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RANDOMIZED SCHEDULING ALGORITHM FOR QUEUEING NETWORKS[1]

BY DEVAVRAT SHAH AND JINWOO SHIN

Massachusetts Institute of Technology

There has recently been considerable interest in design of low-complexity, myopic, distributed and stable scheduling algorithms for constrained queueing network models that arise in the context of emerging communication networks. Here we consider two representative models. One, a queueing network model that captures randomly varying number of packets in the queues present at a collection of wireless nodes communicating through a shared medium. Two, a buffered circuit switched network model for an optical core of future internet to capture the randomness in calls or flows present in the network. The maximum weight scheduling algorithm proposed by Tassiulas and Ephremides [*IEEE Trans. Automat. Control* **37** (1992) 1936–1948], leads to a myopic and stable algorithm for the packet-level wireless network model. But computationally it is expensive (NP-hard) and centralized. It is not applicable to the buffered circuit switched network due to the requirement of nonpreemption of the calls in the service. As the main contribution of this paper, we present a stable scheduling algorithm for both of these models. The algorithm is myopic, distributed and performs few logical operations at each node per unit time.

1. Introduction. The primary task of a communication network architect is to provision as well as utilize network resources efficiently to satisfy the demands imposed on it. The main algorithmic problem is that of allocating or scheduling resources among various entities or data units, for example, packets, flows, that are contending to access them. In recent years the question of designing a simple, myopic, distributed and high-performance (aka stable) scheduling algorithm has received considerable interest in the context of emerging communication network models. Two such models that we consider in this paper are that of a wireless network and a buffered circuit switched network.

The wireless network consists of wireless transmission capable nodes. Each node receives exogenous demand in the form of packets. These nodes communicate these packets through a shared wireless medium. Hence, their simultaneous

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transmissions may contend with each other. The purpose of a scheduling algorithm is to resolve these contentions among transmitting nodes so as to utilize the wireless network bandwidth efficiently while keeping the queues at nodes finite. Naturally, the desired scheduling algorithm should be distributed, simple/low-complexity and myopic (i.e., utilize only the network state information like queue-sizes).

The buffered circuit switched network can be utilized to model the dynamics of flows or calls in an optical core of future internet. Here a link capacitated network is given with a collection of end-to-end routes. At the ingress (i.e., input or entry point) of each route, calls arriving as per exogenous process are buffered or queued. Each such call desires resources on each link of its route for a random amount of time duration. Due to link capacity constraints, calls of routes sharing links contend for resources. And a scheduling algorithm is required to resolve this contention so as to utilize the network links efficiently while keeping buffers or queues at ingress of routes finite. Again, the scheduling algorithm is desired to be distributed, simple and myopic.

An important scheduling algorithm is the maximum weight algorithm that was proposed by Tassiulas and Ephremides [31]. It was proposed in the context of a packet queueing network model with generic scheduling constraints. It is primarily applicable in a scenario where scheduling decisions are synchronized or made every discrete time. It suggests scheduling queues, subject to constraints, that have the maximum net weight at each time step with the weight of a queue being its queue-size. They established stability or throughput optimality (precisely, positive recurrence and subsequently ergodicity of the associated network Markov process) property of this algorithm for this general class of networks. Further, this algorithm, as the description suggests, is myopic. Due to the general applicability and myopic nature, this algorithm and its variants have received a lot of attention in recent years (see, e.g., [3, 4, 19, 25, 26, 28]).

The maximum weight algorithm provides a myopic and stable scheduling algorithm for the wireless network model. However, it requires solving a combinatorial optimization problem, the maximum weight independent set problem, to come up with a schedule every time. And the problem of finding a maximum weight independent set is known to be NP-hard as well as hard to approximate in general [32]. To address this concern, there has been a long line of research conducted to devise implementable approximations of the maximum weight scheduling algorithm (e.g., [6, 10, 18, 22, 29]). A comprehensive survey of such maximum weight inspired and other algorithmic approaches that have been studied over more than four decades in the context of wireless networks can be found in [14, 24].

In the context of buffered circuit switched network, calls have random service requirement. Therefore, scheduling decisions cannot be synchronized and the maximum weight scheduling algorithm is not applicable. In [30], a "batching-like" modification of the maximum weight algorithm was proposed for such a network model. However, no entirely myopic (nonbatching) and distributed algorithm is known to be stable for this network model.

1.1. *Contributions*. We propose a scheduling algorithm for both wireless and buffered circuit switched network model. The algorithm utilizes only local queue-size information to make scheduling decisions. That is, the algorithm is myopic and distributed. It is randomized and requires each queue (or node) in the network to perform few (literally, constant) logical operations per scheduling decision. We establish that it is stable or throughput optimal. That is, the associated network Markov process is ergodic as long as the network is under-loaded.

The basic idea behind the algorithm design is simple. The randomized scheduling algorithm can be seen as a distributed (reversible) Markovian dynamics over the space of schedules with the transition probabilities dependent on the queue-sizes. For the wireless network, it corresponds to the known Glauber dynamics; cf. [17] over the space of independent sets of the wireless network interference graph; for the buffered circuit switched network, it corresponds to the known stochastic loss network; cf. [15].

The stationary distribution of this reversible dynamics over schedules, assuming queue-sizes are fixed, has a product-form. The variational characterization of this product-form stationary distribution suggests that the expected weight of schedule (with respect to this product-form stationary distribution) is close to that of the maximum weight schedule which is essentially sufficient for establishing stability. Therefore, assuming that queue-sizes are essentially fixed or, change over a much slower time-scale compared to the time-scale over which scheduling dynamics reaches stationarity, the algorithm is effectively the maximum weight.

The main technical contribution of this paper is in effectively establishing the validity of such a "time-scale separation" assumption between the network queueing dynamics and the scheduling dynamics induced by the algorithm. To make this possible, we use an appropriately slowly changing function such as $f(x) = \log\log(x+e)$ of queue-size as weight in contrast to the standard $f(x) = x$ weight function; cf. [31]. Selection of such a weight function helps because, even though the queue-sizes changes at a constant rate, the function [like $\log\log(\cdot + e)$] of queue-sizes change very slowly. This effectively induces the desired "time-scale separation." Technically, establishing validity of time-scale separation requires studying the mixing property of time varying Markov chain over the space of schedules.

To establish the stability (ergodicity of associated network Markov process) property of the algorithm, we exhibit an appropriate Lyapunov function. We note that use of Lyapunov function for establishing stability is somewhat standard now (see, e.g., [25, 28, 31]). Usually difficulty lies in finding an appropriate candidate function followed by establishing that it is indeed a Lyapunov function.

1.2. *Organization*. We start by describing two network models, the wireless network and the buffered circuit switched network in Section 2. We formally introduce the problem of scheduling and performance metric for scheduling algorithms. The maximum weight scheduling algorithm is described as well. Our randomized algorithm and its throughput optimality for both network models are presented in Section 3. The paper beyond Section 3 is dedicated to establishing the throughput optimality. Necessary technical preliminaries are presented in Section 4. Here we relate our algorithm for both models with appropriate reversible Markov chains on the space of schedules and state useful properties of these Markov chains. We also describe known sufficient conditions, including Lyapunov drift criteria, for establishing ergodicity. Detailed proofs of our main results are presented in Section 5.

2. Setup.

2.1. *Wireless network.* We consider a *single-hop* wireless network of n (≥ 2) queues. Queues receive work as per exogenous arrivals and work leaves the system upon receiving service. Specifically, let $Q_i(t) \in \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ denote the amount of work in the ith queue at time $t \in \mathbb{R}_+$ and $\mathbf{Q}(t) = [Q_i(t)]_{1 \leq i \leq n}$; initially $t = 0$ and $\mathbf{Q}(0) = \mathbf{0}$.[2] Work arrives at each queue in terms of unit-sized packets as per a discrete-time process. Let $A_i(s,t)$ denote the amount of work arriving at queue i in time interval $[s,t]$ for $0 \leq s < t$. For simplicity, assume that for each i, $A_i(\cdot)$ is an independent Bernoulli process with parameter λ_i, where $A_i(\tau) \stackrel{\Delta}{=} A_i(0,\tau)$. That is, $A_i(\tau+1) - A_i(\tau) \in \{0,1\}$ and $\Pr(A_i(\tau+1) - A_i(\tau) = 1) = \lambda_i$ for all i and $\tau \in \mathbb{Z}_+ = \{k \in \mathbb{Z} : k \geq 0\}$. Throughout this paper we will use $\tau \in \mathbb{Z}_+$ and $t \in \mathbb{R}_+$ to denote discrete and continuous time, respectively. Denote the arrival rate vector as $\boldsymbol{\lambda} = [\lambda_i]_{1 \leq i \leq n}$. We assume that arrivals happen at the end of a time slot.

The work from queues is served at the unit rate, but subject to *interference* constraints. Specifically, let $G = (V, E)$ denote the inference graph between the n queues, represented by vertices $V = \{1, \dots, n\}$ and edges E; an $(i,j) \in E$ implies that queues i and j cannot transmit simultaneously since their transmission *interferes* with each other. Formally, let $\sigma_i(t) \in \{0,1\}$ denotes whether the queue i is transmitting at time t, that is, work in queue i is being served at unit rate at time t and $\boldsymbol{\sigma}(t) = [\sigma_i(t)]$. Then, it must be that for $t \in \mathbb{R}_+$,

$$\boldsymbol{\sigma}(t) \in \mathcal{I}(G) \stackrel{\Delta}{=} \{\boldsymbol{\rho} = [\rho_i] \in \{0,1\}^n : \rho_i + \rho_j \leq 1 \text{ for all } (i,j) \in E\}.$$

The total amount of work served at queue i in time interval $[s,t]$ is

$$D_i(s,t) = \int_s^t \sigma_i(y) \mathbf{I}_{\{Q_i(y) > 0\}}\, dy,$$

where $\mathbf{I}_{\{x\}}$ denotes the indicator function.

In summary, the above model induces the following queueing dynamics: for any $0 \leq s < t$ and $1 \leq i \leq n$,

$$Q_i(t) = Q_i(s) - \int_s^t \sigma_i(y) \mathbf{I}_{\{Q_i(y) > 0\}}\, dy + A_i(s,t).$$

[2]Bold letters are reserved for vectors; $\mathbf{0}$, $\mathbf{1}$ represent vectors of all 0's and all 1's, respectively.

2.2. *Buffered circuit switched network.* We consider a buffered circuit switched network. Here the network is represented by a capacitated graph $G = (V, E)$ with V being vertices, $E \subset V \times V$ being links (or edges) with each link $e \in E$ having a finite integral capacity $C_e \in \mathbb{N}$. This network is accessed by a fixed set of n (≥ 2) routes $R_1, \dots, R_n$; each route is a collection of interconnected links. At each route R_i, flows arrive as per an exogenous arrival process. For simplicity, we assume it to be an independent Poisson process of rate λ_i and let $A_i(s, t)$ denote total number of flow arrivals at route R_i in time interval $[s, t]$. Upon arrival of a flow at route R_i, it joins the queue or buffer at the ingress of R_i. Let $Q_i(t)$ denote the number of flows in this queue at time t; initially $t = 0$ and $Q_i(0) = 0$.

Each flow arriving at R_i, comes with the service requirement of using unit capacity simultaneously on all the links of R_i for a time duration—it is assumed to be distributed independently as per exponential of unit mean. Now a flow in the queue of route R_i can get simultaneous possession of links along route R_i in the network at time t, if there is a unit capacity available at *all* of these links. To this end, let $z_i(t)$ denote the number of flows that are *active* along route R_i, that is, posses links along the route R_i. Then, by capacity constraints on the links of the network, it must be that $z(t) = [z_i(t)]$ satisfies

$$\mathbf{z}(t) \in \mathcal{X} \stackrel{\Delta}{=} \left\{ \mathbf{z} = [z_i] \in \mathbb{Z}_+^n : \sum_{i\,:\,e \in R_i} z_i \leq C_e, \forall e \in E \right\}.$$

This represents the scheduling constraints of the circuit switched network model similar to the interference constraints of the wireless network model.

Finally, a flow active on route R_i departs the network after the completion of its service requirement and frees unit capacity on the links of R_i. Let $D_i(s, t)$ denote the number of flows which are served (hence, leave the system) in time interval $[s, t]$.

2.3. *Scheduling algorithm and performance metric.* In both models described above, the scheduling is the key operational question. In the wireless network, queues need to decide which of them transmit subject to interference constraints. In the circuit switched network, queues need to agree on which flows becomes active subject to network capacity constraints. And, a *scheduling algorithm* is required to make these decisions every time.

In the wireless network, the scheduling algorithm decides the schedule $\boldsymbol{\sigma}(t) \in \mathcal{I}(G)$ at each time t. We are interested in *distributed* scheduling algorithms, that is, queue i decides $\sigma_i(t)$ using its local information such as its queue-size $Q_i(t)$. We assume that queues have instantaneous *carrier sensing* information, that is, if a queue (or node) j starts transmitting at time t, then all neighboring queues can *listen* to this transmission immediately.

In buffered circuit switched network, the scheduling algorithm decides active flows or schedules $\mathbf{z}(t)$ at time t. Again, our interest is in *distributed* scheduling algorithms, that is, queue at ingress of route R_i decides $z_i(t)$ using its local information. Each queue (or route) can obtain instantaneous information on whether all links along its route have unit capacity available or not.

In summary, both models need scheduling algorithms to decide when each queue (or its ingress port) will request the network for availability of resources; upon a positive answer (or successful request) from the network, the queue acquires network resources for a certain amount of time. And these decisions need to be based on local information.

From the perspective of network performance, we would like the scheduling algorithm to be such that the queues in network remain as small as possible for the largest possible range of arrival rate vectors. To formalize this notion of performance, we define the capacity regions for both of these models. Let $\mathbf{\Lambda}_w$ be the capacity region of the wireless network model defined as

$$\begin{aligned}\mathbf{\Lambda}_w &= \mathsf{Conv}(\mathcal{I}(G)) \\ &= \Big\{\mathbf{y} \in \mathbb{R}_+^n : \mathbf{y} \leq \sum_{\boldsymbol{\sigma} \in \mathcal{I}(G)} \alpha_{\boldsymbol{\sigma}} \boldsymbol{\sigma}, \text{ with } \alpha_{\boldsymbol{\sigma}} \geq 0, \text{ and } \sum_{\boldsymbol{\sigma} \in \mathcal{I}(G)} \alpha_{\boldsymbol{\sigma}} \leq 1\Big\}.\end{aligned} \tag{1}$$

And let $\mathbf{\Lambda}_{cs}$ be the capacity region of the buffered circuit switched network defined as

$$\begin{aligned}\mathbf{\Lambda}_{cs} &= \mathsf{Conv}(\mathcal{X}) \\ &= \Big\{\mathbf{y} \in \mathbb{R}_+^n : \mathbf{y} \leq \sum_{\mathbf{z} \in \mathcal{X}} \alpha_{\mathbf{z}} \mathbf{z}, \text{ with } \alpha_{\mathbf{z}} \geq 0, \text{ and } \sum_{\mathbf{z} \in \mathcal{X}} \alpha_{\mathbf{z}} \leq 1\Big\}.\end{aligned} \tag{2}$$

Intuitively, these bounds of capacity regions come from the fact that any algorithm produces the "service rate" from $\mathcal{I}(G)$ (or $\mathcal{X}$) each time and hence, the time average of the service rate induced by any algorithm must belong to its convex hull. Therefore, if arrival rates $\boldsymbol{\lambda}$ can be "served well" by any algorithm, then it must belong to $\mathsf{Conv}(\mathcal{I}(G))$ [or $\mathsf{Conv}(\mathcal{X})$].

Motivated by this, we call an arrival rate vector $\boldsymbol{\lambda}$ admissible if $\boldsymbol{\lambda} \in \mathbf{\Lambda}$ and say that an arrival rate vector $\boldsymbol{\lambda}$ is strictly admissible if $\boldsymbol{\lambda} \in \mathbf{\Lambda}^o$, where $\mathbf{\Lambda}^o$ is the interior of $\mathbf{\Lambda}$ formally defined as

$$\mathbf{\Lambda}^o = \{\boldsymbol{\lambda} \in \mathbb{R}_+^n : \boldsymbol{\lambda} < \boldsymbol{\lambda}^* \text{ componentwise, for some } \boldsymbol{\lambda}^* \in \mathbf{\Lambda}\}.$$

Equivalently, we may say that the network is *under-loaded*. Now we are ready to define a performance metric for a scheduling algorithm. Specifically, we desire the scheduling algorithm to be throughput optimal as defined below.

DEFINITION 1 (Throughput optimal). A scheduling algorithm is called throughput optimal, or stable, or providing 100% throughput, if for any $\boldsymbol{\lambda} \in \mathbf{\Lambda}^o$ the (appropriately defined) underlying network Markov process is ergodic.

In the above definition and throughout this paper, by ergodic we mean that (a) the network Markov process has a unique stationary distribution and (b) starting from any initial state, the distribution of the Markov process converges to this stationary distribution.

2.4. *The* MW *algorithm.* Here we describe a popular algorithm known as the maximum weight, or in short, MW, algorithm that was proposed by Tassiulas and Ephremides [31]. It is throughput optimal for a large class of network models. The algorithm readily applies to the wireless network model. However, it does not apply (exactly) in the case of circuit switched network. This algorithm requires solving a hard combinatorial problem each time slot, for example, maximum weight independent set for wireless network, which is NP-hard in general. Therefore, it is far from being practically useful. In a nutshell, the randomized algorithm proposed in this paper will overcome these drawbacks of the MW algorithm while retaining the throughput optimality property. For completeness, next we provide a brief description of the MW algorithm.

In the wireless network model, the MW algorithm chooses a schedule $\boldsymbol{\sigma}(\tau) \in \mathcal{I}(G)$ every time step $\tau \in \mathbb{Z}_+$ as follows[3]:

$$\boldsymbol{\sigma}(\tau) \in \arg\max_{\boldsymbol{\rho} \in \mathcal{I}(G)} \mathbf{Q}(\tau) \cdot \boldsymbol{\rho}.$$

In other words, the algorithm changes its decision once in unit time utilizing the information $\mathbf{Q}(\tau)$. The maximum weight property allows one to establish positive recurrence by means of Lyapunov drift criteria (see Lemma 5) when the arrival rate is admissible, that is, $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_w^o$. However, as indicated above, picking such a schedule every time is computationally burdensome. A natural generalization of this, called MW-f algorithm, that uses weight $f(Q_i(\cdot))$ instead of $Q_i(\cdot)$ for an increasing nonnegative function f also leads to throughput optimality; cf. see [25, 26, 28].

For the buffered circuit switched network model, the MW algorithm is not applicable. To understand this, consider the following. The MW algorithm would require the network to schedule active flows as $\mathbf{z}(\tau) \in \mathcal{X}$ where

$$\mathbf{z}(\tau) \in \arg\max_{\mathbf{z} \in \mathcal{X}} \mathbf{Q}(\tau) \cdot \mathbf{z}.$$

This will require the algorithm to possibly preempt some of active flows without the completion of their service requirement. This is not allowed in this model.

[3]Here and everywhere else, we use notation $\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{d} u_i v_i$ for any d-dimensional vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$. That is, $\mathbf{Q}(\tau) \cdot \boldsymbol{\rho} = \sum_i Q_i(\tau) \cdot \rho_i$.

3. Main result. As stated above, the MW algorithm is not practical for wireless network and is not applicable to circuit switched network. However, it has the desirable throughput optimality property. As the main result of this paper, we provide a simple, randomized algorithm that is applicable to both wireless and circuit switched network as well as throughput optimal. The algorithm requires each node (or queue) to perform only a few logical operations at each time step, it is distributed and effectively it "simulates" the MW-f algorithm for an appropriate choice of f. In that sense, it is a simple, randomized, distributed implementation of the MW algorithm.

In what follows, we shall describe algorithms for wireless network and buffered circuit switched network, respectively. We will state their throughput optimality property. While these algorithms seem different, philosophically they are very similar—also, witnessed in the commonality in their proofs.

3.1. *Algorithm for wireless network.* Let $t \in \mathbb{R}_+$ denote the time index and $\mathbf{W}(t) = [W_i(t)] \in \mathbb{R}_+^n$ be the vector of weights at the n queues. The $\mathbf{W}(t)$ will be a function of $\mathbf{Q}(t)$ to be determined later. In a nutshell, the algorithm described below will choose a schedule $\boldsymbol{\sigma}(t) \in \mathcal{I}(G)$ so that the weight, $\mathbf{W}(t) \cdot \boldsymbol{\sigma}(t)$, is as large as possible.

The algorithm is randomized and asynchronous. Each node (or queue) has an independent exponential clock of rate 1 (i.e., Poisson process of rate 1). Let the kth tick of the clock of node i happen at time T_k^i; $T_0^i = 0$ for all i. By definition $T_{k+1}^i - T_k^i$, $k \geq 0$, are i.i.d. mean 1 exponential random variables. Each node changes its scheduling decision only at its clock ticks. That is, for node i the $\sigma_i(t)$ remains constant for $t \in (T_k^i, T_{k+1}^i]$. Clearly, with probability 1 no two clock ticks across nodes happen at the same time.

Initially, we assume that $\sigma_i(0) = 0$ for all i. The node i at the kth clock tick, $t = T_k^i$, *listens* to the medium and does the following:

- If any neighbor of i is transmitting, that is, $\sigma_j(t) = 1$ for some $j \in \mathcal{N}(i) = \{j' : (i, j') \in E\}$, then set $\sigma_i(t^+) = 0$.
- Else, set

$$\sigma_i(t^+) = \begin{cases} 1, & \text{with probability } \dfrac{\exp(W_i(t))}{1 + \exp(W_i(t))}, \\ 0, & \text{otherwise.} \end{cases}$$

Here, we assume that if $\sigma_i(t) = 1$, then node i will always transmit data irrespective of the value of $Q_i(t)$ so that the neighbors of node i can infer $\sigma_i(t)$ by *listening* to the medium.

3.1.1. *Throughout optimality.* The above described algorithm for wireless network is throughput optimal for an appropriate choice of weight $\mathbf{W}(t)$. Define weight $W_i(t)$ at node i in the algorithm for wireless network as

$$W_i(t) = \max\{f(Q_i(\lfloor t \rfloor)), \sqrt{f(Q_{\max}(\lfloor t \rfloor))}\}, \tag{3}$$

where[4] $f(x) = \log\log(x+e)$ and $Q_{\max}(\cdot) = \max_i Q_i(\cdot)$. The nonlocal information of $Q_{\max}(\lfloor t \rfloor)$ can be replaced by its approximate estimation that can be computed through a very simple distributed algorithm. This does not alter the throughput optimality property of the algorithm. A discussion is provided in Section 6. We state the following property of the algorithm.

THEOREM 1. *Suppose the algorithm of Section 3.1 uses the weight as per (3). Then, for any $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_w^o$ and Bernoulli arrival process, the network Markov process is ergodic.*

In this paper, Theorem 1 (as well as Theorem 2) is established for the choice of $f(x) = \log\log(x+e)$. However, the proof technique of this paper extends naturally for any choice of $f : \mathbb{R}_+ \to \mathbb{R}_+$ that satisfies the following conditions: $f(0) = 0$, f is a monotonically strictly increasing function, $\lim_{x\to\infty} f(x) = \infty$ and

$$\lim_{x\to\infty} \exp(f(x)) f'(f^{-1}(\delta f(x))) = 0 \qquad \text{for any } \delta \in (0,1).$$

Examples of such functions include $f(x) = \varepsilon(x)\log(x+1)$, where $\varepsilon(0) = 1$ and as $x \to \infty$, $\varepsilon(x)$ monotonically decreases to 0, but $\varepsilon(x)\log(x+1)$ monotonically increases to ∞; $f(x) = \sqrt{\log(x+1)}$; $f(x) = \log\log\log(x+e^e)$, etc.

3.2. *Algorithm for buffered circuit switched network.* In a buffered circuit switched network, the scheduling algorithm decided when each of the ingress node (or queue) should request the network for availability of resources (links) along its route and upon positive response from the network, it acquires the resources. Our algorithm to make such a decision at each node is described as follows:

- Each ingress node of a route, say R_i, generates request as per a time varying Poisson process whose rate at time t is equal to $\exp(W_i(t))$.
- If the request generated by an ingress node of route, say R_i, is accepted, a flow from the head of its queue leaves the queue and acquires the resources in the network. Else, do nothing.

In the above, like the algorithm for wireless network, we assume that if the request of ingress node i is accepted, a new flow will acquire resources in the network along its route. This is irrespective of whether queue is empty or not—if queue is empty, a *dummy* flow is generated. This is merely for technical reasons.

[4]Unless stated otherwise, here and everywhere else the $\log(\cdot)$ is natural logarithm, that is, base e.

3.2.1. *Throughput optimality.* We describe a specific choice of weight $\mathbf{W}(t)$ for which the algorithm for circuit switched network as described above is throughput optimal. Specifically, for route R_i its weight at time t is defined as

$$W_i(t) = \max\{f(Q_i(\lfloor t \rfloor)), \sqrt{f(Q_{\max}(\lfloor t \rfloor))}\}, \tag{4}$$

where $f(x) = \log\log(x + e)$. The remark about distributed estimation of $Q_{\max}(\lfloor t \rfloor))$ after (3) applies here as well. We state the following property of the algorithm.

THEOREM 2. *Suppose the algorithm of Section* 3.2 *uses the weight as per* (4). *Then, for any $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_{cs}^{o}$ and Poisson arrival process, the network Markov process is ergodic.*

4. Technical preliminaries.

4.1. *Finite state Markov chain.* Consider a discrete-time, time-homogeneous Markov chain over a finite state space Ω. Let its probability transition matrix be $P = [P_{ij}] \in \mathbb{R}_+^{|\Omega| \times |\Omega|}$. If P is irreducible and aperiodic, then the Markov chain is known to have a unique stationary distribution $\pi = [\pi_i] \in \mathbb{R}_+^{|\Omega|}$ and it is ergodic, that is,

$$\lim_{\tau \to \infty} P_{ji}^{\tau} \to \pi_i \qquad \text{for any } i, j \in \Omega.$$

The adjoint of P, also known as the time-reversal of P, denoted by P^*, is defined as

$$\pi_i P_{ij}^* = \pi_j P_{ji} \qquad \text{for any } i, j \in \Omega. \tag{5}$$

By definition, P^* has π as its stationary distribution as well. If $P = P^*$ then P is called *reversible* or *time reversible*.

Similar notions can be defined for a continuous time Markov process over Ω. To this end, let $P(s,t) = [P_{ij}(s,t)] \in \mathbb{R}_+^{|\Omega| \times |\Omega|}$ denote its transition matrix over time interval $[s,t]$. The Markov process is called time-homogeneous if $P(s,t)$ is stationary, that is, $P(s,t) = P(0, t-s)$ for all $0 \le s < t$ and is called reversible if $P(s,t)$ is reversible for all $0 \le s < t$. Further, if $P(0,t)$ is irreducible and aperiodic for all $t > 0$, then this time-homogeneous reversible Markov process has a unique stationary distribution π and it is ergodic, that is,

$$\lim_{t \to \infty} P_{ji}(0,t) \to \pi_i \qquad \text{for any } i, j \in \Omega.$$

4.2. *Mixing time of Markov chain.* Given an ergodic finite state Markov chain, the distribution at time τ converges to the stationary distribution starting from any initial condition as described above. We will need quantitative bounds on the time it takes for them to reach "close" to the stationary distribution. This time to reach stationarity is known as the *mixing time* of the Markov chain. Here we introduce necessary preliminaries related to this notion. We refer an interested reader to survey papers [16, 23]. We start with the definition of distances between probability distributions.

DEFINITION 2 (Distance of measures). Given two probability distributions ν and μ on a finite space Ω, we define the following two distances. The total variation distance, denoted as $\|\nu - \mu\|_{\mathrm{TV}}$ is

$$\|\nu - \mu\|_{\mathrm{TV}} = \frac{1}{2}\sum_{i\in\Omega} |\nu_i - \mu_i|.$$

The χ^2 *distance*, denoted as $\|\frac{\nu}{\mu} - 1\|_{2,\mu}$ is

$$\left\|\frac{\nu}{\mu} - 1\right\|_{2,\mu}^2 = \|\nu - \mu\|_{2,1/\mu}^2 = \sum_{i\in\Omega} \mu_i \left(\frac{\nu_i}{\mu_i} - 1\right)^2.$$

More generally, for any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}_+^{|\Omega|}$, we define

$$\|\mathbf{v}\|_{2,\mathbf{u}}^2 = \sum_{i\in\Omega} u_i v_i^2.$$

We make note of the following relation between the two distances defined above: using the Cauchy–Schwarz inequality, we have

$$\left\|\frac{\nu}{\mu} - 1\right\|_{2,\mu} \geq 2\|\nu - \mu\|_{\mathrm{TV}}. \tag{6}$$

Next, we define a matrix norm that will be useful in determining the rate of convergence or the mixing time of a finite-state Markov chain.

DEFINITION 3 (Matrix norm). Consider a $|\Omega| \times |\Omega|$ nonnegative valued matrix $A \in \mathbb{R}_+^{|\Omega|\times|\Omega|}$ and a given vector $\mathbf{u} \in \mathbb{R}_+^{|\Omega|}$. Then, the matrix norm of A with respect to $\mathbf{u}$ is defined as

$$\|A\|_{\mathbf{u}} = \sup_{\mathbf{v}:\, \mathbb{E}_{\mathbf{u}}[\mathbf{v}]=0} \frac{\|A\mathbf{v}\|_{2,\mathbf{u}}}{\|\mathbf{v}\|_{2,\mathbf{u}}},$$

where $\mathbb{E}_{\mathbf{u}}[\mathbf{v}] = \sum_i u_i v_i$.

The following are known properties (most of them are easily verifiable) of the defined matrix norm (see, e.g., [12]).

(P1) For matrices $A, B \in \mathbb{R}_+^{|\Omega|\times|\Omega|}$ and $\pi \in \mathbb{R}_+^{|\Omega|}$,

$$\|A+B\|_\pi \le \|A\|_\pi + \|B\|_\pi.$$

(P2) For matrix $A \in \mathbb{R}_+^{|\Omega|\times|\Omega|}$, $\pi \in \mathbb{R}_+^{|\Omega|}$ and $c \in \mathbb{R}$,

$$\|cA\|_\pi = |c|\|A\|_\pi.$$

(P3) Let A and B be transition matrices of reversible Markov chains, that is, $A = A^*$ and $B = B^*$. Let both of them have π as their unique stationary distribution. Then,

$$\|AB\|_\pi \le \|A\|_\pi \|B\|_\pi.$$

(P4) Let A be the transition matrix of an irreducible and aperiodic Markov chain that is reversible, that is, $A = A^*$. Then, A has n real eigenvalues $1 = \lambda_1 > \lambda_2 \ge \cdots \ge \lambda_n > -1$ and

$$\|A\|_\pi \le \lambda_{\max} = \max\{\lambda_2, |\lambda_n|\},$$

where π is the stationary distribution of the Markov chain.

For a probability matrix P, we will mostly be interested in the matrix norm of P with respect to its stationary distribution π, that is, $\|P\|_\pi$. Therefore, in this paper if we use a matrix norm for a probability matrix without mentioning the reference measure, then it is with respect to the stationary distribution.

With these definitions, it follows that for any distribution μ on Ω

$$\left\|\frac{\mu P}{\pi} - 1\right\|_{2,\pi} \le \|P^*\| \left\|\frac{\mu}{\pi} - 1\right\|_{2,\pi}, \tag{7}$$

since $\mathbb{E}_\pi[\frac{\mu}{\pi} - 1] = 0$, where $\frac{\mu}{\pi} = [\mu_i/\pi_i]$. The Markov chains of our interest will be reversible, that is, $P = P^*$. Therefore, for a reversible Markov chain starting with initial distribution $\mu(0)$, the distribution $\mu(\tau)$ at time τ is such that

$$\left\|\frac{\mu(\tau)}{\pi} - 1\right\|_{2,\pi} \le \|P\|^\tau \left\|\frac{\mu(0)}{\pi} - 1\right\|_{2,\pi}. \tag{8}$$

Now starting from any state i, that is, probability distribution with unit mass on state i, the initial distance $\|\frac{\mu(0)}{\pi} - 1\|_{2,\pi}$ in the worst case is bounded above by $\sqrt{1/\pi_{\min}}$ where $\pi_{\min} = \min_i \pi_i$. Therefore, for any $\delta > 0$ we have $\|\frac{\mu(\tau)}{\pi} - 1\|_{2,\pi} \le \delta$ for any τ such that[5]

$$\tau \ge \frac{\log 1/\pi_{\min} + \log 1/\delta}{\log 1/\|P\|} = \Theta\left(\frac{\log 1/\pi_{\min} + \log 1/\delta}{1 - \|P\|}\right).$$

[5]Throughout this paper, we shall utilize the standard order-notation: for two functions $g, f : \mathbb{R}_+ \to \mathbb{R}_+$, $g(x) = \omega(f(x))$ means $\liminf_{x\to\infty} g(x)/f(x) = \infty$; $g(x) = \Omega(f(x))$ means $\liminf_{x\to\infty} g(x)/f(x) > 0$; $g(x) = \Theta(f(x))$ means $0 < \liminf_{x\to\infty} g(x)/f(x) \le \limsup_{x\to\infty} g(x)/f(x) < \infty$; $g(x) = O(f(x))$ means $\limsup_{x\to\infty} g(x)/f(x) < \infty$; $g(x) = o(f(x))$ means $\limsup_{x\to\infty} g(x)/f(x) = 0$.

This suggests that the "mixing time," that is, time to reach (close to) the stationary distribution of the Markov chain scales inversely with $1 - \|P\|$. Therefore, we will define the "mixing time" of a (reversible) Markov chain with transition matrix P as $1/(1 - \|P\|)$.

4.3. *Glauber dynamics and algorithm for wireless network.* We will describe the relation between the algorithm for wireless network; cf. Section 3.1 and a specific irreducible, aperiodic, reversible Markov chain on the space of independent sets $\mathcal{I}(G)$ or schedules for wireless network with graph $G = (V, E)$. It is also known as the Glauber dynamics, which is used by the standard Metropolis and Hastings [11, 20] sampling mechanism that is described next.

4.3.1. *Glauber dynamics and its mixing time.* We shall start off with the definition of the Glauber dynamics followed by a useful bound on its mixing time.

DEFINITION 4 (Glauber dynamics). Consider a graph $G = (V, E)$ of $n = |V|$ nodes with node weights $\mathbf{W} = [W_i] \in \mathbb{R}_+^n$. The Glauber dynamics based on weight $\mathbf{W}$, denoted by GD($\mathbf{W}$), is a Markov chain on the space of independent sets of G, $\mathcal{I}(G)$. The transitions of this Markov chain are described next. Suppose the Markov chain is currently in the state $\boldsymbol{\sigma} \in \mathcal{I}(G)$. Then, the next state, say $\boldsymbol{\sigma}'$ is decided as follows: pick a node $i \in V$ uniformly at random and

- set $\sigma'_j = \sigma_j$ for $j \neq i$;
- if $\sigma_k = 0$ for all $k \in \mathcal{N}(i)$, then set

$$\sigma'_i = \begin{cases} 1, & \text{with probability } \dfrac{\exp(W_i)}{1 + \exp(W_i)}, \\ 0, & \text{otherwise,} \end{cases}$$

- else set $\sigma'_i = 0$.

It can be verified that the Glauber dynamics GD($\mathbf{W}$) is reversible with stationary distribution π given by

$$\pi_{\boldsymbol{\sigma}} \propto \exp(\mathbf{W} \cdot \boldsymbol{\sigma}) \qquad \text{for any } \boldsymbol{\sigma} \in \mathcal{I}(G). \tag{9}$$

Now we describe bound on the mixing time of Glauber dynamics.

LEMMA 3. *Let P be the transition matrix of the Glauber dynamics* GD($\mathbf{W}$) *with n nodes. Then,*

$$\|P\| \leq 1 - \frac{1}{n^2 2^{2n+3} \exp(2(n+1)W_{\max})}, \tag{10}$$

$$\|e^{n(P-I)}\| \leq 1 - \frac{1}{n 2^{2n+4} \exp(2(n+1)W_{\max})}. \tag{11}$$

PROOF. By the property (P4) of the matrix norm, it is sufficient to establish that

$$\lambda_2 \le 1 - \frac{1}{n^2 2^{2n+3}\exp(2(n+1)W_{\max})},$$
(12)
$$\lambda_N \ge -1 + \frac{1}{n^2 2^{2n+3}\exp(2(n+1)W_{\max})},$$

where λ_2, λ_N are the second largest and the smallest eigenvalues of P, respectively, with $N = |\mathcal{I}(G)|$.

First, an upper bound on λ_2. By Cheeger's inequality [2, 5, 7, 13, 27], it is well known that $\lambda_2 \le 1 - \frac{\Phi^2}{2}$ where Φ is the conductance of P, defined as

$$\Phi = \min_{S \subset \mathcal{I}(G)\,:\,\pi(S)\le 1/2} \frac{Q(S, S^c)}{\pi(S)\pi(S^c)},$$

where $S^c = \mathcal{I}(G) \setminus S$, $Q(S, S^c) = \sum_{\boldsymbol{\sigma}\in S, \boldsymbol{\sigma}'\in S^c} \pi_{\boldsymbol{\sigma}} P_{\boldsymbol{\sigma}\boldsymbol{\sigma}'}$. Now we have

$$\begin{aligned}
\Phi &\ge \min_{S\subset\mathcal{I}(G)} Q(S, S^c) \\
&\ge \min_{P_{\boldsymbol{\sigma}\boldsymbol{\sigma}'}\neq 0} \pi_{\boldsymbol{\sigma}} P_{\boldsymbol{\sigma}\boldsymbol{\sigma}'} \\
&\ge \pi_{\min} \cdot \min_i \frac{1}{n}\frac{1}{1+\exp(W_i)} \\
&\ge \frac{1}{2^n \exp(nW_{\max})} \cdot \frac{1}{n}\frac{1}{1+\exp(W_{\max})} \\
&\ge \frac{1}{n2^{n+1}\exp((n+1)W_{\max})}.
\end{aligned}$$

This implies the desired bound on λ_2.

Now, we lower bound λ_N. For this, note that for any $\boldsymbol{\sigma} \neq \mathbf{0}$, under GD($\mathbf{W}$) with $\mathbf{W} \in \mathbb{R}_+^n$,

$$P_{\boldsymbol{\sigma}\boldsymbol{\sigma}} \ge \frac{1}{2n}.$$

For $\boldsymbol{\sigma} = \mathbf{0}$,

$$P_{\boldsymbol{\sigma}\boldsymbol{\sigma}} = P_{\mathbf{00}} \ge \frac{1}{1+\exp(W_{\max})}.$$

Therefore, it follows that for $n \ge 2$ and $\mathbf{W} \in \mathbb{R}_+^n$, P can be decomposed as

$$P = \eta I + (1-\eta)Q, \tag{13}$$

where

$$\begin{aligned}\eta &= \frac{1}{2}\min_{\boldsymbol{\sigma}} P_{\boldsymbol{\sigma}\boldsymbol{\sigma}} \\ &\geq \frac{1}{2\max\{2n, 1+\exp(W_{\max})\}},\end{aligned} \tag{14}$$

and by construction Q corresponds to the transition matrix of an irreducible, aperiodic, reversible Markov chain on $\mathcal{I}(G)$ with the same stationary distribution π. More generally, Q and P have identical eigenvectors. Since all eigenvalues of Q belong to $[-1, 1]$, from decomposition (13), it follows that the smallest eigenvalue of P must satisfy

$$\begin{aligned}\lambda_N &\geq -1 + 2\eta \\ &\geq -1 + \frac{1}{\max\{2n, 1+\exp(W_{\max})\}} \\ &\geq -1 + \frac{1}{n^2 2^{2n+3}\exp(2(n+1)W_{\max})}.\end{aligned} \tag{15}$$

Thus, we have established that

$$\|P\| \leq 1 - \frac{1}{n^2 2^{2n+3}\exp(2(n+1)W_{\max})}.$$

Now consider $e^{n(P(\tau)-I)}$. Using properties (P1), (P2) and (P3) of matrix norm, we have

$$\begin{aligned}\|e^{n(P-I)}\| &= \left\| e^{-n}\sum_{k=0}^{\infty}\frac{n^k P^k}{k!}\right\| \\ &\leq e^{-n}\sum_{k=0}^{\infty}\frac{n^k\|P\|^k}{k!} = e^{n(\|P\|-1)} \\ &\leq e^{-n/(n^2 2^{2n+3}\exp(2(n+1)W_{\max}))} \\ &\leq 1 - \frac{1}{n 2^{2n+4}\exp(2(n+1)W_{\max})},\end{aligned}$$

where we have used the bound of $\|P\|$ and the fact that $e^{-x} \leq 1 - x/2$ for all $x \in [0, 1]$. This completes the proof of Lemma 3. □

4.3.2. *Relation to algorithm.* Now we relate our algorithm for wireless network scheduling described in Section 3.1 with an appropriate continuous time version of the Glauber dynamics with time-varying weights. Recall that $\mathbf{Q}(t)$ and $\boldsymbol{\sigma}(t)$ denote the queue-size vector and schedule at time t. The algorithm changes its scheduling decision, $\boldsymbol{\sigma}(t)$, when a node's exponential clock of rate 1 ticks. Due to memoryless property of exponential distribution and independence of clocks of all nodes, this is equivalent to having a global exponential clock of rate n and upon clock tick one of the n nodes gets chosen. This node decides its transition as explained in Section 3.1. Thus, the effective dynamics of the algorithm upon a global clock tick is such that the schedule $\boldsymbol{\sigma}(t)$ evolves exactly as per the Glauber dynamics $\mathrm{GD}(\mathbf{W}(t))$. Here recall that $\mathbf{W}(t)$ is determined based on $\mathbf{Q}(\lfloor t \rfloor)$. With abuse of notation, let the transition matrix of this Glauber dynamics be denoted by $\mathrm{GD}(\mathbf{W}(t))$.

Consider any $\tau \in \mathbb{Z}_+$. Let $\mathbf{Q}(\tau), \boldsymbol{\sigma}(\tau)$ be the states at time τ. Then,

$$\mathbb{E}\big[\boldsymbol{\delta}_{\boldsymbol{\sigma}(\tau+1)} | \mathbf{Q}(\tau), \boldsymbol{\sigma}(\tau)\big] = \sum_{k=0}^{\infty} \boldsymbol{\delta}_{\boldsymbol{\sigma}(\tau)} \Pr(\zeta = k) \, \mathrm{GD}(\mathbf{W}(\tau))^k,$$

where we have used notation $\boldsymbol{\delta}_{\boldsymbol{\sigma}}$ for the distribution with singleton support $\{\boldsymbol{\sigma}\}$ and ζ is a Poisson random variable of mean n. In the above, the expectation is taken with respect to the distribution of $\boldsymbol{\sigma}(\tau+1)$ given $\mathbf{Q}(\tau), \boldsymbol{\sigma}(\tau)$ and the notation $\mathbb{E}[\mathbf{u}]$ for a d-dimensional random vector $\mathbf{u} = [u_i]_{1 \le i \le d} \in \mathbb{R}^d$ denotes

$$\mathbb{E}[\mathbf{u}] = [\mathbb{E}[u_i]]_{1 \le i \le d}.$$

Therefore, $\mathbb{E}[\boldsymbol{\delta}_{\boldsymbol{\sigma}(\tau+1)} | \mathbf{Q}(\tau), \boldsymbol{\sigma}(\tau)]$ is interpreted as the distribution of $\boldsymbol{\sigma}(\tau+1)$ (i.e., schedule at time $\tau+1$) given $\mathbf{Q}(\tau), \boldsymbol{\sigma}(\tau)$.

Now it follows that

$$\begin{aligned} \mathbb{E}\big[\boldsymbol{\delta}_{\boldsymbol{\sigma}(\tau+1)} | \mathbf{Q}(\tau), \boldsymbol{\sigma}(\tau)\big] &= \boldsymbol{\delta}_{\boldsymbol{\sigma}(\tau)} e^{n(\mathrm{GD}(\mathbf{W}(\tau)) - I)} \\ &= \boldsymbol{\delta}_{\boldsymbol{\sigma}(\tau)} P(\tau), \end{aligned} \tag{16}$$

where $P(\tau) \stackrel{\triangle}{=} e^{n(\mathrm{GD}(\mathbf{W}(\tau)) - I)}$. In general, for any $\delta \in [0, 1]$

$$\mathbb{E}\big[\boldsymbol{\delta}_{\boldsymbol{\sigma}(\tau+\delta)} | \mathbf{Q}(\tau), \boldsymbol{\sigma}(\tau)\big] = \boldsymbol{\delta}_{\boldsymbol{\sigma}(\tau)} P^{\delta}(\tau), \tag{17}$$

where $P^{\delta}(\tau) \stackrel{\triangle}{=} e^{\delta n(\mathrm{GD}(\mathbf{W}(\tau)) - I)}$.

4.4. *Loss network and algorithm for circuit switched network.* For the buffered circuit switched network, the Markov chain of interest is related to the classical stochastic loss network model. This model has been popularly utilized to study the performance of various systems including the telephone networks, human resource allocation, etc.; cf. see [15]. The stochastic loss network model is very similar to the model of the buffered circuit switched network with the only difference that it does not have any buffers at the ingress nodes.

4.4.1. *Loss network and its mixing time.* A loss network is described by a network graph $G = (V, E)$ with capacitated links $[C_e]_{e \in E}$, n (≥ 2) routes $\{R_i : R_i \subset E, 1 \le i \le n\}$ and without any buffer or queues at the ingress of each route. For each route R_i, there is a dedicated exogenous, independent Poisson arrival process with rate ϕ_i. Let $z_i(t)$ be the number of active flows on route i at time t, with notation $\mathbf{z}(t) = [z_i(t)]$. Clearly, $\mathbf{z}(t) \in \mathcal{X}$ due to network capacity constraints. At time t when a new exogenous flow arrives on route R_i, if it can be accepted by the network, that is, $\mathbf{z}(t) + e_i \in \mathcal{X}$, then it is accepted with $z_i(t) \to z_i(t) + 1$, or else, it is dropped (and hence, lost forever). Each flow holds unit amount of capacity on all links along its route for time that is distributed as exponential distribution with mean 1, independent of everything else. Upon the completion of holding time, the flow departs and frees unit capacity on all links of its own route.

Therefore, effectively this loss network model can be described as a finite state Markov process with state space $\mathcal{X}$. Given state $\mathbf{z} = [z_i] \in \mathcal{X}$, the possible transitions and corresponding rates are given as

$$z_i \to \begin{cases} z_i + 1, & \text{with rate } \phi_i \text{ if } \mathbf{z} + e_i \in \mathcal{X}, \\ z_i - 1, & \text{with rate } x_i. \end{cases} \tag{18}$$

It can be verified that this Markov process is irreducible, aperiodic, and time-reversible. Therefore, it is positive recurrent (due to the finite state space) and has a unique stationary distribution. Its stationary distribution π is known (cf. [15]) to have the following product-form: for any $\mathbf{z} \in \mathcal{X}$,

$$\pi_{\mathbf{z}} \propto \prod_{i=1}^{n} \frac{\phi_i^{z_i}}{z_i!}. \tag{19}$$

We will be interested in the discrete-time (or *embedded*) version of this Markov processes, which can be defined as follows.

DEFINITION 5 (Loss network). A loss network Markov chain with capacitated graph $G = (V, E)$, capacities $C_e, e \in E$ and n routes $R_i, 1 \le i \le n$, denoted by $\mathbf{LN}(\boldsymbol{\phi})$ is a Markov chain on $\mathcal{X}$. The transition probabilities of this Markov chain are described next. Given a current state $\mathbf{z} \in \mathcal{X}$, the next state $\mathbf{z}^* \in \mathcal{X}$ is decided by first picking a route R_i uniformly at random and performing the following: $z_j^* = z_j$ for $j \neq i$ and z_i^* is decided by

$$z_i^* = \begin{cases} z_i + 1, & \text{with probability } \dfrac{\phi_i}{\mathcal{R}} \cdot \mathbf{1}_{\{\mathbf{z} + e_i \in \mathcal{X}\}}, \\ z_i - 1, & \text{with probability } \dfrac{z_i}{\mathcal{R}}, \\ z_i, & \text{otherwise}, \end{cases}$$

where $\mathcal{R} = \sum_i \phi_i + C_{\max}$.

$\mathbf{LN}(\boldsymbol{\phi})$ has the same stationary distribution as in (19) and it is also irreducible, aperiodic, and reversible. Next, we state a bound on the mixing time of the loss network Markov chain $\mathbf{LN}(\boldsymbol{\phi})$ as follows.

LEMMA 4. *Let P be the transition matrix of* $\mathbf{LN}(\boldsymbol{\phi})$ *with n routes. If* $\boldsymbol{\phi} = \exp(\mathbf{W})$ *with*[6] $W_i \geq 0$ *for all i, then,*

$$\|P\| \leq 1 - \frac{1}{8n^4 C_{\max}^{2nC_{\max}+2n+2} \exp(2(nC_{\max}+1)W_{\max})}, \tag{20}$$

$$\|e^{n\mathcal{R}(P-I)}\| \leq 1 - \frac{1}{16n^3 C_{\max}^{2nC_{\max}+2n+2} \exp(2(nC_{\max}+1)W_{\max})}. \tag{21}$$

PROOF. Similar to the proof of Lemma 3, it is sufficient to establish

$$\begin{aligned}
\lambda_2 &\leq 1 - \frac{1}{8n^4 C_{\max}^{2nC_{\max}+2n+2} \exp(2(nC_{\max}+1)W_{\max})}, \\
\lambda_N &\geq -1 + \frac{1}{8n^4 C_{\max}^{2nC_{\max}+2n+2} \exp(2(nC_{\max}+1)W_{\max})},
\end{aligned} \tag{22}$$

where λ_2, λ_N are the second largest and the smallest eigenvalue of reversible transition matrix P, respectively, with $N = |\mathcal{X}|$.

First, we shall bound λ_2 using Cheeger's inequality as in the proof of Lemma 3. A simple lower bound for the conductance Φ of P is given by

$$\Phi \geq \pi_{\min} \cdot \min_{P_{\mathbf{z}\mathbf{z}'} \neq 0} P_{\mathbf{z}\mathbf{z}'}. \tag{23}$$

To obtain the lower bound of $\pi_{\min}$, recall the equation (19),

$$\pi_{\mathbf{z}} = \frac{1}{Z} \prod_{i=1}^{n} \frac{\phi_i^{z_i}}{z_i!},$$

where $Z = \sum_{\mathbf{z} \in \mathcal{X}} \prod_{i=1}^{n} \frac{\phi_i^{z_i}}{z_i!}$, and consider the following:

$$Z \leq |\mathcal{X}| \phi_{\max}^{nC_{\max}} \leq C_{\max}^{n} \exp(nC_{\max} W_{\max})$$

and

$$\prod_{i=1}^{n} \frac{\phi_i^{z_i}}{z_i!} \geq \frac{1}{(C_{\max}!)^n} \geq \frac{1}{C_{\max}^{nC_{\max}}}.$$

By combining the above inequalities, we obtain

$$\pi_{\min} \geq \frac{1}{C_{\max}^{nC_{\max}+n} \exp(nC_{\max} W_{\max})}. \tag{24}$$

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[6] We use the following notation: given a function $g : \mathbb{R} \to \mathbb{R}$ and a d-dimensional vector $\mathbf{u} \in \mathbb{R}^d$, let $g(\mathbf{u}) = [g(u_i)] \in \mathbb{R}^d$.

On the other hand, one can bound $\min_{P_{\mathbf{z}\mathbf{z}'}\neq 0} P_{\mathbf{z}\mathbf{z}'}$ as follows:

$$P_{\mathbf{z}\mathbf{z}'} \geq \frac{1}{n}\cdot\frac{1}{\mathcal{R}} \geq \frac{1}{n}\cdot\frac{1}{n\phi_{\max}+C_{\max}} \geq \frac{1}{2n^2 C_{\max}\exp(W_{\max})}, \tag{25}$$

where we use the fact that $x+y \leq 2xy$ if $x, y \geq 1$. Now, by combining (24) and (25), we have

$$\Phi \geq \frac{1}{2n^2 C_{\max}^{nC_{\max}+n+1}\exp((nC_{\max}+1)W_{\max})}.$$

Therefore, bound on λ_2 as desired in (22) follows through Cheeger's inequality.

For λ_N, like Lemma 3, we shall lower bound the diagonal entries of the transition matrix P. Specifically, for any state $\mathbf{z}$, it follows that for $n \geq 2$, $\boldsymbol{\phi} = \exp(\mathbf{W})$ with $\mathbf{W} \in \mathbb{R}_+^n$ and $C_{\max} \geq 1$,

$$\begin{aligned} P_{\mathbf{z}\mathbf{z}} &\geq \frac{1}{n}\frac{(n-1)\phi_{\min}}{\mathcal{R}} \\ &\geq \frac{1}{2C_{\max}+2n\exp(W_{\max})} \\ &\geq \frac{1}{8n^4 C_{\max}^{2nC_{\max}+2n+2}\exp(2(nC_{\max}+1)W_{\max})}. \end{aligned} \tag{26}$$

From above and arguments similar to those in Lemma 3, we obtain the desired conclusion as

$$\|P\| \leq 1 - \frac{1}{8n^4 C_{\max}^{2nC_{\max}+2n+2}\exp(2(nC_{\max}+1)W_{\max})}.$$

Furthermore, using this bound and arguments similar to those in the proof of Lemma 3, we have

$$\|e^{n\mathcal{R}(P-I)}\| \leq 1 - \frac{1}{16n^3 C_{\max}^{2nC_{\max}+2n+2}\exp(2(nC_{\max}+1)W_{\max})}. \qquad \square$$

4.4.2. *Relation to algorithm.* The scheduling algorithm for buffered circuit switched network described in Section 3.2 effectively simulates a stochastic loss network with time-varying arrival rates $\boldsymbol{\phi}(t)$ where $\boldsymbol{\phi}_i(t) = \exp(W_i(t))$. That is, the relation of the algorithm in Section 3.2 with loss network is similar to the relation of the algorithm in Section 3.1 with Glauber dynamics that we explained in the previous section. To this end, for a given $\tau \in \mathbb{Z}_+$, let $\mathbf{Q}(\tau)$ and $\mathbf{z}(\tau)$ be queue-size vector and active flows at time τ. With abuse of notation, let $\mathbf{LN}(\exp(\mathbf{W}(\tau)))$ be the transition matrix of the corresponding loss network with $\mathbf{W}(\tau)$ dependent on $\mathbf{Q}(\tau)$. Then, for any $\delta \in [0, 1]$

$$\mathbb{E}[\boldsymbol{\delta}_{\mathbf{z}(\tau+\delta)}|\mathbf{Q}(\tau),\mathbf{z}(\tau)] = \boldsymbol{\delta}_{\mathbf{z}(\tau)}e^{n\delta\mathcal{R}(\tau)(\mathbf{LN}(\exp(\mathbf{W}(\tau)))-I)}, \tag{27}$$

where $\mathcal{R}(\tau) = \sum_i \exp(W_i(\tau)) + C_{\max}$.[7]

[7] See Section 4.3.2 for a detailed explanation of notation such as $\boldsymbol{\delta}_{\mathbf{z}}$.

4.4.3. *Background on ergodic Markov processes.* Here we introduce necessary background for establishing ergodicity of the network Markov processes of interest. In this paper, we will be concerned with discrete-time, time-homogeneous Markov process or chain evolving over a complete, separable metric (Polish) space X. Let $\mathcal{B}_{\mathsf{X}}$ denote the Borel σ-algebra on X. Let $X(\tau)$ denote the state of Markov process at time $\tau \in \mathbb{Z}_+$.

Let $\mu_{\mathbf{x}}(\tau)$ be distribution of $X(\tau)$ at time $\tau \geq 1$ given $X(0) = \mathbf{x} \in \mathsf{X}$. As noted earlier, we shall call Markov process $X(\cdot)$ ergodic, if there exists a unique stationary distribution π such that $\mu_{\mathbf{x}}(\tau)$ converges to π as $\tau \to \infty$, for any initial state $\mathbf{x} \in \mathsf{X}$. The following are sufficient conditions that will imply ergodicity of such a Markov process (see [1], pages 198–202, and [8], Section 4.2, for details).

(C1) There exists a bounded set $A \in \mathcal{B}_{\mathsf{X}}$ such that

$$\mathbb{E}_{\mathbf{x}}[T_A] < \infty \qquad \text{for any } \mathbf{x} \in \mathsf{X}, \tag{28}$$

$$\sup_{\mathbf{x} \in A} \mathbb{E}_{\mathbf{x}}[T_A] < \infty. \tag{29}$$

In the above, the stopping time $T_A = \inf\{\tau \geq 1 : X(\tau) \in A\}$; notation $\Pr_{\mathbf{x}}(\cdot) \equiv \Pr(\cdot | X(0) = \mathbf{x})$ and $\mathbb{E}_{\mathbf{x}}[\cdot] \equiv \mathbb{E}[\cdot | X(0) = \mathbf{x}]$.

(C2) Given A satisfying (28) and (29), there exists $\mathbf{x}^* \in \mathsf{X}$, finite $\ell \geq 1$ and $\delta > 0$ such that

$$\Pr_{\mathbf{x}}\bigl(X(\ell) = \mathbf{x}^*\bigr) \geq \delta \qquad \text{for any } \mathbf{x} \in A, \tag{30}$$

$$\Pr_{\mathbf{x}^*}\bigl(X(1) = \mathbf{x}^*\bigr) > 0. \tag{31}$$

Given this, the following path-wise ergodic property is satisfied (cf. [1, 21]): for any $\mathbf{x} \in \mathsf{X}$ and nonnegative measurable function $f : \mathsf{X} \to \mathbb{R}_+$,

$$\lim_{T \to \infty} \frac{1}{T} \sum_{\tau=0}^{T-1} f(X(\tau)) \to \mathbb{E}_\pi[f], \qquad \Pr_{\mathbf{x}}\text{-almost surely.}$$

Here $\mathbb{E}_\pi[f] = \int f(z)\pi(z)$. Note that $\mathbb{E}_\pi[f]$ may not be finite.

Our interest will be in verifying conditions (C1) and (C2). As we shall see, condition (C2) will follow easily due to the structure of the Markov process. The condition (C1) will be established using the Lyapunov drift criteria, also known as the Lyapunov–Foster criteria. Specifically, we shall utilize the following lemma; cf. [8], Theorem 1.

LEMMA 5. *Let $L : \mathsf{X} \to \mathbb{R}_+$ be a function such that $L(\mathbf{x}) \to \infty$ as $|\mathbf{x}| \to \infty$. For any $\kappa > 0$, let $B_\kappa = \{\mathbf{x} : L(\mathbf{x}) \leq \kappa\}$. And let there exist functions $h, g : \mathsf{X} \to \mathbb{Z}_+$ such that for any $\mathbf{x} \in \mathsf{X}$,*

$$\mathbb{E}_{\mathbf{x}}[L(X(g(\mathbf{x}))) - L(X(0))] \leq -h(\mathbf{x}),$$

that satisfy the following conditions:

(a) $\inf_{\mathbf{x}\in \mathsf{X}} h(\mathbf{x}) > -\infty$.
(b) $\liminf_{L(\mathbf{x})\to\infty} h(\mathbf{x}) > 0$.
(c) $\sup_{L(\mathbf{x})\le\gamma} g(\mathbf{x}) < \infty$ *for all* $\gamma > 0$.
(d) $\limsup_{L(\mathbf{x})\to\infty} g(\mathbf{x})/h(\mathbf{x}) < \infty$.

Then, *there exists constant* $\kappa_0 > 0$ *so that for all* $\kappa_0 < \kappa$, *the following holds*:

$$\mathbb{E}_{\mathbf{x}}[T_{B_\kappa}] < \infty \qquad \textit{for any } \mathbf{x} \in \mathsf{X}, \tag{32}$$

$$\sup_{\mathbf{x}\in B_\kappa} \mathbb{E}_{\mathbf{x}}[T_{B_\kappa}] < \infty. \tag{33}$$

5. Proofs of Theorems 1 and 2. This section provides proofs of Theorems 1 and 2. We shall start by introducing necessary formalism, summarize the key steps of the proof and then provide detailed proof.

5.1. *Network Markov process*. We describe discrete-time network Markov processes under both algorithms that we shall utilize throughout. Let $\tau \in \mathbb{Z}_+$ be the time index. Let $\mathbf{Q}(\tau) = [Q_i(\tau)]$ be the queue-size vector at time τ, $\mathbf{x}(\tau)$ be the schedule at time τ with $\mathbf{x}(\tau) = \boldsymbol{\sigma}(\tau) \in \mathcal{I}(G)$ for the wireless network and $\mathbf{x}(\tau) = \mathbf{z}(\tau) \in \mathcal{X}$ for the circuit switched network. It can be checked that the tuple $X(\tau) = (\mathbf{Q}(\tau), \mathbf{x}(\tau))$ is the Markov state of the network for both setups. Here $X(\tau) \in \mathsf{X}$ where $\mathsf{X} = \mathbb{R}_+^n \times \mathcal{I}(G)$ or $\mathsf{X} = \mathbb{Z}_+^n \times \mathcal{X}$. Clearly, X is a Polish space endowed with the natural product topology. Let $\mathcal{B}_{\mathsf{X}}$ be the Borel σ-algebra of X with respect to this product topology. For any $\mathbf{x} = (\mathbf{Q}, \mathbf{x}) \in \mathsf{X}$, we define norm of $\mathbf{x}$ denoted by $|\mathbf{x}|$ as

$$|\mathbf{x}| = |\mathbf{Q}| + |\mathbf{x}|,$$

where $|\mathbf{Q}|$ denotes the standard ℓ_1 norm while $|\mathbf{x}|$ is defined as its index in $\{0, \dots, |\Omega| - 1\}$, which is assigned arbitrarily. Since $|\mathbf{x}|$ is always bounded, $|\mathbf{x}| \to \infty$ if and only if $|\mathbf{Q}| \to \infty$. Theorems 1 and 2 wish to establish that the Markov process $X(\tau)$ is ergodic.

5.2. *Proof plan*. To establish ergodicity of $X(\tau)$, we will verify conditions (C1) and (C2) stated in Section 4.4.3. To verify condition (C1), using Lemma 5, we shall establish the existence of an appropriate Lyapunov function implied by our randomized scheduling algorithms. In a nutshell, we shall show that our scheduling algorithms are *simulating* the maximum weight scheduling algorithm with respect to an appropriate weight function of the queue-size. This will lead to the desired Lyapunov function and a drift criteria. The condition (C2) follows by showing that the "empty" state of the network verifies it. Next, we give an overview of the detailed proof which is stated in four steps: the first three steps are concerned with the verification of (C1) or Lyapunov drift criteria while the fourth step is concerned with (C2).

Recall that the randomized algorithms for wireless or circuit switched network are effectively asynchronous, continuous versions of the time-varying $\mathrm{GD}(\mathbf{W}(t))$ or $\mathbf{LN}(\exp(\mathbf{W}(t)))$, respectively. Let $\pi(t)$ be the stationary distribution of the Markov chain $\mathrm{GD}(\mathbf{W}(t))$ or $\mathbf{LN}(\exp(\mathbf{W}(t)))$; $\mu(t)$ be the distribution of the schedule, either $\boldsymbol{\sigma}(t)$ or $\mathbf{z}(t)$, under our algorithm at time t. In the first step, roughly speaking, we argue that the weight of schedule sampled as per the stationary distribution $\pi(t)$ is close to the weight of maximum weight schedule for both networks (with an appropriately defined weight). In the second step, roughly speaking, we argue that indeed the distribution $\mu(t)$ is close enough to that of $\pi(t)$ for all time t. In the third step, using these two properties we establish the Lyapunov drift criteria for appropriately defined Lyapunov function; cf. Lemma 5. This concludes the verification of (C1) for both models. In the fourth and final step, we establish that Markov processes are ergodic by showing that the "empty" state of the system satisfies condition (C2) under both models.

5.3. *Formal proof.* To this end, we are interested in establishing Lyapunov drift criteria; cf. Lemma 5. For this, consider Markov process starting at time 0 in state $X(0) = (\mathbf{Q}(0), \mathbf{x}(0))$ and as per hypothesis of both theorems, let $\boldsymbol{\lambda} \in (1 - \varepsilon)\,\mathsf{Conv}(\Omega)$ with some $\varepsilon > 0$ and $\Omega = \mathcal{I}(G)$ (or $\mathcal{X}$).

5.3.1. *Step one.* Let $\pi(0)$ be the stationary distribution of $\mathrm{GD}(\mathbf{W}(0))$ or $\mathbf{LN}(\exp(\mathbf{W}(0)))$. Lemma 6 states that the average weight of schedule as per $\pi(0)$ is essentially as good as that of the maximum weight schedule with respect to weight $f(\mathbf{Q}(0))$.

LEMMA 6. *Let* $\mathbf{x}$ *be distributed over* Ω *as per* $\pi(0)$ *given* $\mathbf{Q}(0)$. *Then*,

$$\mathbb{E}_{\pi(0)}[f(\mathbf{Q}(0)) \cdot \mathbf{x}] \geq \left(1 - \frac{\varepsilon}{4}\right)\left(\max_{\mathbf{y}\in\Omega} f(\mathbf{Q}(0)) \cdot \mathbf{y}\right) - O(1). \tag{34}$$

In the above and throughout the paper, the order notation subsumes constants that do not depend on the scaling of queue-sizes. However, it may depend on all other (nonscaling, constant) system parameters such as n, ε, etc.

The proof of Lemma 6 is based on the variational characterization of distribution in the exponential form. Specifically, we state the following proposition which is a direct adaptation of the known results in literature; cf. [9].

PROPOSITION 7. *Let* $T : \Omega \to \mathbb{R}$ *and let* $\mathcal{M}(\Omega)$ *be space of all distributions on* Ω. *Define* $F : \mathcal{M}(\Omega) \to \mathbb{R}$ *as*

$$F(\mu) = \mathbb{E}_\mu(T(\mathbf{x})) + H_{\mathrm{ER}}(\mu),$$

where $H_{\mathrm{ER}}(\mu)$ *is the standard discrete entropy of* μ. *Then*, F *is uniquely maximized by the distribution* ν, *where*

$$\nu_{\mathbf{x}} = \frac{1}{Z}\exp(T(\mathbf{x})) \qquad \textit{for any } \mathbf{x} \in \Omega,$$

where Z is the normalization constant (or partition function). Further, with respect to ν, we have

$$\mathbb{E}_\nu[T(\mathbf{x})] \ge \Big[\max_{\mathbf{x}\in\mathcal{X}} T(\mathbf{x})\Big] - \log|\Omega|.$$

PROOF. Observe that the definition of distribution ν implies that for any $\mathbf{x} \in \Omega$,

$$T(\mathbf{x}) = \log Z + \log \nu_{\mathbf{x}}.$$

Using this, for any distribution μ on Ω, we obtain

$$\begin{aligned}
F(\mu) &= \sum_{\mathbf{x}} \mu_{\mathbf{x}} T(\mathbf{x}) - \sum_{\mathbf{x}} \mu_{\mathbf{x}} \log \mu_{\mathbf{x}} \\
&= \sum_{\mathbf{x}} \mu_{\mathbf{x}} (\log Z + \log \nu_{\mathbf{x}}) - \sum_{\mathbf{x}} \mu_{\mathbf{x}} \log \mu_{\mathbf{x}} \\
&= \sum_{\mathbf{x}} \mu_{\mathbf{x}} \log Z + \sum_{\mathbf{x}} \mu_{\mathbf{x}} \log \frac{\nu_{\mathbf{x}}}{\mu_{\mathbf{x}}} \\
&= \log Z + \sum_{\mathbf{x}} \mu_{\mathbf{x}} \log \frac{\nu_{\mathbf{x}}}{\mu_{\mathbf{x}}} \\
&\le \log Z + \log\Big(\sum_{\mathbf{x}} \mu_{\mathbf{x}} \frac{\nu_{\mathbf{x}}}{\mu_{\mathbf{x}}}\Big) = \log Z
\end{aligned}$$

with equality if and only if $\mu = \nu$. To complete other claim of proposition, consider $\mathbf{x}^* \in \arg\max T(\mathbf{x})$. Let μ be Dirac distribution $\mu_{\mathbf{x}} = \mathbf{1}_{[\mathbf{x}=\mathbf{x}^*]}$. Then, for this distribution

$$F(\mu) = T(\mathbf{x}^*).$$

But, $F(\nu) \ge F(\mu)$. Also, the maximal entropy of any distribution on Ω is $\log|\Omega|$. Therefore,

$$\begin{aligned}
T(\mathbf{x}^*) &\le F(\nu) \\
&= \mathbb{E}_\nu[T(\mathbf{x})] + H_{\mathrm{ER}}(\nu) \qquad (35)\\
&\le \mathbb{E}_\nu[T(\mathbf{x})] + \log|\Omega|.
\end{aligned}$$

Rearrangement of terms in (35) will imply the second claim of Proposition 7. This completes the proof of Proposition 7. □

PROOF OF LEMMA 6. The proof is based on known observations in the context of classical loss networks literature; cf. [15]. In what follows, for simplicity, we use $\pi = \pi(0)$ for a given $\mathbf{Q} = \mathbf{Q}(0)$. From (9) and (19), it follows that for both network models, the stationary distribution π has the following form: for any $\mathbf{x} \in \Omega$,

$$\pi_{\mathbf{x}} \propto \prod_i \frac{\exp(W_i x_i)}{x_i!} = \exp\biggl(\sum_i W_i x_i - \log(x_i!)\biggr).$$

To apply Proposition 7, this suggests the choice of function $T : \mathcal{X} \to \mathbb{R}$ as

$$T(\mathbf{x}) = \sum_i W_i x_i - \log(x_i!) \qquad \text{for any } \mathbf{x} \in \Omega.$$

Observe that for any $\mathbf{x} \in \Omega$, x_i takes one of the finitely many values in wireless or circuit switched network for all i. Therefore, it easily follows that

$$0 \le \sum_i \log(x_i!) \le O(1),$$

where the constant may depend on n and the problem parameter (e.g., $C_{\max}$ in circuit switched network). Therefore, for any $\mathbf{x} \in \Omega$,

$$\begin{aligned} T(\mathbf{x}) &\le \sum_i W_i x_i \\ &\le T(\mathbf{x}) + O(1). \end{aligned} \tag{36}$$

Define $\widehat{\mathbf{x}} = \arg\max_{\mathbf{x}\in\Omega} \sum_i W_i x_i$. From (36) and Proposition 7, it follows that

$$\begin{aligned} \mathbb{E}_\pi\biggl[\sum_i W_i x_i\biggr] &\ge \mathbb{E}_\pi[T(\mathbf{x})] \\ &\ge \max_{\mathbf{x}\in\Omega} T(\mathbf{x}) - \log|\Omega| \\ &\ge T(\widehat{\mathbf{x}}) - \log|\Omega| \\ &= \biggl(\sum_i W_i \widehat{x}_i\biggr) - O(1) - \log|\Omega| \\ &= \Bigl(\max_{\mathbf{x}\in\Omega} \mathbf{W} \cdot \mathbf{x}\Bigr) - O(1). \end{aligned} \tag{37}$$

From the definition of weight in both algorithms [(3) and (4)] for a given $\mathbf{Q}$, weight $\mathbf{W} = [W_i]$ is defined as

$$W_i = \max\bigl\{f(Q_i), \sqrt{f(Q_{\max})}\bigr\}.$$

Define $\eta \triangleq \frac{\varepsilon}{4\max_{\mathbf{x}\in\Omega}\|x\|_1}$. To establish the proof of Lemma 6, we will consider $Q_{\max}$ such that it is large enough satisfying

$$\eta f(Q_{\max}) \ge \sqrt{f(Q_{\max})}.$$

For smaller $Q_{\max}$ we do not need to argue as that case (34) [due to $O(1)$ term] is straightforward. Therefore, in the remainder we assume $Q_{\max}$ large enough. For this large enough $Q_{\max}$, it follows that for all i,

$$0 \le W_i - f(Q_i) \le \sqrt{f(Q_{\max})} \le \eta f(Q_{\max}). \tag{38}$$

Using (38), for any $\mathbf{x} \in \Omega$,

$$\begin{aligned} 0 &\le \mathbf{W} \cdot \mathbf{x} - f(\mathbf{Q}) \cdot \mathbf{x} = \big(\mathbf{W} - f(\mathbf{Q})\big) \cdot \mathbf{x} \\ &\le \|\mathbf{x}\|_1 \|\mathbf{W} - f(\mathbf{Q})\|_\infty \\ &\le \|\mathbf{x}\|_1 \times \eta f(Q_{\max}) \\ &\overset{\text{(a)}}{\le} \frac{\varepsilon}{4} f(Q_{\max}) \\ &\overset{\text{(b)}}{\le} \frac{\varepsilon}{4}\Big(\max_{\mathbf{y}\in\Omega} f(\mathbf{Q}) \cdot \mathbf{y}\Big), \end{aligned} \tag{39}$$

where (a) is from our choice of $\eta = \frac{\varepsilon}{4 \max_{\mathbf{x}\in\Omega} \|x\|_1}$. For (b), we use the fact that the singleton set $\{i\}$, that is, independent set $\{i\}$ for wireless network and a single active on route i for circuit switched network, is a valid schedule. And, for $i = \arg\max_j Q_j$, it has weight $f(Q_{\max})$. Therefore, the weight of the maximum weighted schedule among all possible schedules in Ω is at least $f(Q_{\max})$. Finally, using (37) and (39) we obtain

$$\begin{aligned} \mathbb{E}_\pi[f(\mathbf{Q}) \cdot \mathbf{x}] &\ge \mathbb{E}_\pi[\mathbf{W} \cdot \mathbf{x}] - \frac{\varepsilon}{4}\Big(\max_{\mathbf{y}\in\Omega} f(\mathbf{Q}) \cdot \mathbf{y}\Big) \\ &\ge \Big(\max_{\mathbf{y}\in\Omega} \mathbf{W} \cdot \mathbf{y}\Big) - O(1) - \frac{\varepsilon}{4}\Big(\max_{\mathbf{y}\in\Omega} f(\mathbf{Q}) \cdot \mathbf{y}\Big) \\ &\ge \Big(\max_{\mathbf{y}\in\Omega} f(\mathbf{Q}) \cdot \mathbf{y}\Big) - O(1) - \frac{\varepsilon}{4}\Big(\max_{\mathbf{y}\in\Omega} f(\mathbf{Q}) \cdot \mathbf{y}\Big) \\ &= \Big(1 - \frac{\varepsilon}{4}\Big)\Big(\max_{\mathbf{y}\in\Omega} f(\mathbf{Q}) \cdot \mathbf{y}\Big) - O(1). \end{aligned}$$

This completes the proof of Lemma 6. □

5.3.2. *Step two.* Let $\mu(t)$ be the distribution of schedule $\mathbf{x}(t)$ over Ω at time t, given initial state $X(0) = (\mathbf{Q}(0), \mathbf{x}(0))$. We wish to show that for any initial condition $\mathbf{x}(0) \in \Omega$, for t large (but not too large) enough, $\mu(t)$ is close to $\pi(0)$ if $Q_{\max}(0)$ is large enough. Formal statement is as follows.

LEMMA 8. *For a large enough $Q_{\max}(0)$,*

$$\|\mu(t) - \pi(0)\|_{\mathrm{TV}} < \varepsilon/4 \tag{40}$$

for $t \in I = [b_1(Q_{\max}(0)), b_2(Q_{\max}(0))]$, *where* b_1, b_2 *are integer-valued functions on* $\mathbb{R}_+$ *such that*

$$b_1, b_2 = \mathsf{polylog}(Q_{\max}(0)) \quad \textit{and} \quad b_2/b_1 = \Theta(\log(Q_{\max}(0))).$$

In the above, *the constants may depend on* ε, $C_{\max}$ *and* n.

The notation $\mathsf{polylog}(z)$ represents a positive real-valued function of z that scales no faster than a finite degree polynomial of $\log z$.

PROOF OF LEMMA 8. We shall prove this lemma for the wireless network. The proof of buffered circuit switch network follows in an identical manner. Hence, we shall skip it. Therefore, we shall assume $\Omega = \mathcal{I}(G)$ and $\mathbf{x}(t) = \boldsymbol{\sigma}(t)$.

First, we establish the desired claim for integral times. The argument for nonintegral times will follow easily as argued near the end of this proof. For $t = \tau \in \mathbb{Z}_+$, we have

$$\begin{aligned}\mu(\tau+1) &= \mathbb{E}\bigl[\boldsymbol{\delta}_{\boldsymbol{\sigma}(\tau+1)}\bigr]\\ &= \mathbb{E}\bigl[\boldsymbol{\delta}_{\boldsymbol{\sigma}(\tau)} \cdot P(\tau)\bigr],\end{aligned}$$

where recall that $P(\tau)=e^{n(GW(\mathbf{W}(\tau))-I)}$ and the last equality follows from (16). Again recall that the expectation is with respect to the joint distribution of $\{\mathbf{Q}(\tau), \boldsymbol{\sigma}(\tau)\}$. Hence, it follows that

$$\begin{aligned}\mu(\tau+1) &= \mathbb{E}\bigl[\boldsymbol{\delta}_{\boldsymbol{\sigma}(\tau)} \cdot P(\tau)\bigr]\\ &= \mathbb{E}\bigl[\mathbb{E}\bigl[\boldsymbol{\delta}_{\boldsymbol{\sigma}(\tau)} \cdot P(\tau)|\mathbf{Q}(\tau)\bigr]\bigr]\\ &\overset{\text{(a)}}{=} \mathbb{E}\bigl[\mathbb{E}\bigl[\boldsymbol{\delta}_{\boldsymbol{\sigma}(\tau)}|\mathbf{Q}(\tau)\bigr] \cdot P(\tau)\bigr]\\ &= \mathbb{E}[\tilde{\mu}(\tau) \cdot P(\tau)],\end{aligned}$$

where

$$\tilde{\mu}(\tau) = \tilde{\mu}(\mathbf{Q}(\tau)) \overset{\Delta}{=} \mathbb{E}\bigl[\boldsymbol{\delta}_{\boldsymbol{\sigma}(\tau)}|\mathbf{Q}(\tau)\bigr].$$

In the above, the expectation is taken with respect to the conditional marginal distribution of $\boldsymbol{\sigma}(\tau)$ given $\mathbf{Q}(\tau)$; (a) follows from the linearity of expectation and the fact that $P(\tau)$ is a function of $\mathbf{Q}(\tau)$. Next, we establish the relation between $\mu(\tau)$ and $\mu(\tau+1)$:

$$\begin{aligned}\mu(\tau+1) &= \mathbb{E}[\tilde{\mu}(\tau) \cdot P(\tau)]\\ &= \mathbb{E}[\tilde{\mu}(\tau) \cdot P(0)] + \mathbb{E}\bigl[\tilde{\mu}(\tau) \cdot \bigl(P(\tau) - P(0)\bigr)\bigr]\\ &= \mathbb{E}[\tilde{\mu}(\tau)] \cdot P(0) + e(\tau)\\ &= \mu(\tau) \cdot P(0) + e(\tau),\end{aligned}$$

where $e(\tau) \stackrel{\Delta}{=} \mathbb{E}[\tilde{\mu}(\tau) \cdot (P(\tau) - P(0))]$. Here the expectation is with respect to $\mathbf{Q}(\tau)$. Similarly,

$$\begin{aligned}\mu(\tau+1) &= \mu(\tau) \cdot P(0) + e(\tau) \\ &= \big(\mu(\tau-1) \cdot P(0) + e(\tau-1)\big) \cdot P(0) + e(\tau) \\ &= \mu(\tau-1) \cdot P(0)^2 + e(\tau-1) \cdot P(0) + e(\tau).\end{aligned}$$

Therefore, recursively we obtain

$$\mu(\tau+1) = \mu(0) \cdot P(0)^{\tau+1} + \sum_{s=0}^{\tau} e(\tau - s) \cdot P(0)^s. \tag{41}$$

We will choose b_1 [which will depend on $Q_{\max}(0)$] such that for $\tau \ge b_1$,

$$\|\mu(0) \cdot P(0)^\tau - \pi(0)\|_{\mathrm{TV}} \le \varepsilon/8. \tag{42}$$

That is, b_1 is the *mixing time* of $P(0)$. Using inequalities (6), (8) and Lemma 3, it follows that

$$b_1 \equiv b_1(Q_{\max}(0)) = \mathsf{polylog}(Q_{\max}(0)). \tag{43}$$

In the above, constants may depend on n and ε. Therefore, from (41) and (42), it suffices to show that

$$\left\| \sum_{s=0}^{\tau-1} e(\tau - 1 - s) \cdot P(0)^s \right\|_1 \le \varepsilon/4 \tag{44}$$

for $\tau \in I = [b_1, b_2]$ with an appropriate choice of $b_2 = b_2(Q_{\max}(0))$.[8] To this end, we choose

$$b_2 \equiv b_2(Q_{\max}(0)) = \lceil b_1 \log(Q_{\max}(0)) \rceil. \tag{45}$$

Thus, $b_2(Q_{\max}(0)) = \mathsf{polylog}(Q_{\max}(0))$ as well. With this choice of b_2, we obtain the following bound on $e(\tau)$ to conclude (44):

$$\begin{aligned}\|e(\tau)\|_1 &= \big\|\mathbb{E}[\tilde{\mu}(\tau) \cdot \big(P(\tau) - P(0)\big)]\big\|_1 \\ &\le \mathbb{E}\big[\big\|\tilde{\mu}(\tau) \cdot \big(P(\tau) - P(0)\big)\big\|_1\big] \\ &\stackrel{\text{(a)}}{\le} O\big(\mathbb{E}[\|P(\tau) - P(0)\|_\infty]\big) \\ &\stackrel{\text{(b)}}{=} O\big(\mathbb{E}[\|GW(\mathbf{W}(\tau)) - GW(\mathbf{W}(0))\|_\infty]\big) \\ &\stackrel{\text{(c)}}{=} O\left(\mathbb{E}\left[\max_i \left| \frac{1}{1+\exp(W_i(\tau))} - \frac{1}{1+\exp(W_i(0))} \right|\right]\right)\end{aligned} \tag{46}$$

---

[8] Note that $\|\mathbf{u}\|_{\mathrm{TV}} = \frac{1}{2}\|\mathbf{u}\|_1$. Hence, $\| \sum_{s=0}^{\tau-1} e(\tau - 1 - s) \cdot P(0)^s \|_{\mathrm{TV}} \le \varepsilon/8$.

$$\stackrel{(d)}{=} O\Big(\mathbb{E}[\max_i |W_i(\tau) - W_i(0)|]\Big)$$

$$\stackrel{(e)}{=} O\Big(\max_i \mathbb{E}[|W_i(\tau) - W_i(0)|]\Big).$$

In the above, (a) follows from the standard norm inequality and the fact that $\|\tilde{\mu}(\tau)\|_1 = 1$, (b) follows from Lemma 10 in the Appendix, (c) follows directly from the definition of transition matrix GD($\mathbf{W}$), (d) follows from 1-Lipschitz[9] property of function $1/(1+e^x)$ and (e) follows from the fact that vector $\mathbf{W}(\tau)$ being $O(1)$ dimensional.[10]

Next, we will show that for all i and $\tau \le b_2$,

$$\mathbb{E}[|W_i(\tau) - W_i(0)|] = O\left(\frac{1}{\mathsf{superpolylog}(Q_{\max}(0))}\right), \tag{47}$$

the notation $\mathsf{superpolylog}(z)$ represents a positive real-valued function of z that scales faster than any finite degree polynomial of $\log z$. This is enough to conclude (44) (hence, complete the proof of Lemma 8) since

$$\begin{aligned}
\left\|\sum_{s=0}^{\tau-1} e(\tau-1-s)\cdot P(0)^s\right\|_1 &\le \sum_{s=0}^{\tau-1} \|e(\tau-1-s)\cdot P(0)^s\|_1 \\
&= \sum_{s=0}^{\tau-1} O\big(\|e(\tau-1-s)\|_1\big) \\
&\stackrel{(a)}{=} O\left(\frac{\tau}{\mathsf{superpolylog}(Q_{\max}(0))}\right) \\
&\stackrel{(b)}{\le} \frac{\varepsilon}{4},
\end{aligned}$$

where we use (46) and (47) to obtain (a), (b) holds for large enough $Q_{\max}(0)$ and $\tau \le b_2 = \mathsf{polylog}(Q_{\max}(0))$.

Now to complete the proof, we only need to establish (47). This is the step that utilizes "slowly varying" property of function $f(x) = \log\log(x+e)$. First, we provide an intuitive sketch of the argument. Somewhat involved details will follow. To explain the intuition behind (47), let us consider a simpler situation where i is such that $Q_i(0) = Q_{\max}(0)$ and $f(Q_i(\tau)) > \sqrt{f(Q_{\max}(\tau))}$ for a given $\tau \in [0, b_2]$. That is, let $W_i(\tau) = f(Q_i(\tau))$. Now, consider the following sequence

[9]A function $f : \mathbb{R} \to \mathbb{R}$ is k-Lipschitz if $|f(s) - f(t)| \le k|s-t|$ for all $s, t \in \mathbb{R}$.

[10]We note here that the $O(\cdot)$ notation means existences of constants that do not depend scaling quantities such as time τ and $\mathbf{Q}(0)$; however it may depend on the fixed system parameters such as number of queues. The use of this terminology is to retain the clarity of exposition.

of inequalities:

$$
\begin{aligned}
|W_i(\tau) - W_i(0)| &= |f(Q_i(\tau)) - f(Q_i(0))| \\
&\overset{(a)}{\le} f'(\zeta)|Q_i(\tau) - Q_i(0)| \qquad \text{for some } \zeta \text{ around } Q_i(0) \\
&\overset{(b)}{\le} f'(\min\{Q_i(\tau), Q_i(0)\})O(\tau) \\
&\overset{(c)}{\le} f'\big(Q_i(0) - O(\tau)\big)O(\tau) \\
&\overset{(d)}{=} O\left(\frac{\tau}{Q_i(0)}\right).
\end{aligned} \tag{48}
$$

In the above, (a) follows from the mean value theorem; (b) follows from monotonicity of f' and Lipschitz property of $Q_i(\cdot)$ (as a function of τ)—which holds deterministically for wireless network due to the assumption of the Bernoulli arrival process and probabilistically for circuit switched network; (c) uses the same Lipschitz property; (d) uses the fact that $\tau \le b_2$ and $b_2 = \mathsf{polylog}(Q_{\max}(0))$, $Q_{\max}(0) = Q_i(0)$. Therefore, effectively, the bound of (48) is $O(1/\mathsf{superpolylog}(Q_{\max}(0))$.

The above explains the gist of the argument that is to follow. However, to make it precise, we will need to provide lots more details. Toward this, we consider the following two cases: (i) $f(Q_i(0)) \ge \sqrt{f(Q_{\max}(0))}$ and (ii) $f(Q_i(0)) < \sqrt{f(Q_{\max}(0))}$. In what follows, we provide detailed arguments for (i). The arguments for case (ii) are similar in spirit and will be provided later in the proof.

Case (i). Consider an i such that $f(Q_i(0)) \ge \sqrt{f(Q_{\max}(0))}$. Then,

$$
\begin{aligned}
&\mathbb{E}[|W_i(\tau) - W_i(0)|] \\
&\quad = \mathbb{E}[|W_i(\tau) - f(Q_i(0))|] \\
&\quad = \mathbb{E}\big[|f(Q_i(\tau)) - f(Q_i(0))| \cdot \mathbf{I}_{\{f(Q_i(\tau)) \ge \sqrt{f(Q_{\max}(\tau))}\}}\big] \\
&\qquad + \mathbb{E}\big[\big|\sqrt{f(Q_{\max}(\tau))} - f(Q_i(0))\big| \cdot \mathbf{I}_{\{f(Q_i(\tau)) < \sqrt{f(Q_{\max}(\tau))}\}}\big],
\end{aligned} \tag{49}
$$

where each equality follows from (3) and recall that $\mathbf{I}_A$ is the indicator function of event A. The first term in (49) can be bounded as

$$
\begin{aligned}
&\mathbb{E}\big[|f(Q_i(\tau)) - f(Q_i(0))| \cdot \mathbf{I}_{\{f(Q_i(\tau)) \ge \sqrt{f(Q_{\max}(\tau))}\}}\big] \\
&\quad \le \mathbb{E}[|f(Q_i(\tau)) - f(Q_i(0))|] \\
&\quad \overset{(o)}{\le} \mathbb{E}[f'(\min\{Q_i(\tau), Q_i(0)\})|Q_i(\tau) - Q_i(0)|] \\
&\quad \overset{(a)}{\le} \sqrt{\mathbb{E}[f'(\min\{Q_i(\tau), Q_i(0)\})^2]} \cdot \sqrt{\mathbb{E}\big[\big(Q_i(\tau) - Q_i(0)\big)^2\big]} \\
&\quad \overset{(b)}{\le} \sqrt{f'\left(\frac{Q_i(0)}{2}\right)^2 + \Theta\left(\frac{\tau}{Q_i(0)}\right)} \cdot O(\tau)
\end{aligned} \tag{50}
$$

$$
\begin{aligned}
&\stackrel{\text{(c)}}{\le} \sqrt{f'\Big(\frac{1}{2}f^{-1}\big(\sqrt{f(Q_{\max}(0))}\big)\Big)^2 + \Theta\Big(\frac{\tau}{f^{-1}(\sqrt{f(Q_{\max}(0))})}\Big)} \\
&\quad \times O(\tau) \\
&\stackrel{\text{(d)}}{=} O\Big(\frac{1}{\mathsf{superpolylog}(Q_{\max}(0))}\Big).
\end{aligned}
$$

In the above, (o) follows from concavity of f. For (a), we use the standard Cauchy–Schwarz inequality $\mathbb{E}[XY] \le \sqrt{\mathbb{E}[X^2]}\sqrt{\mathbb{E}[Y^2]}$. For (b), note that given $Q_i(0)$, $\mathbb{E}[[Q_i(0) - Q_i(\tau)]^2] = O(\tau^2)$ for both network models—for wireless network, it is deterministically true since we assume the Bernoulli arrival process and $\mathbf{Q}(\cdot)$ is Lipschitz; for circuit switched network, it is due to the fact that the arrival as well as (the overall) departure processes are bounded rate Poisson processes. Given this, using Markov's inequality it follows that

$$
\Pr\Big(\min\{Q_i(\tau), Q_i(0)\} \le \frac{Q_i(0)}{2}\Big) = O\Big(\frac{\tau}{Q_i(0)}\Big).
$$

Finally, using the fact that $\sup_{y\in\mathbb{R}_+} f'(y) = O(1)$, we obtain (b). Now (c) follows from the condition of $Q_i(0)$ that $f(Q_i(0)) \ge \sqrt{f(Q_{\max}(0))}$ and (d) is implied by $\tau \le b_2 = \mathsf{polylog}(Q_{\max}(0))$, $f(x) = \log\log(x+e)$.

Next, we bound the second term in (49). We will use notation

$$
\begin{aligned}
A(\tau) &= \big\{f(Q_i(\tau)) < \sqrt{f(Q_{\max}(\tau))} \text{ and } \sqrt{f(Q_{\max}(\tau))} \ge f(Q_i(0))\big\}, \\
B(\tau) &= \big\{f(Q_i(\tau)) < \sqrt{f(Q_{\max}(\tau))} \text{ and } \sqrt{f(Q_{\max}(\tau))} < f(Q_i(0))\big\}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\mathbb{E}\big[\big|\sqrt{f(Q_{\max}(\tau))} - f(Q_i(0))\big| \cdot \mathbf{I}_{\{f(Q_i(\tau))<\sqrt{f(Q_{\max}(\tau))}\}}\big] \\
&\quad = \mathbb{E}\big[\big(\sqrt{f(Q_{\max}(\tau))} - f(Q_i(0))\big) \cdot \mathbf{I}_{A(\tau)}\big] \\
&\qquad + \mathbb{E}\big[\big(f(Q_i(0)) - \sqrt{f(Q_{\max}(\tau))}\big) \cdot \mathbf{I}_{B(\tau)}\big] \\
&\quad \stackrel{\text{(a)}}{\le} \mathbb{E}\big[\big(\sqrt{f(Q_{\max}(\tau))} - \sqrt{f(Q_{\max}(0))}\big) \cdot \mathbf{I}_{A(\tau)}\big] \\
&\qquad + \mathbb{E}\big[\big(f(Q_i(0)) - f(Q_i(\tau))\big) \cdot \mathbf{I}_{B(\tau)}\big] \\
&\quad \stackrel{\text{(b)}}{\le} \mathbb{E}[|f(Q_{\max}(\tau)) - f(Q_{\max}(0))|] \\
&\qquad + \mathbb{E}[|f(Q_i(0)) - f(Q_i(\tau))|] \\
&\quad = O\Big(\frac{1}{\mathsf{superpolylog}(Q_{\max}(0))}\Big).
\end{aligned} \tag{51}
$$

In the above, (a) follows because we are considering case (i) with $f(Q_i(0)) \geq \sqrt{f(Q_{\max}(0))}$ and definition of event $B(\tau)$; (b) follows from 1-Lipschitz property of $\sqrt{\cdot}$ function and appropriate removal of indicator random variables. For the final conclusion, we observe that the arguments used to establish (50) imply the $O(1/\mathsf{superpolylog}(Q_{\max}(0)))$ bound on both the terms in very similar manner; for the term corresponding to $|f(Q_{\max}(\tau)) - f(Q_{\max}(0))|$, one has to adapt arguments of (50) by essentially replacing the index i by max. This concludes the proof of (47) for case (i) of $f(Q_i(0)) \geq \sqrt{f(Q_{\max}(0))}$.

Case (ii). Now consider i such that $f(Q_i(0)) < \sqrt{f(Q_{\max}(0))}$. Then,

$$\begin{aligned}
&\mathbb{E}[|W_i(\tau) - W_i(0)|] \\
&\quad = \mathbb{E}\big[\big|W_i(\tau) - \sqrt{f(Q_{\max}(0))}\big|\big] \\
&\quad = \mathbb{E}\big[\big|f(Q_i(\tau)) - \sqrt{f(Q_{\max}(0))}\big| \cdot \mathbf{I}_{\{f(Q_i(\tau)) \geq \sqrt{f(Q_{\max}(\tau))}\}}\big] \\
&\qquad + \mathbb{E}\big[\big|\sqrt{f(Q_{\max}(\tau))} - \sqrt{f(Q_{\max}(0))}\big| \cdot \mathbf{I}_{\{f(Q_i(\tau)) < \sqrt{f(Q_{\max}(\tau))}\}}\big].
\end{aligned} \tag{52}$$

First observe that by 1-Lipschitz property of $\sqrt{\cdot}$ function, the second term can be bounded as [similar to (51)]

$$\begin{aligned}
&\mathbb{E}\big[\big|\sqrt{f(Q_{\max}(\tau))} - \sqrt{f(Q_{\max}(0))}\big| \cdot \mathbf{I}_{\{f(Q_i(\tau)) < \sqrt{f(Q_{\max}(\tau))}\}}\big] \\
&\quad \leq \mathbb{E}[|f(Q_{\max}(\tau)) - f(Q_{\max}(0))|] \\
&\quad = O\biggl(\frac{1}{\mathsf{superpolylog}(Q_{\max}(0))}\biggr).
\end{aligned} \tag{53}$$

Therefore, we are left with proving the first term of (52). We will follow a similar line of arguments as those used for (51). Define

$$A'(\tau) = \bigl\{f(Q_i(\tau)) \geq \sqrt{f(Q_{\max}(\tau))} \text{ and } \sqrt{f(Q_{\max}(0))} \geq f(Q_i(\tau))\bigr\},$$

$$B'(\tau) = \bigl\{f(Q_i(\tau)) \geq \sqrt{f(Q_{\max}(\tau))} \text{ and } \sqrt{f(Q_{\max}(0))} < f(Q_i(\tau))\bigr\}.$$

Then,

$$\begin{aligned}
&\mathbb{E}\big[\big|f(Q_i(\tau)) - \sqrt{f(Q_{\max}(0))}\big| \cdot \mathbf{I}_{\{f(Q_i(\tau)) \geq \sqrt{f(Q_{\max}(\tau))}\}}\big] \\
&\quad = \mathbb{E}\big[\big(\sqrt{f(Q_{\max}(0))} - f(Q_i(\tau))\big) \cdot \mathbf{I}_{A'(\tau)}\big] \\
&\qquad + \mathbb{E}\big[\big(f(Q_i(\tau)) - \sqrt{f(Q_{\max}(0))}\big) \cdot \mathbf{I}_{B'(\tau)}\big] \\
&\quad \overset{\text{(a)}}{\leq} \mathbb{E}\big[\big(\sqrt{f(Q_{\max}(0))} - \sqrt{f(Q_{\max}(\tau))}\big) \cdot \mathbf{I}_{A'(\tau)}\big] \\
&\qquad + \mathbb{E}\big[\big(f(Q_i(\tau)) - \sqrt{f(Q_{\max}(0))}\big) \cdot \mathbf{I}_{B'(\tau)}\big]
\end{aligned} \tag{54}$$

$$\stackrel{\text{(b)}}{\le} O\left(\frac{1}{\mathsf{superpolylog}(Q_{\max}(0))}\right)$$
$$+ \mathbb{E}\big[\big(f(Q_i(\tau)) - \sqrt{f(Q_{\max}(0))}\big) \cdot \mathbf{I}_{B'(\tau)}\big].$$

In the above, (a) follows because we are considering case (i) with $f(Q_i(\tau)) \ge \sqrt{f(Q_{\max}(\tau))}$ and definition of event $B(\tau)$; (b) follows from 1-Lipschitz property of $\sqrt{\cdot}$ function and appropriate removal of indicator random variables as follows:

$$\begin{aligned}
&\mathbb{E}\big[\big(\sqrt{f(Q_{\max}(0))} - \sqrt{f(Q_{\max}(\tau))}\big) \cdot \mathbf{I}_{A'(\tau)}\big]\\
&\quad \le \mathbb{E}[|f(Q_{\max}(\tau)) - f(Q_{\max}(0))|] \\
&\quad = O\left(\frac{1}{\mathsf{superpolylog}(Q_{\max}(0))}\right).
\end{aligned} \tag{55}$$

Finally, to complete the proof of case (ii) using (52), we wish to establish

$$\mathbb{E}\big[\big(f(Q_i(\tau)) - \sqrt{f(Q_{\max}(0))}\big) \cdot \mathbf{I}_{B'(\tau)}\big] = O\left(\frac{1}{\mathsf{superpolylog}(Q_{\max}(0))}\right). \tag{56}$$

Now suppose $x \in \mathbb{R}_+$ be such that $f(x) = \sqrt{f(Q_{\max}(0))}$. Then,

$$\begin{aligned}
&\mathbb{E}\big[\big(f(Q_i(\tau)) - \sqrt{f(Q_{\max}(0))}\big) \cdot \mathbf{I}_{B'(\tau)}\big]\\
&\quad = \mathbb{E}\big[\big(f(Q_i(\tau)) - f(x)\big) \cdot \mathbf{I}_{B'(\tau)}\big]\\
&\quad \stackrel{\text{(a)}}{\le} \mathbb{E}\big[f'(x)\big(Q_i(\tau) - x\big) \cdot \mathbf{I}_{B'(\tau)}\big]\\
&\quad = f'(x)\mathbb{E}\big[\big(Q_i(\tau) - x\big) \cdot \mathbf{I}_{B'(\tau)}\big]\\
&\quad \stackrel{\text{(b)}}{\le} f'(x)\mathbb{E}\big[\big(Q_i(\tau) - Q_i(0)\big) \cdot \mathbf{I}_{B'(\tau)}\big]\\
&\quad \le f'(x)\mathbb{E}[|Q_i(\tau) - Q_i(0)|]\\
&\quad \stackrel{\text{(c)}}{=} f'(x)O(\tau)\\
&\quad \stackrel{\text{(d)}}{=} O\left(\frac{1}{\mathsf{superpolylog}(Q_{\max}(0))}\right).
\end{aligned} \tag{57}$$

In the above, (a) follows from concavity of f; (b) from $Q_i(0) \le x$ and $Q_i(\tau) \ge x$ implied by case (ii) and $B'(\tau)$, respectively; (c) follows from arguments used earlier that for any i, $\mathbb{E}[(Q_i(\tau) - Q_i(0))^2] = O(\tau^2)$; (d) follows from $\tau \le b_2 = \mathsf{polylog}(Q_{\max}(0))$ and

$$f'(x) = O\left(\frac{1}{\mathsf{superpolylog}(Q_{\max}(0))}\right).$$

This complete the proof of (47) for both cases and the proof of Lemma 8 for integral time steps. A final remark regarding the validity of this result for nonintegral times is in order.

Consider $t \in I$ and $t \notin \mathbb{Z}_+$. Let $\tau = \lfloor t \rfloor$ and $t = \tau + \delta$ for $\delta \in (0, 1)$. Then, it follows that [using formal definition P^δ as in (17)]

$$\begin{aligned}
\mu(t) &= \mu(\tau + \delta) \\
&= \mu(\tau) P^\delta(0) + \mathbb{E}\big[\tilde{\mu}(\tau)\big(P^\delta(\tau) - P^\delta(0)\big)\big] \qquad (58)\\
&= \mu(0) P(0)^\tau P^\delta(0) + e(\tau + \delta).
\end{aligned}$$

Now it can be checked that $P^\delta(0)$ is a probability matrix and has $\pi(0)$ as its stationary distribution for any $\delta > 0$ and we have argued that for τ large enough $\mu(0)P(0)^\tau$ is close to $\pi(0)$. Therefore, $\mu(0)P(0)^\tau P^\delta(0)$ is also equally close to $\pi(0)$. For $e(\tau + \delta)$, it can be easily argued that the bound obtained in (46) for $e(\tau + 1)$ will dominate the bound for $e(\tau + \delta)$. Therefore, the statement of lemma holds for any nonintegral t as well. This completes the proof of Lemma 8. □

5.3.3. *Step three*: *Wireless network*. In this section, we prove Lemma 5 for the wireless network model. For Markov process $X(t) = (\mathbf{Q}(t), \boldsymbol{\sigma}(t))$, we consider Lyapunov function

$$L(X(t)) = \sum_i F(Q_i(t)),$$

where $F(x) = \int_0^x f(y)\,dy$ and recall that $f(x) = \log\log(x + e)$. For this Lyapunov function, it suffices to find appropriate functions h and g as per Lemma 5 for a large enough $Q_{\max}(0)$. Therefore, we assume that $Q_{\max}(0)$ is large enough so that it satisfies the conditions of Lemma 8. To this end, from Lemma 8, we have that for $t \in I$,

$$\begin{aligned}
&\big|\mathbb{E}_{\pi(0)}[f(\mathbf{Q}(0)) \cdot \boldsymbol{\sigma}] - \mathbb{E}_{\mu(t)}[f(\mathbf{Q}(0)) \cdot \boldsymbol{\sigma}]\big| \\
&\qquad \le \frac{\varepsilon}{4}\Big(\max_{\boldsymbol{\rho} \in \mathcal{I}(G)} f(\mathbf{Q}(0)) \cdot \boldsymbol{\rho}\Big).
\end{aligned}$$

Thus, from Lemma 6, it follows that

$$\mathbb{E}_{\mu(t)}[f(\mathbf{Q}(0)) \cdot \boldsymbol{\sigma}] \ge \Big(1 - \frac{\varepsilon}{2}\Big)\Big(\max_{\boldsymbol{\rho} \in \mathcal{I}(G)} f(\mathbf{Q}(0)) \cdot \boldsymbol{\rho}\Big) - O(1). \qquad (59)$$

Now we can bound the difference between $L(X(\tau + 1))$ and $L(X(\tau))$ as

$$\begin{aligned}
&L\big(X(\tau + 1)\big) - L(X(\tau)) \\
&\quad = \big(F\big(\mathbf{Q}(\tau + 1)\big) - F(\mathbf{Q}(\tau))\big) \cdot \mathbf{1} \\
&\quad \le f\big(\mathbf{Q}(\tau + 1)\big) \cdot \big(\mathbf{Q}(\tau + 1) - \mathbf{Q}(\tau)\big) \\
&\quad \le f(\mathbf{Q}(\tau)) \cdot \big(\mathbf{Q}(\tau + 1) - \mathbf{Q}(\tau)\big) + n,
\end{aligned}$$

where the first inequality is from the convexity of F and the last inequality follows from the fact that $f(\mathbf{Q}(\cdot))$ is 1-Lipschitz (as a function of τ).[11] Therefore,

$$
\begin{aligned}
&L\big(X(\tau+1)\big) - L(X(\tau)) \\
&\quad = \big(F\big(\mathbf{Q}(\tau+1)\big) - F(\mathbf{Q}(\tau))\big) \cdot \mathbf{1} \\
&\quad \le f(\mathbf{Q}(\tau)) \cdot \left(A(\tau,\tau+1) - \int_\tau^{\tau+1} \boldsymbol{\sigma}(y)\mathbf{1}_{\{Q_i(y)>0\}}\,dy\right) + n \qquad (60)\\
&\quad \overset{(a)}{\le} f(\mathbf{Q}(\tau)) \cdot A(\tau,\tau+1) - \int_\tau^{\tau+1} f(\mathbf{Q}(y)) \cdot \boldsymbol{\sigma}(y)\mathbf{1}_{\{Q_i(y)>0\}}\,dy + 2n \\
&\quad = f(\mathbf{Q}(\tau)) \cdot A(\tau,\tau+1) - \int_\tau^{\tau+1} f(\mathbf{Q}(y)) \cdot \boldsymbol{\sigma}(y)\,dy + 2n,
\end{aligned}
$$

where, again, (a) follows from the fact that $f(\mathbf{Q}(\cdot))$ is 1-Lipschitz. Given initial state $X(0) = \mathbf{x}$, taking the expectation of (60) for $\tau, \tau+1 \in I$,

$$
\begin{aligned}
&\mathbb{E}_{\mathbf{x}}\big[L\big(X(\tau+1)\big) - L(X(\tau))\big] \\
&\quad \le \mathbb{E}_{\mathbf{x}}[f(\mathbf{Q}(\tau)) \cdot A(\tau,\tau+1)] - \int_\tau^{\tau+1} \mathbb{E}_{\mathbf{x}}[f(\mathbf{Q}(y)) \cdot \boldsymbol{\sigma}(y)]\,dy + 2n \\
&\quad = \mathbb{E}_{\mathbf{x}}[f(\mathbf{Q}(\tau)) \cdot \boldsymbol{\lambda}] - \int_\tau^{\tau+1} \mathbb{E}_{\mathbf{x}}[f(\mathbf{Q}(y)) \cdot \boldsymbol{\sigma}(y)]\,dy + 2n,
\end{aligned}
$$

where the last equality follows from the independence between $\mathbf{Q}(\tau)$ and $A(\tau, \tau + 1)$ (recall, Bernoulli arrival process). Therefore,

$$
\begin{aligned}
&\mathbb{E}_{\mathbf{x}}\big[L\big(X(\tau+1)\big) - L(X(\tau))\big] \\
&\quad \le \mathbb{E}_{\mathbf{x}}[f(\mathbf{Q}(\tau)) \cdot \boldsymbol{\lambda}] - \int_\tau^{\tau+1} \mathbb{E}_{\mathbf{x}}[f(\mathbf{Q}(0)) \cdot \boldsymbol{\sigma}(y)]\,dy \\
&\qquad - \int_\tau^{\tau+1} \mathbb{E}_{\mathbf{x}}\big[\big(f(\mathbf{Q}(y)) - f(\mathbf{Q}(0))\big) \cdot \boldsymbol{\sigma}(y)\big]\,dy + 2n \\
&\quad \overset{(a)}{\le} f\big(\mathbf{Q}(0) + \tau \cdot \mathbf{1}\big) \cdot \boldsymbol{\lambda} - \int_\tau^{\tau+1} \mathbb{E}_{\mathbf{x}}[f(\mathbf{Q}(0)) \cdot \boldsymbol{\sigma}(y)]\,dy \\
&\qquad - \int_\tau^{\tau+1} \big(f\big(\mathbf{Q}(0) - y \cdot \mathbf{1}\big) - f(\mathbf{Q}(0))\big) \cdot \mathbf{1}\,dy + 2n \\
&\quad \overset{(b)}{\le} f(\mathbf{Q}(0)) \cdot \boldsymbol{\lambda} + f(\tau \cdot \mathbf{1}) \cdot \boldsymbol{\lambda} - \left(1 - \frac{\varepsilon}{2}\right)\Big(\max_{\boldsymbol{\rho} \in \mathcal{I}(G)} f(\mathbf{Q}(0)) \cdot \boldsymbol{\rho}\Big) \\
&\qquad + \int_\tau^{\tau+1} f(y \cdot \mathbf{1}) \cdot \mathbf{1}\,dy + O(1)
\end{aligned}
$$

[11] Recall that $\mathbf{Q}(\cdot)$ is 1-Lipschitz since we assume the Bernoulli arrival process for wireless network.

$$\leq f(\mathbf{Q}(0)) \cdot \boldsymbol{\lambda} - \left(1 - \frac{\varepsilon}{2}\right)\left(\max_{\boldsymbol{\rho} \in \mathcal{I}(G)} f(\mathbf{Q}(0)) \cdot \boldsymbol{\rho}\right)$$
$$+ 2nf(\tau + 1) + O(1).$$

In the above, (a) uses Lipschitz property of $\mathbf{Q}(\cdot)$ (as a function of τ); (b) follows from (59) and the inequality that for $f(x) = \log\log(x+e)$, $f(x) + f(y) + \log 2 \geq f(x+y)$ for all $x, y \in \mathbb{R}_+$. The $O(1)$ term is constant, dependent on n and captures the constant from (59).

Now, since $\boldsymbol{\lambda} \in (1-\varepsilon)\,\mathsf{Conv}(\mathcal{I}(G))$, we obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbf{x}}\bigl[L\bigl(X(\tau+1)\bigr) - L(X(\tau))\bigr] \\
&\qquad \leq -\frac{\varepsilon}{2}\left(\max_{\boldsymbol{\rho} \in \mathcal{I}(G)} f(\mathbf{Q}(0)) \cdot \boldsymbol{\rho}\right) + 2nf(\tau+1) + O(1) \\
&\qquad \leq -\frac{\varepsilon}{2} f(Q_{\max}(0)) + 2nf(\tau+1) + O(1).
\end{aligned}$$

Therefore, summing τ from $b_1 = b_1(Q_{\max}(0))$ to $b_2 = b_2(Q_{\max}(0))$ [recall definition of b_1, b_2 from (43) and (45), resp.], we have

$$\begin{aligned}
&\mathbb{E}_{\mathbf{x}}[L(X(b_2)) - L(X(b_1))] \\
&\quad \leq -\frac{\varepsilon}{2}(b_2 - b_1) f(Q_{\max}(0)) + 2n \sum_{\tau=b_1}^{b_2-1} f(\tau+1) + O(b_2 - b_1) \qquad (61)\\
&\quad \leq -\frac{\varepsilon}{2}(b_2 - b_1) f(Q_{\max}(0)) + 2n(b_2 - b_1) f(b_2) + O(b_2 - b_1).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbf{x}}[L(X(b_2)) - L(X(0))] \\
&\quad = \mathbb{E}_{\mathbf{x}}[L(X(b_1)) - L(X(0))] + \mathbb{E}_{\mathbf{x}}[L(X(b_2)) - L(X(b_1))] \\
&\quad \overset{\text{(a)}}{\leq} \mathbb{E}_{\mathbf{x}}\bigl[f(\mathbf{Q}(b_1)) \cdot \bigl(\mathbf{Q}(b_1) - \mathbf{Q}(0)\bigr)\bigr] - \frac{\varepsilon}{2}(b_2 - b_1) f(Q_{\max}(0)) \qquad (62)\\
&\qquad + 2n \sum_{\tau=b_1}^{b_2-1} f(\tau+1) + O(b_2 - b_1) \\
&\quad \overset{\text{(b)}}{\leq} nb_1 f\bigl(Q_{\max}(0) + b_1\bigr) - \frac{\varepsilon}{2}(b_2 - b_1) f(Q_{\max}(0)) \\
&\qquad + 2n(b_2 - b_1) f(b_2) + O(b_2 - b_1),
\end{aligned}$$

where (a) follows from the convexity of L and (b) is due to the 1-Lipschitz property of $\mathbf{Q}(\cdot)$. Now if we choose $g(\mathbf{x}) = b_2$ and

$$\begin{aligned}
h(\mathbf{x}) = -nb_1 f\bigl(Q_{\max}(0) + b_1\bigr) &+ \frac{\varepsilon}{2}(b_2 - b_1) f(Q_{\max}(0)) \\
&- 2n(b_2 - b_1) f(b_2) - O(b_2 - b_1),
\end{aligned}$$

the desired inequality follows:

$$\mathbb{E}_{\mathbf{x}}[L(X(g(\mathbf{x}))) - L(X(0))] \leq -h(\mathbf{x}).$$

The desired conditions of Lemma 5 can be checked as follows. First observe that with respect to $Q_{\max}(0)$, the function h scales as $b_2(Q_{\max}(0)) f(Q_{\max}(0))$ due to $b_2/b_1 = \Theta(\log Q_{\max}(0))$ as per Lemma 8. Further, h is a function that is lower bounded and its value goes to ∞ as $Q_{\max}(0)$ goes to ∞. Therefore, h/g scales as $f(Q_{\max}(0))$. These properties will imply the verification conditions of Lemma 5.

5.3.4. *Step three*: *Buffered circuit switched network.* In this section we prove Lemma 5 for the circuit switched network model. Similar to wireless network, we are interested in large enough $Q_{\max}(0)$ that satisfies condition of Lemma 8. Given the state $X(t) = (\mathbf{Q}(t), \mathbf{z}(t))$ of the Markov process, we shall consider the following Lyapunov function:

$$L(X(t)) = \sum_i F(R_i(t)).$$

Here $\mathbf{R}(t) = [R_i(t)]$ with $R_i(t) = Q_i(t) + z_i(t)$ and, as before, $F(x) = \int_0^x f(y)\,dy$. Now we proceed toward finding appropriate functions h and g as desired in Lemma 5. For any $\tau \in \mathbb{Z}_+$,

$$\begin{aligned}
&L\bigl(X(\tau+1)\bigr) - L(X(\tau)) \\
&\quad = \bigl(F\bigl(\mathbf{R}(\tau+1)\bigr) - F(\mathbf{R}(\tau))\bigr) \cdot \mathbf{1} \\
&\quad \leq f\bigl(\mathbf{R}(\tau+1)\bigr) \cdot \bigl(\mathbf{R}(\tau+1) - \mathbf{R}(\tau)\bigr), \\
&\quad = f\bigl(\mathbf{R}(\tau) + A(\tau,\tau+1) - D(\tau,\tau+1)\bigr) \cdot \bigl(A(\tau,\tau+1) - D(\tau,\tau+1)\bigr) \\
&\quad \leq f(\mathbf{R}(\tau)) \cdot \bigl(A(\tau,\tau+1) - D(\tau,\tau+1)\bigr) + \|A(\tau,\tau+1) - D(\tau,\tau+1)\|_2^2.
\end{aligned}$$

Given initial state $X(0) = \mathbf{x}$, taking expectation for $\tau, \tau+1 \in I$, we have

$$\begin{aligned}
&\mathbb{E}_{\mathbf{x}}\bigl[L\bigl(X(\tau+1)\bigr) - L(X(\tau))\bigr] \\
&\quad \leq \mathbb{E}_{\mathbf{x}}[f(\mathbf{R}(\tau)) \cdot A(\tau,\tau+1)] - \mathbb{E}_{\mathbf{x}}[f(\mathbf{R}(\tau)) \cdot D(\tau,\tau+1)] \\
&\qquad + \mathbb{E}_{\mathbf{x}}[\|A(\tau,\tau+1) - D(\tau,\tau+1)\|_2^2] \\
&\quad = \mathbb{E}_{\mathbf{x}}[f(\mathbf{R}(\tau)) \cdot \boldsymbol{\lambda}] - \mathbb{E}_{\mathbf{x}}[f(\mathbf{R}(\tau)) \cdot D(\tau,\tau+1)] + O(1).
\end{aligned} \tag{63}$$

The last equality follows from the fact that arrival process is Poisson with rate vector $\boldsymbol{\lambda}$ and $\mathbf{R}(\tau)$ is independent of $A(\tau,\tau+1)$. In addition, the overall departure process for any i, $D_i(\cdot)$, is governed by a Poisson process of rate at most $C_{\max}$. Therefore, the second moment of the difference of arrival and departure processes in unit time is $O(1)$.

Now we consider each term in (63) separately as

$$
\begin{aligned}
\mathbb{E}_{\mathbf{x}}[f(\mathbf{R}(\tau)) \cdot \boldsymbol{\lambda}] &= f(\mathbf{R}(0)) \cdot \boldsymbol{\lambda} + \mathbb{E}_{\mathbf{x}}\big[\big(f(\mathbf{R}(\tau)) - f(\mathbf{R}(0))\big) \cdot \boldsymbol{\lambda}\big] \\
&\le f(\mathbf{R}(0)) \cdot \boldsymbol{\lambda} + \mathbb{E}_{\mathbf{x}}[\| f(\mathbf{R}(\tau)) - f(\mathbf{R}(0))\|_1]
\end{aligned} \tag{64}
$$

and

$$
\begin{aligned}
&\mathbb{E}_{\mathbf{x}}[f(\mathbf{R}(\tau)) \cdot D(\tau, \tau+1)] \\
&\quad = \mathbb{E}_{\mathbf{x}}[f(\mathbf{R}(0)) \cdot D(\tau, \tau+1)] + \mathbb{E}_{\mathbf{x}}\big[\big(f(\mathbf{R}(\tau)) - f(\mathbf{R}(0))\big) \cdot D(\tau, \tau+1)\big] \\
&\quad \ge \mathbb{E}_{\mathbf{x}}[f(\mathbf{R}(0)) \cdot D(\tau, \tau+1)] - O\big(\mathbb{E}_{\mathbf{x}}[\| f(\mathbf{R}(\tau)) - f(\mathbf{R}(0))\|_1]\big),
\end{aligned} \tag{65}
$$

where the last inequality in (65) is because $D_i(\cdot)$ is governed by a Poisson process of rate at most $C_{\max} = O(1)$. In what follows, we will bound each term in (64) and (65). The first term on the right-hand side in (64) can be bounded as

$$
\begin{aligned}
f(\mathbf{R}(0)) \cdot \boldsymbol{\lambda} &\le (1-\varepsilon)\Big(\max_{\mathbf{y}\in\mathcal{X}} f(\mathbf{R}(0)) \cdot \mathbf{y}\Big) \\
&\le -\frac{3\varepsilon}{4}\Big(\max_{\mathbf{y}\in\mathcal{X}} f(\mathbf{R}(0)) \cdot \mathbf{y}\Big) + \mathbb{E}_{\pi(0)}[f(\mathbf{R}(0)) \cdot \mathbf{z}] + O(1),
\end{aligned} \tag{66}
$$

where the first inequality is due to $\boldsymbol{\lambda} \in (1-\varepsilon)\,\mathsf{Conv}(\mathcal{X})$ and the second inequality follows from Lemma 6 with the fact that $|f_i(\mathbf{R}(\tau)) - f_i(\mathbf{Q}(\tau))| < f(C_{\max}) = O(1)$ for all i. On the other hand, the first term in the right-hand side of (65) can be bounded below as

$$
\begin{aligned}
\mathbb{E}_{\mathbf{x}}[f(\mathbf{R}(0)) \cdot D(\tau, \tau+1)] &= f(\mathbf{R}(0)) \cdot \mathbb{E}_{\mathbf{x}}[D(\tau, \tau+1)] \\
&\ge f(\mathbf{R}(0)) \cdot \int_{\tau}^{\tau+1} \mathbb{E}_{\mathbf{x}}[\mathbf{z}(s)]\, ds \\
&= \int_{\tau}^{\tau+1} \mathbb{E}_{\mu(s)}[f(\mathbf{R}(0)) \cdot \mathbf{z}]\, ds.
\end{aligned} \tag{67}
$$

In the above, we have used the fact that $D_i(\cdot)$ is a Poisson process with rate given by $z_i(\cdot)$. Further, the second terms in the right-hand side of (64) and (65) can be bounded using

$$
\begin{aligned}
\mathbb{E}_{\mathbf{x}}[\| f(\mathbf{R}(\tau)) - f(\mathbf{R}(0))\|_1] &\le \mathbb{E}_{\mathbf{x}}\big[f\big(|\mathbf{R}(\tau) - \mathbf{R}(0)|\big)\big] + O(1) \\
&\le f\big(\mathbb{E}_{\mathbf{x}}[|\mathbf{R}(\tau) - \mathbf{R}(0)|]\big) + O(1) \\
&\le n f(C_{\max}\tau) + O(1) \\
&= O(f(\tau)),
\end{aligned} \tag{68}
$$

where the first inequality follows from $f(x+y) \le f(x) + f(y) + O(1)$ for any $x, y \in \mathbb{R}_+$ and the second inequality follows by applying Jensen's inequality for concave function f. Combining (63)–(68), we obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbf{x}}\big[L\big(X(\tau+1)\big)-L(X(\tau))\big]\\
&\quad\le -\frac{3\varepsilon}{4}\Big(\max_{\mathbf{y}\in\mathcal{X}} f(\mathbf{R}(0))\cdot\mathbf{y}\Big)+\mathbb{E}_{\pi(0)}[f(\mathbf{R}(0))\cdot\mathbf{z}]\\
&\qquad -\int_{\tau}^{\tau+1}\mathbb{E}_{\mu(s)}[f(\mathbf{R}(0))\cdot\mathbf{z}]\,ds+O(f(\tau))\\
&\quad\le -\frac{3\varepsilon}{4}\Big(\max_{\mathbf{y}\in\mathcal{X}} f(\mathbf{R}(0))\cdot\mathbf{y}\Big)\\
&\qquad +\int_{\tau}^{\tau+1}\Big(\max_{\mathbf{y}\in\mathcal{X}} f(\mathbf{R}(0))\cdot\mathbf{y}\Big)\|\mu(s)-\pi(0)\|_{\mathrm{TV}}\,ds+O(f(\tau))\\
&\quad\overset{(a)}{\le} -\frac{\varepsilon}{2}\Big(\max_{\mathbf{y}\in\mathcal{X}} f(\mathbf{R}(0))\cdot\mathbf{y}\Big)+O(f(\tau))\\
&\quad\le -\frac{\varepsilon}{2}f(Q_{\max}(0))+O(f(\tau)),
\end{aligned}$$

where (a) follows from Lemma 8. Summing this for $\tau\in I=[b_1,b_2-1]$, it follows that

$$\begin{aligned}
\mathbb{E}_{\mathbf{x}}[L(X(b_2))-L(X(b_1))]\le{}& -\frac{\varepsilon}{2}f(Q_{\max}(0))(b_2-b_1)\\
&+O\big((b_2-b_1)f(b_2)\big).
\end{aligned}\tag{69}$$

Therefore, we have

$$\begin{aligned}
&\mathbb{E}_{\mathbf{x}}[L(X(b_2))-L(X(0))]\\
&\quad=\mathbb{E}_{\mathbf{x}}[L(X(b_1))-L(X(0))]+\mathbb{E}_{\mathbf{x}}[L(X(b_2))-L(X(b_1))]\\
&\quad\overset{(a)}{\le}\mathbb{E}_{\mathbf{x}}\big[f(\mathbf{R}(b_1))\cdot\big(\mathbf{R}(b_1)-\mathbf{R}(0)\big)\big]+\mathbb{E}_{\mathbf{x}}[L(X(b_2))-L(X(b_1))]\\
&\quad=\sum_i\mathbb{E}_{\mathbf{x}}\big[f(R_i(b_1))\cdot\big(R_i(b_1)-R_i(0)\big)\big]+\mathbb{E}_{\mathbf{x}}[L(X(b_2))-L(X(b_1))]\\
&\quad\overset{(b)}{\le}\sum_i\sqrt{\mathbb{E}_{\mathbf{x}}[f(R_i(b_1))^2]}\sqrt{\mathbb{E}_{\mathbf{x}}\big[\big(R_i(b_1)-R_i(0)\big)^2\big]}\\
&\qquad+\mathbb{E}_{\mathbf{x}}[L(X(b_2))-L(X(b_1))]\\
&\quad\overset{(c)}{\le}\sum_i\sqrt{f(\mathbb{E}_{\mathbf{x}}[R_i(b_1)])^2+O(1)}\cdot O(b_1)+\mathbb{E}_{\mathbf{x}}[L(X(b_2))-L(X(b_1))]\\
&\quad\overset{(d)}{=}nf\big(Q_{\max}(0)+O(b_1)\big)\cdot O(b_1)-\frac{\varepsilon}{2}f(Q_{\max}(0))(b_2-b_1)\\
&\qquad+O\big((b_2-b_1)f(b_2)\big)\\
&\quad\overset{\triangle}{=}-h(\mathbf{x}).
\end{aligned}$$

Here (a) follows from convexity of L; (b) follows from Cauchy–Schwarz; (c) is due to the bounded second moment of $\mathbb{E}_{\mathbf{x}}[(R_i(b_1) - R_i(0))^2] = O(b_1^2)$ as argued earlier in the proof and observing that there exists a concave function $\tilde{f}$ such that $f^2 = \tilde{f} + O(1)$ over $\mathbb{R}_+$, subsequently Jensen's inequality can be applied; (d) follows from (69). Finally, choose $g(\mathbf{x}) = b_2$.

With these choices of h and g, the desired conditions of Lemma 5 can be checked as follows. First observe that with respect to $Q_{\max}(0)$, the function h scales as $b_2(Q_{\max}(0)) f(Q_{\max}(0))$ due to $b_2/b_1 = \Theta(\log Q_{\max}(0))$ as per (43), (45) in Lemma 8. Further, h is a function that is lower bounded and its value goes to ∞ as $Q_{\max}(0)$ goes to ∞. Therefore, h/g scales as $f(Q_{\max}(0))$. These properties will imply the verification conditions of Lemma 5.

5.3.5. *Step four.* For completing the proof of ergodicity of the Markov process, we need to verify condition (C2) stated in Section 4.4.3. From the first three steps of the proof and Lemma 5, it follows that for large enough $\kappa > 0$, set $B_\kappa = \{\mathbf{x} \in \mathsf{X} : L(\mathbf{x}) \le \kappa\}$ satisfies (C1). For this set, we wish to verify (C2). The following lemma, for wireless network, establishes (C2) for any such B_κ. An identical result for buffered circuit switched network can be established following essentially the same argument and hence, shall be skipped.

LEMMA 9. *Let the network Markov process $X(\cdot)$ start with the state $\mathbf{x} \in B_\kappa$ at time* 0, *that is*, $X(0) = \mathbf{x}$. *Then*, *there exists $T_\kappa \ge 1$ and $\gamma_\kappa > 0$ such that*

$$\sum_{\tau=1}^{T_\kappa} \Pr_{\mathbf{x}}(X(\tau) = \mathbf{0}) \ge \gamma_\kappa \qquad \forall \mathbf{x} \in B_\kappa.$$

Here $\mathbf{0} = (\mathbf{0}, \mathbf{0}) \in \mathsf{X}$ denote the state where all components of $\mathbf{Q}$ are 0 *and the schedule is the empty independent set*. *Further*, $\Pr_{\mathbf{0}}(X(1) = \mathbf{0}) > 0$.

PROOF. Consider any $\mathbf{x} \in B_\kappa$. Then by definition $L(\mathbf{x}) \le \kappa + 1$ for given $\kappa > 0$. Hence, by definition of $L(\cdot)$, it can be easily checked that each queue is bounded above by κ. Consider some large enough (soon to be determined) T_κ. By the property of Bernoulli (it would be Poisson for circuit switched network) arrival process, there is a positive probability $\theta_\kappa^0 > 0$ of no arrivals happening to the system during time interval of length T_κ. Assuming that no arrival happens, we will show that in large enough time t_κ^1, with probability $\theta_\kappa^1 > 0$ each queue receives at least κ amount of service and after that, in additional time t^2 with positive probability $\theta^2 > 0$, the empty set schedule is reached. This will imply that by defining $T_\kappa \stackrel{\triangle}{=} t_\kappa^1 + t^2$ the state $\mathbf{0} \in \mathsf{X}$ is reached with probability at least

$$\gamma_\kappa \stackrel{\triangle}{=} \theta_\kappa^0 \theta_\kappa^1 \theta^2 > 0.$$

And this will immediately imply the desired result of Lemma 9. To this end, we need to show existence of $t^1_\kappa, \theta^1_\kappa$ and t^2, θ^2 with properties stated above to complete the proof of Lemma 9.

First, existence of $t^1_\kappa, \theta^1_\kappa$. For this, note that the Markov chain corresponding to the scheduling algorithm has time varying transition probabilities and is irreducible over the space of all independent sets, $\mathcal{I}(G)$. If there are no new arrivals and initial $\mathbf{x} \in B_\kappa$, then clearly queue-sizes are uniformly bounded by κ. Therefore, the transition probabilities of all feasible transitions for this time varying Markov chain is uniformly lower bounded by a strictly positive constant (dependent on κ, n). It can be easily checked that the transition probability induced graph on $\mathcal{I}(G)$ has diameter at most $2n$ and Markov chain transits as per exponential clock of overall rate n. Therefore, it follows that starting from any initial scheduling configuration, there exists finite time $\widehat{t_\kappa}$ such that a schedule is reached so that any given queue i is scheduled for at least unit amount of time with probability at least $\widehat{\theta_\kappa} > 0$. Here, both $\widehat{t_\kappa}, \widehat{\theta_\kappa}$ depend on n, κ. Therefore, it follows that in time $t^1_\kappa \stackrel{\triangle}{=} \kappa n \widehat{t_\kappa}$ all queues become empty with probability at least $\theta^1_\kappa \stackrel{\triangle}{=} (\widehat{\theta_\kappa})^{n\kappa}$. Next, to establish existence of t^2, θ^2 as desired, observe that once the system reaches empty queues, it follows that, in the absence of new arrivals, the empty schedule $\mathbf{0}$ is reached after some finite time t^2 with probability $\theta^2 > 0$ by similar properties of the Markov chain on $\mathcal{I}(G)$ when all queues are 0. Here t^2 and θ^2 are dependent on n only. Finally, $\Pr_{\mathbf{0}}(X(1) = \mathbf{0}) > 0$ follows from the above arguments easily. This completes the proof of Lemma 9. $\square$

6. Discussion. This paper introduced a new randomized scheduling algorithm for two constrained queueing network models: wireless network and buffered circuit switched network. The algorithm is simple, distributed, myopic and throughput optimal. The main reason behind the throughput optimality property of the algorithm is two-fold: (1) The relation of algorithm dynamics to the Markovian dynamics over the space of schedules that have a certain product-form stationary distribution and (2) choice of slowly increasing weight function $\log\log(\cdot + e)$ that allows for an effective time scale separation between algorithm dynamics and the queueing dynamics. We chose wireless network and buffered circuit switched network model to explain the effectiveness of our algorithm because (a) they are becoming of great interest [33, 34] and (b) they represent two different, general class of network models: synchronized packet network model and asynchronous flow network model.

Now we turn to discuss the distributed implementation of our algorithm. As described in Section 3.1, given the weight information at each wireless node (or ingress of a route), the algorithm completely distributed. The weight, as defined in (3) [or (4)], depends on the local queue-size as well as the $Q_{\max}$ information. As is, $Q_{\max}$ is global information. To keep the exposition simpler, we have used the precise $Q_{\max}$ information to establish the throughput property. However, as remarked earlier in Section 3.1 [soon after (3)], the $Q_{\max}$ can be replaced by its appropriate distributed estimation without altering the throughput optimality property. Such a distributed estimation can be obtained through an extremely simple Markovian-like algorithm that requires each node to perform broadcast of exactly one number in unit time. A detailed description of such an algorithm can be found in Section 3.3 of [24].

On the other hand, consider the algorithm that does not use $Q_{\max}$ information. That is, instead of (3) or (4), let weight be

$$W_i(t) = f(Q_i(\lfloor t \rfloor)).$$

We conjecture that this algorithm is throughput optimal.

APPENDIX: A USEFUL LEMMA

LEMMA 10. *Let $P_1, P_2 \in \mathbb{R}^{N\times N}$. Then,*

$$\|e^{P_1} - e^{P_2}\|_\infty \le e^{NM}\|P_1 - P_2\|_\infty,$$

where $M = \max\{\|P_1\|_\infty, \|P_1\|_\infty\}$.

PROOF. Using mathematical induction, we first establish that for any $k \in \mathbb{N}$,

$$\|P_1^k - P_2^k\|_\infty \le k(NM)^{k-1}\|P_1 - P_2\|_\infty. \tag{70}$$

To this end, the base case $k = 1$ follows trivially. Suppose it is true for some $k \ge 1$. Then, the inductive step can be justified as

$$\begin{aligned}
&\|P_1^{k+1} - P_2^{k+1}\|_\infty \\
&\quad = \|P_1(P_1^k - P_2^k) + (P_1 - P_2)P_2^k\|_\infty \\
&\quad \le \|P_1(P_1^k - P_2^k)\|_\infty + \|(P_1 - P_2)P_2^k\|_\infty \\
&\quad \overset{\text{(a)}}{\le} N\|P_1\|_\infty\|P_1^k - P_2^k\|_\infty + N\|P_1 - P_2\|_\infty\|P_2^k\|_\infty \\
&\quad \overset{\text{(b)}}{\le} NM \times k(NM)^{k-1}\|P_1 - P_2\|_\infty + N\|P_1 - P_2\|_\infty \times N^{k-1}M^k \\
&\quad = (k+1)(NM)^k\|P_1 - P_2\|_\infty.
\end{aligned}$$

In the above, (a) follows from an easily verifiable fact that for any $Q_1, Q_2 \in \mathbb{R}^{N\times N}$,

$$\|Q_1Q_2\|_\infty \le N\|Q_1\|_\infty\|Q_2\|_\infty.$$

We use induction hypothesis to justify (b). Using (70), we have

$$\begin{aligned}
\|e^{P_1} - e^{P_2}\|_\infty &= \left\| \sum_k \frac{1}{k!}(P_1^k - P_2^k) \right\|_\infty \\
&\le \sum_k \frac{1}{k!} \|P_1^k - P_2^k\|_\infty \\
&\le \sum_k \frac{1}{k!} k(NM)^{k-1} \|P_1 - P_2\|_\infty \\
&= e^{NM} \|P_1 - P_2\|_\infty. \qquad \square
\end{aligned}$$

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LABORATORY FOR INFORMATION
AND DECISION SYSTEMS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE, MASSACHUSETTS 02139
USA
E-MAIL: devavrat@mit.edu

DEPARTMENT OF MATHEMATICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE, MASSACHUSETTS 02139
USA
E-MAIL: jinwoos@mit.edu