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Dispersion models for geometric sums

Bent Jørgensen[a] **and Célestin C. Kokonendji**[b]

[a]*University of Southern Denmark*
[b]*Université de Franche-Comté*

Abstract. A new class of geometric dispersion models associated with geometric sums is introduced by combining a geometric tilting operation with geometric compounding, in much the same way that exponential dispersion models combine exponential tilting and convolution. The construction is based on a geometric cumulant function which characterizes the geometric compounding operation additively. The so-called v-function is shown to be a useful characterization and convergence tool for geometric dispersion models, similar to the variance function for natural exponential families. A new proof of Rényi's theorem on convergence of geometric sums to the exponential distribution is obtained, based on convergence of v-functions. It is shown that power v-functions correspond to a class of geometric Tweedie models that appear as limiting distributions in a convergence theorem for geometric dispersion models with power asymptotic v-functions. Geometric Tweedie models include geometric tiltings of Laplace, Mittag-Leffler and geometric extreme stable distributions, along with geometric versions of the gamma, Poisson and gamma compound Poisson distributions.

1 Introduction

We introduce a new class of dispersion models for geometric sums, defined as two-parameter families that combine geometric compounding with an operation called geometric tilting, in much the same way that exponential dispersion models combine convolution and exponential tilting [Jørgensen (1997, Chapter 3)]. A similar class of two-parameter families for extremes and survival data, called extreme dispersion models, was introduced by Jørgensen et al. (2010), combining the minimum operation with location shifts. A common trait for these three types of dispersion models is the use of a particular kind of cumulant generating function that characterizes a convolution-like operation additively in each case. To each such cumulant generating function corresponds a tilting operator, that takes over the role of the conventional exponential tilting operation, the latter being known in applied probability as the Cramér or Esscher transform [cf. Jørgensen et al. (2009)].

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A geometric sum $S(q)$ [Kalashnikov (1997, p. 3)], indexed by the probability parameter $q \in (0, 1]$, is defined by

$$S(q) = \sum_{k=1}^{N(q)} X_k, \tag{1.1}$$

where $X_1, X_2, \ldots$ are i.i.d. random variables independent of the geometric random variable $N(q)$. The latter has probability mass function $\Pr[N(q) = k] = q(1-q)^{k-1}$ for $k = 1, 2, \ldots,$ with the convention that $N(1) \equiv 1$. The geometric cumulant function (cf. Section 2) is designed to be additive with respect to the geometric compounding operation (1.1), and since the average number of terms for the sum is q^{-1}, the parameter q is analogous to the dispersion parameter of exponential and extreme dispersion models. It is hence natural to define a two-parameter dispersion model for geometric sums by introducing a geometric tilting parameter along with q, based on the geometric tilting operation (cf. Section 4). The structure of the parameter domains of geometric dispersion models (Section 5) is intimately related to geometric infinite divisibility, which we discuss in Section 3, along with an exponential mixture representation for geometric infinitely divisible laws.

In Section 6 we introduce the v-function, which is an important characterization and convergence tool for geometric dispersion models, much like the variance and slope functions are for exponential and extreme dispersion models, respectively. The v-function in fact reveals strong mathematical analogies between the three kinds of dispersion models, as evident in the existence of a key set of power and quadratic v-functions, similar to the main types of variance functions and slope functions of exponential and extreme dispersion models, respectively. For example, the constant v-function characterizes the asymmetric Laplace geometric dispersion model, just like the constant variance function characterizes the normal distribution.

In Section 7 we present a convergence theorem for v-functions, which leads to some new asymptotic results for geometric sums with finite variances. Conventional asymptotics for geometric sums give rise to limit laws in the class of geometric stable distributions [cf. Mittnik and Rachev (1991) and Kozubowski and Rachev (1999)] where the only law with finite variance is the Laplace distribution, just like the normal distribution is the only stable law with finite variance. Instead, we introduce a new asymptotic framework for geometric dispersion models, where geometric Tweedie models with power v-functions (cf. Section 8) emerge in the limit, in parallel to the Tweedie convergence framework for exponential dispersion models [Jørgensen et al. (1994)], based on power asymptotics for variance functions. In particular, we obtain a general Laplace convergence result [cf. Blanchet and Glynn (2007)], similar to the central limit theorem, and we obtain a new proof of Rényi's theorem on convergence to the exponential distribution. The class of geometric Tweedie models form a one-parameter class of geometric dispersion models, which are exponential mixtures of ordinary Tweedie models. This class includes geometric versions of the gamma, Poisson, and gamma compound Poisson models, as well as geometric tiltings of certain geometric extreme stable distributions.

2 Geometric compounding and geometric cumulants

We begin by introducing a new type of cumulant function adapted to geometric sums, which is crucial for our treatment of geometric tilting families and geometric dispersion models below. Throughout the paper we rely on Jørgensen (1997, Chapters 3–5) for standard results and notation for natural exponential families and exponential dispersion models.

We denote the ordinary cumulant generating function (CGF) for the random variable X by

$$\kappa(s) = \kappa(s; X) = \log \mathbb{E}(e^{sX}) \qquad \text{for } s \in \mathbb{R}$$

with effective domain $\mathrm{dom}(\kappa) = \{s \in \mathbb{R} : \kappa(s) < \infty\}$. We define the *geometric cumulant function* (GCF) for X by

$$C(s) = C(s; X) = 1 - e^{-\kappa(s)} \qquad \text{for } s \in \mathcal{D}(C) \tag{2.1}$$

with domain $\mathcal{D}(C) = \{s \in \mathbb{R} : C(s) < 1\} = \mathrm{dom}(\kappa)$. We recall that a CGF κ is a real analytic convex function, which is strictly convex unless X is degenerate. Hence, C is also real analytic, and the domain $\mathcal{D}(C)$, like $\mathrm{dom}(\kappa)$, is an interval. In fact, the derivative $\dot{C}(s) = e^{-\kappa(s)}\dot{\kappa}(s)$ has the same sign as $\dot{\kappa}(s)$ on the interior $\mathrm{int}(\mathcal{D}(C))$. Hence, by the convexity of κ, C is either monotone or u-shaped. Let $\mathcal{K}$ denote the set of CGFs κ such that $\mathrm{int}(\mathrm{dom}(\kappa)) \neq \varnothing$, and let $\mathcal{C}$ denote the corresponding set of functions C of the form (2.1), also with $\mathrm{int}(\mathcal{D}(C)) \neq \varnothing$. In these cases, either function κ or C characterizes the distribution of X. From now on, CGF and GCF refer to functions in $\mathcal{K}$ and $\mathcal{C}$, respectively.

We now derive the GCF for the geometric sum (1.1). Let κ and C denote the CGF and GCF for X_1, respectively, and recall that $N(q)$ has moment generating function (MGF)

$$\mathbb{E}[e^{sN(q)}] = [1 - q^{-1}(1 - e^{-s})]^{-1} \qquad \text{for } s < -\log(1-q), \tag{2.2}$$

corresponding to the GCF

$$C(s; N(q)) = q^{-1}(1 - e^{-s}) \qquad \text{for } s < -\log(1-q). \tag{2.3}$$

It follows that the geometric sum $S(q)$ has MGF

$$\begin{aligned} \mathbb{E}[e^{sS(q)}] &= \mathbb{E}[e^{N(q)\kappa(s)}] \\ &= \{1 - q^{-1}[1 - e^{-\kappa(s)}]\}^{-1} \\ &= [1 - q^{-1}C(s)]^{-1} \qquad \text{for } s \in \mathcal{D}(q^{-1}C), \end{aligned} \tag{2.4}$$

where

$$\mathcal{D}(q^{-1}C) = \{s \in \mathbb{R} : C(s) < q\} \subseteq \mathcal{D}(C) \qquad \text{for } q \leq 1. \tag{2.5}$$

Hence the GCF of $S(q)$ is proportional to C,

$$C(s; S(q)) = q^{-1}C(s) \qquad \text{for } s \in \mathcal{D}(q^{-1}C) \tag{2.6}$$

with the average sample size q^{-1} as the constant of proportionality. In particular, if $X_1 \stackrel{d}{=} N(r)$ is geometric with $r \in (0, 1]$, then (2.3) and (2.6) imply that $S(q) \stackrel{d}{=} N(rq)$ is again geometric [Kalashnikov (1997, p. 6)], where $\stackrel{d}{=}$ denotes equality in distribution.

The main reason for our interest in the GCF is that it characterizes the geometric compounding operation additively, as in (2.6), analogously to the way κ characterizes ordinary convolution additively,

$$\kappa(s; X_1 + \cdots + X_n) = n\kappa(s; X_1).$$

This analogy will be explored further in the following.

When $0 \in \operatorname{int}(\mathcal{D}(C))$, the derivatives $C^{(n)}(0) = C^{(n)}(0; X_1)$ are called the *geo-cumulants* of X_1. In particular, the first geo-cumulant is the mean, $\dot{C}(0) = \dot{\kappa}(0) = \mathbb{E}(X_1)$. The second geo-cumulant is

$$\mathbb{G}(X_1) = \ddot{C}(0) = \ddot{\kappa}(0) - \dot{\kappa}^2(0) = \operatorname{Var}(X_1) - \mathbb{E}^2(X_1), \tag{2.7}$$

which satisfies the inequalities

$$-\mathbb{E}^2(X_1) \leq \mathbb{G}(X_1) \leq \operatorname{Var}(X_1). \tag{2.8}$$

The geo-cumulants satisfy the scaling $C^{(n)}(0; cX_1) = c^n C^{(n)}(0; X_1)$, similar to ordinary cumulants. In particular, the second geo-cumulant $\mathbb{G}(X)$ satisfies

$$\mathbb{G}(cX) = c^2\mathbb{G}(X) \qquad \text{for } c \in \mathbb{R}. \tag{2.9}$$

The geo-cumulants of $S(q)$ are $q^{-1}C^{(n)}(0; X_1)$, the first two being

$$\mathbb{E}[S(q)] = q^{-1}\mathbb{E}(X_1), \qquad \mathbb{G}[S(q)] = q^{-1}\mathbb{G}(X_1), \tag{2.10}$$

where the average sample size q^{-1} again plays the same role as the sample size does for ordinary cumulants.

As noted by Klebanov et al. (1985), the exponential distribution plays the role of degenerate distribution for geometric sums. In fact, let $\operatorname{Exp}(\mu)$ denote the distribution with GCF

$$C(s) = s\mu \qquad \text{for } s\mu < 1, \tag{2.11}$$

which for $\mu > 0$ ($\mu < 0$) corresponds to a positive (negative) exponential variable with mean μ, while $\mu = 0$ corresponds to the degenerate distribution at 0. We refer to GCFs of the form (2.11) as the *degenerate case*. Since $\mathbb{G}(X_1) = 0$ for $X_1 \sim \operatorname{Exp}(\mu)$, we may interpret the operator $\mathbb{G}(X_1) = \operatorname{Var}(X_1) - \mathbb{E}^2(X_1)$ as a signed measure of the deviation of the random variable X_1 or its distribution from exponentiality. A combination of (2.6) and (2.11) shows that $X_1 \sim \operatorname{Exp}(\mu)$ implies $S(q) \sim \operatorname{Exp}(q^{-1}\mu)$ [Kalashnikov (1997, p. 7)].

3 Geometric infinite divisibility and exponential mixtures

We shall now consider geometric infinite divisibility and exponential mixtures, which are important prerequisites for our discussion of geometric dispersion models in Section 4.

Following Klebanov et al. (1985), we say that a random variable X (or its distribution) is *geometric infinitely divisible* if for any $q \in (0, 1)$ there exists a geometric sum $S(q)$ such that $X \overset{d}{=} S(q)$. If C denotes the GCF for X, we obtain from (2.6) that $C(s) = q^{-1}C(s; X_1)$ for $s \in \mathcal{D}(C)$. Hence X is geometric infinitely divisible if and only if $qC \in \mathcal{C}$ for any $q \in (0, 1)$. In particular we may extend the domain of C by analytic continuation to the largest interval for which $qC(s) < 1$, which in turn defines $\mathcal{D}(qC)$. We refer to the process of going from C to qC as *geometric division.* Note that all geometric infinite divisible distributions are infinite divisible in the classical sense.

Now let X and $Z > 0$ be random variables such that the conditional MGF of X given Z is

$$\mathbb{E}(e^{sX}|Z) = e^{Z\kappa(s)},$$

where κ is an infinitely divisible CGF. For $Z \sim \text{Exp}(\lambda)$ with $\lambda > 0$, the marginal distribution of X is called an *exponential mixture*, corresponding to the MGF

$$\mathbb{E}[e^{sX}] = \mathbb{E}\big[e^{Z\kappa(s)}\big] = [1 - \lambda\kappa(s)]^{-1}. \tag{3.1}$$

This implies that $\lambda\kappa \in \mathcal{C}$ for any $\lambda > 0$, and hence κ is a geometric infinitely divisible GCF. As shown by Klebanov et al. (1985), all geometric infinitely divisible MGFs are of the form (3.1). The following makes this statement precise.

Theorem 3.1. *Let $C \in \mathcal{C}$ be a given GCF. Then the following conditions are equivalent*:

1. *C is a geometric infinitely divisible GCF*;
2. *C is the GCF for an exponential mixture*;
3. *$\lambda C \in \mathcal{C}$ for any $\lambda > 0$*;
4. *$\lambda C \in \mathcal{K}$ for any $\lambda > 0$.*

Proof. 1. $\Leftrightarrow$ 3.: This equivalence follows by noting that for any $\lambda \geq 1$, the function λC is the GCF for a geometric sum, hence 3. is equivalent to $\lambda C \in C$ for all $\lambda \in (0, 1)$, which, by the above discussion, is equivalent to the definition of geometric infinite divisibility. 1. $\Rightarrow$ 4.: If C is geometric infinitely divisible, 3. implies that $(\lambda/n)C \in \mathcal{C}$ for any $\lambda > 0$ and integer n, and hence $[1 - (\lambda/n)C]^{-n}$ is an MGF. The weak limit as $n \to \infty$ is the function $e^{\lambda C}$, which is hence an MGF for any $\lambda > 0$, implying 4. 4. $\Rightarrow$ 2.: Condition 4. implies that C is an infinitely divisible CGF. The corresponding exponential mixture (3.1) has GCF λC, which implies 2. Finally, the implication 2. $\Rightarrow$ 1. has already seen shown above. This completes the proof. $\square$

Corollary 3.1. *Any geometric infinitely divisible GCF $C \in \mathcal{C}$ is convex, and strictly convex except in the degenerate case.*

Proof. Since $C \in \mathcal{K}$ in the geometric infinitely divisible case, it follows that C is convex, and strictly convex unless $C(s) = \mu s$ for some $\mu \in \mathbb{R}$, corresponding to the degenerate case (2.11). □

The geometric distribution, whose GCF (2.3) is concave, is not geometric infinitely divisible. We note that if C is a geometric infinitely divisible GCF, then so is the function

$$s \longmapsto C(s) + sa \tag{3.2}$$

for each $a \in \mathbb{R}$, which adds a to the mean, but leaves the second and higher-order geo-cumulants unchanged. That (3.2) is a CGF follows from the exponential mixture representation (3.1) with $\lambda = 1$ by noting that if κ is a CGF, then so is $\kappa(s) + as$, which corresponds to a translation by a. The operation (3.2) is called *geometric translation*.

4 Geometric tilting families and dispersion models

We shall now introduce a geometric tilting operation, similar to exponential tilting, which leads to our main definitions of geometric tilting families and geometric dispersion models.

Consider the set $\overline{\mathcal{C}}$ of real analytic functions $C : \mathrm{dom}(C) \to \mathbb{R}$ satisfying $0 \in \mathrm{dom}(C)$ and $C(0) = 0$, where $\mathrm{dom}(C)$ denotes the largest interval containing zero where C is analytic. We define the *tilting* of C by the amount $\theta \in \mathrm{dom}(C)$ as the function $C_\theta : \mathrm{dom}(C) - \theta \to \mathbb{R}$ given by

$$C_\theta(s) = C(\theta + s) - C(\theta) \qquad \text{for } s \in \mathrm{dom}(C) - \theta. \tag{4.1}$$

The tilting operation defines an equivalence relation on $\overline{\mathcal{C}}$. In particular, if $\kappa \in \mathcal{K}$, then κ_θ is the conventional exponential tilting of κ [e.g., Jørgensen (1997, p. 43)]. If we restrict the tilting operation to $\mathcal{K}$, the corresponding set of equivalence classes is the class of natural exponential families, that is, families of CGFs of the form $\{\kappa_\theta \in \mathcal{K} : \theta \in \mathrm{dom}(\kappa)\}$.

We now consider the restriction of the tilting operation to $\mathcal{C}$, by identifying a GCF $C \in \mathcal{C}$ with domain $\mathcal{D}(C)$ with its analytic continuation to $\mathrm{dom}(C) \supseteq \mathcal{D}(C)$. We call this tilting operation *geometric tilting*. The corresponding set of equivalence classes in $\mathcal{C}$ are called *geometric tilting families*, namely families of GCFs of the form

$$\{C_\theta \in \mathcal{C} : \theta \in \Theta\}, \tag{4.2}$$

where the parameter domain $\Theta \subseteq \mathrm{dom}(C)$ will be identified below.

We now introduce geometric dispersion models as two-parameter families of GCFs obtained by combining the operations of geometric tilting and geometric compounding. For a given $C \in \mathcal{C}$, we consider the following two types of GCFs,

$$s \longmapsto \gamma^{-1} C_\theta(s) \qquad \text{(additive case)}, \tag{4.3}$$

$$s \longmapsto \gamma^{-1} C_\theta(\gamma s) \qquad \text{(reproductive case)}. \tag{4.4}$$

The class of *additive* and *reproductive geometric dispersion models* are defined by (4.3) and (4.4), respectively, the two cases being linked by a scale transformation. As we shall see below, we can parametrize a geometric tilting family locally (but not necessarily globally) by the mean $\mu = \dot{C}(\theta)$ of (4.4), in which case we denote the distributions corresponding to (4.3) and (4.4) by $\mathrm{GD}^*(\mu, \gamma)$ and $\mathrm{GD}(\mu, \gamma)$, respectively.

By comparison, the class of additive/reproductive exponential dispersion models is obtained from a CGF $\kappa \in \mathcal{K}$ in a similar way [Jørgensen (1997, Chapter 3)], by considering CGFs of the form

$$s \longmapsto \gamma^{-1} \kappa_\theta(s) \qquad \text{(additive case)}, \tag{4.5}$$

$$s \longmapsto \gamma^{-1} \kappa_\theta(\gamma s) \qquad \text{(reproductive case)}. \tag{4.6}$$

We denote the distributions (4.3) and (4.4) by $\mathrm{ED}^*(\mu, \gamma)$ and $\mathrm{ED}(\mu, \gamma)$, respectively, where $\mu = \dot{\kappa}(\theta)$ denotes the mean.

One of the main characteristics of additive exponential dispersion models is that the n-fold convolution of $\mathrm{ED}^*(\mu, \gamma)$ with itself is $\mathrm{ED}^*(\mu, \gamma/n)$, and hence belongs to the same family. Similarly, an additive geometric dispersion model $\mathrm{GD}^*(\mu, \gamma)$ is closed under geometric compounding. Thus, if $\{X_k\}$ is a sequence of i.i.d. random variables with distribution $\mathrm{GD}^*(\mu, \gamma)$, then the corresponding geometric sum has distribution

$$\sum_{k=1}^{N(q)} X_k \sim \mathrm{GD}^*(\mu, \gamma q) \qquad \text{for } q \in (0, 1], \tag{4.7}$$

where the random variables on the left-hand side are assumed independent. As we shall see in the following, there are many other analogies between geometric and exponential dispersion models.

Example 4.1 (Asymmetric Laplace model). The asymmetric Laplace model [Kotz et al. (2001, Chapter 3)] is the geometric dispersion model generated by the Laplace distribution. It has probability density functions of the form

$$f(x; \mu, \gamma) = \frac{1}{\sqrt{2\gamma + \mu^2}} \exp\left\{ \frac{1}{\gamma} \left[x\mu - |x| \sqrt{2\gamma + \mu^2} \right] \right\} \qquad \text{for } x \in \mathbb{R}, \tag{4.8}$$

where the mean is $\mu \in \mathbb{R}$ and $\gamma > 0$, corresponding to the GCF

$$C(s) = \frac{\gamma}{2} s^2 + \mu s. \tag{4.9}$$

This distribution is denoted $\mathrm{GT}_0(\mu, \gamma)$, conforming with the notation of Section 8. The ordinary Laplace distribution is obtained for $\mu = 0$, whereas for $\mu \neq 0$, (4.8) may be obtained by exponential tilting of the Laplace distribution, in agreement with (5.4) below. Alternatively, (4.9) may be obtained by geometric translation of the Laplace distribution by the amount μ, in much the same way that exponential tilting of the normal distribution is equivalent to a location shift.

5 Parameter domains for geometric dispersion models

We shall now determine the domain for the parameter (θ, γ) of an additive geometric dispersion model, which is crucial in order to understand the structure of the models. First we consider the case where the models are geometric infinitely divisible.

Proposition 5.1. *If $C \in \mathcal{C}$ is geometric infinitely divisible, then $C_\theta \in \mathcal{C}$ for any $\theta \in \mathrm{dom}(C)$, and all such C_θ are geometric infinitely divisible.*

Proof. By Theorem 3.1, a geometric infinitely divisible GCF C is also an infinitely divisible CGF. This implies that C_θ is an infinitely divisible CGF for any $\theta \in \mathrm{dom}(C)$ [Barndorff-Nielsen (1978, p. 136)]. By taking $\kappa = C_\theta$ in the exponential mixture representation (3.1) we conclude that $\lambda C_\theta \in \mathcal{C}$ for all $\lambda > 0$ and $\theta \in \mathrm{dom}(C)$, and hence C_θ is geometric infinitely divisible. □

In the infinitely divisible case, this result implies that the parameter domain for (θ, γ) is the Cartesian product $\mathrm{dom}(C) \times \mathbb{R}_+$. By the trivial observation that $\gamma^{-1}C_\theta(s) = \gamma^{-1}C(\theta + s) - \gamma^{-1}C(\theta)$, it follows that the operations of geometric division/compounding and geometric tilting commute in the infinitely divisible case, in much the same way that convolution and exponential tilting commute in the case of infinitely divisible exponential dispersion models. Outside the geometric infinitely divisible case, the next result can help determine the limit for the geometric division process.

Lemma 5.1. *Let X be a random variable with GCF C satisfying $0 \in \mathrm{int}(\mathcal{D}(C))$. If X is not geometric infinitely divisible, there exists a $\gamma_0 \in [1, \infty)$ such that $\gamma^{-1}C \in \mathcal{C}$ for any $0 < \gamma \leq \gamma_0$, but not for $\gamma > \gamma_0$.*

Proof. We note from (2.6) that if $C \in \mathcal{C}$, then $q^{-1}C \in \mathcal{C}$ for any $q \in (0, 1]$. Let us define $\gamma_0 \in [1, \infty)$ by $\gamma_0 = \sup\{\gamma \geq 1 : \gamma^{-1}C \in \mathcal{C}\}$. It follows that for any $\varepsilon > 0$ there exists a $\gamma \in (\gamma_0 - \varepsilon, \gamma_0)$ such that $\gamma^{-1}C \in \mathcal{C}$, and hence by geometric division $(q\gamma)^{-1}C \in \mathcal{C}$ for any $q \in (0, 1]$. We conclude that $\gamma^{-1}C \in \mathcal{C}$ for any $0 < \gamma < \gamma_0$, but not for $\gamma > \gamma_0$. Let $\{\gamma_n\}$ be a positive sequence such that $\gamma_n \nearrow \gamma_0$, and hence

$\gamma_n^{-1}C(s) \to \gamma_0^{-1}C(s)$ as $n \to \infty$ for $s \in \mathcal{D}(\gamma_1^{-1}C)$. The corresponding sequence of MGFs also converges,

$$[1-\gamma_n^{-1}C(s)]^{-1} \to [1-\gamma_0^{-1}C(s)]^{-1} \qquad \text{as } n \to \infty. \tag{5.1}$$

Since $0 \in \operatorname{int}(\mathcal{D}(\gamma_1^{-1}C))$, we conclude from Theorem 1[1] of Jensen and Nielsen (1997) that there exists a probability measure P such that the sequence of probability measures P_n corresponding to the left-hand side of (5.1) converges weakly to P. It follows that the sequence $[1-\gamma_n^{-1}C(s)]^{-1}$ converges to the MGF of P for $s \in \mathcal{D}(\gamma_1^{-1}C)$, which in view of (5.1) implies that $(1-\gamma_0^{-1}C)^{-1}$ is the MGF for P. Hence, $\gamma_0^{-1}C \in \mathcal{C}$, which completes the proof. □

Based on these results, we define the domain for γ be $\Gamma = \mathbb{R}_+$ and set $\gamma_0 = \infty$ in the geometric infinitely divisible case, and $\Gamma = (0, \gamma_0]$ in the nongeometric infinitely divisible case. For each $\gamma \in \Gamma$ the effective domain of the GCF $\gamma^{-1}C$ is

$$\mathcal{D}(\gamma^{-1}C) = \{s \in \mathbb{R} : C(s) < \gamma\}.$$

The domains are nested, such that $\mathcal{D}(\gamma_1^{-1}C) \subseteq \mathcal{D}(\gamma_2^{-1}C)$ for $\gamma_1 < \gamma_2$ in Γ, in line with (2.5).

The next step is to derive the form of the parameter domain for (θ, γ) when C is not geometric infinitely divisible.

Lemma 5.2. *Consider a geometric dispersion model generated from the GCF $C \in \mathcal{C}$, where C is not geometric infinitely divisible, and hence $\gamma_0 < \infty$. Then the function $\gamma^{-1}C_\theta$ is a GCF if and only if (θ, γ) belongs to the set*

$$\{(\theta, \gamma) \in \mathbb{R} \times \mathbb{R}_+ : 0 < \gamma \le \gamma_0 - C(\theta)\}, \tag{5.2}$$

and such a GCF may be constructed by a combination of exponential tilting and geometric division/compounding.

Proof. Let $0 < \rho \le \gamma_0$, and consider the MGF corresponding to $\rho^{-1}C$,

$$\mathbb{M}(s;\rho) = [1-\rho^{-1}C(s)]^{-1} \tag{5.3}$$

with effective domain $\mathcal{D}(\rho^{-1}C) = \{s \in \mathbb{R} : C(s) < \rho\}$. The exponential tilting of (5.3) by the amount $\theta \in \mathcal{D}(\rho^{-1}C)$ corresponds to the following MGF (as a function of s)

$$\frac{\mathbb{M}(\theta+s;\rho)}{\mathbb{M}(\theta;\rho)} = \left[\frac{1-\rho^{-1}C(\theta+s)}{1-\rho^{-1}C(\theta)}\right]^{-1} = [1-\gamma^{-1}C_\theta(s)]^{-1}, \tag{5.4}$$

[1] The statement of this theorem in Jensen and Nielsen (1997) contains a small typographical error, in that 0 must be assumed to be in the *interior* of the set of convergence points for the sequence of MGFs.

where

$$\gamma = \rho - C(\theta). \tag{5.5}$$

This implies that $\gamma^{-1}C_\theta$ is a GCF if and only if γ belongs to the interval $(0, \gamma_0 - C(\theta))$. This interval is not empty, since the condition $\theta \in \mathcal{D}(\gamma^{-1}C) \subseteq \mathcal{D}(\gamma_0^{-1}C)$ implies that $\gamma_0 - C(\theta) > 0$. Hence, (5.2) describes the largest possible domain for (θ, γ) such that $\gamma^{-1}C_\theta$ is a GCF, and such a GCF is constructed by a combination of the geometric division/compounding (5.3) followed by the exponential tilting (5.4). □

Corollary 5.1. *The parameter domain for the geometric tilting family* $\{C_\theta \in \mathcal{C} : \theta \in \Theta\}$ *is* $\Theta = \mathrm{dom}(C)$ *in the geometric infinitely divisible case*, *and*

$$\Theta = \{\theta \in \mathbb{R} : C(\theta) \le \gamma_0 - 1\} \tag{5.6}$$

in the nongeometric infinitely divisible case.

Proof. In order to determine the set of θ for which $C_\theta \in \mathcal{C}$, we take $\gamma = 1$, or equivalently $\rho = 1 + C(\theta)$, in the proof of Lemma 5.2. The requirement that $\rho \le \gamma_0$ then implies the desired result. □

Example 5.1 (Geometric family). By (2.2), the geometric GCF is $C(s) = q^{-1}(1 - e^{-s})$. Geometric tilting then yields the GCF

$$C_\theta(s) = (qe^\theta)^{-1}(1 - e^{-s}) \qquad \text{for } s \le -\log(1 - qe^\theta), \tag{5.7}$$

which is again a geometric distribution. Without loss of generality, we may take $q = 1$. The geometric distribution is hence seen to be a geometric tilting family with probability parameter $e^\theta \le 1$. Furthermore, the corresponding additive geometric dispersion model defined from (5.7), corresponding to the GCF $\gamma^{-1}C_\theta$, is again geometric, with parameter domain given by $\theta \le -\log \gamma$. In this case, only the parameter γe^θ is identifiable (cf. Theorem 6.2).

The operations of exponential tilting and geometric division/compounding employed in the construction of the GCF (5.4) in the proof of Lemma 5.2 may also be applied in reverse order. Thus, the expression

$$\gamma^{-1}C_\theta(s) = \left[\gamma e^{\kappa(\theta)}\right]^{-1}\left[1 - e^{-\kappa_\theta(s)}\right] \tag{5.8}$$

shows that the GCF $\gamma^{-1}C_\theta$ may be also be obtained by first applying an exponential tilting to κ, which gives the GCF $1 - e^{-\kappa_\theta}$, followed by geometric division/compounding. In the special case of geometric compounding, we shall now derive the corresponding expression for the probability density function of a geometric dispersion model.

Proposition 5.2. *Assume that $\gamma e^{\kappa(\theta)} \le 1$. Then the additive geometric dispersion model with GCF $\gamma^{-1}C_\theta(s)$ has probability density function*

$$f(x;\theta,\gamma) = \gamma e^{\theta x} \sum_{k=1}^{\infty} \left[e^{-\kappa(\theta)} - \gamma\right]^{k-1} g^{*k}(x), \tag{5.9}$$

where g^{*k} denotes the probability density function of the k-fold convolution with itself of the distribution corresponding to the GCF C.

Proof. Note first that the assumption $\gamma e^{\kappa(\theta)} \le 1$ implies that (θ,γ) belongs to the domain (5.2), while at the same time making the right-hand side of (5.8) a geometric sum with probability parameter $\gamma e^{\kappa(\theta)}$. The k-fold convolution of the CGF κ_θ with itself has probability density function $g^{*k}(x)e^{\theta x - k\kappa(\theta)}$, and hence the geometric sum (5.8) has probability density function

$$\begin{aligned} f(x;\theta,\gamma) &= \gamma e^{\kappa(\theta)} \sum_{k=1}^{\infty} \left[1 - \gamma e^{\kappa(\theta)}\right]^{k-1} g^{*k}(x) e^{\theta x - k\kappa(\theta)} \\ &= \gamma e^{\theta x} \sum_{k=1}^{\infty} \left[e^{-\kappa(\theta)} - \gamma\right]^{k-1} g^{*k}(x) \end{aligned}$$

as desired. □

The following example illustrates some of the issues that may be encountered when applying the above results.

Example 5.2 (Normal family). Let us consider generating a geometric tilting family from the standard normal distribution with GCF $C(s) = 1 - e^{-s^2/2}$ for $s \in \mathbb{R}$. Since the value of γ_0 is not immediately obvious for this GCF, we shall discuss some possible scenarios that follow from Corollary 5.1, depending on the actual value of γ_0 in this case. At one extreme, the value $\gamma_0 = 1$ would imply $\Theta = \{0\}$, corresponding to a degenerate family. At the other extreme, a value for γ_0 of 2 or greater would imply $\Theta = \mathbb{R}$, although the following argument rules out this possibility. In fact, the first inequality of (2.8) implies that the derivatives of C must satisfy

$$-\dot{C}^2(\theta) \le \ddot{C}(\theta) \qquad \text{for } \theta \in \Theta,$$

and straightforward calculations show that $\Theta \subseteq [-\theta_0, \theta_0]$, where $\theta_0 \approx 1.3147$ is the positive root of the equation $\theta^2 C(\theta) = 1$. This implies that $\gamma_0 \le 1 + C(\theta_0) \approx 1.5786$. In particular, the standard normal distribution is not geometric infinitely divisible. We also note that C is convex on $(-1, 1)$ and concave outside this interval. Hence, in case Θ is bigger than $[-1, 1]$, the family has both convex and concave subfamilies.

6 The v-function

For a natural exponential family generated from the CGF κ, it is well known that the corresponding variance function $V = \ddot{\kappa} \circ \dot{\kappa}^{-1}$ is a useful characterization and convergence tool. We shall now introduce the analogously defined v-function for geometric dispersion models, and show that it has similar properties.

Let the GCF $C = 1 - e^{-\kappa} \in \mathcal{C}$ be given, and consider the geometric tilting family $\{C_\theta : \theta \in \Theta\}$ generated by C, where Θ is defined by (5.6). All geo-cumulants of C_θ are finite for $\theta \in \operatorname{int}\Theta$, the first two being

$$\mu = \dot{C}_\theta(0) = \dot{C}(\theta) = e^{-\kappa(\theta)}\dot{\kappa}(\theta)$$

and

$$\ddot{C}_\theta(0) = \ddot{C}(\theta) = e^{-\kappa(\theta)}[\ddot{\kappa}(\theta) - \dot{\kappa}^2(\theta)].$$

Let $\Theta_0 \subseteq \Theta$ be a nondegenerate interval for which $\ddot{C}(\theta)$ has constant sign, such that $\dot{C}(\theta)$ is strictly monotone on Θ_0, with μ belonging to the interval $\Psi_0 = \dot{C}(\Theta_0)$. Here we define μ by continuity at the endpoint(s) of Θ_0 that belong to Θ_0 [Jørgensen (1997, p. 46)] allowing infinite values of μ. We say that the family is *locally convex* or *locally concave* on Θ_0, according to the sign of $\ddot{C}(\theta)$. We may then parametrize the family locally by the mean μ, and we denote the corresponding family member by $\mathrm{GE}(\mu)$. For a globally convex or concave family, we may parametrize the family globally by $\mu \in \Psi = \dot{C}(\Theta)$.

Theorem 6.1. *Consider a locally convex (concave) geometric tilting family and define the local v-function $v : \Psi_0 \to \mathbb{R}$ by*

$$v(\mu) = \ddot{C} \circ \dot{C}^{-1}(\mu) \qquad \text{for } \mu \in \Psi_0, \tag{6.1}$$

where v is defined by continuity at endpoints of Ψ_0 belonging to Ψ_0, and where $v(\mu)$ is positive (negative) for all $\mu \in \Psi_0$. Then v characterizes the family among all geometric tilting families.

The fact that a geometric tilting family is characterized by the relations between its two first geo-cumulants provides an example of a family with finitely generated cumulants in the sense of Pistone and Wynn (1999). For a geometric dispersion model generated by C, we refer to C and v as the unit GCF and unit v-function, respectively.

Proof of Theorem 6.1. The proof is similar to the proof that a natural exponential family is characterized by its variance function [Jørgensen (1997, p. 51)]. We first show that the v-function does not depend on the choice of the GCF C representing the family. Thus, for given $\theta \in \Theta$, let us derive the local v-function corresponding to C_θ. For $s \in \operatorname{dom}(C) - \theta$ we obtain

$$\dot{C}_\theta(s) = e^{-\kappa(\theta+s)}\dot{\kappa}(\theta + s) = \dot{C}(\theta + s)$$

so that $\dot{C}_\theta(\Theta_0 - \theta) = \dot{C}(\Theta_0) = \Psi_0$. The second derivative is

$$\ddot{C}_\theta(s) = e^{-\kappa(\theta+s)}[\ddot{\kappa}(\theta+s) - \dot{\kappa}^2(\theta+s)] = \ddot{C}(\theta+s),$$

and hence

$$\ddot{C}_\theta \circ \dot{C}_\theta^{-1}(\mu) = \ddot{C} \circ \dot{C}^{-1}(\mu) = v(\mu) \qquad \text{for } \mu \in \Psi_0.$$

It follows that C_θ yields the same local v-function as C, so that v represents an intrinsic property of the family.

To see that v characterizes the family among all geometric tilting families, we derive an inversion formula for v. If the GCF C satisfies (6.1), then $\dot{C}^{-1}$ satisfies the equation

$$\frac{d\dot{C}^{-1}}{d\mu}(\mu) = \frac{1}{\ddot{C} \circ \dot{C}^{-1}(\mu)} = \frac{1}{v(\mu)}.$$

For given v, the set of solutions to this equation are of the form $\dot{C}^{-1}(\mu) - \theta$, where $-\theta$ is an arbitrary constant. By solving the equation $s = \dot{C}^{-1}(\mu) - \theta$ with respect to μ we obtain $\mu = \dot{C}(\theta + s)$, and integration in turn yields the function $C(\theta+s) - C(\theta) = C_\theta(s)$ satisfying the initial condition $C_\theta(0) = 0$. Since C_θ is a GCF if and only if $\theta \in \Theta$, we have thus recovered the geometric tilting family generated by C, as desired. □

Example 6.1 (Geometric gamma sum). Let us consider the geometric sum based on the gamma distribution, which is different from the geometric gamma distribution of Table 1 below. We start with the unit gamma distribution with MGF $M(s) = (1 - s/\alpha)^{-\alpha}$ for some $\alpha > 0$, and GCF

$$C(s) = 1 - (1 - s/\alpha)^\alpha.$$

Straightforward calculations show that the corresponding v-function is

$$v(\mu) = \frac{1-\alpha}{\alpha}\mu^p \qquad \text{for } \mu > 0,$$

Table 1 *The main types of quadratic unit v-functions, with mean domain Ψ and unit GCF C. GHS refers to the generalized hyperbolic secant distribution*

Family	$C(s)$	$v(\mu)$	Ψ
Asymmetric Laplace	$s^2/2$	1	$\mathbb{R}$
Geometric	$1 - e^{-s}$	$-\mu$	$(1, \infty)$
Geometric Poisson	$e^s - 1$	μ	$\mathbb{R}_+$
Geometric gamma	$-\log(1-s)$	μ^2	$\mathbb{R}_+$
Geometric negative binomial	$-\log(1-e^s)$	$\mu(1+\mu)$	$\mathbb{R}_+$
Geometric GHS	$-\log\cos s$	$1+\mu^2$	$\mathbb{R}$

where

$$p = 1 + (1 - \alpha)^{-1}. \tag{6.2}$$

In the case $0 < \alpha < 1$ ($p > 2$) this v-function is positive, and the corresponding convex geometric tilting families are geometric Mittag-Leffler models (cf. Section 8). The (degenerate) case $\alpha = 1$ ($v(\mu) = 0$) is the exponential distribution $\mathrm{Exp}(\mu)$ with $\mu > 0$. The case $\alpha > 1$ ($p < 1$) corresponds to concave geometric tilting families, which are not geometric infinitely divisible, and we refer to these models as *concave gamma models*. It remains an open problem to find the parameter domains for the corresponding geometric dispersion models; see also Example 5.2.

A further example of a concave geometric tilting family is obtained from the geometric distribution with GCF (5.7), which has v-function $v(\mu) = -\mu$ for $\mu > 1$. However, we shall now see that many important geometric tilting families are exponential mixtures, and hence convex.

Proposition 6.1. *The family of exponential mixtures* (3.1) *generated from a natural exponential family with variance function V yields a convex geometric tilting family with v-function V.*

Proof. Consider the natural exponential family of CGFs κ_θ generated from the CGF $\kappa \in \mathcal{K}$. In view of (3.1), this family of CGFs is identical to the family of GCFs for the corresponding exponential mixtures, which hence form a geometric tilting family, and which is convex due to the convexity of κ. The v-function of this family is identical to the variance function $\ddot{\kappa} \circ \dot{\kappa}^{-1}$ of the natural exponential family. □

Based on this result, we may now derive geometric dispersion models with quadratic and power v-functions (for the latter, see Section 8), which are analogues of the two corresponding types of exponential dispersion models; cf. Morris (1982) and Tweedie (1984). Table 1 shows the main types of families with quadratic v-functions (meaning polynomials of degree at most 2), including the geometric distribution, which is the only family with negative v-function in the table. The remaining five cases in the table are all obtained as exponential mixtures of the form (3.1), based on the normal, Poisson, gamma, negative binomial and generalized hyperbolic secant models, respectively. The first four families in the table have power v-functions, which will be considered further in Section 8. In particular, the asymmetric Laplace geometric dispersion model with unit v-function $v(\mu) = 1$ (cf. Example 4.1) plays the role of the normal distribution in the geometric setting. The geometric gamma distribution was introduced by Jose and Seetha Lakshmy (1999).

Example 6.2 (Geometric Poisson model). We may obtain the geometric Poisson model as an exponential mixture of Poisson distributions. The Poisson distribution has CGF $\kappa(s) = \lambda(e^s - 1)$ for $\lambda > 0$. Hence the corresponding exponential mixture has GCF $C(s) = \lambda(e^s - 1)$, leading to the geometric tilting family with GCF $C_\theta(s) = \lambda e^\theta(e^s - 1)$, which corresponds to a shifted geometric distribution starting at 0. We note in particular that only the mean λe^θ is identifiable, whereas the parameter (θ, γ) is not identifiable.

The duality between the geometric and geometric Poisson families, which is evident in Table 1, also extends to the lack of identifiability of the parameters for the corresponding additive geometric dispersion models (cf. Examples 5.1 and 6.2). However, we shall now show that these two examples are essentially the only cases with this defect.

Theorem 6.2. *Consider a locally convex or concave additive geometric dispersion model* $\mathrm{GD}^*(\mu, \gamma)$. *If the geometric tilting families* $\mathrm{GD}^*(\cdot, \gamma)$ *are identical for an interval of* γ*-values containing* 1, *then* $\mathrm{GD}^*(\mu, \gamma)$ *is a scaled geometric or geometric Poisson family.*

Proof. Let v with domain Ψ_0 be the local v-function of the geometric tilting family $\mathrm{GD}^*(\cdot, 1)$. Then there exists a nondegenerate subinterval $\Psi_1 \subseteq \Psi_0$ satisfying $\Psi_1 \subseteq \gamma\Psi_0$ for all γ in a small enough interval I around 0. The geometric tilting family $\mathrm{GD}^*(\cdot, \gamma)$ has v-function $\gamma^{-1}v(\gamma\mu)$, which by assumption, and in view of Theorem 6.1, is identical to v. By fixing a nonzero $\mu \in \Psi_1$ and taking $m = \gamma\mu \in \mu I$ we conclude that v has the form $v(m) = v(\gamma\mu) = \gamma v(\mu) = m v(\mu)/\mu$, which is proportional to m, corresponding to either a scaled geometric or a scaled geometric Poisson family. □

Theorem 6.2 implies that the parameter (μ, γ) is identifiable for all additive geometric dispersion models outside the two cases identified in the theorem. The situation is hence similar to additive exponential dispersion models, where only the scaled Poisson family has this lack of identifiability [Jørgensen (1997, p. 74)]. No such lack of identifiability occurs for a locally convex or concave reproductive geometric dispersion model, because the parameter (μ, γ) is identifiable from the first two geo-cumulants μ and $\gamma v(\mu)$ in this case, similar to the case of reproductive exponential dispersion models.

To round off the discussion of quadratic v-functions, we now consider the Bernoulli case, whose conspicuous absence from Table 1 is due to the lack of infinite divisibility for this distribution.

Example 6.3 (Bernoulli family). Let us consider the geometric tilting family generated from the Bernoulli distribution with probability parameter $1/2$, corresponding to the GCF

$$C(s) = \frac{e^s - 1}{e^s + 1} \qquad \text{for } s \in \mathbb{R}.$$

By Corollary 5.1 we find that the parameter domain Θ for this family contains the interval $\Theta_0 = \mathbb{R}_-$, on which the family is locally convex. Straightforward calculations show that the corresponding local v-function is given by $v(\mu) = \mu\sqrt{1-2\mu}$ for $\mu \in (0, 1/2)$. Since v is not the variance function for any natural exponential family because $v(\mu) \sim \frac{1}{2}\sqrt{1-2\mu}$ as $\mu \uparrow \frac{1}{2}$, it follows from Theorem 6.1 that the Bernoulli family is not geometric infinitely divisible. However, the value of γ_0 corresponding to C is not known. In view of (5.9) the probability mass function of the Bernoulli family is, for $\theta < 0$,

$$f(x; \theta, 1) = \frac{e^{\theta x}}{2} \sum_{k=1}^{\infty} \binom{k}{x} \left[\frac{1-e^\theta}{2(1+e^\theta)}\right]^{k-1} \qquad \text{for } x = 0, 1, \ldots.$$

7 Convergence of v-functions

We now turn to the topic of convergence of geometric tilting families based on convergence of their v-functions. Our main tool is a new convergence theorem similar to the Mora (1990) convergence theorem for variance functions [Jørgensen (1997, p. 54)]. Mora's theorem says that convergence of a sequence of variance functions, when the convergence is uniform on compact sets, implies weak convergence for the corresponding sequence of natural exponential families. We now present an analogous result for geometric tilting families, whose proof is given in the Appendix. In the theorem we use the notation $\mathrm{GE}(\mu)$ with $\mu \in \Psi_0 \subseteq [-\infty, \infty]$ for a locally convex or concave geometric tilting family, as defined in Section 6. We use the convention $1/\infty = 0$.

Theorem 7.1. *Let* $\{\mathrm{GE}_n(\mu) : n = 1, 2, \ldots\}$ *denote a sequence of locally convex or concave geometric tilting families having local* v*-functions* v_n *with domains* Ψ_n*. Suppose that*:

1. $\bigcap_{n=1}^{\infty} \Psi_n$ *contains a nonempty interval* Ψ_0;
2. $\lim_{n\to\infty} v_n(\mu) = v(\mu)$ *exists uniformly on compact subsets of* $\mathrm{int}\,\Psi_0$;
3. $v(\mu) \neq 0$ *for all* $\mu \in \mathrm{int}\,\Psi_0$ *or* $v(\mu) = 0$ *for all* $\mu \in \mathrm{int}\,\Psi_0$.

In the case $v(\mu) \neq 0$, *there exists a geometric tilting family* $\mathrm{GE}(\mu)$ *whose local* v*-function coincides with* v *on* $\mathrm{int}\,\Psi_0$, *such that for each* μ *in* $\mathrm{int}\,\Psi_0$ *the sequence of distributions* $\mathrm{GE}_n(\mu)$ *converges weakly to* $\mathrm{GE}(\mu)$. *In the case* $v(\mu) = 0$, $\mathrm{GE}_n(\mu)$ *converges weakly for each* μ *in* $\mathrm{int}\,\Psi_0$ *to the exponential distribution* $\mathrm{Exp}(\mu)$ *defined by* (2.11).

Remark 7.1. The proof of Theorem 7.1, which is given in the Appendix, is similar to the proof by Mora (1990); see also Jørgensen (1997, p. 54). However, the case of convergence to a zero v is new. A similar method of proof can be applied to show that convergence of a sequence of variance functions to zero implies weak convergence of the corresponding sequence of natural exponential families to a degenerate distribution. For an exponential dispersion model $\mathrm{ED}(\mu, \gamma)$ with mean μ, this implies that

$$\mathrm{ED}(\mu, \gamma) \xrightarrow{d} \delta_\mu \qquad \text{as } \gamma \downarrow 0, \tag{7.1}$$

where δ_μ denotes the degenerate distribution at μ, and $\xrightarrow{d}$ denotes weak convergence. This implies the weak law of large numbers for random variables with finite MGF, in view of the fact that $\mathrm{ED}(\mu, \gamma_0/n)$ is the distribution of the sample average of n i.i.d. random variables with distribution $\mathrm{ED}(\mu, \gamma_0)$.

We now give a new proof of *Rényi's theorem* [Kalashnikov (1997, p. 3)], based on Theorem 7.1 and convergence of v-functions. We first note that a locally convex or concave reproductive geometric dispersion model $\mathrm{GD}(\mu, \gamma)$ has local v-function of the form $\gamma v(\mu)$, which goes to zero as $\gamma \downarrow 0$. By Theorem 7.1 this implies

$$\mathrm{GD}(\mu, \gamma) \xrightarrow{d} \mathrm{Exp}(\mu) \qquad \text{as } \gamma \downarrow 0, \tag{7.2}$$

similar to (7.1). We may now derive Rényi's theorem as a special case of this result.

Theorem 7.2 (Rényi). *Consider the geometric sum $S(q)$ based on i.i.d. random variables X_i with distribution* $\mathrm{GE}(\mu)$ *for some $\mu \in \Psi_0$. Then*

$$qS(q) \xrightarrow{d} \mathrm{Exp}(\mu) \qquad \textit{as } q \downarrow 0. \tag{7.3}$$

Proof. We may consider $\mathrm{GE}(\mu) = \mathrm{GD}^*(\mu, 1)$ to be a member of the additive geometric dispersion model generated by $\mathrm{GE}(\mu)$. In view of (4.7) we obtain

$$qS(q) \sim q\,\mathrm{GD}^*(\mu, q) = \mathrm{GD}(\mu, q). \tag{7.4}$$

The result now follows from (7.2). □

It is well known that Rényi's theorem has the flavour of a law of large numbers for geometric sums, in the sense that (7.3) involves convergence of the geometric average $qS(q)$ as the average sample size q^{-1} goes to infinity. Our proof is based on the fairly strong assumption of a finite MGF, as compared with the weaker assumption of finite mean in the original proof by Rényi (1956). We also note the analogy with the exponential convergence for extreme dispersion models; cf. Jørgensen et al. (2010). Further applications of Theorem 7.1 will be considered below; see in particular Theorem 8.2.

8 Geometric Tweedie models

We have now investigated the main properties of geometric dispersion models, and examined a few basic examples. We conclude the paper by introducing the class of geometric Tweedie models, which includes several well-known distributions as special cases. Geometric Tweedie models turn out to have many properties in common with ordinary Tweedie models, and appear as limiting distributions in a convergence theorem (Theorem 8.2) similar to the Tweedie convergence theorem of Jørgensen et al. (1994).

8.1 General

We first recall the class of power unit variance functions $V(\mu) = \mu^p$ for $\mu \in \Omega_p$, where $p \in \Delta = \mathbb{R}\diagdown(0, 1)$. Here $\Omega_0 = \mathbb{R}$ and $\Omega_p = \mathbb{R}_+$ for $p \in \Delta\diagdown\{0\}$. The corresponding Tweedie exponential dispersion model with dispersion parameter γ is denoted $\mathrm{Tw}_p(\mu, \gamma)$ [cf. Jørgensen (1997, Chapter 4)]. Tweedie models are infinitely divisible, so in view of Proposition 6.1, the corresponding exponential mixtures (3.1) have power unit v-functions given by $v(\mu) = \mu^p$ for $\mu \in \Omega_p$. This defines the class of *Tweedie geometric dispersion models*, denoted $\mathrm{GT}_p(\mu, \gamma)$ for $p \in \Delta$, $\mu \in \Omega_p$ and $\gamma > 0$. We have already met the three quadratic v-functions above, corresponding to $p = 0$, 1 and 2 (cf. Table 1), which we consider in more detail in Section 8.2. Although of power form, the v-function $v(\mu) = -\mu$ ($\mu > 1$) of the geometric distribution is negative, and hence does not belong to the (locally) convex geometric Tweedie class.

As we shall see below, some geometric Tweedie models are geometric tiltings of geometric α-stable distributions for $\alpha \in (0, 1) \cup (1, 2]$. Here the parameter $\alpha \in [-\infty, 2]$ is defined from $p \in \Delta \cup \{\infty\}$ by

$$\alpha = \alpha(p) = 1 + (1 - p)^{-1} \tag{8.1}$$

[consistent with (6.2)] with the conventions that $\alpha(1) = -\infty$ and $\alpha(\infty) = 1$ [Jørgensen (1997, p. 131)]. The case $p = \infty$ ($\alpha = 1$) corresponds to exponential v-functions, which we will discuss in Section 8.5. Table 2 summarizes the main types of geometric Tweedie models.

The next theorem shows that geometric Tweedie models are characterized by a scaling property, similar to the characterization theorem for Tweedie exponential dispersion models [Jørgensen (1997, p. 128)].

Theorem 8.1. *Let* $\mathrm{GD}(\mu, \gamma)$ *be a nondegenerate locally convex geometric dispersion model on* $\Psi_0 \supseteq \mathbb{R}_+$, *such that for some* $\gamma > 0$

$$c^{-1}\,\mathrm{GD}(c\mu, \varphi_c\gamma) = \mathrm{GD}(\mu, \gamma) \qquad \text{for } c > 0 \text{ and } \mu \in \Psi_0, \tag{8.2}$$

where φ_c *is a positive function of* c. *Then* $\mathrm{GD}(\mu, \gamma)$ *is a geometric Tweedie model for some* $p \in \Delta$, *and* $\varphi_c = c^{2-p}$.

Table 2 *The main types of geometric Tweedie models*

Type	p	α	Support
Geometric extreme stable models	$p<0$	$1<\alpha<2$	$\mathbb{R}$
Asymmetric Laplace models	$p=0$	$\alpha=2$	$\mathbb{R}$
Geometric Poisson models	$p=1$	$\alpha=-\infty$	$\mathbb{N}_0$
Geometric compound Poisson models	$1<p<2$	$\alpha<0$	$\mathbb{R}_0$
Geometric gamma models	$p=2$	$\alpha=0$	$\mathbb{R}_+$
Geometric Mittag-Leffler models	$p>2$	$0<\alpha<1$	$\mathbb{R}_+$
Models with exponential v-functions	$p=\infty$	$\alpha=1$	$\mathbb{R}$

Proof. Calculating the second geo-cumulant on each side of (8.2) gives

$$c^{-2}\varphi_c\gamma v(c\mu)=\gamma v(\mu) \qquad \text{for } \mu, c>0, \tag{8.3}$$

where v is the local unit v-function of $\mathrm{GD}(\mu,\gamma)$. Taking $\mu=1$ in (8.3) gives $\varphi_c=c^2v(1)/v(c)$, which together with (8.3) implies that v satisfies the functional equation $v(1)v(c\mu)=v(c)v(\mu)$ for $c,\mu>0$. By the continuity of v, the solutions to this equation are of the form $v(\mu)=\lambda\mu^p$ for some $p\in\mathbb{R}$, where $\lambda>0$ because the family is locally convex and nondegenerate. This implies that $\varphi_c=c^{2-p}$. In view of Theorem 6.1, $\mathrm{GD}(\mu,\gamma)$ is hence a geometric Tweedie model in the case $p\in\Delta$. For values of p outside Δ (where in particular $p\neq 2$) we find that the domain for the dispersion parameter $\varphi_c\gamma=c^{2-p}\gamma$ on the left-hand side of (8.2) is $\mathbb{R}_+$, implying geometric infinite divisibility. Hence by Proposition 6.1 and Theorem 3.1, $\mathrm{GD}(\mu,\gamma)$ would have to be the exponential mixture of an exponential dispersion model with power unit variance function μ^p, but such models do not exist for $p\notin\Delta$ [Jørgensen (1997, p. 132)]. Hence, we conclude that $p\notin\Delta$ is not possible, concluding the proof. □

The next result shows that geometric dispersion models with power asymptotic v-functions are attracted to geometric Tweedie models via the fixed point (8.2), similar to the Tweedie convergence theorem for exponential dispersion models [Jørgensen (1997, pp. 148–149)]. Since many geometric dispersion models have power asymptotic v-functions, this implies that a large class of geometric dispersion models may be approximated by geometric Tweedie models.

Theorem 8.2. *Let* $\mathrm{GD}(\mu,\gamma)$ *denote a locally convex geometric dispersion model with unit v-function v on Ψ_0, such that either* $\inf\Psi_0\le 0$ *or* $\sup\Psi_0=\infty$. *Assume that for some $p\in\Delta\setminus\{2\}$ and $\varphi>0$ the unit v-function satisfies $v(\mu)\sim\varphi\mu^p$ as either $\mu\downarrow 0$ or $\mu\to\infty$. Then for each $\mu\in\Omega_p$*

$$\gamma^{1/(p-2)}\mathrm{GD}(\gamma^{1/(2-p)}\mu,\gamma)\xrightarrow{d}\mathrm{GT}_p(\mu,\varphi) \qquad \text{as } \gamma\downarrow 0 \text{ or } \gamma\to\infty. \tag{8.4}$$

In the case $\gamma\to\infty$, the model $\mathrm{GD}(\mu,\gamma)$ *is required to be geometric infinitely divisible.*

A similar convergence result for the case $p = 2$ will be considered in Section 8.2.

Proof of Theorem 8.2. We start by noting that for each given value of γ, the left-hand side of (8.4) is a geometric tilting family with mean μ, provided that γ is small (large) enough for $\gamma^{1/(2-p)}\mu$ to belong to Ψ_0. The corresponding v-function satisfies

$$\gamma\gamma^{2/(p-2)}v(\gamma^{1/(2-p)}\mu) \to \varphi\mu^p \qquad \text{as } \gamma \downarrow 0 \text{ or } \gamma \to \infty,$$

and hence converges to the v-function of $\mathrm{GT}_p(\mu, \varphi)$. To show that the convergence is uniform in μ on compact subsets of Ω_p, let us consider the case where $\gamma^{1/(2-p)}\mu \downarrow 0$ (the proof is similar in the case $\gamma^{1/(2-p)}\mu \to \infty$). Let $0 < M_1 \le \mu \le M_2 < \infty$ be given, and let μ_0 be such that

$$\left|\frac{v(\mu)}{\mu^p} - \varphi\right| < \varepsilon$$

for all $0 < \mu < \mu_0$. Then for any $\gamma > 0$ such that $\gamma^{1/(2-p)} < \mu_0/M_2$ we find that

$$\left|\frac{v(\gamma^{1/(2-p)}\mu)}{\gamma^{p/(2-p)}} - \varphi\mu^p\right| = \mu^p\left|\frac{v(\gamma^{1/(2-p)}\mu)}{[\gamma^{1/(2-p)}\mu]^p} - \varphi\right| \le (M_1^p + M_2^p)\varepsilon,$$

which shows that the convergence is uniform on the compact interval $M_1 \le \mu \le M_2$. The result (8.4) now follows from Theorem 7.1. □

Remark 8.1. In the case where γ tends to 0, the result (8.4) in effect concerns weak convergence of a centered and scaled geometric sum $S(q)$ as the average sample size q^{-1} tends to infinity, where the centering is achieved by geometric tilting. Similarly, the case where γ tends to infinity involves carrying the geometric division process of Section 3 to its limit, subject to centering and scaling, a process that requires the model $\mathrm{GD}(\mu, \gamma)$ to be geometric infinitely divisible. These two types of convergence are hence analogous to the central limit and infinitely divisible types of convergence discussed by Jørgensen (1997, p. 149) for ordinary Tweedie convergence. In the following we discuss some examples of the convergence result (8.4) in more detail.

A stronger version of Theorem 8.2 based on regular variation of the v-function can be developed along the same lines as Jørgensen et al. (2009, Theorem 5), but the present version of the theorem suffices for our purpose.

8.2 Geometric Tweedie models with quadratic v-functions

We now discuss the case of quadratic geometric Tweedie models, where the power parameter p is 0, 1 or 2. These models correspond to geometric versions of the normal, Poisson and gamma distributions, respectively, and each will now be discussed in turn.

The asymmetric Laplace model $\mathrm{GT}_0(\mu, \gamma)$ introduced in Example 4.1 is the geometric Tweedie model with $p = 0$ ($\alpha = 2$). This model is the geometric parallel of the normal distribution, and we shall now discuss the corresponding analogue of the central limit theorem. Let us first note that any locally convex geometric dispersion model $\mathrm{GD}(\mu, \gamma)$ with $0 \in \Psi_0$ satisfies $v(\mu) \sim v(0) > 0$ as $\mu \downarrow 0$. By Theorem 8.2, this implies convergence to the asymmetric Laplace geometric dispersion model,

$$\gamma^{-1/2}\,\mathrm{GD}(\gamma^{1/2}\mu, \gamma) \xrightarrow{d} \mathrm{GT}_0(\mu, v(0)) \qquad \text{as } \gamma \downarrow 0 \tag{8.5}$$

for each $\mu \in \mathbb{R}$. In the special case $\mu = 0$, we obtain convergence to the Laplace distribution $\mathrm{GT}_0(0, v(0))$, which corresponds to a result by Kalashnikov (1997, p. 162); see also Blanchet and Glynn (2007). In the case where $\mathrm{GD}(\mu, \gamma)$ is geometric infinitely divisible, a close parallel with the ordinary central limit theorem is obtained as follows [cf. Jørgensen (1997, p. 78)],

$$\gamma^{-1/2}[\mathrm{GD}(\mu_0 + \gamma^{1/2}\mu, \gamma) \overset{g}{-} \mu_0] \xrightarrow{d} \mathrm{GT}_0(\mu, v(\mu_0)) \qquad \text{as } \gamma \downarrow 0 \tag{8.6}$$

for $\mu_0 \in \Psi_0$, where $\overset{g}{-}\mu$ denotes geometric translation by the amount $-\mu$, as defined by (3.2). In the special case $\mu = 0$ we may interpret (7.1) as providing the limiting Laplace distribution of a centered and scaled geometric sum as the probability parameter tends to zero (compare with Remark 8.1).

We now turn to the geometric Poisson model introduced in Example 6.2, which is denoted $\mathrm{GT}_1(\mu, 1)$ in the present notation, corresponding to $p = 1$ ($\alpha = -\infty$). The full geometric Tweedie model $\mathrm{GT}_1(\mu, \gamma)$ is a scaled geometric Poisson model, corresponding to GCFs of the form $\gamma^{-1}C_\theta(\gamma s) = \gamma^{-1}e^{\theta}(e^{\gamma s} - 1)$. We shall now develop a convergence theorem similar to the Poisson convergence theorem of Jørgensen et al. (1994).

Theorem 8.3. *Let* $\mathrm{GD}^*(\mu, \gamma)$ *denote an additive geometric dispersion model with support* $\boldsymbol{S}$ *such that* $\inf \boldsymbol{S} = 0$ *and* $\inf[\boldsymbol{S}\backslash\{0\}] = 1$. *Then the model* $\mathrm{GD}^*(\mu, \gamma)$ *is locally convex for* μ *near zero, and*

$$\mathrm{GD}^*(\gamma\mu, \gamma) \xrightarrow{d} \mathrm{GT}_1(\mu, 1) \qquad \textit{as } \gamma \downarrow 0 \tag{8.7}$$

for each $\mu > 0$.

Proof. We can let $\mathrm{GD}^*(\mu, \gamma)$ be the additive geometric dispersion model generated by a GCF of the form

$$C(s) = 1 - \frac{1}{1 + r[\mathbb{M}(s) - 1]},$$

where $\mathbb{M}(s)$ is the MGF for a distribution with support $\boldsymbol{S}_1$ such that $\inf \boldsymbol{S}_1 = 1$, and $1 - r \in (0, 1)$ is the probability mass at zero. The first and second derivatives of C are

$$\dot{C}(s) = r\dot{\mathbb{M}}(s)[1 - C(s)]^2 \quad \text{and} \quad \ddot{C}(s) = \frac{\ddot{\mathbb{M}}(s)}{\dot{\mathbb{M}}(s)}\dot{C}(s) - \frac{2\dot{C}^2(s)}{1 - C(s)}.$$

It follows that the asymptotic behaviour of the unit v-function at zero is

$$v(\mu) \sim \mu \qquad \text{as } \mu \downarrow 0. \tag{8.8}$$

The result (8.7) now follows by applying Theorem 8.2, noting that for $p = 1$, the left-hand side of (8.4) has the form $\gamma^{-1}\,\mathrm{GD}(\gamma\mu, \gamma) = \mathrm{GD}^*(\gamma\mu, \gamma)$. □

The conditions of Theorem 8.3 are clearly satisfied for GCFs corresponding to distributions with support contained in $\mathbb{N}_0$, provided that the probabilities at 0 and 1 are both positive. An example is the Bernoulli geometric tilting family of Example 6.3, whose v-function satisfies (8.8). However, the distribution $\mathrm{GD}^*(\mu, \gamma)$ need not necessarily be discrete in order for (8.7) to apply, as long as $\Pr\{0\} > 0$ and $\Pr\{[1, \delta)\} > 0$ for all $\delta > 0$.

We now discuss the geometric gamma model $\mathrm{GT}_2(\mu, \gamma)$ with $p = 2$ ($\alpha = 0$). This model is an exponential mixture of gamma distributions, which may be generated from the GCF $C(s) = -\log(1 - s)$ for $s < 1$, which has probability density function of the following form [Pillai (1990a)],

$$f(x) = e^{-x}\int_0^\infty \frac{1}{\Gamma(u)} x^{u-1} e^{-u}\, du.$$

The case $p = 2$ is not included in Theorem 8.2, and requires special treatment. First note that in this case the scaling property (8.2) takes the form

$$c^{-1}\,\mathrm{GT}_2(c\mu, \gamma) = \mathrm{GT}_2(\mu, \gamma) \qquad \text{for } c > 0,$$

for all $\mu > 0$ and $\gamma > 0$. The next theorem shows that this fixed point also has a domain of attraction, similar to the convergence of Tauber type of Jørgensen (1997, Theorem 4.5, p. 148).

Theorem 8.4. *Assume that the geometric dispersion model* $\mathrm{GD}(\mu, \gamma)$ *with mean domain* Ψ_0 *is such that either* $\inf \Psi_0 \leq 0$ *or* $\sup \Psi_0 = \infty$. *Assume that for some* $\varphi > 0$ *the unit* v*-function satisfies* $v(\mu) \sim \varphi\mu^2$ *as* $\mu \downarrow 0$ *or* $\mu \to \infty$, *respectively. Then for all* $\mu > 0$

$$c^{-1}\,\mathrm{GD}(c\mu, \gamma) \xrightarrow{d} \mathrm{GT}_2(\mu, \gamma\varphi) \qquad \text{as } c \downarrow 0 \text{ or } c \to \infty, \tag{8.9}$$

respectively.

Proof. The left-hand side of (8.9) is a geometric dispersion model with mean $\mu > 0$, provided that c is small (large) enough for $c\mu$ to belong to Ψ_0. The corresponding v-function is

$$c^{-2}\gamma v(c\mu) \to \gamma\varphi\mu^2 \qquad \text{as } c \to 0 \text{ or } c \to \infty, \tag{8.10}$$

respectively, which converges to the v-function of $\mathrm{GT}_2(\mu, \gamma\varphi)$. The uniform convergence on compact sets can be shown along the same lines as in the proof of Theorem 8.2, and the result hence follows from Theorem 7.1. □

An example of the gamma convergence (8.9) will be considered in Section 8.4.

8.3 Geometric Mittag-Leffler and extreme stable models

We now consider geometric Tweedie models that are geometric tiltings of certain geometric stable distributions. For $\alpha \neq 0, 1,$ we define the function $\kappa^{(\alpha)}$ by

$$\kappa^{(\alpha)}(s) = \frac{\alpha - 1}{\alpha}\left(\frac{s}{\alpha - 1}\right)^{\alpha} \qquad \text{for } s/(\alpha - 1) > 0 \tag{8.11}$$

with corresponding tilting given by

$$\kappa_\theta^{(\alpha)}(s) = \kappa^{(\alpha)}(\theta + s) - \kappa^{(\alpha)}(\theta) = \kappa^{(\alpha)}(\theta)[(1 + s/\theta)^{\alpha} - 1].$$

We now consider the values $\alpha \in (0, 1) \cup (1, 2]$, for which $\kappa^{(\alpha)}$ is the CGF of a positive ($0 < \alpha < 1$) or extreme ($1 < \alpha < 2$) α-stable distribution, or a normal distribution ($\alpha = 2$) [Jørgensen, (1997, p. 136)]. The CGF $\gamma^{-1}\kappa_\theta^{(\alpha)}(\gamma s)$ defines the Tweedie model $\mathrm{Tw}_p(\mu, \gamma)$ for $p \in (-\infty, 0] \cup (2, \infty)$, with unit variance function μ^p. The corresponding GCF $\gamma^{-1}\kappa_\theta^{(\alpha)}(\gamma s)$ defines the geometric Tweedie model $\mathrm{GT}_p(\mu, \gamma)$.

Following Kozubowski (2000), we shall now consider an exponential mixture of the stable distribution (8.11). Let X have CGF $\gamma^{-1}\kappa^{(\alpha)}(s)$ for some $\alpha \in (0, 1) \cup (1, 2]$, and let Z be a unit exponential random variable independent of X. As we know from Kozubowski (2000), $Y = Z^{1/\alpha}X$ is then a geometric α-stable random variable. In fact, the MGF of Y is

$$\begin{aligned}
\mathbb{M}_Y(s) &= \mathbb{E}[\exp(sZ^{1/\alpha}X)] \\
&= \mathbb{E}[\mathbb{M}_X(sZ^{1/\alpha})] \\
&= \mathbb{E}\{\exp[\gamma^{-1}\kappa^{(\alpha)}(sZ^{1/\alpha})]\} \\
&= \mathbb{E}\{\exp[\gamma^{-1}\kappa^{(\alpha)}(s)Z]\} \\
&= [1 - \gamma^{-1}\kappa^{(\alpha)}(s)]^{-1}
\end{aligned} \tag{8.12}$$

which is the MGF of what we may call a positive ($0 < \alpha < 1$) or extreme ($1 < \alpha < 2$) geometric α-stable distribution, with skewness parameter $\beta = 1$ and

$\beta = -1$, respectively. The case $\alpha = 2$ corresponds to the Laplace distribution $\mathrm{GT}_0(0, \gamma)$. In these cases, the additive geometric Tweedie models $\mathrm{GT}_p^*(\mu, \gamma)$ are hence geometric tilting families generated by such positive or extreme geometric α-stable distributions.

In the case $p > 2$ ($0 < \alpha < 1$), the positive geometric α-stable distribution (8.12) may be obtained by division/compounding of a scaled Mittag-Leffler distribution with MGF

$$\mathbb{M}(s) = [1 + (-s)^\alpha]^{-1} \qquad \text{for } s < 0;$$

cf. Pillai (1990b) and Kozubowski and Rachev (1999). We shall hence refer to the corresponding geometric Tweedie models $\mathrm{GT}_p(\mu, \gamma)$ as *geometric Mittag-Leffler models*. These models may also be generated from the geometric gamma sum of Example 6.1. An interesting special case is $p = 3$ ($\alpha = 1/2$), which may be called the *geometric inverse Gaussian distribution*. The geometric α-stable distribution (8.12) is the special (limiting) case $\mathrm{GT}_p^*(\infty, \gamma)$.

In the case $p < 0$ ($1 < \alpha < 2$) the geometric extreme α-stable distributions (8.12) have support $\mathbb{R}$, and so do the corresponding geometric Tweedie models $\mathrm{GT}_p(\mu, \gamma)$. The geometric α-stable distribution (8.12) is now the special (limiting) case $\mathrm{GT}_p^*(0, \gamma)$.

In the Mittag-Leffler and extreme stable cases the geometric Tweedie convergence of Theorem 8.2 may be interpreted as a geometric tilting of the generalized geometric central limit theorem. To make this claim precise, let us take $0 < \alpha < 1$, and consider the limiting case $\mu \to \infty$ of (8.4) (if it exists). Using (7.1) and the relation $\mathrm{GD}(0, \gamma) = \gamma\, \mathrm{GD}^*(0, \gamma)$ we obtain the convergence

$$\gamma^{1/\alpha}\, \mathrm{GD}^*(\infty, \gamma) \xrightarrow{d} \mathrm{GT}_p(\infty, \varphi) \qquad \text{as } \gamma \downarrow 0 \text{ or } \gamma \to \infty. \tag{8.13}$$

Similarly we obtain for $1 < \alpha < 2$ the convergence

$$\gamma^{1/\alpha}\, \mathrm{GD}^*(0, \gamma) \xrightarrow{d} \mathrm{GT}_p(0, \varphi) \qquad \text{as } \gamma \downarrow 0 \text{ or } \gamma \to \infty \tag{8.14}$$

by letting $\mu \to 0$ in (8.4), if possible. However, in view of (7.4) the case $\gamma \downarrow 0$ of (8.13) and (8.14) may be interpreted as saying that the distributions $\mathrm{GD}^*(\infty, 1)$ and $\mathrm{GD}^*(0, 1)$ are in the geometric domains of attraction of the positive or extreme α-stable laws $\mathrm{GT}_p(\infty, \varphi)$ and $\mathrm{GT}_p(0, \varphi)$, respectively. Conversely, if either of (8.13) or (8.14) holds, this implies that the corresponding version of (8.4) applies for all $\mu \in \Omega_p$, so in this sense the geometric Tweedie convergence (8.4) may be obtained by applying the geometric tilting operation to each side of (8.13) and (8.14). This may be compared with Vinogradov (2000), where the corresponding interpretation of the Tweedie convergence theorem is discussed in the positive and extreme stable cases.

8.4 Geometric gamma compound Poisson models

We now consider the case $1 < p < 2$ ($\alpha < 0$), where the corresponding geometric Tweedie models $\mathrm{GT}_p(\mu, \gamma)$ are associated with the gamma compound Poisson form of Tweedie models. Such distributions are nonnegative, continuous on $\mathbb{R}_+$, and with an atom at zero, as illustrated by the following example.

Example 8.1 (Zero-modified exponential distribution). Let $0 < \delta < 1$ and $\mu > 0$, and consider the distribution function

$$F(x) = 1 - \delta e^{-\delta x/\mu} \qquad \text{for } x > 0,$$

which is a zero-modified exponential distribution with mean μ and probability mass $1 - \delta$ at zero; see Kalashnikov (1997, p. 77) and Vinogradov (2007). The corresponding GCF is given by

$$C(s) = \frac{s}{1/\mu - \gamma s/2},$$

where $\gamma = 2(1-\delta)/\delta > 0$. Straightforward calculations show that the corresponding v-function is $\gamma\mu^{3/2}$ for $\mu > 0$, which hence identifies this distribution as the geometric Tweedie model $\mathrm{GT}_{3/2}(\mu, \gamma)$.

To further illustrate this case, let X be a nonnegative random variable with probability mass $\rho = \Pr[X = 0]$ at zero, and probability density function of the form $x^{-\alpha-1} f(x)$ for $x > 0$, where $\alpha < 0$. The following result is due to Jørgensen et al. (1994); see also Jørgensen et al. (2009).

Theorem 8.5 [Cf. Jørgensen (1997, p. 147)]. *Consider the variance function V for the natural exponential family generated by the distribution of X. If f is analytic at 0 and $f(0) > 0$, then $V(\mu) \sim \gamma\mu^p$ as $\mu \downarrow 0$ for some $\gamma > 0$ and $p \in (1, 2]$, where*

$$p = \begin{cases} 1 + (1-\alpha)^{-1} & \textit{for } \rho > 0, \\ 2 & \textit{for } \rho = 0. \end{cases}$$

Under the further assumption that X is infinitely divisible, the corresponding exponential mixture (3.1) has v-function proportional to V, with the same asymptotic behaviour. In the case $\rho > 0$, the corresponding geometric dispersion model $\mathrm{GD}(\mu, \gamma)$, say, hence satisfies the conditions of the geometric Tweedie convergence result (8.4) in the limit $\gamma \downarrow 0$. If there is no atom at zero ($\rho = 0$) the geometric dispersion model $\mathrm{GD}(\mu, \gamma)$ provides an example of the gamma convergence theorem (8.9) in the limit $c \downarrow 0$.

8.5 Models with exponential v-functions

Finally, we now consider the exponential unit v-function defined by $v(\mu) = e^{\beta\mu}$ for $\mu \in \mathbb{R}$, where $\beta \in \mathbb{R}$. Following Jørgensen (1997, Section 4.5), this unit v-function will be considered to be a geometric Tweedie model with $p = \infty$ ($\alpha = 1$), denoted by $\mathrm{GT}_\infty(\mu, \gamma, \beta)$. This geometric dispersion model may be defines as an exponential mixture of the Tweedie exponential dispersion model $\mathrm{Tw}_\infty(\mu, \gamma, \beta)$ corresponding to the exponential variance function $V(\mu) = e^{\beta\mu}$ (Proposition 6.1), where the case $\beta = 0$ corresponds to the asymmetric Laplace model $\mathrm{GT}_0(\mu, \gamma)$.

In order to characterize the geometric Tweedie model $\mathrm{GT}_\infty(\mu, \gamma, \beta)$, it is necessary to consider geometric infinitely divisible distributions, such that we may work with the geometric subtraction used in (8.6). The following characterization result is a parallel to Theorem 8.1.

Theorem 8.6. *Let* $\mathrm{GD}(\mu, \gamma)$ *be a nondegenerate locally convex geometric dispersion model on* $\Psi_0 = \mathbb{R}$, *assumed to be geometric infinitely divisible. Suppose that for some* $\gamma > 0$,

$$[\mathrm{GD}(\mu + a, \varphi_a \gamma) \overset{g}{-} a] = \mathrm{GD}(\mu, \gamma) \qquad \textit{for } a, \mu \in \mathbb{R}, \tag{8.15}$$

where φ_a *is a positive function of* a. *Then* $\mathrm{GD}(\mu, \gamma)$ *is a geometric Tweedie model* $\mathrm{GT}_\infty(\mu, \gamma, \beta)$ *for some* $\beta \in \mathbb{R}$, *and* $\varphi_a = e^{-\beta a}$.

Proof. Calculating the second geo-cumulant on each side of (8.15) gives

$$\varphi_a \gamma v(\mu + a) = \gamma v(\mu) \qquad \text{for } \mu, a \in \mathbb{R}, \tag{8.16}$$

where v is the local unit v-function of $\mathrm{GD}(\mu, \gamma)$. Taking $\mu = 0$ in (8.16) gives $\varphi_a = v(0)/v(a)$, which together with (8.16) implies that v satisfies the functional equation $v(0)v(\mu + a) = v(\mu)v(a)$ for $a, \mu \in \mathbb{R}$. By the continuity of v, the solutions to this equation are of the form $v(\mu) = \lambda e^{\beta\mu}$ for some $\beta \in \mathbb{R}$, where $\lambda > 0$ because the family is locally convex and nondegenerate. This implies that $\varphi_a = e^{-\beta a}$. In view of Theorem 6.1, $\mathrm{GD}(\mu, \gamma)$ is hence a geometric Tweedie model with $p = \infty$. □

The next result, which is a parallel to Theorem 8.2, on geometric Tweedie convergence, shows that geometric dispersion models with asymptotically exponential v-functions are attracted to the geometric Tweedie model $\mathrm{GT}_\infty(\mu, \gamma, \beta)$ via the fixed point (8.15).

Theorem 8.7. *Let* $\mathrm{GD}(\mu, \gamma)$ *denote a geometric infinitely divisible locally convex geometric dispersion model with unit* v*-function* v *on* Ψ_0 *such that either* $\inf \Psi_0 = -\infty$ *or* $\sup \Psi_0 = \infty$. *Assume that for some* $\beta \in \mathbb{R}$ *and* $\varphi > 0$, $v(\mu) \sim \varphi e^{\beta\mu}$ *as either* $\mu \to \infty$ *or* $\mu \to -\infty$, *respectively. Then for each* $\mu \in \mathbb{R}$

$$[\mathrm{GD}(a + \mu, \gamma e^{-\beta a}) \overset{g}{-} a] \xrightarrow{d} \mathrm{GT}_\infty(\mu, \gamma\varphi, \beta) \qquad \textit{as } a \to -\infty \textit{ or } a \to \infty, \tag{8.17}$$

respectively.

Proof. We first note that for given $\mu \in \mathbb{R}$, a in (8.17) must be large positive or large negative enough for $a + \mu$ to belong to Ψ_0. Similar to the proof of Theorem 8.2, we observe that the υ-function for the geometric tilting model on the left-hand side of (8.17) satisfies

$$\gamma e^{-\beta a} \upsilon(a + \mu) \to \gamma \varphi e^{\beta \mu} \qquad \text{as } a \to -\infty \text{ or } a \to \infty,$$

and hence converges to the υ-function of $\mathrm{GT}_\infty(\mu, \gamma\varphi, \beta)$ on the right-hand side. We may show that the convergence is uniform in μ on compact subsets of $\mathbb{R}$ by using the same arguments as Jørgensen (1997, p. 165). The result (8.17) now follows from Theorem 7.1. □

We note that for $\beta = 0$, the convergence result (8.17) shows convergence to the asymmetric Laplace model under the assumption that υ is asymptotically a positive constant φ at $-\infty$ or ∞, that is,

$$[\mathrm{GD}(a + \mu, \gamma) \overset{g}{-} a] \xrightarrow{d} \mathrm{GT}_0(\mu, \gamma\varphi) \qquad \text{as } a \to -\infty \text{ or } a \to \infty,$$

respectively, complementing the Laplace convergence results (8.5) and (8.6).

Similar to Remark 8.1, we note that in the case where $e^{-\beta a}$ tends to 0 in (8.17) this result in effect concerns weak convergence of a centered geometric sum $S(q)$ as the average sample size q^{-1} tends to infinity, where the centering is achieved by a combination of geometric tilting and geometric translation. Similarly, the case where $e^{-\beta a}$ tends to infinity involves carrying the geometric division process of Section 3 to its limit. In both cases, the mean on the left-hand side of (8.17) is kept fixed at the value μ throughout the convergence, similar to what is the case for (8.4).

9 Discussion

We have developed the new class of geometric dispersion models as analogues of exponential dispersion models, showing geometric analogues of many key ideas from exponential dispersion models. We summarize the main analogies between the two cases in Table 3. It is striking that analogous structures were discovered for the recently developed class of extreme dispersion models [Jørgensen et al. (2010)]. In the latter case, the analogues of Tweedie exponential dispersion models are the generalized extreme value distributions (Weibull, Fréchet and Gumbel distributions), and the analogue of the Tweedie convergence theorem is the classical convergence theorem for extremes; see Jørgensen et al. (2010) for details. In the present case, however, the geometric Tweedie convergence theorem (Theorem 8.2) establishes a new class of convergence results for geometric sums, the

Table 3 *The main parallels between exponential and geometric dispersion models*

Exponential dispersion models	Geometric dispersion models
Natural exponential families	Geometric tilting families
Exponential dispersion models	Geometric dispersion models
Tweedie dispersion models	Geometric Tweedie models
Tweedie convergence	Geometric Tweedie convergence

only previously known result being the special case $\mu = 0$ of the Laplace convergence (8.5).

Based on these three examples, it seems likely that there exist further classes of dispersion models with a similar structure, although it is not entirely clear in which direction to look for such models. One possible area is free probability, where Bryc (2009) has introduced so-called free exponential families, and studied an analogue of quadratic variance functions. Another possibility is the setting of geometric minima, based on studying the minimum of a geometric number of i.i.d. random variables.

Appendix: Proof of Theorem 7.1

Let K be a compact subinterval of $\operatorname{int}\Psi_0 \subseteq \operatorname{int}\Psi_n$, and fix a $\mu_0 \in K$. Let $\psi_n : \operatorname{int}\Psi_n \to \mathbb{R}$ be a strictly monotone function defined by $\dot{\psi}_n(\mu) = 1/v_n(\mu)$ and $\psi_n(\mu_0) = 0$. Let $I_n = \psi_n(\operatorname{int}\Psi_n)$ and $J_n = \psi_n(K) \subseteq I_n$, both of which are intervals containing zero. We let the geometric tilting family corresponding to $\mathrm{GE}_n(\mu)$ be generated by the GCF $C_n : I_n \to \Psi_n$ defined by $\dot{C}_n(s) = \psi_n^{-1}(s)$, satisfying $C_n(0) = 0$ and $\dot{C}_n(0) = \mu_0$.

Consider the case where $v(\mu) \neq 0$ (the nonzero case). In this case, we proceed by defining a strictly monotone function $\psi : \operatorname{int}\Psi_0 \to \mathbb{R}$ by $\dot{\psi}(\mu) = 1/v(\mu)$ and $\psi(\mu_0) = 0$. We let $I_0 = \psi(\operatorname{int}\Psi_0)$ and $J = \psi(K) \subseteq I_0$, which are again intervals containing zero. We define the function $C : I_0 \to \Psi_0$ by $\dot{C}(s) = \psi^{-1}(s)$ and $C(0) = 0$, again satisfying $\dot{C}(0) = \mu_0$. Since ψ is strictly monotone and analytic, the same is the case for $\dot{C}$.

For $\mu \in \operatorname{int}\Psi_0$ we observe that

$$|\dot{\psi}_n(\mu) - \dot{\psi}(\mu)| = \left|\frac{v_n(\mu) - v(\mu)}{v_n(\mu)v(\mu)}\right|. \tag{A.1}$$

By the uniform convergence of $v_n(\mu)$ to $v(\mu)$ on K, it follows that $\{v_n(\mu)\}$ is uniformly bounded on K. Since $v(\mu)$ is bounded on K, it follows from the uniform convergence of $v_n(\mu)$ that $\dot{\psi}_n(\mu) \to \dot{\psi}(\mu)$ uniformly on K. This and the fact that $\psi_n(\mu_0) = \psi(\mu_0)$ for all n implies [Rudin (1976, Theorem 7.17, p. 152)] that $\psi_n(\mu) \to \psi(\mu)$ uniformly on K, and since K was arbitrary, we have $\psi_n(\mu) \to \psi(\mu)$ for all $\mu \in \Psi_0$. Note also that $J_n = \psi_n(K) \to \psi(K) = J$.

Let $\mu \in K$ be given and let $s = \psi(\mu) \in J$ and $s_n = \psi_n(\mu) \in J_n$. Since $v_n(\mu)$ is uniformly bounded on K, there exists an m such that $|v_n(\mu)| \leq m$ for all n and $\mu \in K$, which implies that $|\ddot{C}_n(s)| = |v_n(\dot{C}_n(s))| \leq m$ for all $s \in J$, due to the fact that an $s \in J$ belongs to J_n for n large enough, because then $\dot{C}_n(s) \in K$. Since $\mu = \dot{C}(s) = \dot{C}_n(s_n)$ we find, using the mean value theorem, that

$$\begin{aligned}|\dot{C}_n(s) - \dot{C}(s)| &= |\dot{C}_n(s) - \dot{C}_n(s_n)| \\ &\leq m|s - s_n| \\ &= m|\psi(\mu) - \psi_n(\mu)|.\end{aligned}$$

This implies that $\dot{C}_n(s) \to \dot{C}(s)$ uniformly in $s \in J$. Since $C_n(0) = C(0)$ for all n, it follows by the same argument as above that $C_n(s) \to C(s)$ uniformly in $s \in J$. We conclude from the convergence of the sequence of MGFs, $[1 - C_n(s)]^{-1} \to [1 - C(s)]^{-1}$ for $s \in J$, that the sequence of distributions $\mathrm{GE}_n(\mu_0)$ converges weakly to a probability measure P with GCF C. If we let $\mathrm{GE}(\mu)$ denote the geometric tilting family with local v-function v on Ψ_0, then $\mathrm{GE}(\mu)$ may be generated from C and $P = \mathrm{GE}(\mu_0)$. This concluded the proof in the nonzero case.

In the case where $v(\mu) = 0$ (the zero case), we cannot define the function ψ as above. Instead we take $C(s) = s\mu_0$, such that $\dot{C}(s) = \mu_0$ and $\ddot{C}(s) = 0$ for $s \in \mathbb{R}$. For any $\varepsilon > 0$, we may choose an n_0 such that $|v_n(\mu)| \leq \varepsilon$ for any $n \geq n_0$ and $\mu \in K$. For such n and μ we hence obtain, in the case $\mu > \mu_0$,

$$|\psi_n(\mu)| = \int_{\mu_0}^{\mu} \frac{1}{|v_n(t)|}\, dt \geq \frac{\mu - \mu_0}{\varepsilon},$$

which can be made arbitrarily large by choosing ε small, and similarly for $\mu < \mu_0$, where $\psi_n(\mu)$ has the opposite sign. We hence conclude that $J_n = \psi_n(K) \to \mathbb{R}$ as $n \to \infty$.

Now we let J be a compact interval such that $0 \in \operatorname{int} J$, implying that $J \subseteq J_n$ for n large enough. For such n we hence find that $|\ddot{C}_n(s)| = |v_n(\dot{C}_n(s))| \leq \varepsilon$ for all $s \in J$, because then $\dot{C}_n(s) \in K$. Since $\mu_0 = \dot{C}(s) = \dot{C}_n(0)$ we find, again by the mean value theorem, that for $s \in J$,

$$|\dot{C}_n(s) - \dot{C}(s)| = |\dot{C}_n(s) - \dot{C}_n(0)| \leq \varepsilon|s|.$$

This implies that $\dot{C}_n(s) \to \dot{C}(s)$ uniformly in $s \in J$. By the same arguments as above, we conclude that $\mathrm{GE}_n(\mu_0)$ converges weakly to a probability measure P with GCF $C(s) = s\mu_0$, which is the desired conclusion in the zero case.

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Department of Mathematics & Computer Science
University of Southern Denmark
Campusvej 55
DK-5230 Odense M
Denmark
E-mail: bentj@stat.sdu.dk

Laboratoire de Mathématiques de Besançon
Université de Franche-Comté
UMR 6623 CNRS—UFR Sciences & Techn.
16 route de Gray
F-25030 Besançon cedex
France
E-mail: celestin.kokonendji@univ-fcomte.fr