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ON THE SUPERPOSITION OF HETEROGENEOUS TRAFFIC AT LARGE TIME SCALES

By Luis López-Oliveros and Sidney I. Resnick

Cornell University

Various empirical and theoretical studies indicate that cumulative network traffic is a Gaussian process. However, depending on whether the intensity at which sessions are initiated is large or small relative to the session duration tail, [25] and [15] have shown that traffic at large time scales can be approximated by either fractional Brownian motion (fBm) or stable Lévy motion. We study distributional properties of cumulative traffic that consists of a finite number of independent streams and give an explanation of why Gaussian examples abound in practice but not stable Lévy motion. We offer an explanation of how much vertical aggregation is needed for the Gaussian approximation to hold. Our results are expressed as limit theorems for a sequence of cumulative traffic processes whose session initiation intensities satisfy growth rates similar to those used in [25].

1. Introduction. Collection of data network measurements often uses an algorithm for clustering packets with the same source and destination IP addresses. Various criteria for grouping packets yield different entities, e.g. connections, flows (or unidirectional connections), end-to-end streams, etc. [See e.g. 33, Section 4]. These high-order constructs of packet clusters are sometimes termed sessions. For now, think of a session as a user downloading a file, streaming media, or accessing websites. For each session, summary measurements are computed for the *size* (the number of bytes transmitted in a session), the duration of the session and the average transfer rate. Data sets of these summaries show some distinctive properties, such as heavy tails for session size and duration [1, 7, 39] and sometimes rate [23, 29].

Typically, a time resolution or granularity is selected or imposed. Typical resolutions are 1, 10 or 100 milliseconds, 1 second, 1 minute, 1 hour, etc. Once a resolution is fixed, the number of bytes or number of packets per unit time can be recorded and cumulative network loads over stationary time intervals computed. These cumulative loads have been studied from empirical and theoretical perspectives with the objectives of satisfying performance criterion and offering adequate bandwidth provisioning [37] or predicting properties of congestion events [14].

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Conventional wisdom based on empirical studies claims that a heavily loaded network link subject to aggregation over many users should see Gaussian traffic. This wisdom is considered a network *invariant*. Influential examples based on the Bellcore measurements [21] suggest that *horizontal* aggregation, that is, working with a single on/off stream at sufficiently large time scale justifies Gaussian modeling. See also [20] and [41].

However, mathematically it is known that with heavy tailed session durations, cumulative load at large time scales can be approximated by either fractional Brownian motion (fBm) or stable Lévy motion, depending on whether the intensity at which sessions are initiated is large or small relative to the size of the duration tails. See [25, 15, 36]. The stable approximation has not been observed empirically [11] and use of Gaussian cumulative loads has become dominant [17, 34, 13].

But why should traffic be Gaussian? According to the empirical study [38], in addition to horizontal aggregation, the superposition of independent traffic streams, that is, *vertical* aggregation, can justify a Gaussian model and, in fact, the number of traffic streams need not be large to make cumulative loads approximately Gaussian.

In this paper we

- study the distribution of the cumulative load in the presence of a finite number of independent traffic streams;
- give an explanation for why Gaussian examples abound in practice but not stable ones;
- answer how much vertical aggregation is needed to justify the use of fBm.

Our findings suggest that cumulative load for aggregate traffic can be approximated by fBm at large time scales provided the initiation intensity of at least one of the traffic components is large. Network traffic in the wild has several distinct constituents and we claim that in practice there is one or more components with dominant large initiation intensities. For example, this should be the case with web traffic using port 80 and this suggests why Gaussian traffic should be pervasive [38].

Before discussing mathematical details, we illustrate the phenomena of interest with a motivating example of a network trace captured at Cornell University main campus servers during 55 days between November 2, 2009, and January 15, 2010. Cornell's data set is a collection of *netflow* records, where all non-IP traffic has been discarded and only TCP and UPD traffic is present in the trace. A netflow is a collection of packets with the same source and destination IP addresses, source and destination ports, protocol, ingress interface and IP type of service [5]. In our data, TCP traffic accounts for nearly 90% of the bytes, and over 80% of the total number of netflows, mostly port 80 (http traffic) netflows. We have taken the part of the trace corresponding to both outgoing and incoming traffic between 1 and 5 p.m. local time, adding up to 220 hours of traffic. The anonymization procedure used on the data obliterated the distinction between outgoing and incoming flows.

We analyze the distribution of $A^{(TCP)}$ and $A^{(UDP)}$, namely the cumulative load generated by TCP and UDP bytes, respectively. For this purpose, we separate the trace into TCP and UDP netflows and for $k = 1, \dots, 220$ we count

$$A_k^{(TCP)} := \text{total number of TCP bytes captured in the } k\text{th hour},$$
$$A_k^{(UDP)} := \text{total number of UDP bytes captured in the } k\text{th hour}.$$

Due to the dates and times of collection, these counts exhibit both a trend and a daily seasonality. Here we detrend and remove daily seasonality [see e.g. 4, Section 1.4], but our conclusions are the same without this massage.

Figure 1 shows Gaussian QQ plots for $A^{(TCP)}$ (*left*) and $A^{(UDP)}$ (*right*). A straight line fit is evident for the TCP cumulative input. However, the UDP counterpart shows a significant departure from the straight line. Using the p-values of the Anderson-Darling two-sided test also shows no evidence against the normality of $A^{(TCP)}$ ($p = 0.1369$), but strong evidence against a Gaussian model for $A^{(UDP)}$ ($p = 9.8 \times 10^{-16}$).

We also check whether $A^{(UDP)}$ is a heavy-tailed random variable, in the sense of its distribution tail being regularly varying with tail index α [9, 30].

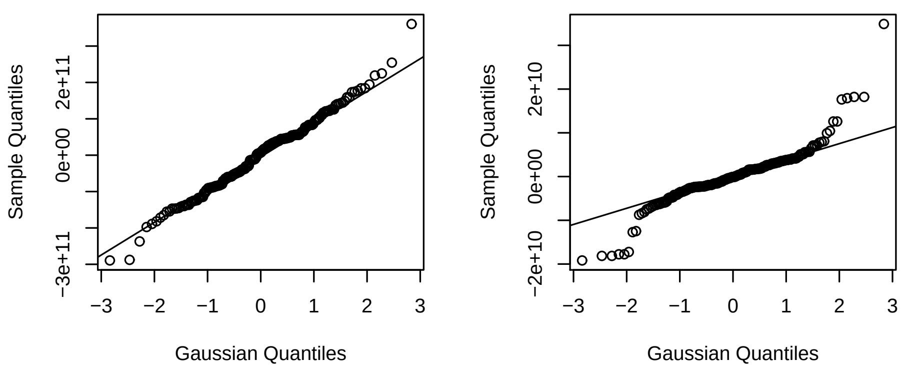

FIG 1. *Normal QQ plots of cumulative inputs. Left: TCP traffic. Right: UDP traffic.*

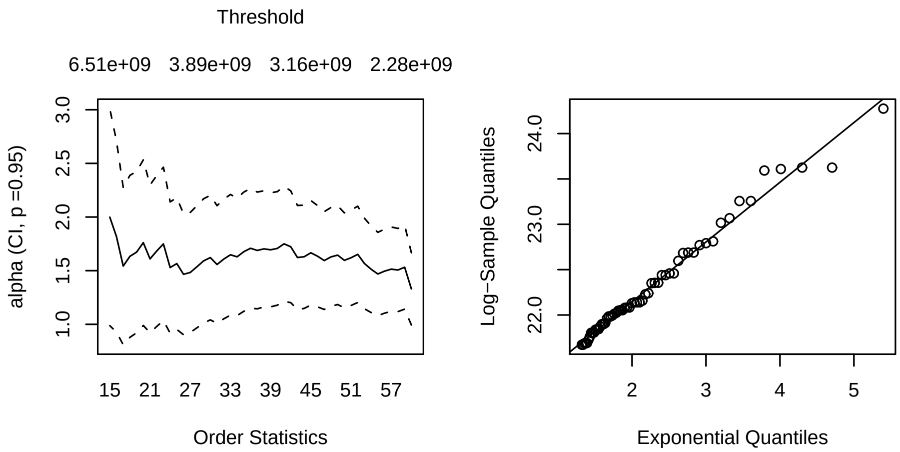

FIG 2. *Plots for the UDP cumulative input. Left: Hill plot of tail index with 95% confidence interval. Right: Exponential QQ plot of the log data.*

For instance, Figure 2 *left* shows a stable regime in the Hill plot of α [for Hill plots, see e.g. 12, 10, 30]. Additionally, in Figure 2 *right* we present the exponential QQ plot of $\log(A^{(UDP)})$ with a straight line fit through the biggest 55 observations. This shows no evidence against approximating the distribution of thresholded values of $A^{(UDP)}$ by a Pareto (Recall that the logarithm of Pareto random variable is exponential; see, for example, [24, 30, 6].)

If we consider the aggregated cumulative load, $A^{(TCP)} + A^{(UDP)}$, the normal QQ plot in Figure 3 exhibits a straight line fit and the Anderson-Darling test $p-$value is 0.2117, showing no evidence to reject normality. Without accounting for centering and scaling, this result is rather counterintuitive due to the nature of the individual tails of $A^{(TCP)}$ and $A^{(UDP)}$.

Our explanation to the above phenomenon starts by modeling the quantity of data in windows of length T in Section 2. Analogously to the slow and fast growths of [25], we define two different scenarios for the aggregated traffic. A third scenario is defined similarly to the boundary case considered in [15]. In Section 3 we obtain approximations and provide clarification of the asymptotic behavior at large time scales. We let $T \to \infty$ and see what limits exist for the aggregated cumulative load. In Section 4 we study extensions to our model and finally Section 5 contains some technical results used to prove our main theorems.

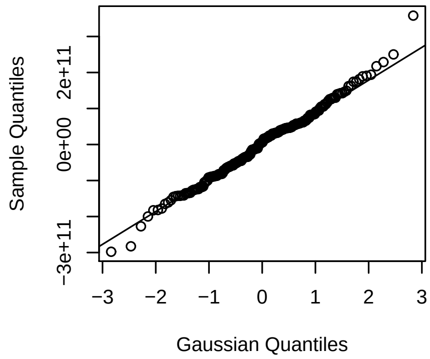

FIG 3. *Normal QQ plot of the aggregated cumulative input.*

1.1. *Notation.* In order to simplify the later presentation, we introduce and collect some notation. References are provided for further reading.

$\bar{F}$	The right tail of the distribution function F, i.e. $\bar{F}=1-F$.
$F^{\leftarrow}$	The left continuous inverse of the distribution function F, i.e. $F^{\leftarrow}(y)=\inf\{x : F(x)\geq y\}$.
$f_1 \sim f_2$	$\lim_{x\to\infty} f_1(x)/f_2(x) = 1$.
$\xrightarrow{fidi}$	Convergence of finite dimensional distributions.
$\xrightarrow{v}$	Vague convergence of measures. See e.g. [16, 28].
$M_+(0,\infty]$	The space of nonnegative Radon measures on $(0,\infty]$.
RV_γ	The class of regularly varying functions with index γ. See e.g. [3, 9, 27].
$\xi=\mathbb{PRM}(E\xi)$	A Poisson random measure ξ with mean measure $E\xi$.
$\overset{\circ}{\xi}$	A compensated Poisson random measure with mean measure $E\xi$, i.e. $\overset{\circ}{\xi}=\xi-E\xi$.
$N^\infty_{\gamma,h_\delta}$	$N^\infty_{\gamma,h_\delta}=\mathbb{PRM}(ds\cdot\gamma u^{-(\gamma+1)}du\cdot h_\delta(dr))$ on $\mathbb{R}\times(0,\infty)^2$, where h_δ is a measure on $(0,\infty)$ such that $h_\delta[\cdot,\infty)\in RV_{-\delta}$. If $h_\delta(dr)=\delta r^{-(\delta+1)}dr$, we may simply write $N^\infty_{\gamma,\delta}$.

$M_{\gamma,m}(dv)$	A $\gamma-$stable random measure with control measure $m(dv)$ and stable index $1<\gamma<2$. For $\xi=\mathbb{PRM}(m(dv)w^{-(\rho+1)}dw)$, we can write $M_{\gamma,m}(A)\stackrel{d}{=}\left(\left(-\cos\frac{\pi\gamma}{2}\right)\frac{2\Gamma(2-\gamma)}{\gamma(\gamma-1)}\right)^{-1/\gamma}\int_A\int_{w=0}^{\infty}w\overset{\circ}{\xi}(dv,dw)$. See e.g. [32, Chapter 3].
$\Lambda_\gamma(\cdot)$	A $\gamma-$stable Lévy motion totally skewed to the right with stable index $1<\gamma<2$. In general, we can write $\Lambda_\gamma(t)\stackrel{d}{=}\left(\left(-\cos\frac{\pi\gamma}{2}\right)\frac{2\Gamma(2-\gamma)}{\gamma(\gamma-1)}\right)^{1/\gamma}\int_0^{\infty}1_{\{0<v<t\}}M_{\rho,m}(dv)$. See e.g. [32, Chapter 3].
$B_H(\cdot)$	The standard fractional Brownian motion with Hurst exponent H.

2. Model description and basic assumptions. Consider a network that has an infinite number of nodes. At certain times, a node begins a transmission session at a random rate that is fixed throughout the session. Suppose network traffic consists of p distinct types which we call *streams*. In practice, such a division of network traffic arises naturally; e.g. traffic can be segmented by application type (web, email, streaming media, file-sharing applications, etc.), by protocol (TCP, UDP, IMTP, etc.), and even by users. We suppose the p streams are independent and that each follows an $M/G/\infty$ input model. The overall load is obtained by aggregating over the p streams. Thus, the basic assumptions are as follows:

- Sessions corresponding to the jth stream are initiated at homogenous Poisson time points $\{\Gamma_k^{(j)},-\infty<k<\infty\}$ with arrival intensity $\lambda^{(j)}>0$. These points are labeled so that $\Gamma_0^{(j)}<0<\Gamma_1^{(j)}$ whence $\{-\Gamma_0^{(j)},\Gamma_1^{(j)},(\Gamma_{k+1}^{(j)}-\Gamma_k^{(j)},k\neq 0)\}$ are iid exponential with parameter $\lambda^{(j)}$. Thus, we have:
$$\sum_k \epsilon_{\Gamma_k^{(j)}}=\mathbb{PRM}(\lambda^{(j)}ds).$$
We assume that these $\mathbb{PRM}$s are independent.
- All the sessions in the network transmit data at positive random rates that are iid with common distribution F_R. Let $\{R_k^{(j)}\}$ be the rate of

the kth session of the jth stream. Assume that either $\bar{F}_R \in RV_{-\alpha_R}$, $1 < \alpha_R < 2$, or $E[(R_1^{(1)})^2] < \infty$. In either case, define $\mu_R := ER_1^{(1)}$.

- Sessions in the jth stream have positive durations $\{D_k^{(j)}\}$, $j = 1, \ldots, p$, that are iid $F_D^{(j)}$, with $\bar{F}_D^{(j)} \in RV_{-\alpha_D^{(j)}}$, $1 < \alpha_D^{(j)} < 2$, and $\mu_D^{(j)} := ED_1^{(j)}$. In general, not all the $\alpha_D^{(j)}$s are equal.
- We also assume mutually independent durations across streams, and that durations and rates are independent.

There is empirical evidence justifying the choices of $\alpha_D^{(j)}$s and α_R: See e.g. [8, 40, 21, 29, 22]. For now, we adopt a network-centric approach by assuming the rate of communication entirely depends on the state and speed of the network. Studies supporting this assumption include [35] and [19].

We will need

$$\lambda = \sum_{j=1}^{p} \lambda^{(j)}, \tag{2.1}$$

$$F_D := \sum_{j=1}^{p} (\lambda^{(j)}/\lambda) F_D^{(j)}, \tag{2.2}$$

and the quantile functions

$$b_D^{(j)}(t) = (1/\bar{F}_D^{(j)})^{\leftarrow}(t) = (F_D^{(j)})^{\leftarrow}(1 - 1/t), \tag{2.3}$$

$$b_D(t) = (1/\bar{F}_D)^{\leftarrow}(t) = F_D^{\leftarrow}(1 - 1/t), \tag{2.4}$$

$$b_R(t) = (1/\bar{F}_R)^{\leftarrow}(t) = F_R^{\leftarrow}(1 - 1/t). \tag{2.5}$$

Notice that F_D is the mixture model of the durations of the p streams, with weights $\lambda^{(j)}/\lambda$, $j = 1, \ldots, p$. In fact, F_D is the distribution of the duration of the sessions of the aggregated stream, and $\lambda^{(j)}/\lambda$ is the proportion of the traffic that consists of sessions from the jth stream. We return to this interpretation later.

Now consider (s, u, r) as a generic Poisson point representing a session that starts at time s, has duration u and rate r. By augmentation, the counting function of the session descriptors $(\Gamma_k^{(j)}, D_k^{(j)}, R_k^{(j)})$ of the jth stream on $\mathbb{R} \times [0, \infty)^2$ is

$$\begin{aligned} N^{(j)} &:= \sum_k \epsilon_{(\Gamma_k^{(j)}, D_k^{(j)}, R_k^{(j)})} \\ &= \mathbb{PRM}(\lambda^{(j)} ds F_D^{(j)}(du) F_R(dr)), \quad j = 1, \ldots, p. \end{aligned} \tag{2.6}$$

By independence, the counting function of the session descriptors of the aggregated stream is

$$
\begin{aligned}
N := \sum_{j=1}^{p} N^{(j)} &= \mathbb{PRM}\left(\lambda ds \sum_{j=1}^{p} (\lambda^{(j)}/\lambda) F_D^{(j)}(du) F_R(dr)\right) \\
&= \mathbb{PRM}(\lambda ds F_D(du) F_R(dr)). \qquad (2.7)
\end{aligned}
$$

Thus, the mean measures of the $N^{(j)}$ and N are given by

$$
\begin{aligned}
EN^{(j)}(ds, du, dr) &:= \lambda^{(j)} ds F_D^{(j)}(du) F_R(dr), \quad j = 1, \ldots, p, \\
EN(ds, du, dr) &:= \lambda ds F_D(du) F_R(dr).
\end{aligned}
$$

In addition, let

$$
L_t(s, u) = |[0, t] \cap [s, s+u]| = \int_0^t 1_{[s,s+u]}(y) dy = \int_0^u 1_{[0,t]}(y+s) dy, \qquad (2.8)
$$

be the length of the subinterval of $[0, t]$ during which the session (s, u, r) transmits data. In Lemma 5.4, we summarize several required properties of $L_t(s, u)$.

For each j, define

$$
\begin{aligned}
A^{(j)}(t) &:= \text{cumulative input in } [0, t] \text{ from the } j\text{th stream} \\
&= \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} r L_t(s, u) N^{(j)}(ds, du, dr), \qquad (2.9)
\end{aligned}
$$

and similarly

$$
\begin{aligned}
A(t) &:= \text{cumulative input in } [0, t] \text{ from the aggregated stream} \\
&= \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} r L_t(s, u) N(ds, du, dr). \qquad (2.10)
\end{aligned}
$$

([15] showed that these integrals are well defined using Campbell's theorem [18, Section 3.2].) Also,

$$
EA^{(j)}(t) = \lambda^{(j)} \mu_D^{(j)} \mu_R t, \quad EA(t) = \sum_{j=1}^{p} \lambda^{(j)} \mu_D^{(j)} \mu_R t = \lambda \mu_D \mu_R t,
$$

where

$$
\mu_D := \sum_{j=1}^{p} (\lambda^{(j)}/\lambda) \mu_D^{(j)} \qquad (2.11)
$$

is the mean of the mixture model of the durations of the different streams.

Observe that we can write the cumulative inputs as linear drift plus compensated random Poisson fluctuation as follows:

$$A^{(j)}(t) := \lambda^{(j)}\mu_D^{(j)}\mu_R t + \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} rL_t(s,u)\mathring{N}^{(j)}(ds,du,dr), \tag{2.12}$$

$$A(t) := \lambda\mu_D\mu_R t + \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} rL_t(s,u)\mathring{N}(ds,du,dr). \tag{2.13}$$

After scaling time by T, we think of $A_T^{(j)} := (A^{(j)}(Tt), t > 0), j = 1,\ldots,p$ and $A_T := (A(Tt), t > 0)$ for large T, as the cumulative inputs on large time scales. Thus, we consider a family of models indexed by the time scale parameter T and from now on we let the arrival intensities depend on T so that $\lambda^{(j)} := \lambda^{(j)}(T)$. If necessary, we let $\lambda_j(T) \to \infty$ as $T \to \infty$ (see (2.15)). Dependence of the arrival intensities on T means λ, F_D, $b_D^{(j)}$ and b_D as defined in (2.1)–(2.4) depend on T as well; however, notice that the tail indices of the distribution of the duration, namely $\alpha_D^{(j)}$, remain independent of T. In practice, the fact that we focus on the stream at a particular time period, say $[0, Tt]$, does not affect the tail index of the distribution of the sessions duration, which is in accordance with our assumptions. For convenience, we often suppress the subscript T.

Fix j, $1 \le j \le p$ and in the Tth model, let $A_{cs}^{(j)}(t)$ be the centered and scaled cumulative input of the jth stream in $[0, Tt]$, that is

$$A_{cs}^{(j)}(t) := \frac{A^{(j)}(Tt) - \lambda^{(j)}\mu_D^{(j)}\mu_R Tt}{a^{(j)}(T)}, \tag{2.14}$$

for a suitable $a_j(T)$ to be made precise below. Assuming $\lim_{T\to\infty}\lambda^{(j)}T\bar{F}_D^{(j)}(T)$ exists, the asymptotic behavior of $A_{cs}^{(j)}(t)$ as $T \to \infty$, depends on whether the arrival rate is large, moderate, or small, relative to the tail of the duration.

THEOREM 2.1. [25, 15]. *For any $1 \le j \le p$, consider the following three growth regimes of the arrival rate:*

$$\lim_{T\to\infty}\lambda^{(j)}T\bar{F}_D^{(j)}(T) = \begin{cases}\infty, & \textit{fast-growth.}\\ c_j^{\alpha_D^{(j)}-1}, & \textit{moderate-growth,}\\ 0, & \textit{slow-growth,}\end{cases} \tag{2.15}$$

where $c_j \in (0,\infty)$. (The form of the moderate-growth limit facilitates a simple expression of the corresponding limit process.) Assume that either $\bar{F}_R \in RV_{-\alpha_R}, \alpha_R > \alpha_D^{(j)}$ or $E[(R_1^{(1)})^2] < \infty$. (If $\alpha_R \le \alpha_D^{(j)}$, the limit process *is the same for all three growth regimes and the distinction among the growth regimes is irrelevant [15, Theorem 4].)*

(a) Under fast-growth, we distinguish two subcases:

(i) If $E[(R_1^{(1)})^2] < \infty$,

$$A_{cs}^{(j)}(\cdot) \xrightarrow{fidi} E[(R_1^{(1)})^2]^{1/2} \sigma^{(j)}_{B_{H^{(j)}}(1)} B_{H^{(j)}}(\cdot), \quad T \to \infty,$$

where

$$a^{(j)}(T) = [\lambda^{(j)} T^3 \bar{F}_D^{(j)}(T)]^{1/2},$$

$$\sigma^{(j)}_{B_{H^{(j)}}(1)} = \frac{2}{(\alpha_D^{(j)} - 1)(2 - \alpha_D^{(j)})(3 - \alpha_D^{(j)})},$$

and $B_{H^{(j)}}$ *is a fractional Brownian motion with Hurst exponent*

$$H^{(j)} = (3 - \alpha_D^{(j)})/2 \in (1/2, 1).$$

(ii) If $\bar{F}_R \in RV_{-\alpha_R}, 1 < \alpha_D^{(j)} < \alpha_R < 2$, *then*

$$A_{cs}^{(j)}(\cdot) \xrightarrow{fidi} Z_{\alpha_D^{(j)}, \alpha_R}(\cdot), \quad T \to \infty,$$

where

$$a^{(j)}(T) = T b_R(\lambda T \bar{F}_D^{(j)}(T)),$$

$$\begin{aligned} Z_{\alpha_D^{(j)}, \alpha_R}(t) &= \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} r L_t(s,u) \mathring{N}^{\infty}_{\alpha_D^{(j)}, \alpha_R}(ds, du), \\ &\stackrel{d}{=} \left(\left(-\cos \frac{\pi \alpha_D^{(j)}}{2} \right) \frac{2\Gamma(2 - \alpha_D^{(j)})}{\alpha_D^{(j)}(\alpha_D^{(j)} - 1)} \right)^{1/\alpha_D^{(j)}} \times \\ &\qquad \int_{-\infty}^{\infty} \int_0^{\infty} L_t(s,u) M_{\alpha_R, m}(ds, du), \end{aligned}$$

and $M_{\alpha_R,m}(ds, du)$ *is a* α_R*-stable random measure with control measure*

$$m(ds, du) = ds \cdot \alpha_D^{(j)} u^{-(\alpha_D^{(j)}+1)} du.$$

Thus, the process $Z_{\alpha_D^{(j)}, \alpha_R}(t)$ *is* α_R*-stable and* $H^{(j)}$*-similar with*

$$H^{(j)} = (\alpha_R + 1 - \alpha_D^{(j)})/\alpha_R \in (1/\alpha_R, 1).$$

(b) Under moderate-growth

$$A_{cs}^{(j)}(\cdot) \xrightarrow{fidi} c_j Y_{\alpha_D^{(j)}}(\cdot/c_j), \quad T \to \infty,$$

where

$$a^{(j)}(T) = T,$$

and

$$Y_{\alpha_D^{(j)}}(t) = \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} r L_t(s,u) \overset{\circ}{N}{}^{\infty}_{\alpha_D^{(j)},F_R}(ds,du,dr).$$

(c) Under slow-growth

$$A_{cs}^{(j)}(\cdot) \xrightarrow{fidi} E[(R_1^{(1)})^{\alpha_D^{(j)}}]^{1/\alpha_D^{(j)}} \Lambda_{\alpha_D^{(j)}}(\cdot), \quad T \to \infty,$$

where

$$a^{(j)}(T) = b_D^{(j)}(\lambda^{(j)} T),$$

$\Lambda_{\alpha_D^{(j)}}$ *is an* $\alpha_D^{(j)}$*-stable Lévy motion totally skewed to the right, which we can write as*

$$\begin{aligned} E[(R_1^{(1)})^{\alpha_D^{(j)}}]^{1/\alpha_D^{(j)}} \Lambda_{\alpha_D^{(j)}}(t) &= \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} u r 1_{\{0<s<t\}} \overset{\circ}{N}{}^{\infty}_{\alpha_D^{(j)},F_R}(ds,du,dr) \\ &\overset{d}{=} \left(\left(-\cos\frac{\pi\alpha_D^{(j)}}{2}\right)\frac{2\Gamma(2-\alpha_D^{(j)})}{\alpha_D^{(j)}(\alpha_D^{(j)}-1)}\right)^{1/\alpha_D^{(j)}} \times \\ &\qquad \int_{-\infty}^{\infty}\int_0^{\infty} r 1_{\{0<s<t\}} M_{\alpha_D^{(j)},m}(ds,dr), \end{aligned}$$

and $M_{\alpha_D^{(j)},m}(ds,dr)$ *is an* $\alpha_D^{(j)}$*-stable random measure with control measure*

$$m(ds,dr) = ds F_R(dr).$$

Real network traffic consists of several distinct types and in this paper we are interested in the centered and scaled cumulative input of the superimposed streams in $[0, Tt]$, namely

$$A_{cs}(t) := \frac{A(Tt) - \lambda\mu_D\mu_R Tt}{a(T)}, \tag{2.16}$$

for a suitable $a(T)$. In order to study the limit distribution of $A_{cs}(t)$ as $T \to \infty$, let $\mathcal{F}$, $\mathcal{M}$, $\mathcal{S}$ be the subsets of indices of streams whose arrival intensities behave under the fast-, moderate-, and slow-growth regimes, respectively.

Assuming that all indices belong to one of these three classes, consider the following scenarios.

Scenario $\mathcal{F}$ There is at least one stream whose arrival intensity satisfies fast-growth; i.e. $\mathcal{F} \neq \emptyset$. In this case, the aggregated stream's arrival intensity also satisfies fast-growth:

$$\lambda T \bar{F}_D(T) \geq \sum_{j \in \mathcal{F}} \lambda^{(j)} T \bar{F}_D^{(j)}(T) \to \infty, \quad T \to \infty. \tag{2.17}$$

Scenario $\mathcal{M}$ No stream's arrival intensity satisfies fast-growth, but at least one stream satisfies moderate-growth; i.e. $\mathcal{F} = \emptyset$ and $\mathcal{M} \neq \emptyset$. Then, the aggregated stream's arrival intensity satisfies moderate growth, since

$$\lambda T \bar{F}_D(T) \to c^{\alpha_D - 1}, \quad T \to \infty, \tag{2.18}$$

where

$$\alpha_D := \bigwedge_{j=1}^{p} \alpha_D^{(j)} \tag{2.19}$$

and

$$c = \left(\sum_{j \in \mathcal{M}} c_j^{\alpha_D^{(j)} - 1} \right)^{1/(\alpha_D - 1)}. \tag{2.20}$$

Scenario $\mathcal{S}$ All the stream's arrival intensities satisfy slow growth, that is $\mathcal{S} = \{1, \ldots, p\}$. In this case, the aggregated stream's arrival intensity also satisfies slow-growth:

$$\lambda T \bar{F}_D(T) = \sum_{j \in \mathcal{S}} \lambda^{(j)} T \bar{F}_D^{(j)}(T) \to 0, \quad T \to \infty. \tag{2.21}$$

The different growth regimes in Theorem 2.1 are specified by the arrival intensity $\lambda^{(j)}$, and the distribution $F_D^{(j)}$ of the duration of the sessions of the jth stream. While $\lambda^{(j)} = \lambda^{(j)}(T) \to \infty$ as $T \to \infty$, $F_D^{(j)}$ does not vary with T. However, the growth regimes described in Scenarios $\mathcal{F}$, $\mathcal{M}$ and $\mathcal{S}$ are given in terms of the arrival rate λ, and the distribution F_D of the duration of the sessions of the aggregated stream and here both λ and F_D vary with T, as seen in (2.2). Therefore, we cannot directly apply Theorem 2.1 for the aggregated stream when

$$\lambda^{(j)}/\lambda = \text{proportion of the sessions that belong to the } j\text{th stream}, \quad j = 1, \ldots, p, \tag{2.22}$$

are functions of T. Nevertheless, in the special case that these proportions are constant, F_D does not vary with T, and a direct application of Theorem 2.1 yields the following result.

COROLLARY 2.2. *Suppose that for all T (or at least for T large enough), the proportions $\lambda^{(j)}/\lambda$ remain constant, $j = 1, \ldots, p$, so that*

$$\bar{F}_D = \sum_{j=1}^{p} (\lambda^{(j)}/\lambda)\bar{F}_D^{(j)} \in RV_{-\alpha_D},$$

where α_D is given in (2.19). Let the Scenarios $\mathcal{F}$, $\mathcal{M}$ and $\mathcal{S}$ take the place of the fast-, moderate- and slow-growth regimes.

If $\bar{F}_R \in RV_{-\alpha_R}, \alpha_R > \alpha_D$ or $E[(R_1^{(1)})^2] < \infty$, then Theorem 2.1 holds for $A_{cs}(\cdot)$, where $\alpha_D^{(j)}$, $F_D^{(j)}$ and c_j are replaced by α_D, F_D, and the constant c in (2.20), respectively.

If $\bar{F}_R \in RV_{-\alpha_R}, \alpha_R \leq \alpha_D$, the distinction among Scenarios $\mathcal{F}$, $\mathcal{M}$ and $\mathcal{S}$ is irrelevant, and limit results are discussed in Section 4.

Implications of Corollary 2.2. This result provides a partial answer to the question of how much aggregation is required for traffic to be Gaussian at large time scales: Suppose that at least one traffic stream falls in the fast-growth regime, thus generating a cumulative input that can be approximated by fractional Brownian motion. When applicable, Corollary 2.2 implies that the superimposed traffic load can also be approximated by fractional Brownian motion.

In the case that the traffic also contains streams that satisfy the slow-growth regime, Corollary 2.2 is somewhat counterintuitive due to the nature of the distribution tails of the two limit processes. Although these slow-growth streams produce cumulative inputs that are approximately stable Lévy-motion when considered individually, with the inclusion of one single stream that behaves under the fast-growth regime, the cumulative aggregated input is approximately Gaussian.

Moreover, a sufficient condition for the fast-growth regime of Scenario $\mathcal{F}$ is that a single stream, say the jth one, satisfies fast-growth, even if all the other streams' arrival intensities do not follow a growth regime at all. In this sense, Scenario $\mathcal{F}$ is a robust assumption. We will see that as long as one $\alpha_D^{(j)} < \alpha_R$, the limit result of Corollary 2.2 is still valid.

In real networks, there are arguably streams with large initiation rates. For instance, the arrival rates of http traffic must be large, since there are a large number of users constantly accessing websites and this translates into

Scenario $\mathcal{F}$. Furthermore, even though some studies report or assume session transmission rates have infinite variance, the assumption $E[(R_1^{(1)})^2] < \infty$ may be justified by rate constraint mechanisms required for congestion control. Although assumptions always deserve rigorous scrutiny, Corollary 2.2 provides a compelling explanation for the data example in Section 1.

We now address more general assumptions which allow the conclusions of Corollary 2.2 to hold. While the assumption of constant proportions $\lambda^{(j)}/\lambda$ may sometimes be reasonable, in general the proportions of sessions corresponding to the p independent streams are not constant over time. We may have that $\lim \lambda^{(j)}/\lambda$ exists or, more generally, that $\lambda^{(j)}/\lambda \in (a,b) \subset (0,1)$ varies with no limit whatsoever. Extending the conclusions of Corollary 2.2 to under weaker assumptions is the focus of the next section.

3. Behavior of cumulative load of aggregated streams.

We prove that the conclusion of Corollary 2.2 is still valid even when the proportion of the sessions corresponding to the p independent streams is not constant. Here is the result:

THEOREM 3.1. *Assume that*

$$\liminf_{T\to\infty} \bigvee_{j:\alpha_D^{(j)}=\alpha_D} \lambda^{(j)}/\lambda > 0, \tag{3.1}$$

Let the Scenarios $\mathcal{F}$, $\mathcal{M}$ and $\mathcal{S}$ take the place of the fast-, moderate- and slow-growth regimes.

If $\bar{F}_R \in RV_{-\alpha_R}, \alpha_R > \alpha_D$ or $E[(R_1^{(1)})^2] < \infty$, then Theorem 2.1 holds for $A_{cs}(\cdot)$, where $\alpha_D^{(j)}$, $F_D^{(j)}$ and c_j are respectively replaced by

$$\alpha_D = \bigwedge_{j=1}^{p} \alpha_D^{(j)}, \qquad F_D = \sum_{j=1}^{p} (\lambda^{(j)}/\lambda) F_D^{(j)}, \qquad c = \left(\sum_{j\in\mathcal{M}} c_j^{\alpha_D^{(j)}-1} \right)^{1/(\alpha_D-1)}.$$

If $\bar{F}_R \in RV_{-\alpha_R}, \alpha_R \le \alpha_D$, the distinction among Scenarios $\mathcal{F}$, $\mathcal{M}$ and $\mathcal{S}$ is irrelevant, and limit results are discussed in Section 4.

Condition 3.1 implies that there exists $d > 0$ such that for all T sufficiently large, there is at least one $k = k(T)$ such that $\alpha_D^{(k)} = \alpha_D$ and $\lambda^{(k)}/\lambda > d$. Roughly speaking, this means that the proportion of the traffic with the heaviest-tailed duration always remains greater than a positive quantity.

All the limits in Theorem 3.1 follow from the convergence of the characteristic function of the finite-dimensional distributions (*fidi chf*) of the processes. Thus, let $m \geq 1$ represent the dimension, $0 \leq t_1, \dots, t_m$ the times, and $z_1, \dots, z_m$ arbitrary real numbers; we need

$$g(s,u,r) = \exp\left\{ i \sum_{j=1}^{m} z_j r L_{t_j}(s,u) \right\} - 1 - i \sum_{j=1}^{m} z_j r L_{t_j}(s,u),$$

as defined in Proposition 5.6.

From the second integral in (2.8), we can compute the partial derivative of $L_t(s,u)$ with respect to u, which yields

$$\begin{aligned} g_u(s-u,u,r) &:= \frac{\partial}{\partial u} g\big|_{(s-u,u,r)} \\ &= i \left(\exp\left\{ i \sum_{j=1}^{m} z_j r L_{t_j}(s-u,u) \right\} - 1 \right) \sum_{k=1}^{m} z_k r 1_{[0,t_k]}(s), \end{aligned} \tag{3.2}$$

where g_u is the partial derivative of $g(s,u,r)$ with respect to u. Moreover, putting together (5.4), (5.16), (5.18), the bounds in Lemmas 5.4 and 5.7, we get

$$\left| \exp\left\{ i \sum_{j=1}^{m} z_j r L_{t_j}(s-u,u) \right\} - 1 \right| \leq 2 \sum_{j=1}^{m} |z_j|^{\zeta} (t_j \wedge u)^{\zeta} r^{\zeta}, \quad 0 \leq \zeta \leq 1. \tag{3.3}$$

We will use three more relations in the proof of Theorem 3.1: For $0 < \eta < 1$, there exists a number $T_0 = T_0(\eta) > 0$ such that for $T \geq T_0$ and $b_D(\lambda T) \geq T_0$,

$$2u^{-\alpha_D} \left\{ u^{-\eta} \vee u^{\eta} \right\} \geq \begin{cases} \bar{F}_D(Tu)/\bar{F}_D(T), & u \geq T_0/T, \\ \bar{F}_D(b_D(\lambda T)u)/\bar{F}_D(b_D(\lambda T)), & u \geq T_0/b(\lambda T), \end{cases} \tag{3.4}$$

$$\frac{\bar{F}_D(Tu)}{\bar{F}_D(T)} \leq \mu_D T^{\alpha_D - 1 + \eta} u^{-1}, \quad u < T_0/T, \tag{3.5}$$

and

$$\frac{\bar{F}_D(b_D(\lambda T)u)}{\bar{F}_D(b_D(\lambda T))} \leq \mu_D b_D(\lambda T)^{\alpha_D - 1 + \eta} u^{-1}, \quad u < T_0/b_D(\lambda T). \tag{3.6}$$

We can readily derive (3.4) from Lemma 5.3. Both (3.5) and (3.6) follow from Markov's Inequality and, for example, [28, Proposition 0.8] or [3, Proposition 1.3.6].

PROOF OF THEOREM 3.1. We use the same framework as in [15]. However, generalizing [15]'s proofs from Pareto to the regularly varying case induces technical difficulties.

First, we will prove parts (b) and (c). For both parts, set $0 < \eta < \alpha_D - 1$ and $0 < \zeta < 1$ such that

$$\alpha_D + \eta < 1 + \zeta < \begin{cases} \alpha_R, & \text{if } \bar{F}_R \in RV_{-\alpha_R}, 1 < \alpha_R < 2, \\ 2, & \text{if } E[(R_1^{(1)})^2] < \infty. \end{cases}$$

Part (b). Under Scenario $\mathcal{M}$, use $a(T) = T$, apply Proposition 5.6, yielding as $T \to \infty$

$$\begin{aligned} \ln E \exp & \left\{ i \sum_{j=1}^{m} z_m A_{cs}(t_j) \right\} \\ &= \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} g_u(s-u,u,r) \lambda T \bar{F}_D(T) \frac{\bar{F}_D(Tu)}{\bar{F}_D(T)} ds du F_R(dr) \\ &\to \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} g_u(s-u,u,r) c^{\alpha_D - 1} u^{-\alpha_D} ds du F_R(dr) \\ &= c^{\alpha_D - 1} \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} g(s,u,r) E N^{\infty}_{\alpha_D^{(j)}, F_R}(ds, du, dr), \end{aligned} \tag{3.7}$$

which is the log fidi chf of $cY_{\alpha_D}(\cdot/c)$, assuming we can take the limit inside the integral, performing afterwards an integration by parts in u. Thus, it suffices to justify taking the limit inside the integral.

First observe there exists a number $T_0 > 0$ such that for $T \geq T_0$,

$$\lambda T \bar{F}_D(T) \leq c^{\alpha_D - 1} + \eta, \tag{3.8}$$

by the moderate-growth assumption. Together with (3.3)–(3.5) and a possibly larger T_0, the above implies that the integrand in the left side of (3.7) is bounded in $\{u \geq T_0/T\}$ by

$$\begin{aligned} & B_{\mathcal{M},(>)}(s,u,r) := \\ & 4 \left(c^{\alpha_D - 1} + \eta \right) u^{-\alpha_D} (u^{-\eta} \vee u^{\eta}) \sum_{j=1}^{m} \sum_{k=1}^{m} |z_j|^{\zeta} |z_k| (t_j \wedge u)^{\zeta} r^{1+\zeta} 1_{[0,t_k]}(s), \end{aligned}$$

and bounded in $\{u < T_0/T\}$ by

$$\begin{aligned} & B_{\mathcal{M},(<)}(s,u,r) := \\ & 2 \left(c^{\alpha_D - 1} + \eta \right) T_0^{\alpha_D - 1 + \eta} \mu_D \sum_{j=1}^{m} \sum_{k=1}^{m} |z_j|^{\zeta} |z_k| u^{\zeta - \alpha_D - \eta} r^{1+\zeta} 1_{[0,t_k]}(s) 1_{(0,1)}(u), \end{aligned}$$

whenever $T \geq T_0$. Here we used the bound

$$u^{\zeta} \leq (T_0/T)^{\alpha_D - 1 + \eta} u^{1+\zeta-\alpha_D-\eta}, \quad 0 < u < T_0/T. \tag{3.9}$$

Therefore, (3.7) follows by the dominated convergence theorem, since for all $T \geq T_0$

$$\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} B_{\mathcal{M},(>)}(s,u,r)\,ds\,du\,F_R(dr)$$
$$\leq 4\left(c^{\alpha_D-1}+\eta\right) E[(R_1^{(1)})^{1+\zeta}]\times$$
$$\sum_{k=1}^{m}\sum_{j=1}^{m} |z_j|^{\zeta}|z_k| t_k \left\{\int_0^1 u^{\zeta-\alpha_D-\eta}du + t_j^{\zeta}\int_1^{\infty} u^{-\alpha_D+\eta}du\right\},$$

and

$$\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} B_{\mathcal{M},(<)}(s,u,r)\,ds\,du\,F_R(dr)$$
$$\leq 2\left(c^{\alpha_D-1}+\eta\right) T_0^{\alpha_D-1+\eta}\mu_D\times$$
$$\sum_{j=1}^{m}\sum_{k=1}^{m} |z_j|^{\zeta}|z_k| t_k E[(R_1^{(1)})^{1+\zeta}]\int_0^1 u^{\zeta-\alpha_D-\eta}du,$$

which are both finite by our choice of η and ζ.

Part (c). Under Scenario $\mathcal{S}$, $a(T) = b_D(\lambda T)$, so we use Lemma 5.1 and [25, Lemma 1] to get

$$\lim_{T\to\infty} T/a(T) = \infty.$$

Thus, it follows from the definition of $L_t(s,u)$ in (2.8) that

$$\lim_{T\to\infty} L_{tT/a(T)}(sT/a(T)-u,u) = u1_{[0,t]}(s).$$

Now, apply Proposition 5.6, perform the change of variables $r \mapsto ra(T)/T$, $u \mapsto uT/a(T)$, and use the scaling property in Lemma 5.4 to get as $T \to \infty$

$$\ln E\exp\left\{i\sum_{j=1}^{m} z_m A_{cs}(t_j)\right\}$$
$$= \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} g_u\left(s-\frac{ua(T)}{T},\frac{ua(T)}{T},\frac{rT}{a(T)}\right)\lambda a(T)\bar{F}_D(a(T)u)\,ds\,du\,F_R(dr)$$

$$= \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} i\left(\exp\left\{\sum_{j=1}^m z_j r L_{t_jT/a(T)}(sT/a(T)-u,u)\right\}-1\right)\times$$
$$\sum_{k=1}^m z_k r 1_{(0,t_k)}(s)\lambda T\bar F_D(a(T)u)\,ds\,du\,F_R(dr),$$

$$\to \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} i\left(\exp\left\{\sum_{j=1}^m z_j u r 1_{(0,t_j)}(s)\right\}-1\right)\times \tag{3.10}$$
$$\sum_{k=1}^m z_k r 1_{(0,t_k)}(s)u^{-\alpha_D}\,ds\,du\,F_R(dr),$$

which is the log fidi chf of $E[(R_1^{(1)})^{\alpha_D}]^{1/\alpha_D}\Lambda_{a_D}(\cdot)$, assuming limit and integration can be interchanged to get the last line of (3.10). We justify passing the limit inside the integral as follows.

First, by Lemma 5.2, there exists a number $T_0 > 0$ such that for $T \ge T_0$,

$$\lambda T\bar F_D(a(T)) \le 2. \tag{3.11}$$

Hence, by taking a possibly larger T_0, (3.3), (3.4) and (3.6) imply that the integrand in the left side of (3.10) is bounded in $\{u \ge T_0/a(T)\}$ by

$$B_{\mathcal{S},(>)}(s,u,r) := 8u^{-\alpha_D}(u^{-\eta}\vee u^{\eta})\sum_{j=1}^m\sum_{k=1}^m |z_j|^{\zeta}|z_k|(t_j\wedge u)^{\zeta}r^{1+\zeta}1_{[0,t_k]}(s),$$

and bounded in $\{u < T_0/a(T)\}$ by

$$B_{\mathcal{S},(<)}(s,u,r) := 4T_0^{\alpha_D-1+\eta}\mu_D\sum_{j=1}^m\sum_{k=1}^m |z_j|^{\zeta}|z_k|u^{\zeta-\alpha_D-\eta}r^{1+\zeta}1_{[0,t_k]}(s)1_{(0,1)}(u),$$

whenever $T \ge T_0$ and $a(T) \ge T_0$, using

$$u^{\zeta} \le (T_0/a(T))^{\alpha_D-1+\eta}u^{1+\zeta-\alpha_D-\eta},\quad 0<u<T_0/a(T).$$

Therefore, (3.10) follows exactly as in part (b) from the dominated convergence theorem.

Part (a). Under Scenario $\mathcal{F}$ and $E[(R_1^{(1)})^2]<\infty$, set $a(T)=[\lambda T^3\bar F_D(T)]^{1/2}$. Use Proposition 5.6 and the change of variables $r\mapsto ra(T)/T$, to write

$$\ln E\exp\left\{i\sum_{j=1}^m z_m A_{cs}(t_j)\right\}$$
$$= \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} g_u(s-u,u,rT/a(T))(a(T)/T)^2\frac{\bar F_D(Tu)}{\bar F_D(T)}\,ds\,du\,F_R(dr), \tag{3.12}$$

where

$$(a(T)/T)^2 = \lambda T \bar{F}_D(T) \to \infty, \quad T \to \infty,$$

by the fast-growth assumption.

By (3.2), as $T \to \infty$

$$g_u(s-u, u, rT/a(T))(a(T)/T)^2$$
$$= i\left(i \sum_{j=1}^m z_j r \frac{T}{a(T)} L_{t_j}(s-u, u) + o\left(\frac{T}{a(T)}\right)\right) \sum_{k=1} z_k r 1_{[0,t_k]}(s) \frac{a(T)}{T}.$$

Hence, assuming we can pass the limit inside the integral, we use Lemma 5.4 (iii) to write

$$\lim_{T\to\infty} \ln E \exp\{i \sum_{j=1}^m z_m A_{cs}(t_j)\}$$
$$= -\sum_{j=1}^m \sum_{k=1}^m z_j z_k \int_{-\infty}^\infty \int_0^\infty \int_0^\infty r^2 L_{t_j}(s-u, u) 1_{[0,t_k]}(s) u^{-\alpha_D} ds du F_R(dr)$$
$$= -E[(R_1^{(1)})^2] \left(\frac{1}{(\alpha_D - 1)(2-\alpha_D)(3-\alpha_D)} \right) \times$$
$$\left\{ \sum_{j=1}^m \sum_{k=1}^j z_j z_k t_k^{3-\alpha_D} + \sum_{j=1}^m \sum_{k=j+1}^m z_j z_k \left(t_k^{3-\alpha_D} - (t_k - t_j)^{3-\alpha_D} \right) \right\}$$
$$= -\frac{1}{2} E[(R_1^{(1)})^2] \sigma^2_{B_H(1)} \sum_{j=1}^m \sum_{k=1}^m z_j z_k \frac{1}{2} \left\{ |t_j|^{2H} + |t_k|^{2H} - |t_j - t_k|^{2H} \right\},$$

where the last line follows by rearranging of the terms in the sum, $\sigma^2_{B_H(1)}$ is given in (5.10) and $H = (3-\alpha_D)/2$. It remains to prove that we can take the limit inside the integral.

Let $0 < \eta < \alpha_D - 1$. We use (3.3)–(3.5) with $\zeta = 1$ and a possibly larger T_0, which imply that the integrand in (3.12) is bounded in $\{u \geq T_0/T\}$ by

$$B_{\mathcal{F},(>)} := 4u^{-\alpha_D}(u^{-\eta} \vee u^{\eta}) \sum_{j=1}^m \sum_{k=1}^m |z_j z_k| (t_j \wedge u) r^2 1_{[0,t_k]}(s),$$

and bounded in $\{u < T_0/T\}$ by

$$B_{\mathcal{F},(<)} := 2T_0^{\alpha_D - 1 + \eta} \mu_D \sum_{j=1}^m \sum_{k=1}^m |z_j z_k| u^{1-\alpha_D-\eta} r^2 1_{[0,t_k]}(s) 1_{(0,1)}(u),$$

whenever $T \geq T_0$. Here we used

$$u \leq (T_0/T)^{\alpha_D - 1 + \eta} u^{2 - \alpha_D - \eta}, \quad 0 < u < T_0/T.$$

The result now follows by the dominated convergence theorem, since

$$\int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} B_{\mathcal{F},(>)}(s,u,r) ds du F_R(dr)$$
$$\leq 4E[(R_1^{(1)})^2] \sum_{j=1}^m \sum_{k=1}^m |z_j z_k| t_k \left\{ \int_0^1 u^{1-\alpha_D-\eta} du + t_j E R_1^{(1)} \int_1^{\infty} u^{-\alpha_D+\eta} du \right\},$$

and

$$\int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} B_{\mathcal{F},(<)}(s,u,r) ds du F_R(dr)$$
$$\leq 2T_0^{\alpha_D - 1 + \eta} \mu_D \sum_{j=1}^m \sum_{k=1}^m |z_j z_k| t_k E[(R_1^{(1)})^2] \int_0^1 u^{1-\alpha_D-\eta} du,$$

and both bounds are finite by our choice of η.

Finally, still under Scenario $\mathcal{F}$, assume $\bar{F}_R \in RV_{-\alpha_R}, 1 < \alpha_R < 2$. Set $a(T) = T b_R(\lambda T \bar{F}_D(T))$. By Proposition 5.6, an integration by parts in u and the change of variables $s \mapsto s + u$:

$$\ln E \exp\left\{ i \sum_{j=1}^m z_m A_{cs}(t_j) \right\}$$
$$= \lambda T \bar{F}_D(T) \bar{F}_R(b_R(\lambda T \bar{F}_D(T))) \times$$
$$\int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} g(s,u,r) ds \frac{F_D(T du)}{\bar{F}_D(T)} \frac{F_R(b_R(\lambda T \bar{F}_D(T)) dr)}{\bar{F}_R(b_R(\lambda T \bar{F}_D(T)))}$$

$$= \lambda T \bar{F}_D(T) \bar{F}_R(b_R(\lambda T \bar{F}_D(T))) \left\{ I_{(u>\epsilon, r>\epsilon)} + I_{(u<\epsilon, r>\epsilon)} + I_{(r<\epsilon)} \right\}, \tag{3.13}$$

where $\epsilon > 0$ and we split the integral into three parts according to the domains of integration $\{u > \epsilon, r > \epsilon\}$, $\{u < \epsilon, r > \epsilon\}$ and $\{r < \epsilon\}$, respectively. To establish the limit result, we will take $\lim_{\epsilon \to 0} \lim_{T \to \infty}$ on both sides of (3.13).

Fix $\epsilon > 0$ and start with the first integral. Let

$$\nu_T(du, dr) := \left(u \frac{F_D(T du)}{\bar{F}_D(T)} \right) \left(r \frac{F_R(b_R(\lambda T \bar{F}_D(T)) dr)}{\bar{F}_R(b_R(\lambda T \bar{F}_D(T)))} \right),$$
$$\nu(du, dr) := \alpha_D u^{-\alpha_D} du \, \alpha_R r^{-\alpha_R} dr,$$
$$G(u, r) := \frac{1}{ur} \int_{-\infty}^{\infty} g(s,u,r) ds,$$

which allows writing

$$I_{(u>\epsilon,r>\epsilon)} = \int_\epsilon^\infty \int_\epsilon^\infty G(u,r)\nu_T(du,dr).$$

The fast-growth regime, regular variation of $\bar{F}_R$, (5.3) and [2, Theorem 2.8] imply $\nu_T \xrightarrow{v} \nu$ as $T \to \infty$. Moreover, $G(u,r)$ is jointly continuous and it follows from Lemmas 5.4 and 5.7 that $|G(u,r)| \le d_0 \sum_{j=1}^m |z_j| t_j < \infty$, where d_0 is a positive constant. Therefore:

$$\begin{aligned} \lim_{T\to\infty} I_{(u>\epsilon,r>\epsilon)} &= \int_\epsilon^\infty \int_\epsilon^\infty G(u,r)\nu(du,dr) \\ &= \int_{-\infty}^\infty \int_\epsilon^\infty \int_\epsilon^\infty g(s,u,r)ds \cdot \alpha_D u^{-(\alpha_D+1)} du \cdot \alpha_R r^{-(\alpha_R+1)} dr \end{aligned} \tag{3.14}$$

Now let $0 \le \zeta, \eta \le 1$ such that $\alpha_D + \eta < 1 + \zeta < \alpha_R$. By Lemmas 5.4 and 5.7, there exists $d_\zeta > 0$ such that

$$\left|I_{(u<\epsilon,r>\epsilon)}\right| \le d_\zeta \sum_{j=1}^m |z_j| t_j \int_\epsilon^\infty r^{1+\zeta} \frac{F_R(b_R(\lambda T \bar{F}_D(T))dr)}{\bar{F}_R(b_R(\lambda T \bar{F}_D(T)))} \int_0^\epsilon u^{1+\zeta} \frac{F_D(T du)}{\bar{F}_D(T)}.$$

Furthermore, by fast-growth and regular variation of $\bar{F}_R$, as $T \to \infty$

$$\int_\epsilon^\infty r^{1+\zeta} \frac{F_R(b_R(\lambda T \bar{F}_D(T))dr)}{\bar{F}_R(b_R(\lambda T \bar{F}_D(T)))} \to \alpha_R \int_\epsilon^\infty r^{\zeta-\alpha_R} dr = \frac{\alpha_R}{\alpha_R - 1 - \zeta} \epsilon^{1+\zeta-\alpha_R}.$$

Similarly, integration by parts and (3.5) with $T \ge T_0$ such that $\epsilon < T_0/T$ yields

$$\begin{aligned} \int_0^\epsilon u^{1+\zeta} \frac{F_D(T du)}{\bar{F}_D(T)} &= (1+\zeta) \int_0^\epsilon u^\zeta \frac{\bar{F}_D(Tu)}{\bar{F}_D(T)} du \\ &\le (1+\zeta)\mu_D T_0^{\alpha_D - 1 + \eta} \int_0^\epsilon u^{\zeta - \alpha_D - \eta} du \\ &\le \frac{(1+\zeta)\mu_D T_0^{\alpha_D-1+\eta}}{1+\zeta-\alpha_D-\eta} \epsilon^{1+\zeta-\alpha_D-\eta}, \end{aligned}$$

where we used the bound (3.9). Thus

$$\limsup_{T\to\infty} \left|I_{(u<\epsilon,r>\epsilon)}\right| \le \mathit{constant} \cdot \epsilon^{2(1+\zeta)-\alpha_R-\alpha_D-\zeta}, \tag{3.15}$$

where the exponent of ϵ is positive if we additionally let $(\alpha_R + \alpha_D + \eta)/2 < 1 + \zeta$.

For $I_{(r<\epsilon)}$, use again Lemmas 5.4 and 5.7 to get a $d_1 > 0$ such that

$$|I_{(r<\epsilon)}| \leq d_1 \sum_{j=1}^{m} \sum_{k=1}^{m} |z_j z_k| t_j \int_0^{\epsilon} r^2 \frac{F_R(b_R(\lambda T \bar{F}_D(T))dr)}{\bar{F}_R(b_R(\lambda T \bar{F}_D(T)))} \int_0^{\infty} u(t_k \wedge u) \frac{F_D(T du)}{\bar{F}_D(T)}.$$

Analogously to the bound for $I_{(u<\epsilon, r>\epsilon)}$, it can be readily shown that there exists $T_0 > 0$ such that for $T \geq T_0$

$$\int_0^{\epsilon} r^2 \frac{F_R(b_R(\lambda T \bar{F}_D(T))dr)}{\bar{F}_R(b_R(\lambda T \bar{F}_D(T)))} \leq \frac{3\mu_R T_0^{\alpha_R}}{2 - \alpha_R} \epsilon^{2-\alpha_R},$$

and

$$\begin{aligned}\int_0^{\infty} u(t_k \wedge u) \frac{F_D(T du)}{\bar{F}_D(T)} &\leq \int_0^1 u^2 \frac{F_D(T du)}{\bar{F}_D(T)} + t_k \int_1^{\infty} u \frac{F_D(T du)}{\bar{F}_D(T)} \\ &\leq \frac{3\mu_D T_0^{\alpha_D}}{2 - \alpha_D} + t_k \frac{\alpha_D}{\alpha_D - 1},\end{aligned}$$

whence

$$\limsup_{T\to\infty} \left| I_{(r<\epsilon)} \right| \leq constant \cdot \epsilon^{2-\alpha_R}. \tag{3.16}$$

Also, by fast growth and the regular variation of $\bar{F}_R$

$$\lambda T \bar{F}_D(T) \bar{F}_R(b_R(\lambda T \bar{F}_D(T))) \to 1, \qquad T \to \infty. \tag{3.17}$$

Finally, we can put together (3.13)–(3.17) to write:

$$\begin{aligned}\lim_{T\to\infty} \ln E \exp\left\{ i \sum_{j=1}^{m} z_j A_{cs}(t_j) \right\} &= \lim_{\epsilon\to 0} \lim_{T\to\infty} \ln E \exp\left\{ i \sum_{j=1}^{m} z_j A_{cs}(t_j) \right\} \\ &= \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} g(s,u,r) ds \cdot \alpha_D u^{-(\alpha_D+1)} du \cdot \alpha_R r^{-(\alpha_R+1)} dr.\end{aligned}$$

□

4. Remaining choices of α_D and α_R. We first study what happens if $\alpha_R < \alpha_D$, namely, $\alpha_R < \alpha_D^{(j)}$ for $j = 1, \ldots, p$. Consider the centered and scaled cumulative input of any of the streams, say the first one, for simplicity. Set the normalizing term to $a^{(1)}(T) = b_R(\lambda T)$ (and the same for all other streams). Write

$$\begin{aligned}A_{cs}^{(1)}(t) &= \frac{1}{b_R(\lambda T)} \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} r L_{Tt}(s,u) \mathring{N}^{(1)}(ds, du, dr) \\ &= \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} r L_{Tt}(Ts,u) \mathring{N}^{(1)}(T ds, du, b_R(\lambda T) dr).\end{aligned}$$

First, note from the definition of $L_t(s,u)$ in (2.8) that

$$\lim_{T\to\infty} L_{Tt}(Ts,u) = u1_{[0,t]}(s).$$

Thus, analogous to the proof of Proposition 5.6, it can be shown that the log fidi chf of $A_{cs}^{(1)}$ is

$$\ln E \exp\left\{ i \sum_{j=1}^{m} z_j A_{cs}^{(1)}(t_j) \right\} = \frac{\lambda^{(1)}}{\lambda}\lambda T \bar{F}_R(b_R(\lambda T)) \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} i\left(\exp\left\{ ir\sum_{j=1}^{m} z_j L_{Tt_j}(Ts,u)\right\} - 1\right) \times \sum_{k=1}^{m} z_k L_{Tt_j}(Ts,u)\, ds F_D^{(1)}(du) \frac{\bar{F}_R(b_R(\lambda T)r)}{\bar{F}_R(b_R(\lambda T))} dr. \tag{4.1}$$

Now observe that, provided we can take the limit inside the integral

$$\lim_{T\to\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} i\left(\exp\left\{ ir\sum_{j=1}^{m} z_j L_{Tt_j}(Ts,u)\right\} - 1\right) \times \sum_{k=1}^{m} z_k L_{Tt_j}(Ts,u)\, ds F_D^{(1)}(du) \frac{\bar{F}_R(b_R(\lambda T)r)}{\bar{F}_R(b_R(\lambda T))} dr$$
$$= \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} i\left(\exp\left\{ ir\sum_{j=1}^{m} z_j u1_{[0,t_j]}(s)\right\} - 1\right) \times \sum_{k=1}^{m} z_k u1_{[0,t_j]}(s)\, ds F_D^{(1)}(du) r^{-\alpha_R} dr. \tag{4.2}$$

Since $|\lambda^{(1)}/\lambda| \le 1$, then

$$\epsilon^{(1)}(T) := \ln E \exp\left\{ i \sum_{j=1}^{m} z_j A_{cs}^{(1)}(t_j) \right\} - \frac{\lambda^{(1)}}{\lambda}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} i\left(\exp\left\{ ir\sum_{j=1}^{m} z_j u1_{[0,t_j]}(s)\right\} - 1\right) \times \sum_{k=1}^{m} z_k u1_{[0,t_j]}(s)\, ds F_D^{(1)}(du) r^{-\alpha_R} dr \to 0, \qquad T\to\infty, \tag{4.3}$$

which yields the following result.

THEOREM 4.1. *Let $\Psi := \Psi_T$ be the fidi chf of*

$$\sum_{j=1}^{p} \frac{\lambda^{(j)}}{\lambda} E[(D_1^{(j)})^{\alpha_R}]^{1/\alpha_R} \Lambda_{\alpha_R}(t)$$

$$\stackrel{d}{=} \left(\left(-\cos \frac{\pi\alpha_R}{2} \right) \frac{2\Gamma(2-\alpha_R)}{\alpha_R(\alpha_R - 1)} \right)^{-1/\alpha_R} \int_{-\infty}^{\infty} \int_0^{\infty} 1_{[0,t]}(s) u M_{\alpha_R}(ds, du), \tag{4.4}$$

where for each T, $\Lambda_{\alpha_R}(\cdot)$ is an α_R-stable Lévy motion totally skewed to the right with index α_R and $M_{\alpha_R}(ds, du)$ is α_R-stable with control measure $m(ds, du) = dsF_D(du)$. Then,

$$\lim_{T\to\infty} \left\{ \ln E \exp \left\{ i \sum_{j=1}^{m} z_j A_{cs}^{(1)}(t_j) \right\} - \ln \Psi_{t_1,\ldots,t_m}(z_1, \ldots, z_m) \right\} = 0. \tag{4.5}$$

In addition, if for $j = 1, \ldots, p$, the limits

$$w^{(j)} := \lim_{T\to\infty} \lambda^{(j)}/\lambda \tag{4.6}$$

exist, then the fidi chf of $A_{cs}(\cdot)$ converges to the fidi chf of the process defined by (4.4), with $w^{(j)}$ and $\sum_{j=1}^{p} w^{(j)} F_D^{(j)}(\cdot)$ replacing $\lambda^{(j)}/\lambda$ and $F_D(\cdot)$.

PROOF. By the independence of $N^{(j)}, j = 1, \ldots, p$,

$$\ln E \exp \left\{ i \sum_{j=1}^{m} z_j A_{cs}(t_j) \right\} -$$

$$\int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} i \left(\exp \left\{ ir \sum_{j=1}^{m} z_j u 1_{[0,t_j]}(s) \right\} - 1 \right) \times$$

$$\sum_{k=1}^{m} z_k u 1_{[0,t_j]}(s) ds F_D(du) r^{-\alpha_R} dr$$

$$= \sum_{j=1}^{p} \epsilon^{(j)}(T) \to 0 \qquad T \to \infty. \tag{4.7}$$

Analogously to the proof of (5.13), the second integral in (4.7) is equal to $\ln \Psi_{t_1,\ldots,t_m}(z_1, \ldots, z_m)$. Thus, it only remains to justify taking the limit (4.3).

Let $0 < \zeta < \zeta' < 1$ and $0 < \eta < 1$ such that $1 + \zeta < \alpha_R - \eta$ and $\alpha_R + \eta < 1 + \zeta' < \alpha_D$. Similarly to (3.4) and (3.6), there exists $T_0 := T_0(\eta) > 0$ such that for $T \geq T_0$ and $b_R(\lambda T) \geq T_0$,

$$\frac{\bar{F}_R(b_R(\lambda T)r)}{\bar{F}_R(b_R(\lambda T))} \leq \begin{cases} 2r^{-\alpha_R}\left\{r^{-\eta} \vee r^{\eta}\right\}, & r \geq T_0/b_R(\lambda T), \\ \mu_R b_R(\lambda T)^{\alpha_R - 1+\eta} r^{-1}, & r \in \mathbb{R}. \end{cases}$$

Together with Lemmas 5.4 and 5.7, this implies that the integrand in the left side of (4.2) is bounded in $\{r \geq 1\}$

$$B_{(>)} := 2^{1-\zeta} u^{1+\zeta} r^{\zeta-\alpha_R+\eta} \sum_{j=1}^{m}\sum_{k=1}^{m} |z_j|^{\zeta}|z_k| 1_{[0,t_k]}(s) 1_{[1,\infty)}(u),$$

and bounded in $\{r < 1\}$ by

$$B_{(<)} := 2^{1-\zeta'} \mu_R T_0^{\alpha_R-1+\eta} u^{1+\zeta'} r^{\zeta'-\alpha_R-\eta} \sum_{j=1}^{m}\sum_{k=1}^{m} |z_j|^{\zeta'}|z_k| 1_{[0,t_k]}(s) 1_{(0,1)}(r),$$

whenever $b_R(\lambda T) > T_0$. Here we used

$$r^{\zeta'} \leq (T_0/(b_R(\lambda T))^{\alpha_R-1+\eta} r^{1+\zeta'-\alpha_D-\eta}.$$

By our choice of ζ, ζ' and η, both bounds are integrable and we can use dominated convergence to prove the result. □

In principle, it also is possible to have $\alpha_D = \alpha_R$. However, we cannot say much except in the special case $\alpha_D = \alpha_R = 2$, in which case the limit process is a Brownian motion provided (4.6) holds. We refer the reader to [15, Theorem 4] for the formal statement of this case.

5. Technical proofs.

This section contains a collection of technical results needed for our proofs. The first lemma establishes bounds for $b_D(\cdot) = (1/\bar{F}_D)^{\leftarrow}(\cdot)$ which yield $b_D(\lambda T) \to \infty$. This is not immediate since the function b_D depends on T.

LEMMA 5.1. *The quantile functions given* (2.3) *and* (2.4) *satisfy the following inequality.*

$$\bigvee_{j=1}^{p} b_D^{(j)}(p\lambda^{(j)}T) \geq b_D(\lambda T) \geq \bigvee_{j=1}^{p} b_D^{(j)}(\lambda^{(j)}T), \quad T > 0. \tag{5.1}$$

Hence

$$b_D(\lambda T) \to \infty, \quad T \to \infty.$$

PROOF. Since $\bar{F}_D^{(j)}$ is decreasing for all j, then

$$\bar{F}_D\left(\bigvee_{j=1}^{p} b_D^{(j)}(p\lambda^{(j)}T)\right) \leq \sum_{j=1}^{p}(\lambda^{(j)}/\lambda)\bar{F}_D^{(j)}(b_D^{(j)}(p\lambda^{(j)}T))$$
$$\leq \sum_{j=1}^{p}(\lambda^{(j)}/\lambda)(p\lambda^{(j)}T)^{-1}$$
$$= (\lambda T)^{-1}.$$

Thus, the left side of (5.1) follows.

On the other hand, since F_D is right continuous, we have for each $j = 1, \dots, p$:

$$(\lambda^{(j)}/\lambda)\bar{F}_D^{(j)}(b_D(\lambda T)) \leq \sum_{k=1}^{p}(\lambda^{(k)}/\lambda)\bar{F}_D^{(k)}(b_D(\lambda T)) \leq (\lambda T)^{-1},$$

whence

$$\bar{F}_D^{(j)}(b_D(\lambda T)) \leq (\lambda^{(j)}T)^{-1}.$$

Therefore, the right side of (5.1) follows. □

The distribution $F_D = \sum_{j=1}^{p}(\lambda^{(j)}/\lambda)F_D^{(j)}$ of session durations of superimposed streams is a function of T since $\lambda^{(j)}$ and λ depend on T. Nevertheless, $\bar{F}_D$ behaves as a regularly varying function.

LEMMA 5.2. *Under the assumption* (3.1)

$$\lim_{T\to\infty}\frac{\bar{F}_D(Tu)}{\bar{F}_D(T)} = \lim_{T\to\infty}\lambda T\bar{F}_D(b_D(\lambda T)u) = u^{-\alpha_D}, \quad u > 0, \tag{5.2}$$

and therefore, in $M_+(0,\infty]$,

$$\frac{\bar{F}_D(T\,du)}{\bar{F}_D(T)} \xrightarrow{v} \alpha_D u^{-(\alpha_D+1)}\,du, \qquad T\to\infty. \tag{5.3}$$

PROOF. Note that $\bar{F}_D \in RV_{-\alpha_D}$ for each fixed T. However, because F_D varies with T, the limit is not straightforward.

Fix an arbitrary $u > 0$. We start by writing

$$\frac{\bar{F}_D(Tu)}{\bar{F}_D(T)} = \frac{\sum_{j:\alpha_D^{(j)}=\alpha_D}(\lambda^{(j)}/\lambda)\bar{F}_D^{(j)}(Tu)}{\bar{F}_D(T)} + \sum_{j:\alpha_D^{(j)}>\alpha_D}(\lambda^{(j)}/\lambda)\frac{\bar{F}_D^{(j)}(Tu)}{\bar{F}_D(T)}$$
$$=: B + \sum_{j:\alpha_D^{(j)}>\alpha_D}(\lambda^{(j)}/\lambda)C_j,$$

and additionally write

$$B^{-1} = \frac{\sum_{j:\alpha_D^{(j)}=\alpha_D} (\lambda^{(j)}/\lambda)\bar{F}_D^{(j)}(T)}{\sum_{j:\alpha_D^{(j)}=\alpha_D} (\lambda^{(j)}/\lambda)\bar{F}_D^{(j)}(Tu)} + \sum_{j:\alpha_D^{(j)}>\alpha_D} (\lambda^{(j)}/\lambda) \frac{\bar{F}_D^{(j)}(T)}{\sum_{k:\alpha_k=\alpha_D} (\lambda_k/\lambda)\bar{F}_D^{(k)}(Tu)}$$
$$=: B_1 + \sum_{j:\alpha_D^{(j)}>\alpha_D} (\lambda^{(j)}/\lambda)B_{2,j}.$$

Thus, the first limit in (5.2) will follow by proving $B_1 \to u^{\alpha_D}$, $B_{2,j} \to 0$ and $C_j \to 0$ as $T \to \infty$, for all j such that $\alpha_D^{(j)} > \alpha_D$.

First, by Potter's bounds applied to the regular variation of each $\bar{F}_D^{(j)}$, we have as $T \to \infty$,

$$B_1 \sim \frac{\sum_{j:\alpha_D^{(j)}=\alpha_D} (\lambda^{(j)}/\lambda)\bar{F}_D^{(j)}(T)}{\sum_{j:\alpha_D^{(j)}=\alpha_D} (\lambda^{(j)}/\lambda)\bar{F}_D^{(j)}(T)u^{-\alpha_D}} = u^{\alpha_D}.$$

Now, consider $B_{2,j}$ for $\alpha_D^{(j)} > \alpha_D$. Choose an arbitrarily large $z > \min\{u^{-\alpha_D^{(j)}}, u^{\alpha_D^{(j)}}\}$. By regular variation, for T sufficiently large:

$$\frac{\bar{F}_D^{(k)}(Tu)}{\bar{F}_D^{(j)}(Tu)} > z, \qquad \frac{\bar{F}_D^{(k)}(T)}{\bar{F}_D^{(j)}(T)} > z,$$

for all k such that $\alpha_D^{(k)} = \alpha_D$. In addition

$$\frac{\bar{F}_D^{(j)}(Tu)}{\bar{F}_D^{(j)}(T)} > u^{-a_D^{(j)}} - z^{-1}, \qquad \frac{\bar{F}_D^{(j)}(T)}{\bar{F}_D^{(j)}(Tu)} > u^{a_D^{(j)}} - z^{-1}.$$

Furthermore, the assumption (3.1) means there exists $d > 0$ such that for all T sufficiently large, there is some $k' := k'(T)$ such that $\alpha_D^{(k')} = \alpha_D$ and $\lambda^{(k')}/\lambda > d$. Hence for T sufficiently large:

$$B_{2,j}^{-1} = \sum_{k:\alpha_D^{(k)}=\alpha_D} (\lambda^{(k)}/\lambda)\frac{\bar{F}_D^{(k)}(Tu)}{\bar{F}_D^{(j)}(T)} = \sum_{k:\alpha_D^{(k)}=\alpha_D} (\lambda^{(k)}/\lambda)\frac{\bar{F}_D^{(k)}(Tu)}{\bar{F}_D^{(j)}(Tu)}\frac{\bar{F}_D^{(j)}(Tu)}{\bar{F}_D^{(j)}(T)}$$
$$> dz(u^{-\alpha_D} - z^{-1}).$$

This shows that $B_{2,j}^{-1}$ can be made arbitrarily large for T sufficiently large, whence $B_{2,j} \to 0$ as $T \to \infty$.

Similarly, consider C_j, and

$$C_j^{-1} \geq \sum_{k:\alpha_D^{(k)}=\alpha_D} (\lambda^{(k)}/\lambda) \frac{\bar{F}_D^{(k)}(T)}{\bar{F}_D^{(j)}(Tu)} = \sum_{k:\alpha_D^{(k)}=\alpha_D} (\lambda^{(k)}/\lambda) \frac{\bar{F}_D^{(k)}(T)}{\bar{F}_D^{(j)}(T)} \frac{\bar{F}_D^{(j)}(T)}{\bar{F}_D^{(j)}(Tu)}$$
$$> dz(u^{\alpha_D} - z^{-1}).$$

This shows that C_j^{-1} can be made arbitrarily large for T sufficiently large, which completes the first part of the lemma.

For the second limit in (5.2), recall that $z < b_D(\lambda T)$ iff $1/\bar{F}_D(z) < \lambda T$ for each T. For $\epsilon > 0$, setting $z = b_D(\lambda T)(1-\epsilon)$ and $z = b_D(\lambda T)(1+\epsilon)$ yields

$$\frac{\bar{F}_D(b_D(\lambda T)(1+\epsilon))}{\bar{F}_D(b_D(\lambda T))} \leq \frac{1}{\lambda T \bar{F}_D(b_D(\lambda T))} \leq \frac{\bar{F}_D(b_D(\lambda T)(1-\epsilon))}{\bar{F}_D(b_D(\lambda T))}.$$

Letting $T \to \infty$ and using Lemma 5.1 and the first limit gives

$$(1+\epsilon)^{-\alpha_D} \leq \frac{1}{\lambda T \bar{F}_D(b_D(\lambda T))} \leq (1-\epsilon)^{-\alpha_D}.$$

Because ϵ is arbitrary, then

$$\lim_{T\to\infty} \lambda T \bar{F}_D(b_D(\lambda T)) = 1.$$

Therefore

$$\lim_{T\to\infty} \lambda T \bar{F}_D(b_D(\lambda T)u) = \lim_{T\to\infty} \lambda T \bar{F}_D(b_D(\lambda T)) \lim_{T\to\infty} \frac{\bar{F}_D(b_D(\lambda T)u)}{\bar{F}_D(b_D(\lambda T))} = u^{-\alpha_D}.$$

The final statement about vague convergence follows the proof of [30, Theorem 3.6]. □

Even though F_D depends on T, a version of Potter's bounds holds.

LEMMA 5.3. *Let $\delta > 0$. Under the assumption (3.1), there exists $T_0 = T_0(\delta) > 0$ such that for all $T \geq T_0$, $Tu \geq T_0$:*

$$\frac{\bar{F}_D(Tu)}{\bar{F}_D(T)} \leq (1+\delta)u^{-\alpha_D} \max\{u^{-\delta}, u^{\delta}\}.$$

PROOF. Observe

$$\frac{\bar{F}_D(Tu)}{\bar{F}_D(T)} = \frac{\sum_{j=1}^{p}(\lambda^{(j)}/\lambda)\bar{F}_D^{(j)}(T)\frac{\bar{F}_D^{(j)}(Tu)}{\bar{F}_D^{(j)}(T)}}{\sum_{j=1}^{p}(\lambda^{(j)}/\lambda)\bar{F}_D^{(j)}(T)} \leq \bigvee_{j=1}^{p} \frac{\bar{F}_D^{(j)}(Tu)}{\bar{F}_D^{(j)}(T)}.$$

By Potter bounds [See e.g. 3, Theorem 1.5.6], for all $j = 1, \dots, p$ there exists $T_j = T_j(\delta)$ such that

$$\frac{\bar{F}_D^{(j)}(Tu)}{\bar{F}_D^{(j)}(T)} \leq (1+\delta)u^{-\alpha_D^{(j)}} \max\{u^{-\delta}, u^{\delta}\} \leq (1+\delta)u^{-\alpha_D} \max\{u^{-\delta}, u^{\delta}\},$$

for $T \geq T_j$, $Tu \geq T_j$. Therefore, the result holds for $T_0 = \bigvee_{j=1}^{p} T_j$. □

We now study $L_t(s, u)$, as defined in (2.8).

LEMMA 5.4. *The length of the subinterval of* $[0, t]$ *during which the session* (s, u, r) *transmits data, namely* $L_t(s, u)$ *in* (2.8)*, satisfies the following properties:*

(i) ***Scaling property:*** *For* $C > 0$,

$$CL_t(s, u) = L_{Ct}(Cs, Cu).$$

(ii) ***Bounds:***

$$L_t(s, u) \leq t \wedge u. \tag{5.4}$$

(iii) ***Integrals:*** *For* $1 < \gamma < 2$ *and nonnegative* t_1, t_2,

$$\int_{-\infty}^{\infty} L_{t_1}(s, u)\,ds = ut_1,$$

and

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} L_{t_1}(s-u, u)1_{[0,t_2]}(s)u^{-\gamma}\,du\,ds$$
$$= \frac{1}{(\gamma-1)(2-\gamma)(3-\gamma)}\left\{(t_2^{3-\gamma} - (t_2-t_1)^{3-\gamma})1_{t_1<t_2} + t_2^{3-\gamma}1_{t_1\geq t_2}\right\}.$$

PROOF. The scaling property and the bounds follow directly from (2.8). Now, the first part of Property (iii) is readily checked by using the first integral in (2.8) after reversing the order of integration. Finally, the second part of Property (iii) can be derived by writing

$$L_{t_1}(s-u,u) = \begin{cases} 0, & s < 0 \text{ or } s > u + t_1, \\ s, & 0 \le s \le u \wedge t_1, \\ t_1, & t_1 \le s \le u, \\ u, & u \le s \le t_1, \\ t_1 - s + u, & u \vee t_1 \le s \le u + t_1, \end{cases}$$

and integrating accordingly. Observe that the four regions in which $L_{t_1}(s-u,u)$ is nonzero correspond to those of the basic decomposition in [25, (4.1)]. □

The next lemma helps obtain approximations to the cumulative input of the aggregated streams.

LEMMA 5.5. *For any $a, T > 0$, we have*

$$\begin{aligned} \frac{1}{a}(A(Tt) - \lambda\mu_D\mu_R Tt) &= \frac{1}{a}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} rL_{Tt}(Ts,u)\mathring{N}(Tds,du,dr) \\ &= \frac{T}{a}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} rL_t(s,u)\mathring{N}(Tds,Tdu,dr) \\ &= \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} rL_t(s,u)\mathring{N}(Tds,Tdu,(a/T)dr). \end{aligned}$$

PROOF. All the relations here follow from several ways to change variables in (2.13) and using the scaling property of $L_t(s,u)$. See Lemma 5.4. □

Our limit theorems are proved by verifying convergence of finite dimensional distributions for various processes. The following is required.

PROPOSITION 5.6. *For arbitrary $m \ge 1$, $0 \le t_1, \dots, t_m$, and real $z_1, \dots, z_m$, define*

$$g(s,u,r) = \exp\left\{ i\sum_{j=1}^m z_j rL_{t_j}(s,u)\right\} - 1 - i\sum_{j=1}^m z_j rL_{t_j}(s,u), \tag{5.5}$$

and

$$h(s,u,r) = i\left(\exp\left\{ i\sum_{j=1}^m z_j ur 1_{(0,t_j)}(s)\right\} - 1\right)\sum_{k=1}^m z_k 1_{[0,t_k]}(s) ru^{-\alpha_D}. \tag{5.6}$$

(a) *For any* $a, T > 0$, *the characteristic function of the finite-dimensional distributions* (fidi chf) *of the process* $\{(1/a)(A(Tt) - \lambda\mu_D\mu_R Tt); t \geq 0\}$ *is given by*

$$\ln E \exp\left\{ i \sum_{j=1}^{m} z_j \left[\frac{1}{a}(A(Tt_j) - \lambda\mu_D\mu_R Tt_j) \right] \right\}$$

$$= \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} g(s,u,r) EN(Tds, Tdu, (a/T)dr) \tag{5.7}$$

$$= \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} g_u(s-u,u,r) \lambda T \bar{F}_D(Tu) ds du F_R((a/T)dr). \tag{5.8}$$

where g_u *is the partial derivative of* g *with respect to* u.

(b) *The fidi chf of the limit processes in Corollary* 2.2 *are given as follows.*

(i) *The fidi chf of the limit process under Scenario* $\mathcal{F}$ *and* $E[(R_1^{(1)})^2] < \infty$ *is given by*

$$\ln E \exp\left\{ i \sum_{j=1}^{m} z_j E[(R_1^{(1)})^2]^{1/2} \sigma_{B_H(1)} B_H(t_j) \right\}$$

$$= -\frac{1}{2} E[(R_1^{(1)})^2] \sum_{j=1}^{m} \sum_{k=1}^{m} z_i z_j \sigma^2_{B_H(1)} \frac{1}{2} \left(t_i^{2H} + t_j^{2H} - |t_i - t_j|^2 \right), \tag{5.9}$$

where B_H *is fractional Brownian motion with*

$$\sigma^2_{B_H(1)} = \frac{2}{(\alpha_D - 1)(2-\alpha_D)(3-\alpha_D)}, \tag{5.10}$$

and $H = (3-\alpha_D)/2$.

(ii) *The fidi chf of the limit process under Scenario* $\mathcal{F}$ *and* $\bar{F}_R \in RV_{-\alpha_R}, 1 < \alpha_R < 2$, *is given by*

$$\ln E \exp\left\{ i \sum_{j=1}^{m} z_j Z_{\alpha_D,\alpha_R}(t_j) \right\}$$

$$= \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} g(s,u,r) EN^{\infty}_{\alpha_D,\alpha_R}(ds, du, dr). \tag{5.11}$$

(iii) *The fidi chf of the limit process under Scenario $\mathcal{M}$ is given by*

$$\ln E \exp\left\{ i \sum_{j=1}^{m} z_j c Y_{\alpha_D}(t_j/c) \right\} \tag{5.12}$$
$$= c^{\alpha_D - 1} \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} g(s,u,r) E N^{\infty}_{\alpha_D, F_R}(ds, du, dr).$$

(iv) *Finally, the fidi chf of the limit process under Scenario $\mathcal{S}$ is given by*

$$\ln E \exp\left\{ i \sum_{j=1}^{m} z_j E[(R_1^{(1)})^{\alpha_D}]^{1/\alpha_D} \Lambda_{a_D}(t_j) \right\}$$
$$= \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} h(s,u,r) ds du F_R(dr). \tag{5.13}$$

Proof. Given (5.7), (5.8) is readily derived using integration by parts and the change of variables $s \mapsto s+u$. Moreover, (5.9) follows from the fact that B_H is fractional Brownian motion. The remaining parts are a consequence of the following property of Poisson random measures [See e.g. 31]:

$$\ln E \exp\left\{ i \int f(x) \mathring{\xi}(dx) \right\} = \int \left(e^{if(x)} - 1 - if(x) \right) E\xi(dx),$$

if

$$\int \left(f^2(x) \wedge |f(x)| \right) E\xi(dx) < \infty.$$

For now, let us focus on (5.7), (5.11) and (5.12). By Lemma 5.5, the exponent in the left side of (5.7) and (5.11) is of the form

$$i \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} \sum_{j=1}^{m} z_j r L_{t_j}(s,u) \mathring{\xi}(ds, du, dr), \tag{5.14}$$

for a $\mathbb{PRM}$ ξ, while the exponent in the left side of (5.12) is $c^{\alpha_D - 1}$ times (5.14), using Lemma 5.4 and the change of variables $s \mapsto s/c, u \mapsto u/c$. Thus, it suffices to check that

$$\int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} \left(\sum_{j=1}^{m} z_j r L_{t_j}(s,u) \right)^2 \bigwedge \left| \sum_{k=1}^{m} z_k r L_{t_k}(s,u) \right| E\xi(ds, du, dr) < \infty. \tag{5.15}$$

Bounds and integral results for $L_t(s,u)$ in Lemma 5.4 (ii) and (iii) needed. First observe that

$$\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}\left|\sum_{j=1}^m z_j r L_{t_j}(s,u)\right| EN(T ds, T du, (a/T) dr) \leq \frac{T}{a}\sum_{j=1}^m |z_j|\lambda\mu_D\mu_R t_j,$$

which proves (5.7).

In order to prove (5.11), split the corresponding integral (5.15) into two parts $I_{(<)}$ and $I_{(>)}$, according to the two domains of integration $D_{(<)} = \{ur < 1\}$ and $D_{(>)} = \{ur > 1\}$. This yields

$$\begin{aligned} I_{(<)} &\leq \sum_{j=1}^m\sum_{k=1}^m |z_j z_k| \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{1/u} r^2 L_{t_j}(s,u)L_{t_k}(s,u) EN^{\infty}_{\alpha_D,\alpha_R}(ds,du,dr) \\ &\leq \sum_{j=1}^m\sum_{k=1}^m \frac{|z_j z_k| t_j \alpha_D\alpha_R}{2-\alpha_R}\left(\frac{1}{\alpha_R-\alpha_D} + \frac{t_k}{1-\alpha_R+\alpha_D}\right), \end{aligned}$$

and

$$\begin{aligned} I_{(>)} &\leq \sum_{j=1}^m |z_j| \int_{-\infty}^{\infty}\int_0^{\infty}\int_{1\vee u^{-1}}^{\infty} r L_{t_j}(s,u) EN^{\infty}_{\alpha_D,\alpha_R}(ds,du,dr) \\ &\quad + \sum_{j=1}^m\sum_{k=1}^m |z_j z_k| \int_{-\infty}^{\infty}\int_0^{\infty}\int_{u^{-1}}^{1\vee u^{-1}} r^2 L_{t_j}(s,u)L_{t_k}(s,u) EN^{\infty}_{\alpha_D,\alpha_R}(ds,du,dr) \\ &\leq \sum_{j=1}^m \frac{|z_j| t_j\alpha_D\alpha_R}{\alpha_R-1}\left(\frac{1}{\alpha_R-\alpha_D}+\frac{1}{\alpha_D}\right) + \sum_{j=1}^m\sum_{k=1}^m \frac{|z_j z_k t_j t_k|}{(\alpha_D-1)(2-\alpha_R)}, \end{aligned}$$

whence (5.11) holds.

Similarily, split the integral (5.15) corresponding to the process (5.12) into two parts $J_{(<)}$ and $J_{(>)}$, according to the two domains of integration $D_{(<)}$ and $D_{(>)}$, which yields

$$\begin{aligned} J_{(<)} &\leq \sum_{j=1}^m\sum_{k=1}^m |z_j z_k| \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{r^{-1}} r^2 L_{t_j}(s,u)L_{t_k}(s,u) EN^{\infty}_{\alpha_D,F_R}(ds,du,dr) \\ &\leq \sum_{j=1}^m\sum_{k=1}^m \frac{|z_j z_k| t_j\alpha_D}{2-\alpha_D} E[(R_1^{(1)})^{\alpha_D}], \end{aligned}$$

and

$$J_{(>)} \leq \sum_{j=1}^{m} |z_j| \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{r^{-1}}^{\infty} r L_{t_j}(s,u) E N_{\alpha_D, F_R}^{\infty}(ds, du, dr)$$
$$\leq \sum_{j=1}^{m} \frac{|z_j| t_j \alpha_D}{\alpha_D - 1} E[(R_1^{(1)})^{\alpha_D}].$$

This proves (5.12).

Finally, the exponent in the left side of (5.13) is

$$i \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \sum_{j=1}^{m} z_j u r 1_{[0,t_j]}(s) \mathring{N}_{\alpha_D, F_R}^{\infty}(ds, du, dr).$$

Analogous to the proof of (5.12), the finiteness of

$$\int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \left(\sum_{j=1}^{m} z_j u r 1_{[0,t_j]}(s) \right)^2 \bigwedge \left| \sum_{k=1}^{m} z_k u r 1_{[0,t_j]}(s) \right| E N_{\alpha_D, F_R}^{\infty}(ds, du, dr)$$

can be readily shown, whence the left side of (5.13) is equal to

$$\int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \left(\exp\left\{ i \sum_{j=1}^{m} z_j u r 1_{[0,t_j]}(s) \right\} - 1 - \sum_{j=1}^{k} z_j u r 1_{[0,t_j]}(s) \right) \times$$
$$E N_{\alpha_D, F_R}^{\infty}(ds, du, dr).$$

The result now follows after an integration by parts in the variable u. □

Finally, the following result is used to get upper bounds for some integrands throughout the proof of Theorem 3.1.

LEMMA 5.7. *For $0 \leq \zeta \leq 1$ and $x \in \mathbb{R}$:*

(5.16) $$|e^{ix} - 1| \leq 2^{1-\zeta} |x|^{\zeta},$$

(5.17) $$|e^{ix} - 1 - ix| \leq d_{\zeta} |x|^{\zeta+1},$$

where $d_\zeta > 0$, and for real numbers $x_1, \dots, x_m$:

(5.18) $$\left(\sum_{j=1}^{m} |x_j| \right)^{\zeta} \leq \sum_{j=1}^{m} |x_j|^{\zeta}.$$

PROOF. Without loss of generality, fix $x \neq 0$. Define $f : [0,1] \to \mathbb{R}$, $f(\zeta) = (1-\zeta)\ln 2 + \zeta \ln|x|$. We can readily check that

$$\ln|e^{ix} - 1| \leq f(\zeta), \quad \zeta = 0, 1,$$

by taking logarithms in both sides of $|e^{ix} - 1| \leq 2 \wedge |x|$. Since $f(\zeta)$ is linear in ζ, $f(\zeta)$ is either nondecreasing or nonincreasing on $[0,1]$. Hence, (5.16) holds. Using a similar strategy, we can prove (5.17).

For (5.18), assume without loss of generality that $0 < |x_1| \leq |x_2|$, thus $0 < |x_1/x_2| \leq 1$. By Bernoulli's inequality [see e.g. 26, p. 36]:

$$(1 + |x_1/x_2|)^\zeta \leq 1 + \zeta|x_1/x_2| \leq 1 + |x_1/x_2|^\zeta.$$

Multiplying both sides by $|x_2|^\zeta$ proves (5.18) for $m = 2$ and the proof for general m follows by induction. □

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Department of Statistical Science
Cornell University
Ithaca, NY 14853
E-mail: ll278@cornell.edu

School of Operations Research and
Information Engineering
Cornell University
Ithaca, NY 14853
E-mail: sir1@cornell.edu