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PORTFOLIO REBALANCING ERROR WITH JUMPS AND MEAN REVERSION IN ASSET PRICES

By Paul Glasserman* and Xingbo Xu

Columbia University

We analyze the error between a discretely rebalanced portfolio and its continuously rebalanced counterpart in the presence of jumps or mean-reversion in the underlying asset dynamics. With discrete rebalancing, the portfolio's composition is restored to a set of fixed target weights at discrete intervals; with continuous rebalancing, the target weights are maintained at all times. We examine the difference between the two portfolios as the number of discrete rebalancing dates increases. With either mean reversion or jumps, we derive the limiting variance of the relative error between the two portfolios. With mean reversion and no jumps, we show that the scaled limiting error is asymptotically normal and independent of the level of the continuously rebalanced portfolio. With jumps, the scaled relative error converges in distribution to the sum of a normal random variable and a compound Poisson random variable. For both the mean-reverting and jump-diffusion cases, we derive "volatility adjustments" to improve the approximation of the discretely rebalanced portfolio by the continuously rebalanced portfolio, based on on the limiting covariance between the relative rebalancing error and the level of the continuously rebalanced portfolio. These results are based on strong approximation results for jump-diffusion processes.

1. Introduction. The analysis of a portfolio's dynamics is often simplified by assuming that the constituent assets can be traded continuously. For a trading strategy defined by portfolio weights, meaning the fraction of the portfolio held in each asset, continuous trading leads to an idealized model in which the actual weights match the target weights at each instant. For highly liquid stocks bought and sold on electronic exchanges, continuous trading is often a close approximation of reality. But for many other asset classes the practical reality of discrete trading cannot be entirely ignored. A portfolio manager may not be able to maintain an ideal set of portfolio weights continuously in time; transactions costs and liquidity constraints may limit the portfolio manager to rebalancing the portfolio to target weights at discrete intervals.

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In this paper, we analyze the error in approximating a discretely rebalanced portfolio with one that is continuously rebalanced and thus more convenient to model. Our focus is on the effect of jumps and mean reversion in the dynamics of the underlying assets. For both features, we examine the limiting difference between the continuous and discrete portfolios as the rebalancing frequency increases. Our main results are as follows. With either mean reversion or jumps, we derive the limiting variance of the relative error between the two portfolios. With mean reversion and no jumps, we show that the limiting error, scaled by the square root of the number of rebalancing dates, is asymptotically normal and independent of the level of the continuously rebalanced portfolio; moreover, the limiting distribution is identical to the one achieved without mean reversion. In the presence of jumps, we show that the scaled relative error converges to the sum of a normal random variable and a compound Poisson random variable, based on an argument provided by a referee. For both the mean-reverting and jump-diffusion cases, we derive "volatility adjustments" to improve the approximation of the discretely rebalanced portfolio by the continuously rebalanced portfolio. These adjustments are based on the limiting covariance between the relative rebalancing error and the level of the continuously rebalanced portfolio.

The simpler case in which the underlying assets are modeled as a multivariate geometric Brownian motion is analyzed in Glasserman [12]. The analysis there is motivated by the *incremental risk charge* (IRC) introduced by the Basel Committee on Banking Supervision [2, 3]. The IRC is intended to capture the effect of potential illiquidity of assets in a bank's trading portfolio. It models illiquidity by imposing a fixed rebalancing frequency for each asset class: some bonds, for example, might have a liquidity interval of two weeks, and tranches of asset backed securities might have liquidity intervals of a month or even a quarter. The IRC is thus based on the difference between discrete and continuous rebalancing.

The possibility of jumps in asset prices is clearly relevant to portfolio risk and to the modeling of less liquid assets. One would also expect jumps to have a qualitatively different effect on rebalancing error than pure diffusion — adding jumps should cause the discretely rebalanced portfolio to stray farther from the target weights — and this is confirmed in our results. The potential impact of mean reversion is less evident: one might expect mean reversion to offset part of the effect of discrete rebalancing if it helps restore a portfolio's weights to their targets. We will see that this is the case, but only for the volatility adjustment that comes from the covariance between the rebalancing error and the portfolio level. The distribution of the relative rebalancing error itself is, in the limit, unaffected by the presence of mean reversion.

Discretely rebalanced portfolios arise in models of transaction costs and discrete hedging, including Bertsimas, Kogan, and Lo [4], Boyle and Emanuel [5], Duffie and Sun [11], Leland [21], and Morton and Pliska [22]. Sepp [24] examines the asymptotic error of delta hedging with proportional transaction costs under a jump-diffusion model with lognormal jump sizes. Guasoni, Huberman, and Wang [14] analyze the effect of discrete rebalancing on the measurement of tracking error and portfolio alpha. In their analysis of leveraged ETFs, Avellaneda and Zhang [1] examine the impact of discrete rebalancing and derive an asymptotic relation between the behavior of the fund and the underlying asset as the rebalancing frequency increases. Jessen [16] studies the discretization error for CPPI portfolio strategies using simulation. Although these applications do not fit precisely within the specifics of our setting, we nevertheless view our analysis as potentially relevant to extending work on these applications. In Glasserman and Xu [13], we use a continuously rebalanced portfolio to design an importance sampling procedure to estimate the tail of a discretely rebalanced portfolio in a pure-diffusion setting, and the results we develop here suggest potential extensions to models with jumps.

The distribution of the difference between a diffusion process and its discrete-time approximation has received extensive study motivated by simulation methods, as in Kurtz and Protter [20]. Jacod and Protter [15] study this error for more general processes, including processes with jumps. Tankov and Voltchkova [26] apply the results of Jacod and Protter [15] to analyze the error in discrete delta-hedging, thus extending the results of Bertsimas et al. [4] to models with jumps. In their analysis of discretization methods, Kloeden and Platen [19] develop strong approximation results for stochastic Taylor expansions; Bruti-Liberati and Platen [6, 7] derive corresponding expansions for jump-diffusion processes. These results provide very useful tools for our investigation of rebalancing error.

The rest of the paper is organized as follows. Section 2 introduces the mean-reverting and jump-diffusion models and states our main results on the limiting rebalancing error. Section 3 derives our volatility adjustments for discretely rebalanced portfolios. Numerical examples are given in Section 4. The rest of the paper is then devoted to proving our main results. In Section 5, we provide background on strong approximation and then apply these tools to our results for the jump-diffusion model. Section 6 covers the mean-reverting case. Proofs for the volatility adjustments are given in Section 7. Section 8 addresses complications that arise from the possibility of portfolio values becoming negative, which we interpret as a default.

2. Model dynamics and main results.

We begin by introducing two models of the dynamics of the d underlying assets in the portfolio, one with mean reversion and one with jumps. The first model is as follows:

Exponential Ornstein-Uhlenbeck (EOU) model:

$$\frac{d\underline{S}_i(t)}{\underline{S}_i} = \mu_i dt + dU_i(t), \quad i = 1, \dots, d,$$
$$dU_i(t) = \beta(\theta_i - U_i)dt + \sigma_i^\top dW(t), \quad U_i(0) = 0.$$

For each $i = 1, \dots, d$, the drift μ_i and volatility vector $\sigma_i = (\sigma_{i1}, \dots, \sigma_{id})$ are constants. The model is driven by $W = (W_1, \dots, W_d)^\top$, a d-dimensional standard Brownian motion, and each U_i is a Ornstein-Uhlenbeck process. We recover geometric Brownian motion as a special case by taking $\beta = 0$.

We also investigate portfolios under the following dynamics for asset prices:

Jump-Diffusion (JD) model:

$$\frac{dS_i(t)}{S_i(t-)} = \mu_i dt + \sum_{j=1}^{d} \sigma_{ij} dW_j(t) + d\left(\sum_{j=1}^{N(t)} (Y_j^i - 1)\right), \quad i = 1, \dots, d.$$

Here, N is a Poisson process with intensity $0 < \lambda < \infty$, and $Y_j^i > 0$ is the jump size associated with the i^{th} asset at the j^{th} jump of N. The $\{Y_j^i\}_i$ are i.i.d. across different values of j. All of W, N and $\{Y_j^i\}$ are mutually independent. Each S_i is right-continuous, so the left limit $S_i(t-)$ is the value of S_i just prior to a possible jump at t.

The two models could be combined to introduce both mean reversion and jumps in the asset dynamics. However, our interest lies in analyzing the impact of each of these features, so we keep them separate. To avoid confusion between the two models, we underline variables that are specific to the EOU case.

Given a model of asset dynamics, we consider portfolios defined by a fixed vector of weights $w = (w_1, \dots, w_d)^\top$, such that $\sum_{i=1}^d w_i = 1$. Interpret w_i as the fraction of value invested in the i^{th} asset. The weights could be the result of a portfolio optimization, but we do not model the portfolio selection problem. In considering only fixed weights, we exclude portfolios in which the weights themselves change with asset prices, and this is a restriction on the scope of our results. Kallsen [17] showed that under an exponential

Levy model such as our JD model, constant weights are in fact optimal for investors with power and logarithmic utilities. There is a sizeable literature that argues the merits of rebalancing to fixed weights. Kim and Omberg [18] studied portfolio optimization with mean reversion, but their framework does not fit our setting. See, e.g., Chapters 4–6 of Dempster, Mitra, and Pflug [10] and the many references cited there.

With continuous rebalancing to target weights $w_1, \dots, w_d$, the value of the portfolio in the EOU model evolves as

$$\frac{d\underline{V}(t)}{\underline{V}(t)} = \sum_{i=1}^{d} w_i \mu_i dt + \sum_{i=1}^{d} w_i \, dU_i(t),$$

and thus

(1)
$$\underline{V}(t) = \underline{V}(0) \exp\left\{ \left(\mu_w - \frac{1}{2}\sigma_w^2\right) t + \sum_{i=1}^{d} w_i \sigma_i^\top \int_0^t e^{-\beta(t-s)} dW_s + (1 - e^{-\beta t})\bar{\theta} \right\},$$

where $\bar{\theta} = \sum_i w_i \theta_i$, $\mu_w = \sum_i w_i \mu_i$, $\Sigma = (\Sigma_{ij})$ with $\Sigma_{ij} = \sum_{k=1}^{d} \sigma_{ik}\sigma_{jk}$ and $\sigma_w = \sqrt{w^\top \Sigma w}$.

In the jump-diffusion model, portfolio value evolves as

$$\begin{aligned} \frac{dV(t)}{V(t-)} = \sum_{i=1}^{d} w_i \frac{dS_i(t)}{S_i(t-)} &= \mu_w dt + \sum_{i=1}^{d} w_i \sigma_i^\top dW(t) + \sum_{i=1}^{d} w_i d\left(\sum_{j=1}^{N(t)} Y_j^i - 1 \right) \\ &= \mu_w dt + \sigma_w d\tilde{W}(t) + d\left(\sum_{j=1}^{N(t)} \sum_{i=1}^{d} w_i (Y_j^i - 1) \right), \end{aligned}$$

where $\tilde{W}$ is a scalar Brownian motion, $\tilde{W}(t) = \sum_{i,j} w_i \sigma_{ij} W_j(t)/\sigma_w$. This expression assumes that V remains strictly positive, a requirement we will return to shortly. The solution to this equation is then given by

$$V(t) = \exp\left\{ \left(\mu_w - \frac{1}{2}\sigma_w^2\right) t + \sigma_w \tilde{W}(t) \right\} \prod_{j=1}^{N(t)} \left[\sum_{i=1}^{d} w_i Y_j^i \right]. \tag{2}$$

We fix a horizon T over which we analyze the evolution of the portfolio. For the discretely rebalanced case, we fix a rebalancing interval $\Delta t = T/N$, corresponding to a fixed number N of rebalancing dates in $(0, T]$. Denote the value of the discretely rebalanced portfolio by $\hat{V}$ (or $\underline{\hat{V}}$ in the EOU case). With discrete rebalancing, the portfolio composition is restored to the target weights at each rebalancing opportunity. Thus, the portfolio value evolves as

$$\hat{V}((n+1)\Delta t) = \hat{V}(n\Delta t)\sum_{i=1}^{d} w_i \frac{S_i((n+1)\Delta t)}{S_i(n\Delta t-)}, \quad n = 1, \ldots, N-1,$$

and similarly for $\underline{\hat{V}}$. We normalize the initial portfolio value to $V(0) = \hat{V}(0) = \underline{\hat{V}}(0) = 1$.

To ensure that the continuously rebalanced portfolio preserves strictly positive value (i.e., to rule out bankruptcy), we impose the requirement that, almost surely,

$$\sum_{i=1}^{d} w_i Y^i > 0, \tag{3}$$

where $Y^1, \ldots, Y^d$ have the distribution of the jump sizes associated with the d assets. That this condition is sufficient can be seen from (2), and differentiating (2) reproduces the stochastic differential equation that precedes it. This condition still allows jumps to decrease portfolio value to levels arbitrarily close to zero. It holds automatically if all portfolio weights are positive. The condition is crucial for our analysis because we work with the relative error between the discrete and continuous portfolios, and the denominator in the relative error is the value of the continuous-time portfolio. We also make the following technical assumption on the jump sizes:

$$\left\| \frac{Y^k}{\sum_i w_i Y^i} \right\|_3 < \infty \text{ and } \|Y^k\| < \infty \text{ for } k = 1, \ldots, d; \tag{4}$$

and later,

$$\left\| \log\left(\frac{Y^k}{\sum_i w_i Y^i} \right) \right\| < \infty. \tag{5}$$

Here, $\|.\|_3$ indicates the L_3-norm of a random variable, and $\|.\|$ indicates the L_2-norm. Assumptions (3)–(4) will be in force whenever we consider the jump-diffusion model; we use (5) in Section 3.

Even under these assumptions, we cannot rule out the possibility that the discretely rebalanced portfolio value drops to zero and lower. We therefore adopt the convention that the portfolio value is absorbed at zero if it would otherwise become less than or equal to zero; we refer to this event as bankruptcy. We will show (in Section 8) that we can ignore the possibility of bankruptcy for our limiting results because the effect becomes negligible asymptotically. Thus, in most of our discussion, we treat the discretely rebalanced portfolio as a positive process.

We now proceed to state our main results for the EOU model. Our first result approximates the relative error between the discrete and continuous portfolios with a sum of independent random variables and identifies the limiting variance of the relative error.

THEOREM 2.1. *For the EOU model, there exist random variables* $\{\epsilon_{n,N}, n = 1, \ldots, N, N = 1, 2, \ldots\}$, *with* $\{\epsilon_{1,N}, \ldots, \epsilon_{N,N}\}$ *i.i.d. for each* N, *such that*

$$E\left[\left(\frac{\hat{\underline{V}}(T) - \underline{V}(T)}{\underline{V}(T)} - \sum_{n=1}^{N} \epsilon_{n,N}\right)^2\right] = O(\Delta t^2); \tag{6}$$

in particular, with $\bar{\sigma} = \sum_{i=1}^{d} w_i \sigma_i$

$$\epsilon_{n,N} = \sum_{i=1}^{d} w_i \int_{(n-1)\Delta t}^{n\Delta t} \int_{(n-1)\Delta t}^{s} (\sigma_i - \bar{\sigma})^\top dW(r)(\sigma_i - \bar{\sigma})^\top dW(s),$$

and

$$Var[\epsilon_{n,N}] = \sigma_L^2 \Delta t^2 := \left[\frac{1}{2}(w^\top(\Sigma \circ \Sigma)w - 2w^\top \Sigma\Omega\Sigma w + (w^\top \Sigma w)^2)\right]\Delta t^2, \tag{7}$$

where "$\circ$" denotes elementwise multiplication of matrices, Ω *is a diagonal matrix with* $\Omega_{ii} = w_i$.

Thus,

$$N Var\left[\frac{\hat{\underline{V}}(T) - \underline{V}(T)}{\underline{V}(T)}\right] \to \sigma_L^2 T^2.$$

The variance parameter in this result can be understood as

$$\sigma_L^2 = Var\left[\frac{1}{2}\left(\sum_{i=1}^{d} w_i(\sigma_i^\top Z)^2 - \left(\sum_{i=1}^{d} w_i \sigma_i^\top Z\right)^2\right)\right],$$

where $Z \sim N(0, I)$ in $\mathbb{R}^d$. We now supplement this characterization of the limiting variance with the limiting distribution of the error:

THEOREM 2.2. *As* $N \to \infty$,

$$\sqrt{N}\left(\hat{\underline{V}}(T) - \underline{V}(T), \frac{\hat{\underline{V}}(T) - \underline{V}(T)}{\underline{V}(T)}\right) \Rightarrow (\underline{V}(T)\underline{X}, \underline{X}),$$

where $\underline{X} \sim N(0, \sigma_L^2 T^2)$ *is independent of* $\underline{V}(T)$*, and* $\Rightarrow$ *denotes convergence in distribution.*

The limits in Theorems 2.1 and 2.2 coincide with those proved in Glasserman [12] for asset prices modeled by geometric Brownian motion. Thus, we may paraphrase these results as stating that the presence of mean-reversion does not change the relative rebalancing error, as measured by its limiting distribution. The absolute error $\hat{\underline{V}}(T) - \underline{V}(T)$ does change. In both cases, its limiting distribution is that of the independent product of the continuous portfolio ($V(T)$ or $\underline{V}(T)$) and X, but the distribution of the continuous portfolio is itself changed by the presence of mean-reversion.

A key feature of Theorem 2.2 is the asymptotic independence between the portfolio value and the relative error. We will see, however, that with appropriate scaling there is a non-trivial covariance between these terms, and the strength of the limiting covariance depends on the speed of mean-reversion. We take up this issue when we consider volatility adjustments in the next section.

We proceed to the limiting variance of the relative error in the jump-diffusion model. For each asset $i = 1, \ldots, d$, introduce the compound Poisson process

$$J_t^i = \sum_{j=1}^{N(t)} \left(\frac{Y_j^i}{\sum_k w_k Y_j^k} - 1 \right).$$

To simplify notation, we define

$$\bar{Y}_j^i = \frac{Y_j^i}{\sum_k w_k Y_j^k} - 1,$$

and then the compensated version of J_t^i becomes $\tilde{J}_t^i = J_t^i - \lambda \mu_i^y t$, where $\mu_i^y = E[\bar{Y}^i]$. Let $\Delta \tilde{J}_n^i = \tilde{J}^i(n\Delta t) - \tilde{J}^i((n-1)\Delta t)$ and $\Delta W_n = W(n\Delta t) - W((n-1)\Delta t)$. Denote $X_N := (\hat{V}(T) - V(T))/V(T)$.

THEOREM 2.3. *For the JD model, under assumptions (3) and (4),*

$$E\left[\left(\frac{\hat{V}(T) - V(T)}{V(T)} - \sum_{n=1}^{N} \tilde{\epsilon}_{n,N} \right)^2 \right] = O(\Delta t^2), \tag{8}$$

where

$$\tilde{\epsilon}_{n,N} = \epsilon_{n,N} + \sum_{i=1}^{d} w_i \left[b_i^\top \Delta W_n \Delta \tilde{J}_n^i + \int_{(n-1)\Delta t}^{n\Delta t} \int_{(n-1)\Delta t}^{s-} d\tilde{J}^i(r) d\tilde{J}^i(s) \right], \tag{9}$$

and $b_i = \sigma_i - \bar{\sigma}$, $i = 1, \ldots, d$. *And*

$$\begin{aligned} Var[\tilde{\epsilon}_{n,N}] &= \tilde{\sigma}_L^2 \Delta t^2 \\ &= Var[\epsilon_{n,N}] + \Delta t^2 (w^\top (b^\top b \circ M) w) + \frac{\Delta t^2}{2} w^\top M \circ M w, \end{aligned}$$

where $Var[\epsilon_{n,N}]$ *is as in (7),* $b = [b_1, b_2, \ldots, b_d]$, *and* M *is the* $d \times d$ *matrix with entries*

$$m_{ij} := \lambda E[\bar{Y}^i \bar{Y}^j]. \tag{10}$$

Thus

$$Var(X_N) \to \tilde{\sigma}_L^2 T^2.$$

In (9), the $\epsilon_{n,N}$ are the error terms that arise in the case of geometric Brownian motion (i.e., with $\lambda = 0$ in the JD model and, equivalently, with $\beta = 0$ in the EOU model). As in the EOU model, the relative error has a limit distribution. In the original version of this paper, we showed that the limit could not be normal. The following result uses an argument due to a referee.

THEOREM 2.4. *Under assumptions (3) and (4), if the jump part is not degenerate, i.e.* $\lambda \neq 0$ *and* $P(Y^i = 1, i = 1, \ldots, d) \neq 1$, *then*

$$\sqrt{N} \frac{\hat{\underline{V}}(T) - \underline{V}(T)}{\underline{V}(T)} \Rightarrow X,$$

where $X \stackrel{d}{=} \underline{X} + \sqrt{T} \sum_{j=1}^{N(t)} \sum_{i=1}^{d} w_i b_i^\top \xi_j \bar{Y}_j^i$ *and* $\xi_j \sim N(0, I)$ *are i.i.d. d-dimensional standard normal vectors for* $j \geq 1$, *independent of everything else. The limit does not hold in the* L_2 *sense.*

The jump-diffusion model produces a heavier-tailed distribution for the relative error, resulting in the failure to converge to a limiting normal distribution. One can get some intuition from the asymptotics of $\tilde{\epsilon}_{n,N}$ in (9), where the third term is nonzero only when there are at least two jumps in the period. Though the third term in (9) converges to zero in probability, it does contribute to the limiting variance as well as the third absolute moment, both of which are of order $\Theta(\Delta t^2)$.

Because of the presence of $\bar{Y}$ in the limit distribution, we do not have an asymptotic independence result for the JD case, but $\log V(T)$ and X_N are asymptotically uncorrelated, as shown later in Proposition 3.2.

3. Volatility adjustments. We now apply and extend the limiting results of the previous section to develop volatility adjustments that approximate the effect of discrete rebalancing. To motivate this idea, consider the continuous-time dynamics of the portfolio value in (2), and consider first the case without mean reversion, $\beta = 0$. In this setting, $\underline{V}$ is a geometric Brownian motion with volatility σ_w, with $\sigma_w^2 = w^\top \Sigma w$, as defined following (2). The parameter σ_w is a useful measure of portfolio risk under continuous rebalancing. The corresponding parameter for horizon T in the EOU model is (the square root of)

$$\sigma_{w,\beta}^2 := \frac{1}{T} Var[\log \underline{V}(T)] = \sigma_w^2 \frac{1-\exp(-2\beta T)}{2\beta T}, \tag{11}$$

and, in the jump-diffusion model, under assumption (5)

$$\sigma_{w,J}^2 := \frac{1}{T} Var[\log V(T)] = \sigma_w^2 + \lambda E\left[\left(\log \sum_{i=1}^d w_i Y^i\right)^2\right]. \tag{12}$$

In practice, $\sigma_{w,\beta}$ and $\sigma_{w,J}$ serve reasonably well for large N as an approximation for discretely rebalanced portfolio. Our objective is to correct these parameters to capture the impact of discrete rebalancing.

3.1. *Volatility adjustment with mean reversion.* From the definition of X_N, we can write value of the discretely rebalanced portfolio as

$$\hat{\underline{V}}(T) = \underline{V}(T)(1 + X_N/\sqrt{N}),$$

which shows that $\hat{\underline{V}}(T)$ is the product of the continuously rebalanced portfolio value and a correction factor that is asymptotically normal and independent of $\underline{V}(T)$. We would like to calculate the "volatility" of $\hat{\underline{V}}(T)$ — the standard deviation of its logarithm, normalized by $\sqrt{T}$ — but because $\hat{\underline{V}}(T)$ is potentially negative, we cannot do this directly. Instead, we note that

$$\bar{\underline{V}}(T) := \underline{V}(T)\exp(X_N/\sqrt{N}) = \hat{\underline{V}}(T) + O_p(1/N),$$

which yields a strictly positive approximation. The $O_p(1/N)$ error in this approximation is negligible compared to the $O_p(1/\sqrt{N})$ difference between the discrete and continuous portfolios, and we will confirm that making this approximation does not change the limiting variance.

For $\bar{\underline{V}}(T)$ we have

$$\begin{aligned}\frac{Var[\log \bar{\underline{V}}(T)]}{T} &= \frac{1}{T} Var\left[\log \underline{V}(T) + \frac{X_N}{\sqrt{N}}\right] \\ &= \sigma_w^2 \frac{1-e^{-2\beta T}}{2\beta T} + \frac{Var[X_N]}{TN} + \frac{2Cov[\log \underline{V}(T), X_N]}{T\sqrt{N}} \\ &= \sigma_{w,\beta}^2 + \sigma_L^2 T\Delta t + o(\Delta t) + \frac{2Cov[\log \underline{V}(T), X_N]}{T\sqrt{N}}, \end{aligned} \tag{13}$$

with $\sigma_{w,\beta}$ as in (11) and σ_L^2 the variance parameter in (7). Although X_N is asymptotically independent of $\underline{V}(T)$, the covariance term does not vanish fast enough to be negligible. In the following proposition, we find the limit of the third term, and verify the validity of replacing $\hat{\underline{V}}$ with $\bar{\underline{V}}$:

PROPOSITION 3.1. *(i) The limiting covariance is given by*

$$\sqrt{N} Cov[\log \underline{V}(T), X_N] \to \underline{\gamma}_L T^2,$$

where

$$\underline{\gamma}_L = e^{-\beta}(\gamma_L + \sum_i w_i(\bar{\sigma}^\top \sigma_i)\beta(\theta_i - \bar{\theta})),$$

with

$$\gamma_L = \mu^\top \Omega \Sigma w - \mu_w \sigma_w^2 + \sigma_w^4 - w^\top \Sigma \Omega \Sigma w. \tag{14}$$

(ii) Moreover, $E[(\bar{\underline{V}}(T) - \hat{\underline{V}}(T))^2] = O(N^{-2})$*, and*

$$N(Var[\log \bar{\underline{V}}(T)] - Var[\log \underline{V}(T)]) \to (\sigma_L^2 + 2\underline{\gamma}_L)T^2.$$

This result applied to (13) suggests the following adjustment to the volatility for the discretely rebalanced portfolio:

$$\sigma_{adj}^2 = \sigma_{w,\beta}^2 + (\sigma_L^2 + 2\underline{\gamma}_L)\Delta t. \tag{15}$$

At $\Delta t = 0$, we recover the volatility for the continuously rebalanced portfolio, but for small $\Delta t > 0$, the adjusted volatility includes a correction for discrete rebalancing. The parameter γ_L in (14) is the limiting covariance derived in Glasserman [12] for assets modeled by multivariate geometric Brownian motion; thus, at $\beta = 0$ we recover the volatility adjustment derived there in the absence of mean reversion, as expected. The second part of the proposition confirms that the difference between $\bar{\underline{V}}(T)$ and $\hat{\underline{V}}(T)$ is negligible. In Section 4.2, we present numerical results illustrating the performance of the volatility adjustment (15) in approximating the effect of discrete rebalancing.

3.2. *Volatility adjustment in the jump-diffusion model.* We follow similar steps in the jump-diffusion model. We set $\bar{V}(T) := V(T)\exp(X_N/\sqrt{N})$ with

$$X_N = \sqrt{N}\sum_{n=0}^{N-1}\left(\frac{\hat{V}((n+1)\Delta t)}{V((n+1)\Delta t)} - \frac{\hat{V}(b\Delta t)}{V(n\Delta t)}\right),$$

and then

$$\frac{Var[\log\bar{V}(T)]}{T} = \sigma_{w,J}^2 + \frac{Var[X_N]}{TN} + \frac{2Cov[\log V(T), X_N]}{T\sqrt{N}}, \tag{16}$$

with $\sigma_{w,J}$ as defined in (12).

Proposition 3.2. *(i) The limiting covariance is given by*

$$\sqrt{N}Cov[\log V(T), X_N] \to \tilde{\gamma}_L T^2,$$

where

$$\tilde{\gamma}_L := \gamma_L + \lambda\left[\sum_i w_i\bar{\sigma}^\top\sigma_i\mu_i^y\right] + \lambda\sum_i w_i(\mu_i - \sigma_i^\top\bar{\sigma} + \lambda\mu_i^y)E\left[\bar{Y}^i\left(\log\sum_l w_l Y^l - \mu_J\right)\right]$$

and

$$\mu_J = E\left[\log\sum_i w_i Y_j^i\right].$$

(ii) Moreover, $E[(\hat{V}(T) - \bar{V}(T))^2] = O(N^{-2})$ *and*

$$N(Var[\log\bar{V}(T)] - Var[\log V(T)]) \to (\tilde{\sigma}_L + \tilde{\gamma}_L)T^2.$$

The resulting volatility adjustment is

$$\tilde{\sigma}_{adj}^2 = \sigma_{w,J}^2 + (\tilde{\sigma}_L^2 + 2\tilde{\gamma}_L)\Delta t. \tag{17}$$

The asymptotic variance parameters for the relative error (σ_L^2 and $\tilde{\sigma}_L^2$) do not depend on the drift parameters μ_i, but, interestingly, the drifts do appear in the asymptotic covariance γ_L (and $\underline{\gamma}_L$ and $\tilde{\gamma}_L$). We will see that in a stochastic Taylor expansion of the relative error, the μ_i appear only in those terms with norms of order $O(\Delta t^{3/2})$. For the variance, it turns out that only terms with norms up to order $O(\Delta t)$ are relevant, but the covariance involves terms of norm $O(\Delta t^{3/2})$, and these involve the μ_i.

Since the volatility adjustments are explicitly related to the weights, one could reverse the approximation as a guideline for adjusting portfolio weights to control the portfolio volatility σ with discrete rebalancing.

Table 1
Parameters estimated from S&P 500, FTSE 100, Nikkei 225, DAX, Swiss Market Index, CAC 40, FTSE Straits Times Index for Singapore, Hang Seng, Mexico IPC, Thai Set 50 and Argentina Merval

SP500	FTSE	NIK	DAX	SSMI	CAC	STI	HSI	MXX	SET50	MERV
λ										
3.0142										
w										
−1.22	−0.22	0.22	0.87	−3.30	0.82	0.44	−0.47	1.32	1.17	1.38
μ										
0.15	0.13	0.12	0.25	0.09	0.12	0.17	0.21	0.25	0.35	0.40
$\mu_J \times 10^{-2}$										
−0.74	0.24	−1.71	−0.10	0.22	1.28	0.00	0.18	−0.85	−0.01	0.46
$\sigma_J \times 10^{-2}$										
2.91	2.65	1.47	2.92	2.24	4.68	2.46	2.87	2.58	3.56	4.69
$\Sigma \times 10^{-2}$										
3.14	2.00	0.27	2.35	1.52	2.56	0.50	0.44	2.14	0.41	3.17
2.00	2.84	0.75	2.94	1.93	3.26	0.92	1.00	1.72	1.05	2.48
0.27	0.75	4.53	0.60	0.77	1.03	1.76	2.74	0.33	1.81	0.30
2.35	2.94	0.60	3.83	2.29	3.87	0.92	0.98	2.00	1.12	2.93
1.52	1.93	0.77	2.29	2.02	2.54	0.65	0.68	1.14	0.72	1.78
2.56	3.26	1.03	3.87	2.54	4.37	1.05	1.14	2.08	1.16	3.02
0.50	0.92	1.76	0.92	0.65	1.05	2.54	2.47	0.68	1.86	0.78
0.44	1.00	2.74	0.98	0.68	1.14	2.47	4.65	0.88	2.58	0.68
2.14	1.72	0.33	2.00	1.14	2.08	0.68	0.88	2.74	0.73	2.65
0.41	1.05	1.81	1.12	0.72	1.16	1.86	2.58	0.73	4.88	0.91
3.17	2.48	0.30	2.93	1.78	3.02	0.78	0.68	2.65	0.91	6.23

4. Numerical experiments and further discussion of the limits.

4.1. *Example for the jump-diffusion model.* We begin with the JD model model and examine the approximation for the relative error provided by Theorem 2.4.

We calibrated the JD model from the daily returns of global equity indices based on the method introduced in Das [9]. The weights are computed as the optimal weights for power utility with risk aversion parameter $\gamma = 2$ following the results of [9][1]. The data used is from March 2009 to March 2011, and the calibrated results are as in Table 1. Jump sizes are modeled by Merton's jump model with $\log(Y^i) \sim N(\mu^i_J, \sigma^i_J)$. We calibrate the parameters by assuming the jump sizes are perfectly correlated as in [9]. However, perfectly correlated jumps would have the same effect as constant jump sizes because

[1]The negative weights could cause defaults, even in the continuous portfolio, though this occurs very rarely with out estimated value of σ_J. In our numerical examples, we exclude paths with defaults. We address this issue in Section 8.

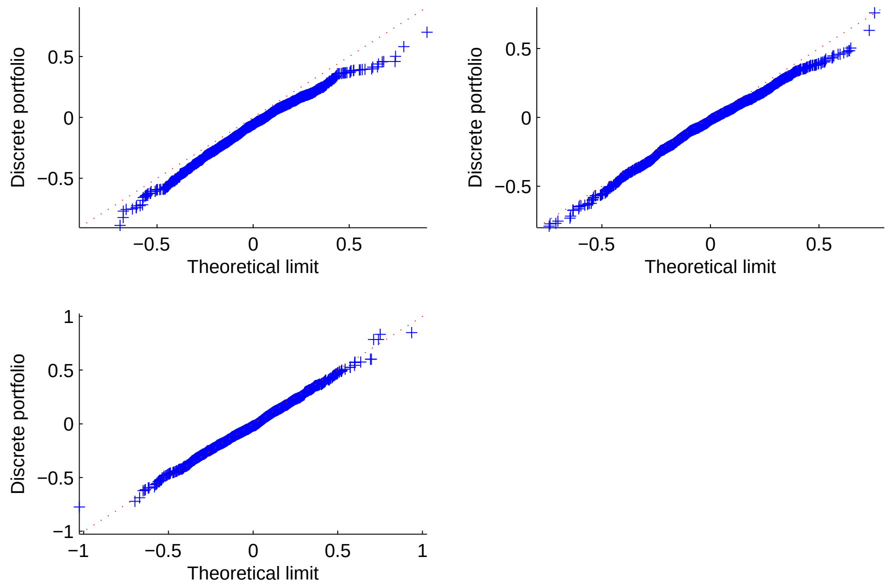

FIG 1. *Jump-diffusion model: QQ plots of X_N versus X at $N = 4$ (upper left), $N = 12$ (upper right), $N = 360$ (lower left).*

we are considering relative error. To make the example more interesting, we simulate independent jumps sizes instead.

Figure 1 shows QQ plots of the value of discrete portfolios versus the limit as described in Theorem 2.4, both simulated over 2500 replications. We choose N to be 4, 12 and 360 to represent quarterly, monthly and daily rebalancings. As the number of steps N gets larger, the figure indicates convergence to the theoretical limit, though relatively slower than in the EOU model.

Since the limiting distribution is not normal, we do not have an asymptotic independence result of the type in Theorem 2.2. But the numerical results in Table 2 still show the correlation between $\log V(T)$ and X_N decreasing toward zero as N increases. This is to be expected because part (i) of Proposition 3.2 shows the covariance of $\log V(T)$ and X_N converging to zero at rate $O(1/\sqrt{N})$, and X_N has a non-degenerate limiting variance. In separate experiments, we have found large discrepancies in the QQ plots when σ_J^i are doubled. Estimation of m_{ij} in (10) becomes unstable, and condition (4) may be violated. Table 3 shows the error reduction of volatility as

$$1 - \left| \frac{\tilde{\sigma}_{adj} - \hat{\sigma}_N}{\sigma_{w,J} - \hat{\sigma}_N} \right|, \tag{18}$$

TABLE 2
Correlations for JD model and EOU model, between $\log V(T)$ *(or* $\log \underline{V}(T)$*) and* X_N*, with 2500 replicates*

N	4	12	360
JD	-12%	-13%	-4%
EOU	-85%	-61%	-13%

TABLE 3
Volatility error reductions for JD model and EOU model, with 50,000 replications. Formula (18) and (19) are used for JD model and EOU model, respectively

N	4	12	360
JD	87%	46%	2%
EOU	69%	55%	18%

where $\tilde{\sigma}_{adj}$ is defined in (17). This measure shows the relative improvement achieved in approximating the volatility using the adjustment; a small value indicates small improvement, and a value close to 1 indicates good improvement. These estimates are based on 50,000 replications. When the correlation between $V(T)$ and X_N is small, the error reduction tends to be unstable. As suggested by (16), when N is small and the covariance term in (16) is negative, the error reduction can be small, or even negative. In this situation, numerical errors, especially from computing the required expectation of the $\bar{Y}^i$, can contaminate the results.

4.2. *Example for the EOU model.* For the purpose of illustration, we use the same parameters w, μ and Σ from Section 4.1. We use the mean-reversion rate $\beta = 1$ and long-run levels $\theta_i = 0.1 \times i/d$, $i = 1, \dots, d$. Figure 2 illustrates the convergence to normality as N increases, using 2500 replicates.

Table 2 reports estimated correlations between $\log \underline{V}(T)$ and X_N using the same parameters as Figure 2. As expected, the correlation decreases toward zero as N increases.

Table 3 evaluates the volatility adjustment by reporting the estimated error reduction using the adjustment, calculated as

$$1 - \left| \frac{\sigma_{adj} - \hat{\sigma}_N}{\sigma_{w,\beta} - \hat{\sigma}_N} \right|, \tag{19}$$

where σ_{adj} is defined in (15) and $\hat{\sigma}_N$ is the volatility of the discretely rebalanced portfolio as estimated by simulation. The results in Table 3 show appreciable error reduction, especially when the number of rebalancing dates N is small. When N becomes large, the denominator $\sigma_{w,\beta} - \hat{\sigma}$ will become very small. The magnitude of the reduction is not necessarily monotone in N. More examples for the diffusion case without mean reversion can be found in Glasserman [12].

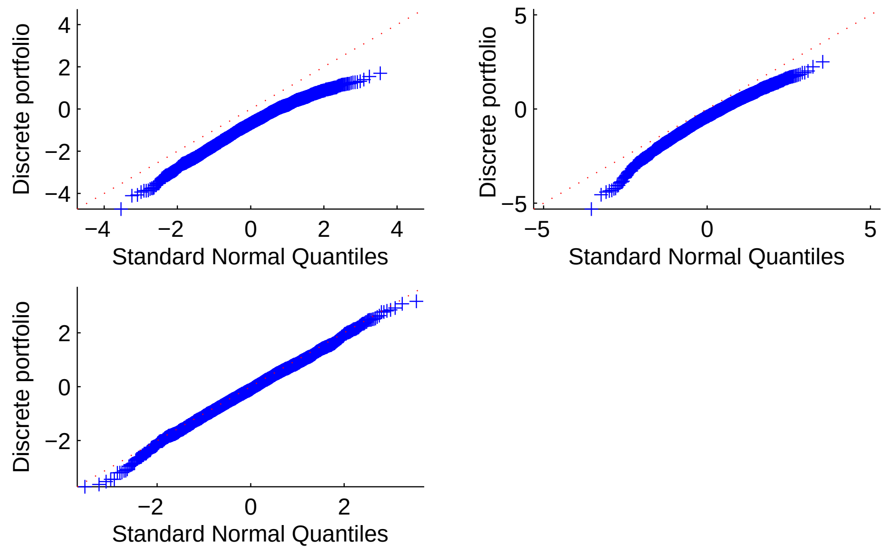

FIG 2. *EOU model: QQ plots of $X_N/\underline{\sigma}_L T$ versus standard normal at $N = 4$ (upper left), $N = 12$ (upper right) and $N = 360$ (lower left).*

5. Asymptotic error via strong approximation. In this section, we develop tools for the strong approximation of jump-diffusion models which we will need to prove our results for that case.

If X solves $dX_t = \tilde{a}(X_t)dt + \tilde{b}(X_t)dW_t + \tilde{c}(X_t)dJ_t$, and $\|X_N - X\|_2 = O(\Delta t^k)$, then we call X_N a *strong approximation* of order k. In the absence of jumps, Kloeden and Platen [19] show the same order then applies to almost sure convergence. Bruti-Liberati and Platen [6] and [7] treat strong approximation for the jump-diffusion case. In following their approach it is convenient to think of dt as having order 1, and dW and dJ as each having order 1/2, in terms of their L_2-norm. Approximations of order k then involve keeping all terms of order k or lower.

We use the following representations of the continuous and discrete portfolios. We set

$$V(1) = \prod_{n=1}^{N} \frac{V(n\Delta t)}{V((n+1)\Delta t)} = \prod_{n=1}^{N} R_{n,N},$$

and

$$\hat{V}(1) = \prod_{n=1}^{N} \frac{\hat{V}(n\Delta t)}{\hat{V}((n+1)\Delta t)} = \prod_{n=1}^{N} \hat{R}_{n,N},$$

where

$$\begin{aligned}\hat{R}_{n,N} &:= \frac{\hat{V}(n\Delta t)}{\hat{V}((n-1)\Delta t)}\\ &= \sum_{i=1}^{d} w_i \exp\left\{\left(\mu_i - \frac{1}{2}\sum_{j=1}^{d}\sigma_{ij}^2\right)\Delta t + \sigma_i^\top \Delta W_n\right\} \prod_{j=N((n-1)\Delta t)+1}^{N(n\Delta t)} Y_j^i\end{aligned}$$

and

$$\begin{aligned}R_{n,N} &:= \frac{V(n\Delta t)}{V((n-1)\Delta t)}\\ &= \exp\left\{\left(\mu_w - \frac{1}{2}\sigma_w^2\right)\Delta t + \bar{\sigma}^\top \Delta W_n\right\} \prod_{j=N((n-1)\Delta t)+1}^{N(n\Delta t)} \left(\sum_{i=1}^{d} w_i Y_j^i\right).\end{aligned}$$

Then

$$\begin{aligned}\frac{\hat{R}_{n,N}}{R_{n,N}} &= \sum_{i=1}^{d} w_i \exp\left\{\left(\mu_i - \mu_w - \frac{1}{2}\|\sigma_i\|^2 + \frac{1}{2}\sigma_w^2\right)\Delta t + (\sigma_i - \bar{\sigma})^\top \Delta W_n\right\}\\ &\times \prod_{j=N((n-1)\Delta t)+1}^{N(n\Delta t)} \frac{Y_j^i}{\sum_{i=1}^d w_i Y_j^i}\\ &= \frac{\hat{R}_{n,N}^c}{R_{n,N}^c} \frac{\sum_{i=1}^d w_i \exp\{(\mu_i - \mu_w - \frac{1}{2}\|\sigma_i\|^2 + \frac{1}{2}\sigma_w^2)\Delta t + (\sigma_i - \bar{\sigma})^\top \Delta W_n\} \prod_j Y_j^i}{\sum_i w_i \exp\{(\mu_i - \mu_w - \frac{1}{2}\|\sigma_i\|^2 + \frac{1}{2}\sigma_w^2)\Delta t + (\sigma_i - \bar{\sigma})^\top \Delta W_n\} \prod_j \sum_i w_i Y_j^i},\end{aligned}$$

where $\hat{R}_{n,N}^c / R_{n,N}^c$ is the ratio of returns in the absence of jumps, as in Glasserman [12],

$$\frac{\hat{R}_{n,N}^c}{R_{n,N}^c} = \sum_{i=1}^{d} w_i \exp\left\{\left(\mu_i - \mu_w - \frac{1}{2}\|\sigma_i\|^2 + \frac{1}{2}\sigma_w^2\right)\Delta t + (\sigma_i - \bar{\sigma})^\top \Delta W_n\right\}.$$

5.1. *Background on strong approximations.* As in Kloeden and Platen [19] and Platen [23], we use the following notation. For a string $\alpha = (i_1, \dots, i_{k-1}, i_k)$ of indices, let $\alpha- := (i_1, \dots, i_{k-1})$ and $-\alpha := (i_2, \dots, i_k)$, for $k > 0$. The length of the string is given by $l(\alpha) = k$, and $n(\alpha)$ denotes the number of zeros in the string α. Define the hierarchical sets $\mathcal{A}_l = \{\alpha | l(\alpha) + n(\alpha) \leq 2l\}$, and the corresponding remainder sets $\mathcal{B}(\mathcal{A}_l) = \{\alpha \notin \mathcal{A}_l, -\alpha \in \mathcal{A}_l\}$, for $l = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$.

For a predictable g satisfying certain regularity and integrability conditions in the main theorem of Platen [23], an iterated integral I_α is defined as follows:

$$I_\alpha[g]_t = \begin{cases} g(t) & \text{if } l(\alpha) = 0; \\ \int_0^t I_{\alpha-}[g]_z dz & \text{if } i_{l(\alpha)} = 0 \text{ and } l(\alpha) > 0; \\ \int_0^t I_{\alpha-}[g]_z dW_z^i & \text{if } i_{l(\alpha)} = i > 0 \text{ and } l(\alpha) > 0; \\ \int_0^t I_{\alpha-}[g]_z d\tilde{J}_z^i & \text{if } i_{l(\alpha)} = i < 0 \text{ and } l(\alpha) > 0. \end{cases}$$

To have a better understanding of the notation, one can interpret the string $\alpha = (i_1, \ldots, i_k)$ as the order for iterated integration, with the direction from left to right corresponding to the order of integration from innermost to outermost integral. Each entry i_k indicates the process against which the integral is taken. For example, $i_k > 0$ indicates an integral against the i_k^{th} component of the Brownian Motion, while $i_k < 0$ indicates an integral against $\tilde{J}^{i_k}$.

The main result of Platen [23] shows that under our particular setting where all coefficient functions are linear, we have the Ito-Taylor expansion

$$f(t, X_t) = \sum_{\alpha \in \mathcal{A}_l} I_\alpha[f_\alpha(0, X_0)]_t + \sum_{\alpha \in \mathcal{B}(\mathcal{A}_l)} I_\alpha[f_\alpha(\cdot\,, X.)]_t.$$

Here we choose $f(x) = x$ and coefficients are defined by

$$f_\alpha(t, x) = \begin{cases} x & \text{if } l(\alpha) = 0; \\ \tilde{a}(x) & \text{if } l(\alpha) = 1, i_1 = 0\ ; \\ \tilde{b}_{i_1}(x) & \text{if } l(\alpha) = 1, i_1 > 0\ ; \\ \tilde{c}(x) & \text{if } l(\alpha) = 1, i_1 < 0\ ; \\ L^{i_1} f_{-\alpha} & \text{if } l(\alpha) > 1; \end{cases}$$

where

$$L^i f(t, x) = \begin{cases} \frac{\partial f}{\partial t} + \tilde{a}\frac{\partial f}{\partial x} + \frac{1}{2}\sum_j \tilde{b}_j^2 \frac{\partial^2 f}{\partial x} & \text{if } i = 0; \\ \tilde{b}_i \frac{\partial f}{\partial x} & \text{if } i > 0; \\ f(t, x + \tilde{c}(x)) - f(t, x) & \text{if } i < 0. \end{cases}$$

A more detailed treatment of strong approximations and this notation can be found in Platen[23].

For our application, we need to approximate $\sum w_i X_i(\Delta t) := \hat{R}_{n,N}/R_{n,N}$, where

$$X_{i,N}(t) = \exp\left\{\left(\mu_i - \mu_w - \frac{1}{2}\|\sigma_i\|^2 + \frac{1}{2}\sigma_w^2\right)t + (\sigma_i - \bar{\sigma})^\top W(t)\right\} \prod_{j=1}^{N(t)} \frac{Y_j^i}{\sum w_k Y_j^k}.$$

Each $X_{i,N}$ satisfies the following SDE:

$$\begin{aligned}\frac{dX_{i,N}(t)}{X_{i,N}(t-)} &= \left(\mu_i - \mu_w - \frac{1}{2}\|\sigma_i\|^2 + \frac{1}{2}\sigma_w^2 + \frac{1}{2}\|\sigma_i - \bar{\sigma}\|^2\right)dt \\ &\quad + (\sigma_i - \bar{\sigma})^\top dW_t + dJ_t^i \\ &= a_i dt + b_i^\top dW_t + d\tilde{J}_t^i,\end{aligned}$$

where $a_i = \mu_i - \mu_w - \frac{1}{2}\|\sigma_i\|^2 + \frac{1}{2}\sigma_w^2 + \frac{1}{2}\|\sigma_i - \bar{\sigma}\|^2 + \lambda\mu_i^y$ and $b_i = \sigma_i - \bar{\sigma}$.

For our analysis, we need some standard properties of predictable quadratic variations: $< t, t >= 0$, $< t, W_t^i >= 0$ and $< t, \tilde{J}_t^j >= 0$ for all i and j; $< W^i, W^j >_t = \delta_{ij} t$, and $< \tilde{J}^j, \tilde{J}^i >_t = m_{ij} t$, for constants m_{ij}. To derive the appropriate constants, we observe that

$$\begin{aligned}E[\tilde{J}_t^i \tilde{J}_t^j] &= E[[\tilde{J}^i, \tilde{J}^j]_t] \\ &= \frac{1}{4}E[[\tilde{J}^i + \tilde{J}^j, \tilde{J}^i + \tilde{J}^j]_t - [\tilde{J}^i - \tilde{J}^j, \tilde{J}^i - \tilde{J}^j]_t] \\ &= \frac{1}{4}E\left[\sum_{0<s<t}(\tilde{J}_s^i - \tilde{J}_{s-}^i + \tilde{J}_s^j - \tilde{J}_{s-}^j)^2 - \sum_{0<s<t}(\tilde{J}_s^i - \tilde{J}_{s-}^i - \tilde{J}_s^j + \tilde{J}_{s-}^j)^2\right] \\ &= E\left[\sum_{0<s<t}((\tilde{J}_s^i - \tilde{J}_{s-}^i)(\tilde{J}_s^j - \tilde{J}_{s-}^j))\right] \\ &= \lambda t E[\bar{Y}^i \bar{Y}^j].\end{aligned}$$

The third equality is due to the fact that a compound Poisson process $\sum_{i=1}^{N(t)} Z_i$ has quadratic variation $\sum_{i=1}^{N(t)} Z_i^2$ (Cont and Tankov [8, Example 8.4]). Thus, we need $m_{ij} = \lambda E[\bar{Y}^i \bar{Y}^j]$.

5.2. *Strong approximation for the jump-diffusion model.* We now use the strong approximation scheme of order 3/2 to prove Theorem 2.3 and 2.4. First we write

$$X_{i,N}(\Delta t) = 1 + \zeta_{1/2,N}^i + \zeta_{1,N}^i + \zeta_{3/2,N}^i + r_N^i,$$

where $\zeta_{\cdot,N}^i$ are defined as follows. First,

$$\zeta_{1/2,N}^i = \int_0^{\Delta t} b_i^\top dW + \int_0^{\Delta t} d\tilde{J}^i = b_i^\top \Delta W + \Delta\tilde{J}^i.$$

(From now on we drop the limits of integration for iterated integrals taken over $[0, \Delta t]$. An integral of the form $\int g \, d\tilde{J}^i$ should be understood as

$\int g(t-)d\tilde{J}^i(t)$.) Continuing, we have

$$\begin{aligned}\zeta^i_{1,N} &= a^i\int dt + \int\int b_i^\top dW b_i^\top dW + \int\int b_i^\top dW d\tilde{J}^i \\ &\quad + \int\int d\tilde{J}^i b_i^\top dW + \int\int d\tilde{J}^i d\tilde{J}^i\end{aligned} \tag{20}$$

and

$$\begin{aligned}\zeta^i_{3/2,N} &= a^i\int\int b_i^\top dW dt + a^i\int\int dt b_i^\top dW + a^i\int\int d\tilde{J}^i dt + a^i\int\int dt d\tilde{J}^i \\ &+ \int\int\int b_i^\top dW b_i^\top dW b_i^\top dW + \int\int\int b_i^\top dW b_i^\top dW d\tilde{J}^i \\ &+ \int\int\int b_i^\top dW d\tilde{J}^i b_i^\top dW + \int\int\int d\tilde{J}^i b_i^\top dW b_i^\top dW + \int\int\int b_i^\top dW d\tilde{J}^i d\tilde{J}^i \\ &+ \int\int\int d\tilde{J}^i d\tilde{J}^i b_i^\top dW + \int\int\int d\tilde{J}^i b_i^\top dW d\tilde{J}^i + \int\int\int d\tilde{J}^i d\tilde{J}^i d\tilde{J}^i.\end{aligned} \tag{21}$$

By observing that $\sum w_i b_i = 0$ and $\sum w_i \tilde{J}^i = 0$, we find that $\sum w_i \zeta^i_{1/2,N} = 0$. For the next term, we have

$$\sum_i w_i \zeta^i_{1,N} = \sum_i w_i[\epsilon_{n,N} + b_i^\top \Delta W \Delta\tilde{J}^i + \int\int d\tilde{J}^i d\tilde{J}^i].$$

Here, $\epsilon_{n,N}$ is the corresponding error term in the absence of jumps; the last two terms are the difference between the continuous and jump-diffusion cases.

It is now easy to see that $\|\sum w_i \zeta^i_{1,N}\| = O(\Delta t)$, and similarly $\|\sum w_i \times \zeta^i_{3/2,N}\| = O(\Delta t^{3/2})$. Now we need to show that the remainder r^i_N satisfies $\|r^i_N\| = O(\Delta t^2)$.

LEMMA 5.1. *(Modified from Studer[25, Lemma 3.42].) Given an adapted caglad (left continuous with right limits) process $g(t)$, with $\int_0^t E[g(s)^2]ds = K < \infty$, then*

$$E\left[\left(\int_0^t g(s)dM_s\right)^2\right] \le \begin{cases} tK, & \text{if } M_t = t; \\ K, & \text{if } M_t = W^i_t; \\ m_{ii}K, & \text{if } M_t = \tilde{J}^i_t. \end{cases}$$

(The integrand should be understood as $g^\top$ when $M = W$.)

PROOF. The result and proof are the same as in Studer [25]. $\square$

To bound the error when we truncate a strong approximation, we can apply a result of Studer [25, Proposition 3.43], or s similar result of Bruti-Liberati and Platen [6, Theorem 6.1]. Out setting is simpler than theirs because of the special form of the dynamics in the JD model.

LEMMA 5.2. *(Modified from Studer [25, Proposition 3.43].) Under our assumptions (3) and (4) for the JD model, there exist some constants C_1 and C_2 such that for any $i = 1, \dots, d$*

$$E[(X_{i,N}(t) - \sum_{\beta \in \mathcal{A}_k} I_\beta[f_\beta(0, X_{i,N}(0))])^2] \le C_1(C_2 t)^{2k+1}.$$

PROOF. Since $f(t, x) = x$, the conditions in Studer [25, Proposition 3.43] (and those in Bruti-Liberati and Platen [6, Theorem 6.1]) are satisfied. Thus, for any $\alpha \in \mathcal{B}(\mathcal{A}_k)$, we can find some constant C_3

$$\sup_{0 \le t \le T} E[(f_\alpha(t, X_{i,N}(t)))^2] \le C_3.$$

Denote $n_i(\alpha)$ be the number of components for $\tilde{J}^i$ in α. By induction and the previous lemma, we have for any $\alpha \in \mathcal{B}(\mathcal{A}_k)$, we can find some constant C_4

$$\begin{aligned} E[I_\alpha[f_\alpha(., X_{i,N}(.))]_t^2] &\le t^{n(\alpha)}(m_{ii})^{n_i(\alpha)} C_3 t^{l(\alpha)} \\ &\le C_3 C_4^{2k+1} t^{l(\alpha)+n(\alpha)}; \end{aligned}$$

and $|\mathcal{B}(\mathcal{A}_k)| \le (3d+3)^{k+1}$, therefore,

$$\begin{aligned} E\left[\left(X_t - \sum_{\beta \in \mathcal{A}_k} I_\beta[f_\beta(0, X_0)]\right)^2\right] &\le \left(\sum_{\alpha \in \mathcal{B}(\mathcal{A}_k)} (E[I_\alpha[f_\alpha(., X)]])^{1/2}\right)^2 \\ &\le \left(\sum_{\alpha \in \mathcal{B}(\mathcal{A}_k)} (C_3 C_4^{2k+1} t^{l(\alpha)+n(\alpha)})^{1/2}\right)^2 \\ &\le C_1(C_2 t)^{2k+1}. \end{aligned}$$

$\square$

As a consequence, for our setting we get

LEMMA 5.3. $\|r_N^i\| = \|X_{i,N} - 1 - \zeta_{1/2,N}^i - \zeta_{1,N}^i - \zeta_{3/2,N}^i\| = O(\Delta t^2)$.

5.3. *Correlation between ζ_1^i and $\zeta_{3/2}^i$.* In this section, we show that the terms $\sum w_i\zeta_{1,N}^i$ and $\sum w_i\zeta_{3/2,N}^i$ are uncorrelated. Before specializing to our setting, we derive some general properties used extensively in this subsection.

To calculate the covariance between iterated integrals, from Cont and Tankov [8, Proposition 8.11] we have (using the notation of Lemma 5.1)

$$
\begin{aligned}
E[I_{\alpha_1}I_{\alpha_2}] &= E\left[\int I_{\alpha_1-}dM_1\int I_{\alpha_2-}dM_2\right]\\
&= E\left[\int I_{\alpha_1-}I_{\alpha_2}dM_1+\int I_{\alpha_2-}I_{\alpha_1}dM_2+\int I_{\alpha_1-}I_{\alpha_2-}d[M_1,M_2]\right]\\
&= E\left[\int I_{\alpha_1-}I_{\alpha_2}dM_1+\int I_{\alpha_2-}I_{\alpha_1}dM_2+\int I_{\alpha_1-}I_{\alpha_2-}d<M_1,M_2>\right], \qquad (22)
\end{aligned}
$$

where

$$
M_i(t)=\begin{cases} t & \text{if } r(\alpha_i)=0;\\ W_t & \text{if } r(\alpha_i)=1;\\ \tilde{J}_t^k & \text{if } r(\alpha_i)=k<0,\end{cases}
$$

with $r(\alpha_i)$ the rightmost element of α_i. As before, when $M_i=\tilde{J}^k$ for some i and k, we use the left-continuous version of the integrand. When $M_i=W$, we take its transpose in the integrand. Here we use the square bracket and sharp bracket to denote quadratic variation and predictable quadratic variation as introduced towards the end of Section 5.1.

When $r(\alpha_i)\neq 0$ for both $i=1$ and 2, $M_{r(\alpha_i)}$ is a martingale, so after taking expectations, the first two terms in (22) vanish. Assumption (4) implies square integrability of these iterated integrals, which contain jump terms. Otherwise, when they consist of only dt or dW, their integrability is immediate. Thus, we have the following possible combinations:

When $r(\alpha_1)>0$ and $r(\alpha_2)=-j<0$, M_1 and M_2 are uncorrelated martingales, so the expectation of their product is 0. Thus, we have:

$$
E\left[\int I_{\alpha_1-}^\top dW\int I_{\alpha_2-}d\tilde{J}^j\right]=0. \qquad (23)
$$

When $r(\alpha_1)=r(\alpha_2)=1$,

$$
E\left[\int I_{\alpha_1-}^\top dW\int I_{\alpha_2-}^\top dW\right]=\int E[I_{\alpha_1-}^\top I_{\alpha_2-}]dt
$$

and when $r(\alpha_1)=-i$, $r(\alpha_2)=-j$,

$$
E\left[\int I_{\alpha_1-}d\tilde{J}^i\int I_{\alpha_2-}d\tilde{J}^j\right]=\int E[I_{\alpha_1-}I_{\alpha_2-}]m_{ij}dt. \qquad (24)
$$

When $r(\alpha_1) = 0$ and $r(\alpha_2) \neq 0$,the second and the third term in (22) vanish, leaving

$$E\left[\int I_{\alpha_1-}dt \int I_{\alpha_2-}dM_2\right] = \int E[I_{\alpha_1-}I_{\alpha_2}]dt.$$

Now we apply these results to analyze the correlation between ζ_1 and $\zeta_{3/2}$. Let $\mathcal{B}_{l,n} = \{\gamma | l(\gamma) = l, n(\gamma) = n\}$. All strings in $\mathcal{B}_{l,n}$ are of the same length l and have the same number of zeros n. We observe from (20) and (21) that $\zeta^i_{1,N}$ is a linear combination of elements in $\mathcal{B}(\mathcal{A}_1)$ and $\zeta^i_{3/2,N}$ is a linear combination of elements of $\mathcal{B}(\mathcal{A}_{3/2})$. From here until the end of this subsection, we let α and β be strings with $l(\alpha) = 1$ and $l(\beta) = 3/2$, and we treat all possible combinations of values of $n(\alpha)$ and $n(\beta)$:

(a) If $n(\alpha) = 0$ and $n(\beta) = 0$ — that is, neither contains dt integrals — then (23)–(24) show that $E[I_\alpha I_\beta]$ equals to an integral against dt with its integrand either zero or $E[I_{\alpha-}I_{\beta-}]$. Applying the same argument again, so we can say that $E[I_{\alpha-}I_{\beta-}]$ is again an integral against dt with its integrand either zero or $E[I_{\alpha--}I_{\beta--}]$, which is zero, since $l(\alpha) = 1$. So $E[I_\alpha I_\beta] = 0$ for any $\alpha \in \mathcal{B}_{1,0}$ and $\beta \in \mathcal{B}_{3/2,0}$. Hence any linear combination of elements of $\{I_\alpha : \alpha \in \mathcal{B}_{1,0}\}$ and any linear combination of elements of $\{I_\beta : \alpha \in \mathcal{B}_{3/2,0}\}$ are uncorrelated.

(b) If $l(\alpha) = n(\alpha) = 1$, but $n(\beta) = 0$, then I_α is actually deterministic. So $E[I_\alpha I_\beta] = I_\alpha E[I_\beta] = 0$, since I_β is a martingale. Hence any linear combination of elements of $\{I_\alpha : \alpha \in \mathcal{B}_{1,1}\}$ and any linear combination of elements of $\{I_\beta : \alpha \in \mathcal{B}_{3/2,0}\}$ are uncorrelated.

(c) For the case $n(\alpha) = 0$ and $n(\beta) = 1$, we observe that in our particular setting, for any $i \neq 0$, $I_{(i,0)}$ and $I_{(0,i)}$ always appear in pairs in ζ^i and have the same coefficients. Using integration by parts we can consider them in pairs, for $i \neq 0$, to get

$$I_{(i,0)} + I_{(0,i)} = \int d(tM_i) = \Delta t \Delta M_i,$$

so

$$E[I_\alpha(I_{(i,0)} + I_{(0,i)})] = \Delta t E[I_\alpha \Delta M_i] = 0,$$

the last equality following from the same argument as (a). Hence, any linear combination of elements of $\{I_\alpha : \alpha \in \mathcal{B}_{1,0}\}$ and any linear combination of elements of $\{I_\beta : \alpha \in \mathcal{B}_{3/2,1}\}$ are uncorrelated.

(d) If $n(\alpha) = 1$, and $n(\beta) = 1$, then $I_\alpha = \Delta t$, which is deterministic, and $I_{(i,0)} + I_{(0,i)} = \Delta t \Delta M_i$ has zero mean. Hence any linear combination of elements of $\{I_\alpha : \alpha \in \mathcal{B}_{1,1}\}$ and any linear combination of elements of $\{I_\beta : \alpha \in \mathcal{B}_{3/2,1}\}$ are uncorrelated.

To summarize, we have proved

LEMMA 5.4. *$\sum w_i \zeta^i_{1,N}$ and $\sum w_i \zeta^i_{3/2,N}$ are uncorrelated.*

5.4. *Convergence Proofs.* Using our analysis of the strong approximation for the jump-diffusion case, we can now prove Theorems 2.3 and 2.4.

PROOF. (Theorem 2.3): We have

$$\frac{\hat{R}_{n,N}}{R_{n,N}} = 1 + \sum_i w_i \zeta^i_{1,N} + \sum_i w_i \zeta^i_{3/2,N} + \sum_i w_i r^i_N.$$

We have shown that $\|\sum w_i \zeta^i_{1,N}\| = O(\Delta t)$, $\|\sum w_i \zeta^i_{3/2,N}\| = O(\Delta t^{3/2})$, $\|\sum w_i r^i_N\| = O(\Delta t^2)$, that $E[\sum w_i \zeta^i_{1,N}] = 0$ and $E[\sum w_i \zeta^i_{3/2,N}] = 0$, and that $\sum w_i \zeta^i_{1,N}$ and $\sum w_i \zeta^i_{3/2,N}$ are uncorrelated. We can now follow the argument used in Glasserman [12, Proposition 1] to prove (8).

Next we calculate the variance of the relative error. To condense (9), let $A = \sum w_i [b_i^\top \Delta W \Delta \tilde{J}^i]$, and $B = \sum w_i [\int\int d\tilde{J}^i d\tilde{J}^i]$. By following steps similar to those used to prove Lemma 5.4, we can show that the pairwise correlations between $\epsilon_{n,N}$, A, and B are all zero. Thus,

$$Var[\tilde{\epsilon}_{n,N}] = Var[\epsilon_{n,N}] + Var[A] + Var[B].$$

We need to calculate the last two terms on the right. For A, we have

$$\begin{aligned}
Var[A] = E[A^2] &= E\left[\left(\sum w_i b_i^\top \Delta W \Delta \tilde{J}^i\right)^2\right] \\
&= E[(\Delta W^\top b \Omega \Delta \tilde{J})^2] \\
&= E\left[\left(\sum_i \Delta W_i^2\right) \Delta \tilde{J}^\top (\Omega b^\top b \Omega) \Delta \tilde{J}\right] \\
&= \Delta t^2 (w^\top (b^\top b \circ M) w).
\end{aligned}$$

For B, we have

$$\begin{aligned}
Var[B] = E[B^2] &= E\left[\left(\sum w_i \int\int d\tilde{J}^i d\tilde{J}^i\right)^2\right] \\
&= \sum_{i,j} w_i w_j \int E[< \tilde{J}^i, \tilde{J}^j >_s] m_{ij} ds \\
&= \frac{\Delta t^2}{2} w^\top M \circ M w.
\end{aligned}$$

□

PROOF. (Theorem 2.4): First, from the expression of the asymptotics of the relative error in (9), the contribution of the compensation terms in the jump terms are of lower order, so we can replace $\tilde{J}_n^i$ and $\tilde{J}^i$ with J_n^i and J^i respectively throughout (9) and (8) still holds. That is,

$$E\left[\left(\frac{\hat{V}(T)-V(T)}{V(T)}-\sum_{n=1}^{N}\bar{\epsilon}_{n,N}\right)^2\right]=O(\Delta t^2)$$

(25)

$$\text{where } \bar{\epsilon}_{n,N}=\epsilon_{n,N}+\sum_{i=1}^{d}w_i[b_i^\top\Delta W_n\Delta J_n^i+\int_{(n-1)\Delta t}^{n\Delta t}\int_{(n-1)\Delta t}^{s-}dJ^i(r)dJ^i(s)].$$

The last term in (25) is nonzero only when there are at least two jumps in the period $[(n-1)\Delta t, n\Delta t]$, which has probability $O(\Delta t^2)$. Since the number of jumps in different periods are i.i.d., the probability that none of the time intervals has more than one jump is of order $1-O(\Delta t)$, so

$$\sqrt{N}\sum_{n=1}^{N}\sum_{i=1}^{d}w_i\int_{(n-1)\Delta t}^{n\Delta t}\int_{(n-1)\Delta t}^{s-}dJ^i(r)dJ^i(s)\Rightarrow 0.$$

For the same reason, we can ignore multiple jumps in each Δt interval in (25). More precisely,

$$\sqrt{N}\sum_{j=1}^{N(T)}\left(\sum_{i=1}^{d}w_ib_i^\top\Delta W_{n(j)}\bar{Y}_{n(j)}^i-\sum_{i=1}^{d}w_ib_i^\top\Delta W_{n(j)}\Delta J_{n(j)}^i\right)\Rightarrow 0, \tag{26}$$

where $n(j)$ is the index of the interval when j^{th} jump takes place.

To analyze the limit of (26), we rewrite it as

(27)

$$\sqrt{N}\sum_{\substack{n\neq n(j)\\ j=1,\ldots,N(T)}}\epsilon_{n,N}+\sqrt{N}\sum_{\substack{n=n(j)\\ j=1,\ldots,N(T)}}\epsilon_{n,N}+\sqrt{N}\sum_{j=1}^{N(T)}\sum_{i=1}^{d}w_ib_i^\top\Delta W_{n(j)}\bar{Y}_{n(j)}^i.$$

Let $N\to\infty$, noting that $N(T)$ remains fixed. In (27), the first term is independent of the other two terms, and it converges to $\underline{X}\sim N(0,\sigma_L^2T)$, as shown in Theorem 2.1. The second term in (27) converges to zero in L_2 and thus in probability. Thus (27) converges in distribution to

$$\underline{X}+\sum_{j=1}^{N(T)}\sum_{i=1}^{d}w_ib_i^\top\xi_j\bar{Y}_j^i,$$

where ξ_j are i.i.d. standard normal random variables independent of everything else. The limit does not hold in L_2, since the L_2-norm of the third term in (25) has order $O(\Delta t^2)$, as shown in the proof of Theorem 2.3. □

6. Strong approximation for the mean-reverting case. In this section, we prove Theorem 2.1. We build on the strong approximation technique introduced in Section 5.2, but the argument will be somewhat simpler because we no longer have jump terms.

PROOF. (Theorem 2.1). The value of the discretely rebalanced portfolio at Δt is given by

$$\underline{\hat{V}}(\Delta t) = \sum_i w_i \exp\left\{\left(\mu_i - \frac{1}{2}\|\sigma_i\|^2\right)\Delta t + \sigma_i \int_0^{\Delta t} e^{-\beta(\Delta t - s)} dW_s \right.$$
$$\left. + (1 - e^{-\beta\Delta t})\theta_i\right\},$$

and the ratio of the discrete portfolio value to the continuous portfolio value is given by

$$\frac{\underline{\hat{R}}_N}{\underline{R}_N} = \frac{\sum_i w_i \exp\{(\mu_i - \frac{1}{2}\sigma_i^2)\Delta t + \sigma_i^\top \int_0^{\Delta t} e^{\beta(s-\Delta t)} dW_s + (1 - e^{-\beta\Delta t})\theta_i\}}{\exp\{(\mu_w - \frac{1}{2}\sigma_w^2)\Delta t + \bar{\sigma}\int_0^{\Delta t} e^{\beta(s-\Delta t)} dW_s + (1 - e^{-\beta\Delta t})\bar{\theta}\}}$$
$$= \sum_i w_i \exp\left\{\left(\mu_i - \mu_w - \frac{1}{2}(\|\sigma_i\|^2 - \sigma_w^2)\right)\Delta t + (\sigma_i - \bar{\sigma})^\top \int_0^{\Delta t} e^{\beta(s-\Delta t)} dW_s \right.$$
$$\left. + (1 - e^{-\beta\Delta t})(\theta_i - \bar{\theta})\right\}$$
$$=: \sum_i w_i C_i(\Delta t),$$

where each C_i satisfies

$$dC_i = C_i\left[\left(\mu_i - \mu_w - \frac{1}{2}(\|\sigma_i\|^2 - \sigma_w^2 - \|\sigma_i - \bar{\sigma}\|^2)\right)dt + d\bar{U}_i\right]$$
$$d\bar{U}_i = \beta(\theta_i - \bar{\theta} - \bar{U}_i)dt + (\sigma_i - \bar{\sigma})^\top dW.$$

Using strong approximation as introduced in Section 5.2, we get (with all iterated integrals taken from 0 to Δt):

$$C_i(\Delta t) = 1 + \left(\mu_i - \mu_w - \frac{1}{2}(\|\sigma_i\|^2 - \sigma_w^2 - \|\sigma_i - \bar{\sigma}\|^2)\right)\left(\Delta t + \int\int d\bar{U}_i dt \right.$$
$$\left. + \int\int dt d\bar{U}_i\right) + \Delta\bar{U}_i + \int\int d\bar{U}_i d\bar{U}_i + \int\int\int d\bar{U}_i d\bar{U}_i d\bar{U}_i + O(\Delta t^2),$$

where

$$\Delta\bar{U}_i = (\sigma_i - \bar{\sigma})^\top e^{-\beta\Delta t}\int_0^{\Delta t} e^{\beta s} dW_s + (1 - e^{-\beta\Delta t})(\theta_i - \bar{\theta})$$
$$= (\sigma_i - \bar{\sigma})^\top \left(\Delta W - \beta\int_0^{\Delta t} W_s ds\right) + \beta(\theta_i - \bar{\theta})\Delta t + O(\Delta t^2).$$

Expanding the iterated integrals of $\bar{U}_i$ and substituting, we get

$$\begin{aligned}
C_i(\Delta t) = 1 &+ (\sigma_i - \bar{\sigma})^\top \Delta W + \left[\left(\mu_i - \mu_w - \frac{1}{2}(\|\sigma_i\|^2 - \sigma_w^2)\right)\Delta t\right. \\
&\left. + \frac{1}{2}\Delta W^\top \bar{B}_i \Delta W\right] + \left[\frac{1}{6}\Delta W^\top \bar{B}_i \Delta W (\sigma_i - \bar{\sigma})^\top \Delta W\right. \\
&+ (\mu_i - \mu_w - \frac{1}{2}(\|\sigma_i\|^2 - \sigma_w^2))(\sigma_i - \bar{\sigma})^\top \Delta W \Delta t \\
&\left. - (\sigma_i - \bar{\sigma})^\top \left(\beta\int_0^{\Delta t} W_s ds\right)\right] \\
&+ \beta(\theta_i - \bar{\theta})\Delta t + \beta(\theta_i - \bar{\theta})(\sigma_i - \bar{\sigma})^\top \Delta W \Delta t + O(\Delta t^2),
\end{aligned}$$

where we drop the term $\Delta W^\top B_i \int_0^{\Delta t} s dW_s$ because its L_2-norm is $O(\Delta t^2)$.

Now taking the weighted sum of the C_i, we get

$$\begin{aligned}
\sum_i w_i C_i(\Delta t) = 1 + 0 &+ \left[-\frac{1}{2}\left(\sum_i w_i\|\sigma_i\|^2 - \sigma_w^2\right)\Delta t + \frac{1}{2}\Delta W^\top \bar{B}\Delta W\right] \\
&+ \sum_i w_i\left[\frac{1}{6}\Delta W^\top \bar{B}_i \Delta W (\sigma_i - \bar{\sigma})^\top \Delta W\right. \\
&\left. + \left(\mu_i - \mu_w - \beta(\theta_i - \bar{\theta}) - \frac{1}{2}(\|\sigma_i\|^2 - \sigma_w^2)\right)(\sigma_i - \bar{\sigma})^\top \Delta W \Delta t\right] \\
&+ O(\Delta t^2) \\
&=: 1 + \zeta_1^N + \zeta_{3/2}^N + \underline{r}
\end{aligned}$$

where $\bar{B} = \sum_i w_i \bar{B}_i$ and $\|\underline{r}\| = O(\Delta t^2)$.

Following essentially the same arguments used in the jump-diffusion case, it is now easy to show that $\|\zeta_1^N\| = O(\Delta t)$ and $\|\zeta_{3/2}^N\| = O(\Delta t^{3/2})$, and also that ζ_1^N and $\zeta_{3/2}^N$ are uncorrelated, leading to

$$\left\|\frac{\hat{\underline{V}}(T)}{\underline{V}(T)} - 1 - \sum_{n=1}^N \zeta_{1,n}^N\right\| = O(\Delta t).$$

At the same time,

$$\zeta_{1,n}^N = \frac{1}{2}(\Delta W^\top \bar{B} \Delta W - Tr(\bar{B})\Delta t) = \epsilon_{n,N},$$

coincides with the $\epsilon_{n,N}$ in the case of multivariate geometric Brownian motion considered in Glasserman [12]. The same limit therefore applies here. $\square$

Given the representation in Theorem 2.1, the proof of Theorem 2.2 is the same as that of Theorem 1 in Glasserman [12].

7. Analysis of the volatility adjustments.

7.1. *The jump-diffusion case.*

PROOF. (Proposition 3.2) With

$$X_N = \sqrt{N} \sum_{n=0}^{N-1} \left(\frac{\hat{V}((n+1)\Delta t)}{V((n+1)\Delta t)} - \frac{\hat{V}(n\Delta t)}{V(n\Delta t)} \right)$$

we can write $Cov[\log V(T), X_N]$ as

$$\begin{aligned} &Cov[\log V(T), X_N] \\ &= \sqrt{N} \sum_{k=1}^{N} \sum_{n=0}^{N-1} E\Bigg[\Bigg(\bar{\sigma}^\top \Delta W_k \\ &\quad + \sum_{j=N(k)+1}^{N(k+1)} \Big(\log \sum_i w_i Y_j^i - \mu_J\Big)\Bigg)\left(\frac{\hat{V}((n+1)\Delta t)}{V((n+1)\Delta t)} - \frac{\hat{V}(n\Delta t)}{V(n\Delta t)}\right)\Bigg] \end{aligned}$$

where, as before, $\mu_J = E[\log \sum_i w_i Y_j^i]$. If we interchange the order of summation and fix a value of n, we need to evaluate

(28)
$$E\left[\left(\bar{\sigma}^\top \Delta W_k + \sum_{j=N(k)+1}^{N(k+1)} \left(\log \sum_i w_i Y_j^i - \mu_J\right)\right)\left(\frac{\hat{V}((n+1)\Delta t)}{V((n+1)\Delta t)} - \frac{\hat{V}(n\Delta t)}{V(n\Delta t)}\right)\right],$$

for which we have three cases:

(1) $k \geq n+2$. In this case, we have

$$\begin{aligned} E\Bigg[&\Bigg(\bar{\sigma}^\top \Delta W_k + \sum_{j=N(k)+1}^{N(k+1)} \Big(\log \sum_i w_i Y_j^i - \mu_J\Big)\Bigg) \\ &\times \left(\frac{\hat{V}((n+1)\Delta t)}{V((n+1)\Delta t)} - \frac{\hat{V}(n\Delta t)}{V(n\Delta t)}\right)\Bigg] = 0, \end{aligned}$$

because $W(k)$ and $\sum_{j=N(k)+1}^{N(k+1)}(\log\sum_i w_i Y_j^i)$ are both independent of $(\hat{V}(n\Delta t), V(n\Delta t), \hat{V}((n+1)\Delta t), V((n+1)\Delta t))$.

(2) $k = n+1$. (28) becomes

$$E\left[\frac{\hat{V}(n\Delta t)}{V(n\Delta t)}\right] E\left[\left(\bar{\sigma}^\top \Delta W_{n+1} + \sum_{j=N(k)+1}^{N(k+1)}\left(\log\sum_i w_i Y_j^i - \mu_J\right)\right)\frac{\hat{R}_{n+1}}{R_{n+1}}\right]. \tag{29}$$

Multiplying the factors inside the last expectation produces two terms. For the first, we have

$$\begin{aligned}
&E\left[\bar{\sigma}^\top \Delta W_{n+1}\frac{\hat{R}_{n+1}}{R_{n+1}}\right]\\
&= \sum_i w_i E\Bigg[\bar{\sigma}^\top \Delta W_{n+1}\exp\bigg\{\left(\mu_i - \mu_w - \frac{1}{2}\|\sigma_i\|^2 + \frac{1}{2}\sigma_w^2\right)\Delta t\\
&+ (\sigma_i - \bar{\sigma})^\top \Delta W_{n+1}\bigg\}\prod_{j=N(n+1)+1}^{N(n+2)}\frac{Y_j^i}{\sum w_l Y_j^l}\Bigg]\\
&= \sum_i w_i(\bar{\sigma}^\top\sigma_i - \sigma_w^2)\Delta t\exp\{(\mu_i - \mu_w + \sigma_w^2 - \sigma_i^\top\bar{\sigma})\Delta t + \lambda\Delta t(\mu_i^y)\}\\
&= \gamma_L\Delta t^2 + \sum_i w_i\bar{\sigma}^\top\sigma_i\lambda\mu_i^y\Delta t^2 + O(\Delta t^3).
\end{aligned}\tag{30}$$

For the other term, from (29) we have

$$\begin{aligned}
&E\left[\sum_{j=N(k)+1}^{N(k+1)}\left(\log\sum_i w_i Y_j^i - \mu_J\right)\frac{\hat{R}_{n+1}}{R_{n+1}}\right]\\
&= \sum_i w_i\exp\{(\mu_i - \mu_w + \sigma_w^2 - \sigma_i^\top\bar{\sigma})\Delta t\}\\
&\quad\times E\left[\left(\prod_{r=N(k)+1}^{N(k+1)}(\bar{Y}^i + 1)\right)\left(\sum_{j=N(k)+1}^{N(k+1)}\left(\log\sum_i w_i Y_j^i - \mu_J\right)\right)\right],
\end{aligned}\tag{31}$$

where

$$E\left[\left(\prod_{r=N(k)+1}^{N(k+1)}(\bar{Y}_r^i + 1)\right)\left(\sum_{j=N(k)+1}^{N(k+1)}\left(\log\sum_i w_i Y_j^i - \mu_J\right)\right)\right]$$

$$= \sum_{n=1}^{\infty} e^{-\lambda\Delta t} \frac{(\lambda\Delta t)^n}{n!} \sum_{j=1}^{n} E\left[\prod_{k=1}^{n} (\bar{Y}^i + 1)\left(\log \sum_l w_l Y_j^l - \mu_J\right)\right]$$

$$= \exp\{\lambda\Delta t \mu_i^y\}\Delta t \lambda E[(\bar{Y}^i + 1)(\log \sum_l w_l Y^l - \mu_J)]. \tag{32}$$

Substituting (32) into (31), we get

$$E\left[\sum_{j=N(k)+1}^{N(k+1)} \left(\log \sum_i w_i Y_j^i - \mu_J\right)\left(\frac{\hat{R}_{n+1}}{R_{n+1}}\right)\right]$$

$$= \sum_i w_i \exp\{(\mu_i - \mu_w + \sigma_w^2 - \sigma_i^\top \bar{\sigma})\Delta t\} \exp\{\lambda\Delta t \mu_i^y\}$$

$$\times \Delta t \lambda E\left[(\bar{Y}^i + 1)\left(\log \sum_l w_l Y^l - \mu_J\right)\right]. \tag{33}$$

Applying a Taylor expansion to the exponential part under assumptions (4) and (5), (33) becomes

$$\sum_i w_i \lambda(\mu_i - \sigma_i^\top \bar{\sigma} + \lambda\mu_i^y) E\left[\bar{Y}^i\left(\log \sum_l w_l Y^l - \mu_J\right)\right]\Delta t^2 + O(\Delta t^3). \tag{34}$$

Using (30) and (34) we have for (29)

$$E\left[\left(\bar{\sigma}^\top \Delta W_{n+1} + \sum_{j=N(k)+1}^{N(k+1)} \left(\log \sum_i w_i Y_j^i - E\right)\right)\left(\frac{\hat{R}_{n+1}}{R_{n+1}}\right)\right] = \tilde{\gamma}_L \Delta t^2 + O(\Delta t^3).$$

(3) $k < n+1$. The same argument applies in this case, and we have

$$E\left[\left(\bar{\sigma}^\top \Delta W_{n+1} + \sum_{j=N(k)+1}^{N(k+1)} \left(\log \sum_i w_i Y_j^i - \mu_J\right)\right)\left(\frac{\hat{V}((n+1)\Delta t)}{V((n+1)\Delta t)} - \frac{\hat{V}(n\Delta t)}{V(n\Delta t)}\right)\right]$$
$$= O(\Delta t^4).$$

Hence we have

$$N^{-1/2} Cov[\log T(T), X_N] = \frac{\tilde{\gamma}_L T^2}{N} + O(N^{-2}).$$

(ii) For the second part of the proposition, we need to show that

$$E[(\bar{V}(T) - \hat{V}(T))^2] = O(N^{-2}).$$

By following the steps of a similar proof in Glasserman [12], it suffices to show $E[V(T)^2 X_N^2] < \infty$.

We can write

$$V^2(T) = \exp\{2\mu_w T + \sigma_w^2 T\}\exp\{2\bar\sigma^\top W(T) - 2\sigma_w^2 T\}\exp\{-(\lambda - \tilde\lambda)T\} \times \exp\{(\lambda - \tilde\lambda)T\}\prod_{j=1}^{N(t)}\left(\sum_i w_i Y_j^i\right)^2,$$

and now we would like to use the following as a Radon-Nikodym derivative:

$$\exp\{2\bar\sigma^\top W(T) - 2\sigma_w^2 T\}\exp\{(\lambda - \tilde\lambda)T\}\prod_{j=1}^{N(t)}\left(\sum_i w_i Y_j^i\right)^2. \tag{35}$$

The first exponential term is itself a Radon-Nikodym derivative for the diffusion process. From assumption (4), we have $E[(Y^i)^2] < \infty$, so we can choose an appropriate $\tilde\lambda$ such that $\tilde f(y) = \lambda y^2 f(y)/\tilde\lambda$ is a well-defined density function, where $f(.)$ and $\tilde f(.)$ are the density functions for $\sum_i w_i Y^i$ under the original probability and the new probability measure, respectively. Therefore, (35) is indeed a Radon-Nikodym derivative, and, under the probability measure it defines, each asset's drift is changed from μ_i to $\mu_i + 2\sigma_i^\top\bar\sigma$, and the $\sum_i w_i Y^i$ now have density $\tilde f$.

From Theorem 2.3, the convergence of the second moment of X_N holds as long as the drifts and Poisson rate are constant, and assumption (3) and the first inequality of (4) hold under the new measure. Because of absolute continuity, (3) will still hold. For (4)

$$\begin{aligned}\tilde E[|\bar Y^k + 1|^3] &= \exp\{(\lambda - \tilde\lambda T)\}E[|Y^k|^2|\bar Y^k + 1|]\\ &\le \exp\{(\lambda - \tilde\lambda T)\}\|\bar Y^k + 1\|_3\|Y^k\|_3^2 < \infty.\end{aligned}$$

Hence we have proved the second part of the proposition. □

7.2. *The mean-reverting case.*

Proof. (Proposition 3.1): (i) With

$$X_N = \sqrt{N}\sum_{n=0}^{N-1}\left(\frac{\underline{\hat V}((n+1)\Delta t)}{\underline{V}((n+1)\Delta t)} - \frac{\underline{\hat V}(n\Delta t)}{\underline{V}(n\Delta t)}\right),$$

we have

$$Cov[\log\underline{V}(T), X_N] = \sqrt{N}\sum_{k=1}^{N}\sum_{n=0}^{N-1}E\left[\bar\sigma^\top e^{-\beta}\int_{(k-1)\Delta t}^{k\Delta t}e^{\beta s}dW_s\left(\frac{\underline{\hat V}((n+1)\Delta t)}{\underline{V}((n+1)\Delta t)} - \frac{\underline{\hat V}(n\Delta t)}{\underline{V}(n\Delta t)}\right)\right]. \tag{36}$$

For $k \geq n+2$,

$$E\left[\bar{\sigma}^\top e^{-\beta}\int_{(k-1)\Delta t}^{k\Delta t} e^{\beta s} dW_s\left(\frac{\hat{\underline{V}}((n+1)\Delta t)}{\underline{V}((n+1)\Delta t)} - \frac{\hat{\underline{V}}(n\Delta t)}{\underline{V}(n\Delta t)}\right)\right] = 0,$$

For $k = n+1$, we have

$$\begin{aligned}
&E\left[\bar{\sigma}^\top e^{-\beta}\int_{(n+1)\Delta t}^{(n+2)\Delta t} e^{\beta s} dW_s\left(\frac{\hat{\underline{V}}((n+1)\Delta t)}{\underline{V}((n+1)\Delta t)} - \frac{\hat{\underline{V}}(n\Delta t)}{\underline{V}(n\Delta t)}\right)\right] \\
&= E\left[\frac{\hat{\underline{V}}(n\Delta t)}{\underline{V}(n\Delta t)}\right] E[\bar{\sigma}^\top e^{-\beta}\int_{(n+1)\Delta t}^{(n+2)\Delta t} e^{\beta s} dW_s \frac{\hat{\underline{R}}_{n+1}}{\underline{R}_{n+1}}].
\end{aligned}$$

$$\begin{aligned}
&E\left[\bar{\sigma}^\top e^{-\beta}\int_{(n+1)\Delta t}^{(n+2)\Delta t} e^{\beta s} dW_s \frac{\hat{\underline{R}}_{n+1}}{\underline{R}_{n+1}}\right] \\
&= \sum_i w_i \bar{\sigma}^\top(\sigma_i - \bar{\sigma}) e^{-\beta(1+\Delta t)} \\
&\quad \times \int_0^{\Delta t} e^{\beta s} ds \ \exp\bigg\{\left(\mu_i - \mu_w - \frac{1}{2}(\|\sigma_i\|^2 - \sigma_w^2)\right)\Delta t \\
&\quad + \frac{1}{2}\|\sigma_i - \bar{\sigma}\|^2 e^{-2\beta\Delta t}\int_0^{\Delta t} e^{2\beta s} ds + (1 - e^{-\beta\Delta t})(\theta_i - \bar{\theta})\bigg)\bigg\}. \qquad (37)
\end{aligned}$$

We only need its coefficient on Δt^2, which is

$$\begin{aligned}
&\sum_i w_i(\bar{\sigma}^\top\sigma_i) e^{-\beta}(\mu_i - \mu_w - \frac{1}{2}(\|\sigma_i\|^2 - \sigma_w^2) + \frac{1}{2}\|\sigma_i - \bar{\sigma}\|^2 + \beta(\theta_i - \bar{\theta})) \\
&= \sum_i w_i(\bar{\sigma}^\top\sigma_i) e^{-\beta}(\mu_i - \mu_w + \sigma_w^2 - \sigma_i^\top\bar{\sigma} + \beta(\theta_i - \bar{\theta})) \\
&= e^{-\beta}\left(\gamma_L + \sum_i w_i(\bar{\sigma}^\top\sigma_i)\beta(\theta_i - \bar{\theta})\right).
\end{aligned}$$

For the first factor in (37), we have

$$E\left[\frac{\hat{\underline{V}}(n\Delta t)}{\underline{V}(n\Delta t)}\right] = \prod_{k=1}^n E\left[\frac{\hat{\underline{R}}_{n+1}}{\underline{R}_{n+1}}\right] = \prod_{k=1}^n (1 + O(\Delta t^2)) = 1 + O(\Delta t).$$

So, we have

$$E\left[\bar{\sigma}^\top e^{-\beta}\int_{(n+1)\Delta t}^{(n+2)\Delta t} e^{\beta s} dW_s\left(\frac{\hat{\underline{V}}((n+1)\Delta t)}{\underline{V}((n+1)\Delta t)} - \frac{\hat{\underline{V}}(n\Delta t)}{\underline{V}(n\Delta t)}\right)\right] = \underline{\gamma}_L \Delta t^2 + O(\Delta t^3).$$

For the case $k \le n$, following the same argument as in the proof of Glasserman[12, Prop. 4], we get

$$E\left[\bar{\sigma}^\top e^{-\beta}\int_{k\Delta t}^{(k+1)\Delta t} e^{\beta s}dW_s\left(\frac{\hat{\underline{V}}((n+1)\Delta t)}{\underline{V}((n+1)\Delta t)} - \frac{\hat{\underline{V}}(n\Delta t)}{\underline{V}(n\Delta t)}\right)\right] = O(\Delta t^4),$$

and then (36) becomes

$$N^{-1/2}Cov(\log \underline{V}(T), X_N) = \frac{\gamma_L T^2}{N} + O(N^{-2}).$$

The proof for part (ii) follows the same line as the one in Glasserman [12]. The only modification needed is that now the Girsanov transformation is a little more general, the change of measure now changing the standard Brownian motion $W(T)$ to a Gaussian process $\int_0^T e^{\beta s}W(s)$. □

8. Dealing with defaults. As explained in Section 2, jumps in asset values can produce negative portfolio values, even under continuous rebalancing. Here we address this issue in greater detail.

Assume that once a portfolio defaults (i.e., drops to zero or below), it is absorbed at zero forever. It follows from (2) that such a default occurs in a continuously rebalanced portfolio if and only if there is a jump before time T with $\sum_i w_i Y^i \le 0$. Under assumption (3), the continuously rebalanced portfolio will therefore never default.

The discretely rebalanced portfolio will default at time t in the n^{th} time interval if and only if t is the first time that $t \in [(n-1)\Delta t, n\Delta t]$ with $\tilde{t} = t - \Delta t\lfloor \frac{t}{\Delta t}\rfloor$ and

$$\begin{aligned}\hat{R}_{n,N}(t) &= \left(\frac{\hat{V}(t)}{\hat{V}((n-1)\Delta t)}\right)\\ &= \sum_{i=1}^d w_i \exp\left\{\left(\mu_i - \frac{1}{2}\sum_{j=1}^d \sigma_{ij}^2\right)\tilde{t} + \sigma_i^\top W(\tilde{t})\right\}\prod_{j=1}^{N(\tilde{t})} Y_j^i \le 0.\end{aligned} \tag{38}$$

Let I_d^n denote the indicator of default for the discrete portfolio, where $I_d^n = 1$ means that the portfolio defaults in n^{th} time interval, while $I_d^n = 0$ if not.

LEMMA 8.1. *Given assumption (3), $P(I_d^n = 0) = O(\Delta t^2)$.*

PROOF. Under assumption (3), first we focus on the case of only one jump

$$
\begin{aligned}
P(I_d^n = 1) &\le P(I_d^n = 1, N(\Delta t) = 1) + P(N(\Delta t) > 1) \\
&= P\Bigg(\sum_i w_i \exp\Bigg\{\Bigg(\mu_i - \frac{1}{2}\sum_{j=1}^d \sigma_{ij}^2\Bigg)s + \sigma_i^\top W(s)\Bigg\} \prod_{j=1}^{N(s)} Y_j^i < 0, \\
&\qquad \text{for some } s \in [0, \Delta t] | N(\Delta t) = 1\Bigg) P(N(\Delta t) = 1) + O(\Delta t^2).
\end{aligned} \tag{39}
$$

The last term $O(\Delta t^2)$ is from the probability of more than one jump within the time interval. Now we simplify the first term by using the fact of having only one jump, and also apply a first-order Taylor expansion to the exponential:

$$
\begin{aligned}
&P\Bigg(\sum_i w_i \exp\Bigg\{\Bigg(\mu_i - \frac{1}{2}\sum_{j=1}^d \sigma_{ij}^2\Bigg)s + \sigma_i^\top W(s)\Bigg\} \prod_{j=1}^{N(s)} Y_j^i < 0, \\
&\text{for some } s \in [0, \Delta t] | N(\Delta t) = 1\Bigg) \\
&= P\Bigg(\sum_i w_i(1 + \sigma_i^\top W(s) + \bar{r}_i(s))Y^i < 0, \text{ for some } s \in [0, \Delta t]\Bigg),
\end{aligned}
$$

where $\bar{r}_i$ is the remainder in the Taylor approximation, with L_2-norm $O(\Delta t)$. Then

$$
\begin{aligned}
&P\Bigg(\sum_i w_i(1 + \sigma_i^\top W(s) + \bar{r}_i(s))Y^i < 0, \text{ for some } s \in [0, \Delta t]\Bigg) \\
&\le P\Bigg(\max(|\sigma_i^\top W(s) + \bar{r}_i(s)|)\sum_i |w_i Y^i| > \sum_i w_i Y^i, \text{ for some } s \in [0, \Delta t]\Bigg) \\
&\le P\Bigg(\sum_i |\sigma_i^\top W(s) + \bar{r}_i(s)| > \frac{\sum_i w_i Y^i}{\sum_i |w_i Y^i|}, \text{ for some } s \in [0, \Delta t]\Bigg).
\end{aligned}
$$

Conditioning on the Y^i and applying Chebyshev's inequality yields

$$
\begin{aligned}
&P\Bigg(\sum_i |\sigma_i^\top W(s) + \bar{r}_i(s)| > \frac{\sum_i w_i Y^i}{\sum_i |w_i Y^i|}, \text{ for some } s \in [0, \Delta t]\Bigg) \\
&\le E\Bigg[Var\Bigg(\sum_i |\sigma_i^\top W(s) + \bar{r}_i(s)|\Bigg)\Bigg(\frac{\sum_i w_i Y^i}{\sum_i |w_i Y^i|}\Bigg)^2, \text{ for some } s \in [0, \Delta t]\Bigg] \\
&\le Var\Bigg(\sum_i |\sigma_i^\top W(s) + \bar{r}_i(s)|\Bigg) E\Bigg[\Bigg(\frac{\sum_i w_i Y^i}{\sum_i |w_i Y^i|}\Bigg)^2\Bigg] \\
&= O(\Delta t).
\end{aligned}
$$

Substituting these results in (39) concludes the proof. □

PROPOSITION 8.2. *Under conditions (3) and (4), we have*

$$\left\|\frac{\hat{V}(T)-V(T)}{V(T)}\right\| - \left\|\frac{\hat{V}(T)-V(T)}{V(T)} I_{\{I_d^n=0 \text{ for all } n=1,\dots,N\}}\right\| = O(\Delta t).$$

PROOF. With N fixed, since $\{\hat{R}_{n,N} : n = 1, \dots, N\}$ are i.i.d., from (38) and the surrounding discussion, the number of intervals n until $I_d^n = 1$ has a geometric distribution, and

$$P(I_d^n = 1 \text{ for some } n = 1, \dots, N) = O(\Delta t).$$

If the discrete portfolio defaults, $\hat{V}(T) = 0$ and $\frac{\hat{V}(T)-V(T)}{V(T)} = -1$, so

$$\left\|\frac{\hat{V}(T)-V(T)}{V(T)} I_{\{I_d^n=1 \text{ for some } n=1,\dots,N\}}\right\| = O(\Delta t).$$

□

Proposition 8.2 confirms that we can ignore possible defaults in the discretely rebalanced portfolio, because the limits in Theorem 2.2 and 2.4 are scaled by $\sqrt{N} = \Delta t^{-1/2}$, while the errors introduced by ignoring defaults are of order $O(\Delta t)$. In fact, we can even weaken our assumptions to allow $\sum_i w_i Y^i \le 0$, replacing (3) with the condition

$$E\left[\left(\frac{w_j Y^j}{\sum_i w_i Y^i}\right)^2 \Bigg| \sum_i w_i Y^i < 0\right] < \infty, \text{ for all } j = 1, \dots, d,$$

This suffices to show that defaults have a negligible effect on the relative error using a similar argument.

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Paul Glasserman
3022 Broadway, Uris Hall
New York, NY 10027, U.S.A.
E-mail: pg20@columbia.edu

Xingbo Xu
500 West 120th Street, S.W. Mudd Building
New York, NY 10027, U.S.A.
E-mail: xx2126@columbia.edu