}(t)$.

COROLLARY 19. *Under the same hypothesis of Proposition 18, meaning in particular that $\mathcal{E}(0) < 1/\nu_\infty$, the energy goes to zero at least exponentially fast pathwise under P,*

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathcal{E}(t) \le -\frac{1}{\nu_\infty}, \qquad P\text{-a.s.}$$

PROOF. Just a direct consequence of Propositions 17 and 18. □

The same condition on the smallness of $\mathcal{E}(0)$ arises when we want to establish an exponential decay for the mean value of $\mathcal{E}(t)$ under P.

PROPOSITION 20. *If*

$$\mathbb{E}^Q[\mathcal{E}(t)] \le \mathcal{E}(0)e^{-\alpha t+h}$$

then

$$\mathbb{E}^P[\mathcal{E}(t)] \le \mathcal{E}(0)\exp\bigl((1-1/p)\bigl[h+\bigl(p\mathcal{E}(0)-\alpha\bigr)t\bigr]\bigr) \tag{21}$$

for every $p > 1$. In particular, under the same hypothesis of Proposition 18,

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}^P[\mathcal{E}(t)] \le -\frac{1}{\nu_\infty}\bigl(1-\sqrt{\mathcal{E}(0)\nu_\infty}\bigr)^2. \tag{22}$$

PROOF. The density f_t of P with respect to Q on F_t is

$$f_t = \exp\bigl(M_t + \tfrac{1}{2}[M]_t\bigr), \tag{23}$$

where

$$M_t := \sum_{n=1}^\infty \int_0^t X_n(s)\,dW_n(s), \qquad [M]_t = \int_0^t \sum_{n=1}^\infty X_n^2(s)\,ds.$$

From (6) we have

$$\exp(\lambda[M]_t) \le e^{2\lambda\mathcal{E}(0)t} \tag{24}$$

for every $\lambda > 0$.

For every $p, p' > 1$ with $\frac{1}{p} + \frac{1}{p'} = 1$, from the a.s. condition $\mathcal{E}(t) \le \mathcal{E}(0)$ and the assumption of the proposition we have

$$\begin{aligned}
\mathbb{E}^P[\mathcal{E}(t)] = \mathbb{E}^Q[f_t\mathcal{E}(t)] &\le \mathbb{E}^Q[f_t^p]^{1/p}\mathbb{E}^Q[\mathcal{E}(t)^{p'}]^{1/p'} \\
&\le \mathbb{E}^Q[f_t^p]^{1/p}\mathbb{E}^Q[\mathcal{E}(t)\mathcal{E}(0)^{p'-1}]^{1/p'} \\
&\le \mathcal{E}(0)^{1-1/p'}\mathbb{E}^Q[f_t^p]^{1/p}\mathcal{E}(0)^{1/p'}e^{-(\alpha/p')t+h/p'} \\
&= \mathcal{E}(0)\mathbb{E}^Q[f_t^p]^{1/p}e^{-(\alpha/p')t+h/p'}.
\end{aligned}$$

From (23) we have

$$\begin{aligned}\mathbb{E}^Q[f_t^p] &= \mathbb{E}^P[f_t^{p-1}] \\ &= \mathbb{E}^P\left[\exp\left((p-1)M_t + \frac{(p-1)}{2}[M]_t\right)\right] \\ &= \mathbb{E}^P\left[\exp\left((p-1)M_t - \frac{(p-1)^2}{2}[M]_t\right)\exp\left(\frac{(p-1)p}{2}[M]_t\right)\right]\end{aligned}$$

and now we use (24) to get

$$\le e^{(p-1)p\mathcal{E}(0)t}\mathbb{E}^P\big[e^{(p-1)M_t-((p-1)^2/2)[M]_t}\big] = e^{(p-1)p\mathcal{E}(0)t} = e^{(p^2/p')\mathcal{E}(0)t}$$

by Girsanov's theorem. To summarize:

$$\mathbb{E}^P[\mathcal{E}(t)] \le \mathcal{E}(0)e^{(1/p')p\mathcal{E}(0)t}e^{-(\alpha/p')t+h/p'}$$

which implies the first claim of the proposition.

To prove the last statement, let $\alpha = 1/\nu_\infty$. Then optimization on p under the condition $\mathcal{E}(0)\nu_\infty < 1$ gives that the right-hand side of (21) is minimum when p is equal to $\phi(t) = \sqrt{\frac{1/\nu_\infty - h/t}{\mathcal{E}(0)}}$. [We notice that $\phi(t) > 1$ for $t > \frac{h}{1/\nu_\infty - \mathcal{E}(0)}$.] With a simple computation, (21) becomes

$$\mathbb{E}^P[\mathcal{E}(t)] \le \mathcal{E}(0)\exp\left\{-(p-1)\left(\frac{\phi^2(t)}{p} - 1\right)\mathcal{E}(0)t\right\}.$$

Letting $p = \phi(t) > 1$, we get

$$\frac{1}{t}\log\mathbb{E}^P[\mathcal{E}(t)] \le \frac{1}{t}\log\mathcal{E}(0) - (\phi(t)-1)^2\mathcal{E}(0), \qquad t > \frac{h}{1/\nu_\infty - \mathcal{E}(0)}.$$

Taking the limsup for $t \to \infty$ leads to (22). □

6.1. *Proof of Theorem* 1. We have $\nu_\infty = \sum_{n=1}^\infty nk_n^{-2} < \infty$, so by virtue of Proposition 8, $\mathcal{P}(\tau > t) < 1$ for some t. Then by Lemma 13, $\mathcal{P}(\tau > t) < 1$ for all t. By equation (13) $\mathbb{E}^Q[\mathcal{E}(t)] < \mathcal{E}(0)$ and since $\mathcal{E}(t) \le \mathcal{E}(0)$ Q-a.s. we get $Q(\mathcal{E}(t) = \mathcal{E}(0)) < 1$, for all t. Equivalence of P and Q on F_t yields the first statement.

By Proposition 15, $Q(\mathcal{E}(t) > \varepsilon) \le Ke^{-t/\nu_\infty}$ for some K not depending on t, hence, for t large enough, this event has Q- (and hence P-) probability less than 1, so we proved the second statement.

The third statement is proved in Corollary 19 and Proposition 20.

The proof of Theorem 1 is complete.

7. Lack of regular solutions. As a consequence of our result on the dissipation of energy we can prove that there exists no regular solution.

Define the space

$$V = \left\{ x \in l^2 : \sum_{n=1}^{\infty} k_n^2 x_n^2 < \infty \right\}$$

which is an Hilbert space under the norm $\|x\|_V^2 = \sum_{n=1}^{\infty} k_n^2 x_n^2$.

PROPOSITION 21. *Assume that* $\{X(t); t \in [0, T]\}$ *is an energy controlled solution. Then*,

$$P\left(\int_0^T \|X(t)\|_V^2 \, dt = \infty\right) > 0.$$

PROOF. We will actually prove the following statement. Assume that $\{X(t); t \in [0, T]\}$ is an energy controlled solution such that

$$P\left(\int_0^T \|X(t)\|_V^2 \, dt < \infty\right) = 1. \tag{25}$$

Then, for every $t \in [0, T]$, $P(\mathcal{E}(t) = \mathcal{E}(0)) = 1$.

Since the latter is in contradiction with Theorem 1, then (25) will be proven to be false.

Let X be a solution as in the claim. By Itô's formula,

$$\begin{aligned} d\left(\sum_{n=1}^{N} X_n^2\right) &= 2\sum_{n=1}^{N}(k_{n-1}X_{n-1}^2 X_n - k_n X_n^2 X_{n+1})\, dt \\ &\quad - \sum_{n=1}^{N}(k_n^2 + k_{n-1}^2) X_n^2 \, dt + \sum_{n=1}^{N}(k_{n-1}^2 X_{n-1}^2 + k_n^2 X_{n+1}^2)\, dt \\ &\quad + 2\sum_{n=1}^{N}(k_{n-1}X_{n-1}X_n \, dW_{n-1} - k_n X_n X_{n+1}\, dW_n) \\ &= -2k_N X_N^2 X_{N+1}\, dt \\ &\quad - k_N^2 X_N^2 \, dt + k_N^2 X_{N+1}^2 \, dt \\ &\quad - 2k_N X_N X_{N+1}\, dW_N. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{n=1}^{N} X_n^2(t) - \sum_{n=1}^{N} x_n^2 &= \int_0^t (-2k_N X_N^2 X_{N+1} + k_N^2 X_{N+1}^2 - k_N^2 X_N^2)\, ds \\ &\quad - \int_0^t 2k_N X_N X_{N+1}\, dW_N(s). \end{aligned} \tag{26}$$

Notice that $|X_{N+1}| \leq \|x\|_{l^2}$, $k_N \geq 1$ and $k_N \leq k_{N+1}$, so that the first integral can be bounded by

$$C \int_0^t (k_N^2 X_N^2 + k_{N+1}^2 X_{N+1}^2 + k_N^2 X_N^2)\, ds.$$

Then the a.s. inequality $\int_0^T \|X(t)\|_V^2\, dt < \infty$ implies that $\int_0^T k_N^2 X_N^2\, ds$ goes to zero a.s. and in probability.

The latter is true also for the stochastic integral in (26) by continuity in probability of stochastic integrals; since $k_N X_N X_{N+1}$ converges to zero in probability in $L^2(0,T)$, its stochastic integral converges to zero in probability. This can be easily proved by applying the following well-known inequality of stochastic integrals. For every $\gamma, \delta > 0$,

$$P\left(\left|\int_0^t k_N X_N X_{N+1}\, dW_N(s)\right| > \gamma\right) \leq P\left(\int_0^t k_N^2 X_N^2 X_{N+1}^2\, ds > \delta\right) + \frac{\delta}{\gamma^2}.$$

Finally, all terms on the RHS of (26) converge to zero in probability. We get

$$\sum_{n=1}^N X_n^2(t) \xrightarrow[N\to\infty]{P} \|x\|_{l^2}^2.$$

But we know, by P-a.s. monotonicity, that

$$\sum_{n=1}^N X_n^2(t) \xrightarrow[N\to\infty]{P} \sum_{n=1}^\infty X_n^2(t).$$

Hence, the claim is proved and the proposition as well. □

REMARK 22. If one could prove local existence (up to some random time) of solutions in V with initial conditions in V, the above proposition would show a blow-up in the V norm. This appears to be a difficult open problem in the stochastic case, while in the deterministic case, it is well known (see, e.g., [7]).

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