The beta power distribution

Gauss Moutinho Cordeiro and Rejane dos Santos Brito

Universidade Federal Rural de Pernambuco

Abstract. The power distribution is defined as the inverse of the Pareto distribution. We study in full detail a distribution so-called the beta power distribution. We obtain analytical forms for its probability density and hazard rate functions. Explicit expressions are derived for the moments, probability weighted moments, moment generating function, mean deviations, Bonferroni and Lorenz curves, moments of order statistics, entropy and reliability. We estimate the parameters by maximum likelihood. The practicability of the model is illustrated in two applications to real data.

1 Introduction

We study the so-called beta power (BP) distribution. Although we can only refer to the book by Balakrishnan and Nevzorov (2003, Chapter 14, pp. 127–132) about the power distribution, we believe that this distribution can have wider applications than the power distribution. Boyce et al. (1999) related the power distribution to environmental policy and stress and public health and Van Dorp and Kotz (2002) presented applications in financial engineering domain. Keeping these applications in mind, we take them as a basis to the power distribution and also to the BP distribution.

The BP distribution stems from the following idea: Eugene et al. (2002) defined the beta G distribution from a quite arbitrary cumulative distribution function (cdf) $G(x)$ by

$$F(x) = I_{G(x)}(a, b), \tag{1.1}$$

where $a > 0$ and $b > 0$ are two extra parameters, $I_y(a, b) = B_y(a, b)/B(a, b)$ is the incomplete beta function ratio, $B_y(a, b) = \int_0^y \omega^{a-1}(1 - \omega)^{b-1} d\omega$ is the incomplete beta function and $B(a, b)$ is the complete beta function. The unknown positive parameters a and b are shape parameters which introduce skewness and vary tail weights.

The class of distributions (1.1) has raised increased attention in recent years after the work by Jones (2004).

Eugene et al. (2002), Nadarajah and Gupta (2004), Nadarajah and Kotz (2004) and Nadarajah and Kotz (2005) defined the beta normal, beta Fréchet, beta Gumbel

and beta exponential distributions by taking $G(x)$ to be the cdf of the normal, Fréchet, Gumbel and exponential distributions, respectively. Another distribution that happens to belong to (1.1) is the log-F (or beta logistic) distribution, which is known for over 20 years (Brown et al., 2002), even if it did not originate directly from equation (1.1). Sepanski and Kong (2007) compared the performance of some generalized beta distributions, such as the beta normal, skewed Student t, log-F, beta exponential and beta Weibull distributions, to the widely used generalized beta distributions of the first and second types in terms of some measures of fit. Recently, Barreto-Souza et al. (2010) proposed the beta generalized exponential distribution motivated by the wide use of the exponential distribution in practice, and also for the fact that the generalization provides more flexibility to analyze more complex situations.

In this article, we provide a comprehensive mathematical treatment of the BP distribution. We begin with the probability density function (pdf) and cdf of the power distribution given by

$$g_{\alpha,\beta}(x) = \alpha\beta^{\alpha}x^{\alpha-1}, \qquad 0 < x < \frac{1}{\beta}, \tag{1.2}$$

and

$$G_{\alpha,\beta}(x) = (\beta x)^{\alpha}, \tag{1.3}$$

respectively, where $\alpha > 0$ is a shape parameter and $\beta > 0$ is a scale parameter. For $\alpha = 1$, we obtain as a special case the uniform distribution defined on the interval $(0, 1/\beta)$, say $X \sim U(0, 1/\beta)$. For $\beta = 1$, we have $g_{\alpha,1}(x) = \alpha x^{\alpha-1}$ and $G_{\alpha,1}(x) = x^{\alpha}$ as defined by Balakrishnan and Nevzorov (2003). The power distribution is related to the Pareto distribution using an inverse transformation (Akinsete et al., 2008). Some other distributions can be obtained from the Pareto distribution using some well-known transformations such as the exponential, logistic and chi-squared distributions.

Using (1.1) and replacing $G(x)$ by the cdf of the power distribution (1.3), the BP cumulative function can be written as

$$F(x) = I_{(\beta x)^{\alpha}}(a, b) = B(a, b)^{-1} \int_{0}^{(\beta x)^{\alpha}} w^{a-1}(1 - w)^{b-1}\,dw \tag{1.4}$$

for $0 < x < 1/\beta$. Here, $a > 0$, $b > 0$ and $\alpha > 0$ are shape parameters and $\beta > 0$ is a scale parameter. For any values of a and b, we can express (1.4) in terms of the well-known hypergeometric function given by

$$F(x) = \frac{(\beta x)^{\alpha a}}{a B(a, b)} {}_2F_1\big(a, 1 - b, a + 1; (\beta x)^{\alpha}\big).$$

In general, the cdf $F(x)$ defined from a baseline cdf $G(x)$ in (1.1) could follow the properties of the hypergeometric function which are well established in the literature; see, for example, Section 9.1 of Gradshteyn and Ryzhik (2000).

The density function corresponding to (1.1) is given by

$$f(x) = \frac{g(x)}{B(a,b)} G(x)^{a-1}[1 - G(x)]^{b-1}, \qquad (1.5)$$

where $g(x) = dG(x)/dx$ is the baseline density function. The pdf $f(x)$ will be most tractable when the functions $G(x)$ and $g(x)$ have simple analytic expressions such as the power distribution. Except for some special choices for $G(x)$ in (1.1), equation (1.5) could be very difficult to deal with in generality.

The hazard rate function defined by $h(x) = f(x)/[1 - F(x)]$ is an important quantity characterizing lifetime phenomena. Correspondingly, the pdf and hazard rate function of the BP distribution are

$$f(x) = \frac{\alpha\beta(\beta x)^{\alpha a-1}[1 - (\beta x)^{\alpha}]^{b-1}}{B(a,b)}, \qquad 0 < x < \frac{1}{\beta} \qquad (1.6)$$

and

$$h(x) = \frac{\alpha\beta(\beta x)^{\alpha a-1}[1 - (\beta x)^{\alpha}]^{b-1}}{B(a,b)[1 - I_{(\beta x)^{\alpha}}(a,b)]}, \qquad (1.7)$$

respectively. Equation (1.6) is not new. McDonald and Richards (1987) defined it as the generalized beta distribution of the second kind. However, they do not study its mathematical properties which we do in this article.

We denote a random variable X having density function (1.6) by $X \sim BP(a, b, \alpha, \beta)$. Simulation of the BP distribution is very easy: if V is a random variable having a beta distribution with parameters a and b, then the random variable $X = \beta^{-1}V^{1/\alpha}$ follows the $BP(a, b, \alpha, \beta)$ distribution.

Figures 1 and 2 illustrate some possible shapes of the density function (1.6) and hazard rate function (1.7), respectively, for selected parameter values, including the power distribution. It is evident that the BP distribution is much more flexible than the power distribution. Figure 2 shows that the BP hazard function can be bathtub shaped, monotonically increasing or decreasing and upside-down bathtub depending basically on the values of the parameters.

The rest of the paper is organized as follows. In Section 2, we give expansions for the pdf and cdf of the BP distribution depending on whether the parameter b is real noninteger and integer. Section 3 provides the moments and a small study on the variation of the skewness and kurtosis measures. Probability weighted moments (PWMs) are determined in Section 4. The moment generating function (mgf) and the quantile function are obtained in Section 5. Mean deviations and Bonferroni and Lorenz curves are derived in Section 6. We show, in Section 7, that the density function of the BP order statistics can be expressed as a mixture of power density functions. The moments of order statistics are also derived in this section for b real noninteger and integer. Sections 8 and 9 are devoted to the entropy and reliability, respectively. In Section 10, we discuss maximum likelihood estimation of the model parameters. Section 11 provides two applications to real data sets. Some conclusions are addressed in Section 12.
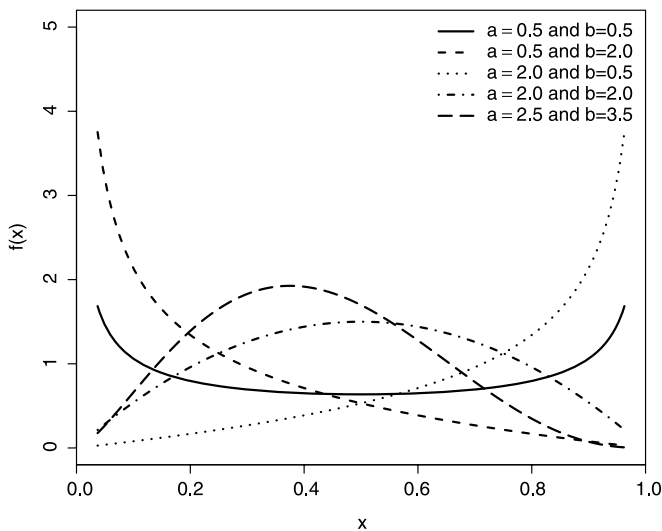
Figure 1 *BP(a, b, 1, 1) density function for selected parameter values.*
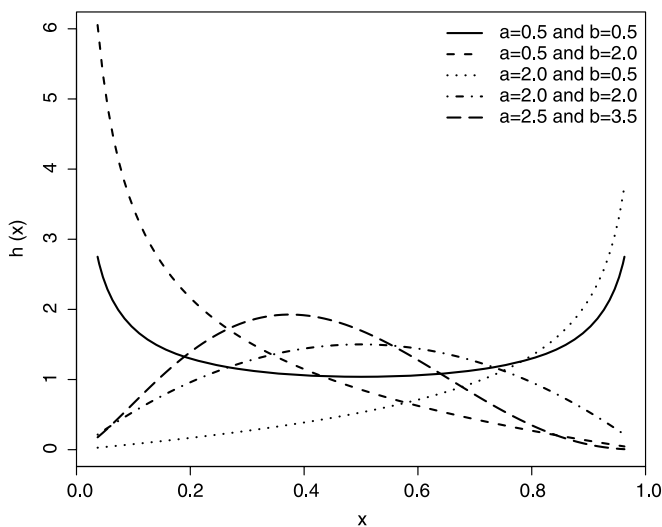

Figure 2 *BP(a, b, 1, 1) hazard function for selected parameter values.*

2 Expansion for the density function

Here, we give a simple expansion for the BP density function. We consider the binomial expansion

$$(1 - z)^{b-1} = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(b)}{\Gamma(b-i)i!} z^i, \tag{2.1}$$

valid for $|z| < 1$ and $b > 0$ real noninteger.

Application of (2.1) to equation (1.4) if b is real noninteger gives

$$F(x) = \frac{\Gamma(a+b)}{\Gamma(a)} \sum_{i=0}^{\infty} \frac{(-1)^i (\beta x)^{\alpha(a+i)}}{\Gamma(b-i)i!(a+i)}. \tag{2.2}$$

Correspondingly, the BP density function can be written as

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)} \sum_{i=0}^{\infty} \frac{(-1)^i}{\Gamma(b-i)i!(a+i)} g_{\alpha(a+i),\beta}(x), \tag{2.3}$$

where $g_{\alpha(a+i),\beta}(x)$ is the power density with shape parameter $\alpha(a+i)$ and scale parameter β. The BP density function (1.6) is easily computed using any statistical software. However, it is clear from (2.3) that it can be expressed as an infinite (or finite) mixture of power densities with increasing shape parameters $\alpha(a+i)$ and a common scale parameter β. This result is important to provide some mathematical properties of the BP distribution (ordinary, central, factorial and inverse moments, mgf, etc.) directly from those of the power distributions. For b integer, the sums in (2.2) and (2.3) stop at $b-1$. These two expansions are the main results of this section.

3 Moments

If X has a BP density (1.6), its rth moment about zero becomes

$$\mu'_r = E(X^r) = \frac{\alpha \beta^{\alpha a}}{B(a,b)} \int_0^{1/\beta} x^{r+\alpha a-1} [1 - (\beta x)^\alpha]^{b-1} \, dx. \tag{3.1}$$

Using the binomial expansion (2.1), (3.1) can be rewritten as

$$\mu'_r = \frac{\Gamma(a+b)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{(-1)^j J(j,r)}{\Gamma(b-j)j!},$$

where $J(j,r)$ denotes the integral

$$J(j,r) = \int_0^{1/\beta} \alpha x^{r-1} G(x)^{a+j} \, dx = \alpha \{\beta^r [r + \alpha(a+j)]\}^{-1}.$$

Equation (3.1) can be expressed as

$$\mu'_r = \frac{\alpha \Gamma(a+b)}{\beta^r \Gamma(a)} \sum_{j=0}^{\infty} (-1)^j \{\Gamma(b-j)[r + \alpha(a+j)]j!\}^{-1}.$$

Hence, a closed-form expression for the moments of X is given by

$$\mu'_r = \frac{B(a+r/\alpha, b)}{\beta^r B(a,b)}. \tag{3.2}$$

Table 1 *Moments of different BP distributions by fixing $\alpha = \beta = 1$*

μ'_k	$BP(1.0, 1.0, 1, 1)$	$BP(1.0, 1.5, 1, 1)$	$BP(1.0, 3.5, 1, 1)$
μ'_1	0.50000	0.40000	0.22222
μ'_2	0.33333	0.22857	0.08081
μ'_3	0.25000	0.15238	0.03730
μ'_4	0.20000	0.11082	0.01989
μ'_5	0.16667	0.08525	0.01170
μ'_6	0.14286	0.06820	0.00739
Variance	0.08333	0.06857	0.03142
Skewness	0	0.33945	0.96428
Kurtosis	1.80000	2.05050	3.40880

Table 2 *Moments of different BP distributions by fixing $\alpha = \beta = 1$*

μ'_k	$BP(1.5, 1.5, 1, 1)$	$BP(1.5, 2.5, 1, 1)$	$BP(2.5, 3.5, 1, 1)$
μ'_1	0.50000	0.375	0.37500
μ'_2	0.31250	0.1875	0.18750
μ'_3	0.21875	0.10938	0.10938
μ'_4	0.16406	0.070313	0.07031
μ'_5	0.12891	0.04834	0.04834
μ'_6	0.10474	0.034912	0.03491
Variance	0.06250	0.046875	0.04687
Skewness	0	0.3849	0.38490
Kurtosis	2.00000	2.3333	2.33330

The proof is given in Appendix B.

Tables 1 and 2 provide the first six ordinary moments, variance, skewness and kurtosis for selected $BP(a, b, 1, 1)$ distributions by fixing $\alpha = \beta = 1$. Figures 3 and 4 illustrate the skewness and kurtosis measures calculated from (3.2) whose forms depend basically on the parameters a and b. The curve for the skewness decreases with a for fixed b and increases with b for fixed a. The curve for the kurtosis first decreases with a (b) for fixed b (a) and then increases, except for the case $b = 1$ ($a = 1$) where the curve always increases when a (b) increases.

The central moments (μ_s) and cumulants (κ_s) of X are easily obtained from the ordinary moments by $\mu_s = \sum_{k=0}^{s} \binom{s}{k}(-1)^k \mu'^s_1 \mu'_{s-k}$ and $\kappa_1 = \mu'_1$, $\kappa_2 = \mu'_2 - \mu'^2_1$, $\kappa_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1$, $\kappa_4 = \mu'_4 - 4\mu'_3\mu'_1 - 3\mu'^2_2 + 12\mu'_2\mu'^2_1 - 6\mu'^4_1$, $\kappa_5 = \mu'_5 - 5\mu'_4\mu'_1 - 10\mu'_3\mu'_2 + 20\mu'_3\mu'^2_1 + 30\mu'^2_2\mu'_1 - 60\mu'_2\mu'^3_1 + 24\mu'^5_1$, $\kappa_6 = \mu'_6 - 6\mu'_5\mu'_1 - 15\mu'_4\mu'_2 + 30\mu'_4\mu'^2_1 - 10\mu'^2_3 + 120\mu'_3\mu'_2\mu'_1 - 120\mu'_3\mu'^3_1 + 30\mu'^3_2 - 270\mu'^2_2\mu'^2_1 + 360\mu'_2\mu'^4_1 - 120\mu'^6_1$, etc., respectively.
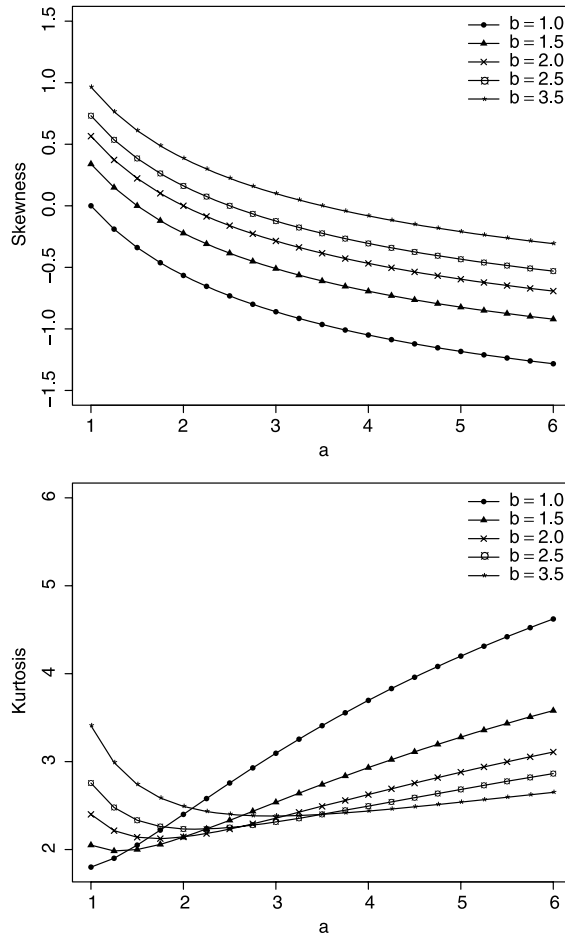
Figure 3 *Skewness and kurtosis of the $BP(a, b, 1, 1)$ distribution for selected values of b as a function of parameter a.*

The rth descending factorial moment of X is

$$\mu'_{(r)} = E\big[X^{(r)}\big] = E[X(X-1) \times \cdots \times (X-r+1)] = \sum_{k=0}^{r} s(r,k)\mu'_k,$$

where $s(r, k)$ is the Stirling number of the first kind that can be defined by $s(r, k) = (k!)^{-1}[\frac{d^k}{dx^k} x^{(r)}]_{x=0}$. They count the number of ways to permute a list of r items into k cycles. Thus, the factorial moments of X are

$$\mu'_{(r)} = \sum_{k=0}^{r} s(r,k) \frac{B(a + k/\alpha, b)}{\beta^k B(a, b)}.$$
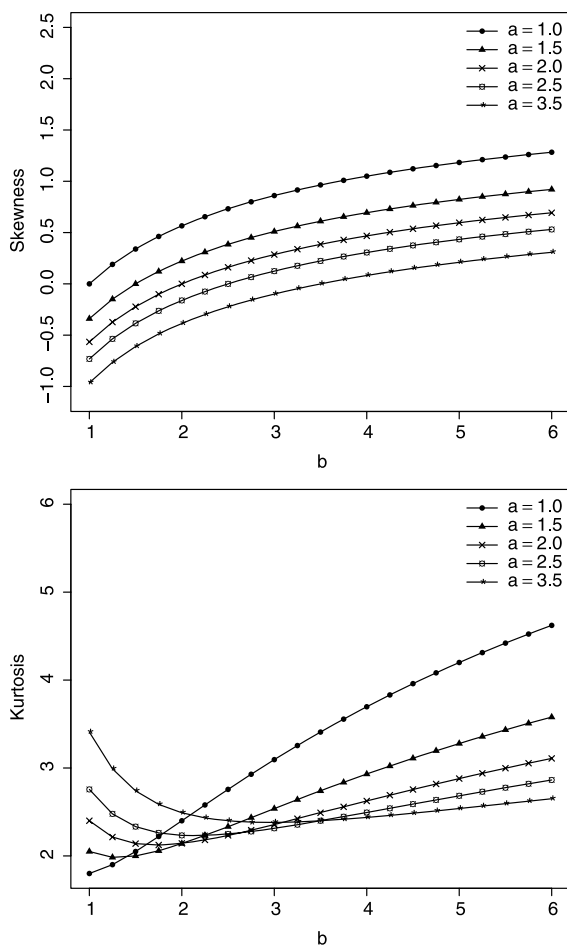
Figure 4 *Skewness and kurtosis of the BP$(a, b, 1, 1)$ distribution for selected values of a as a function of parameter b.*

4 Probability weighted moments

PWMs are expectations of certain functions defined for any random variable whose mean exists and were first fomulated by Greenwood et al. (1979) primarily as an aid to estimate the parameters of the Wakeby distribution. However, the use of the PWMs covers: the summarization and description of theoretical probability distributions and observed data samples, nonparametric estimation of the under-lying distribution of an observed sample, estimation of parameters and quantiles of probability distributions and hypothesis tests for probability distributions. The PWM method can generally be used in estimating parameters of a distribution whose inverse form cannot be given explicitly. For several distributions, such as

Table 3   *PWMs $m_{r,j}$ for the BP(2.0, 2.0, 0.5, 2.0) distribution*

			j			
r	1	2	3	4	5	6
1	0.10714	0.03062	0.00624	0.00098	1.23×10^{-4}	1.30×10^{-5}
2	0.03819	0.00889	0.00159	0.00023	2.67×10^{-5}	2.68×10^{-6}
3	0.01477	0.00287	0.00451	0.00006	6.43×10^{-6}	6.09×10^{-7}
4	0.00601	0.00100	0.00014	0.00002	1.68×10^{-6}	1.50×10^{-7}

normal, log-normal and Pearson type three distributions, the expressions connecting PWMs to the parameters of the model have the same forms. Such expressions may be readily employed in practice for estimating the parameters.

PWMs are formally defined by $m_{r,j} = \int_0^{1/\beta} x^r F(x)^j \, dx$. For the BP distribution, we obtain from (2.2)

$$m_{r,j} = \left[\frac{\Gamma(a+b)}{\Gamma(a)} \right]^j \int_0^{1/\beta} x^r \left[\sum_{i=0}^{\infty} \frac{(-1)^i (\beta x)^{\alpha(a+i)}}{\Gamma(b-i)i!(a+i)} \right]^j dx.$$

Using $(\sum_{i=0}^{\infty} a_i)^j = \sum_{m_1,\ldots,m_j=0} a_{m_1} \cdots a_{m_j}$, for any positive integer j, we can write

$$m_{r,j} = \left[\frac{\Gamma(a+b)}{\Gamma(a)} \right]^j \int_0^{1/\beta} x^r \sum_{m_1,\ldots,m_j=0}^{\infty} a_{m_1} \cdots a_{m_j} (\beta x)^{\alpha(ja+\sum_{q=1}^{j} m_q)} dx,$$

where

$$a_{m_q} = \frac{(-1)^{m_q}}{\Gamma(b-m_q)m_q!(a+m_q)} \qquad \text{for } q=1,\ldots,j.$$

We easily calculate the integral and then obtain

$$m_{r,j} = \frac{1}{\beta^{r+1}} \left[\frac{\Gamma(a+b)}{\Gamma(a)} \right]^j \sum_{m_1,\ldots,m_j=0}^{\infty} \frac{a_{m_1} \cdots a_{m_q}}{[r + \alpha(ja + \sum_{q=1}^{j} m_q) + 1]}. \qquad (4.1)$$

Equation (4.1) can be used numerically in any software with algebraic facilities (Maple, Matlab or Mathematica) by taking in these sums a large positive integer in place of ∞. In Appendix A, we provide an algorithm to calculate the PWMs given in Tables 3–5.

5 Moment generating and quantile functions

Here, we derive a closed form expression for the mgf $M(t)$ of the BP distribution (1.6). Let $M_{\alpha,\beta}(t)$ be the mgf of the power distribution (1.2). Changing the

Table 4 *PWMs $m_{r,j}$ for the BP(1.0, 2.0, 1.0, 1.0) distribution*

			j			
r	1	2	3	4	5	6
1	0.41667	0.11111	0.02080	0.00296	0.00339	3.24×10^{-5}
2	0.30000	0.06612	0.01097	0.00144	0.00015	1.42×10^{-5}
3	0.23333	0.04340	0.00640	0.00077	7.80×10^{-5}	6.76×10^{-6}
4	0.19048	0.03049	0.00403	0.00045	4.21×10^{-5}	3.46×10^{-6}

Table 5 *PWMs $m_{r,j}$ for the BP(2.5, 3.5, 1.0, 1.0) distribution*

			j			
r	1	2	3	4	5	6
1	0.39583	0.08507	0.01184	0.00121	0.97×10^{-4}	6.41×10^{-6}
2	0.29427	0.05610	0.00735	0.00072	0.57×10^{-4}	3.69×10^{-6}
3	0.23210	0.03917	0.00480	0.00045	0.34×10^{-4}	2.19×10^{-6}
4	0.19069	0.02860	0.00326	0.00029	0.22×10^{-4}	1.34×10^{-6}

variable $u = tx$ yields

$$M_{\alpha,\beta}(-t) = \alpha \left(\frac{\beta}{t} \right)^{\alpha} \gamma(\alpha, t/\beta),$$

where $\gamma(\alpha, z) = \int_0^z u^{\alpha-1} e^{-u} \, du$ is the incomplete gamma function defined for any complex z. The mgf of the power distribution comes (for any t) as

$$M_{\alpha,\beta}(t) = \alpha \left(\frac{\beta}{-t} \right)^{\alpha} \gamma(\alpha, -t/\beta).$$

The above equation was also checked using Mathematica. Combining (2.3) and the last equation, we obtain

$$M(t) = \frac{\alpha \Gamma(a+b)}{\Gamma(a)} \sum_{i=0}^{\infty} \frac{(-1)^i}{\Gamma(b-i)i!} \left(\frac{\beta}{-t} \right)^{\alpha(a+i)} \gamma\big(\alpha(a+i), -t/\beta\big). \qquad (5.1)$$

The characteristic function of the BP distribution is

$$M(it) = \frac{\alpha \Gamma(a+b)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(b-j)j!} \left(\frac{\beta}{-it} \right)^{\alpha(a+j)} \gamma\big(\alpha(a+j), -it/\beta\big).$$

It is possible to obtain some expansions for the inverse of the beta incomplete function ratio $I_z^{-1}(a, b)$. One of them can be found in wolfram website.[1] From this

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[1] http://functions.wolfram.com/06.23.06.0004.01.

expansion, we can express the BP quantile function $F^{-1}(x) = I_{(\beta x)^\alpha}^{-1}(a, b)$ as

$$F^{-1}(x) = w + \frac{b-1}{a+1}w^2 + \frac{(b-1)(a^2 + 3ba - a + 5b - 4)}{2(a+1)^2(a+2)}w^3$$
$$+ (((b-1)[a^4 + (6b-1)a^3 + (b+2)(8b-5)a^2$$
$$+ (33b^2 - 30b + 4)a + b(31b - 47) + 18]) \qquad (5.2)$$
$$/3(a+1)^3(a+2)(a+3))w^4$$
$$+ O((\beta x)^{5\alpha/a}),$$

where $w = [aB(a, b)(\beta x)^\alpha]^{1/a}$ for $a > 0$.

Equations (5.1) and (5.2) are the main result of this section.

6 Mean deviations

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. If X has the BP distribution, we can derive the mean deviations about the mean $\mu'_1 = E(X)$ and about the median M from

$$\delta_1 = \int_0^{1/\beta} |x - \mu'_1| f(x)\, dx \quad \text{and} \quad \delta_2 = \int_0^{1/\beta} |x - M| f(x)\, dx,$$

respectively. The median is the solution of the equation $I_{(\beta M)^\alpha}(a, b) = 1/2$. These measures can be calculated using the following relationships

$$\delta_1 = 2\left[\mu'_1 F(\mu'_1) - \int_0^{\mu'_1} x f(x)\, dx\right] \quad \text{and} \quad \delta_2 = \mu'_1 - 2\int_0^M x f(x)\, dx. \quad (6.1)$$

The integrals in (6.1) are easily obtained by the density expansion (2.3). We obtain

$$J(s) = \int_0^s x f(x)\, dx = s\frac{\alpha\Gamma(a+b)}{\Gamma(a)}\sum_{i=0}^\infty \frac{(-1)^i}{\Gamma(b-i)i!}\frac{(\beta s)^{\alpha(a+i)}}{[\alpha(a+i)+1]}. \quad (6.2)$$

Equation (6.2) can be used to determine Bonferroni and Lorenz curves which have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. They are defined by

$$B(p) = \frac{J(q)}{p\mu'_1} \quad \text{and} \quad L(p) = \frac{J(q)}{\mu'_1}, \qquad (6.3)$$

respectively, where $q = F^{-1}(p)$ can be calculated by (5.2) for given p.

7 Order statistics

The density of the ith order statistic $X_{i:n}$, $f_{i:n}(x)$ say, in a random sample of size n from the BP distribution is obtained from the well-known formula

$$f_{i:n}(x) = \frac{f(x)}{B(i, n - i + 1)} F(x)^{i-1}[1 - F(x)]^{n-i},$$

for $i = 1, \ldots, n$. Using formulae (1.4) and (1.6) we can express $f_{i:n}(x)$ in terms of the incomplete beta function ratio

$$f_{i:n}(x) = \frac{\alpha\beta(\beta x)^{\alpha a - 1}[1 - (\beta x)^\alpha]^{b-1}}{B(a, b) B(i, n - i + 1)} I_{(\beta x)^\alpha}(a, b)^{i-1} I_{1-(\beta x)^\alpha}(a, b)^{n-i}.$$

The moments of the order statistics could in principle be determined from the moments of the BP distribution by expressing the density of the order statistics in terms of linear combination of BP densities. However, this method is much more difficult to be developed here. Alternatively, the moments of the BP order statistics can be derived from a result due to Barakat and Abdelkander (2004), applied to the independent and identically distributed case. Then, the kth moment of $X_{i:n}$ can be written as

$$E(X_{i:n}^k) = k \sum_{j=n-i+1}^{n} \sum_{l=0}^{j} (-1)^{j-n+i+l-1} \binom{j-1}{n-i}\binom{n}{j}\binom{j}{l}\left[\frac{\Gamma(a+b)}{\Gamma(a)}\right]^l$$

$$\times \sum_{m_1,\ldots,m_l=0}^{\infty} a_{m_1} \cdots a_{m_l} \tag{7.1}$$

$$\times \left\{\beta^{k+\alpha\sum_{q=1}^{l}(a+m_q)}\left[k + \alpha\sum_{q=1}^{l}(a+m_q)\right]\right\}^{-1}.$$

The proof is given in Appendix C. The sums in (7.1) extend over all l-tuples $(m_1, \ldots, m_l)$ of nonnegative integers and are easily implementable in software such as Matlab, Mathematica and Maple. The moments in Tables 6–9 are produced by a Mathematica script given in Appendix D.

Table 6 *Moments of the order statistics $E(X_{i:n}^k)$ for BP(2.5, 3.5, 1.0, 1.0) and $n = 10$*

			i		
k	1	3	5	7	10
1	−0.60400	−0.33360	0.96739	0.99993	1.00000
2	−0.64119	0.41267	1.96333	1.99992	2.00000
3	−0.48620	1.48566	2.96768	2.99994	3.00000
4	−0.18850	2.64677	3.97367	3.99995	4.00000

Table 7   *Moments of the order statistics $E(X_{i:n}^k)$ for BP(2.5, 3.5, 1.0, 1.0) and $n = 30$*

			i		
k	1	10	20	25	30
1	0.88634	0.99917	1.01445	0.99809	1.00000
2	1.01432	1.99910	1.98135	1.99927	2.00000
3	0.89015	2.99925	2.97655	3.00466	3.00000
4	0.71107	3.99944	3.99073	4.00483	4.00000

Table 8   *Moments of the order statistics $E(X_{i:n}^k)$ for BP(1.5, 1.5, 1.0, 1.0) and $n = 10$*

			i		
k	1	3	5	7	10
1	−0.70728	0.06452	0.98307	0.99997	1.00000
2	−0.79005	0.90955	1.98177	1.99997	2.00000
3	−0.63435	1.96138	2.98429	2.99998	3.00000
4	−0.31361	3.06454	3.98732	3.99998	4.00000

Table 9   *Moments of the order statistics $E(X_{i:n}^k)$ for BP(1.5, 1.5, 1.0, 1.0) and $n = 30$*

			i		
k	1	10	20	25	30
1	0.80966	0.99974	0.97416	1.00014	1.00000
2	0.79096	1.99976	1.98305	2.00025	2.00000
3	0.57228	2.99982	2.99262	2.99880	3.00000
4	0.33587	3.99987	3.99791	3.99654	4.00000

The L-moments are linear functions of expected order statistics defined by Hoskings (1990) as

$$\lambda_{r+1} = (r+1)^{-1} \sum_{k=0}^{r} (-1)^k \binom{r}{k} E(X_{r+1-k:r+1}), \qquad r = 0, 1, \ldots .$$

The first four L-moments are $\lambda_1 = E(X_{1:1})$, $\lambda_2 = \frac{1}{2}E(X_{2:2} - X_{1:2})$, $\lambda_3 = \frac{1}{3}E(X_{3:3} - 2X_{2:3} + X_{1:3})$ and $\lambda_4 = \frac{1}{4}E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4})$. These moments have several advantages over the ordinary moments. For example, they exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are relatively robust to the effects of outliers.

Equation (7.1) applied to the means $(k = 1)$ of the order statistics give the L-moments of the BP distribution. They can also be determined in terms of the PWMs from (4.1) as

$$\lambda_{r+1} = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} \binom{r+k}{k} m_{1,k}, \qquad r = 0, 1, \ldots.$$

In particular, $\lambda_1 = m_{1,0}$, $\lambda_2 = 2m_{1,1} - m_{1,0}$, $\lambda_3 = 6m_{1,2} - 6m_{1,1} + m_{1,0}$, $\lambda_4 = 20m_{1,3} - 30m_{1,2} + 12m_{1,1} - m_{1,0}$.

8 Entropy

The entropy of a random variable X with density function $f(x)$ is a measure of variation of the uncertainty. One of the popular entropy measure is the Rényi entropy given by

$$\mathcal{J}_R(\gamma) = \frac{1}{1 - \gamma} \log\left[\int f^{\gamma}(x)\, dx \right], \qquad \gamma > 0,\ \gamma \neq 1. \tag{8.1}$$

The expansion of the BP density (2.3) yields

$$\mathcal{J}_R(\gamma) = \frac{1}{1 - \gamma} \log\left\{ \left[\alpha \frac{\Gamma(a+b)}{\Gamma(a)} \right]^{\gamma} \int_0^{1/\beta} \left[\sum_{i=0}^{\infty} \frac{(-1)^i \beta^{\alpha(a+i)} x^{\alpha(a+i)-1}}{\Gamma(b-i)i!} \right]^{\gamma} dx \right\}.$$

In order to obtain an expansion for $G(x)^{\gamma}$ for $\gamma > 0$ real noninteger, we can write

$$G(x)^{\gamma} = [1 - \{1 - G(x)\}]^{\gamma} = \sum_{j=0}^{\infty} \binom{\gamma}{j} (-1)^j \{1 - G(x)\}^j$$

and then

$$G(x)^{\gamma} = \sum_{j=0}^{\infty} \sum_{r=0}^{j} (-1)^{j+r} \binom{\gamma}{j} \binom{j}{r} G(x)^r.$$

We can substitute $\sum_{j=0}^{\infty} \sum_{r=0}^{j}$ for $\sum_{r=0}^{\infty} \sum_{j=r}^{\infty}$ to obtain

$$G(x)^{\gamma} = \sum_{r=0}^{\infty} \sum_{j=r}^{\infty} (-1)^{j+r} \binom{\gamma}{j} \binom{j}{r} G(x)^r$$

and then

$$G(x)^{\gamma} = \sum_{r=0}^{\infty} s_r(\gamma) G(x)^r, \tag{8.2}$$

where the coefficients $s_r(\gamma)$ are given by

$$s_r(\gamma) = \sum_{j=r}^{\infty} (-1)^{r+j} \binom{\gamma}{j} \binom{j}{r}. \tag{8.3}$$

For the BP distribution, we have

$$F(x) = \sum_{i=0}^{\infty} \frac{(-1)^i \beta^{\alpha(a+i)} x^{\alpha(a+i)-1}}{\Gamma(b-i)i!}.$$

Thus, from equations (8.2) and (8.3), we obtain

$$\mathcal{J}_R(\gamma) = \frac{\gamma}{1-\gamma}[\log(\alpha) + \delta(a+b) - \delta(a)]$$

$$+ \frac{1}{1-\gamma} \log\left\{ \int_0^{1/\beta} \sum_{r=0}^{\infty} s_r(\gamma) \left[\sum_{i=0}^{\infty} \frac{(-1)^i \beta^{\alpha(a+i)} x^{\alpha(a+i)-1}}{\Gamma(b-i)i!} \right]^r dx \right\},$$

where $\delta(\cdot) = \log\{\Gamma(\cdot)\}$. Hence,

$$\mathcal{J}_R(\gamma) = \frac{\gamma}{1-\gamma}[\log(\alpha) + \delta(a+b) - \delta(a)]$$

$$+ \frac{1}{1-\gamma} \log\left[\int_0^{1/\beta} \sum_{r=0}^{\infty} s_r(\gamma) \right.$$

$$\times \sum_{m_1,\dots,m_q=0}^{r} a_{m_1} \cdots a_{m_q} \beta^{\alpha(ra + \sum_{q=1}^{r} m_q)}$$

$$\left. \times x^{\alpha(ra + \sum_{q=1}^{r} m_q) - r} dx \right],$$

where

$$a_{m_q} = \frac{(-1)^{m_q}}{\Gamma(b - m_q)m_q!}, \qquad q = 1, \dots, r.$$

Then,

$$\mathcal{J}_R(\gamma) = \frac{\gamma}{1-\gamma}[\log(\alpha) + \delta(a+b) - \delta(a)]$$

$$+ \frac{1}{1-\gamma} \log\left[\sum_{r=0}^{\infty} s_r(\gamma) \sum_{[m_1,\dots,m_q]=0}^{r} a_{m_1} \cdots a_{m_q} \beta^{\alpha(ra + \sum_{q=1}^{r} m_q)} \right.$$

$$\left. \times \int_0^{1/\beta} x^{\alpha(ra + \sum_{q=1}^{r} m_q) - r} dx \right].$$

By calculating the integral, we have

$$
\mathcal{J}_R(\gamma) = \frac{\gamma}{1-\gamma}[\log(\alpha) + \delta(a+b) - \delta(a)]
$$

$$
+ \frac{1}{1-\gamma}\log\Biggl\{\sum_{r=0}^{\infty} s_r(\gamma)
$$

$$
\times \sum_{m_1,\ldots,m_q=0}^{r} \left(a_{m_1}\cdots a_{m_q}\beta^{\alpha(ra+\sum_{q=1}^{r}m_q)}\right)
$$

$$
\times \left(\beta^{\alpha(ra+\sum_{q=1}^{r}m_q)-r+1}\right.
$$

$$
\times \left.\left[\alpha\left(ra+\sum_{q=1}^{r}m_q\right)-r+1\right]\right)^{-1}\Biggr\}.
$$

Finally, the entropy of the BP distribution reduces to

$$
\mathcal{J}_R(\gamma) = \frac{\gamma}{1-\gamma}[\log(\alpha) + \delta(a+b) - \delta(a)]
$$

$$
+ \frac{1}{1-\gamma}\log\Biggl\{\sum_{r=0}^{\infty}\beta^{r-1}s_r(\gamma)\sum_{m_1,\ldots,m_q=0}^{r}\frac{a_{m_1}\cdots a_{m_q}}{[\alpha(ra+\sum_{q=1}^{r}m_q)-r+1]}\Biggr\}.
$$

9 Reliability

In the area of stress-strength models there has been a large amount of work as regards estimation of the reliability $R = Pr(X_2 < X_1)$ when X_1 and X_2 are independent random variables belonging to the same univariate family of distributions. The algebraic form for R has been worked out for the majority of the well-known standard distributions. Here, we derive the reliability R when X_1 and X_2 are independent and have the same BP distribution. The definition of the reliability is

$$
R = \int_0^{1/\beta} f(x)F(x)\,dx. \tag{9.1}
$$

Using the expansions (2.2) and (2.3) in (9.1), we have

$$
R = \int_0^{1/\beta}\left[\frac{\Gamma(a+b)}{\Gamma(a)}\right]^2
$$

$$
\times \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left\{\frac{(-1)^{i+j}[\alpha(a+i)\beta(\beta x)^{\alpha(a+i)-1}](\beta x)^{\alpha(a+j)}}{\Gamma(b-i)\Gamma(b-j)i!j!(a+i)(a+j)}\right\}dx.
$$

Interchanging the integral and the sums and calculating the integral, we obtain

$$R = \left[\frac{\Gamma(a+b)}{\Gamma(a)}\right]^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left[\frac{(-1)^{i+j}}{\Gamma(b-i)\Gamma(b-j)i!j!(a+j)(2a+i+j)}\right].$$

The reliability of the power distribution is $1/2$.

10 Estimation

We consider that X has the BP distribution and let $\boldsymbol{\theta} = (a, b, \alpha, \beta)^T$ be the vector of parameters. The log-likelihood $\ell = \ell(\boldsymbol{\theta})$ for a random sample $x_1, \ldots, x_n$ from (1.6) reduces to

$$\ell = n \log(\alpha\beta) - n \log B(a, b)$$
$$+ (\alpha a - 1) \sum_{i=1}^{n} \log(\beta x_i) + (b - 1) \sum_{i=1}^{n} \log[1 - (\beta x_i)^{\alpha}].$$

The components of the score vector $U = U(\boldsymbol{\theta}) = (\partial\ell/\partial a, \partial\ell/\partial b, \partial\ell/\partial\alpha, \partial\ell/\partial\beta)^T$ for n observations are

$$\frac{\partial\ell}{\partial a} = -n\psi(a) + n\psi(a + b) + \alpha \sum_{i=1}^{n} \log(\beta x_i),$$

$$\frac{\partial\ell}{\partial b} = -n\psi(b) + n\psi(a + b) + \sum_{i=1}^{n} \log[1 - (\beta x_i)^{\alpha}],$$

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + a \sum_{i=1}^{n} \log(\beta x_i) - (b - 1) \sum_{i=1}^{n} \frac{(\beta x_i)^{\alpha} \log(\beta x_i)}{1 - (\beta x_i)^{\alpha}},$$

$$\frac{\partial\ell}{\partial\beta} = \frac{\alpha}{\beta}\left[na - (b - 1) \sum_{i=1}^{n} \frac{(\beta x_i)^{\alpha}}{1 - (\beta x_i)^{\alpha}}\right],$$

where $\psi(p) = \partial \log\{\Gamma(p)\}/\partial p$ is the digamma function. We can obtain the maximum likelihood estimate (MLE) of $\boldsymbol{\theta}$ by setting the components of the score vector to zero and solving the nonlinear equations simultaneously. The MLE $\widehat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ can be calculated by the Newton–Raphson method. Since the support of the BP distribution depends on the parameter β, the normal distribution could not be a good approximation for the asymptotic distribution of $\sqrt{n}(\hat{\theta}_r - \theta_r)$ in moderate samples. However, we can construct a confidence interval with significance level γ for each parameter θ_r given by

$$ACI(\theta_r, 100(1 - \gamma)\%) = (\hat{\theta}_r - z_{\gamma/2}\sqrt{\hat{j}^{\theta_r,\theta_r}}, \hat{\theta}_r + z_{\gamma/2}\sqrt{\hat{j}^{\theta_r,\theta_r}}),$$

under the assumption that the normal approximation holds. Here, $\hat{\jmath}^{\theta_r,\theta_r}$ is the rth diagonal element of the estimated inverse observed information matrix for $r = 1, \ldots, 4$ and $z_{\gamma/2}$ is the quantile $1 - \gamma/2$ of the standard normal distribution.

The likelihood ratio (LR) statistic is useful for testing goodness of fit of the BP distribution and for comparing this distribution with some of its special sub-models. If we consider the partition $\boldsymbol{\theta} = (\boldsymbol{\theta}_1^T, \boldsymbol{\theta}_2^T)^T$, tests of hypotheses of the type $H_0 : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_1^{(0)}$ versus $H_A : \boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_1^{(0)}$ can be performed via LR statistics. The LR statistic for testing the null hypothesis H_0 is $w = 2\{\ell(\widehat{\boldsymbol{\theta}}) - \ell(\widetilde{\boldsymbol{\theta}})\}$, where $\widehat{\boldsymbol{\theta}}$ and $\widetilde{\boldsymbol{\theta}}$ are the MLEs of $\boldsymbol{\theta}$ under H_A and H_0, respectively. Under the null hypothesis, w could be approximated by the χ_q^2 distribution, where q is the dimension of the vector θ_1 of interest. The approximation could be poor in moderate samples. The LR test rejects H_0 if $w > \xi_\gamma$, where ξ_γ denotes the upper $100\gamma\%$ point of the χ_q^2 distribution. For example, we can check if the fit of the BP distribution is statistically "superior" to a fit using the power distribution for a given dataset by testing $H_0 : a = b = 1$ versus $H_A : H_0$ is not true.

11 Applications

In this section, we analyze two real datasets in order to illustrate the good performance of the BP distribution.

First dataset

The BP distribution is fitted to a dataset obtained from measurements on petroleum rock samples. The data consist of 48 rock samples from a petroleum reservoir. The dataset corresponds to twelve core samples from petroleum reservoirs that were sampled by four cross-sections. Each core sample was measured for permeability and each cross-section has the following variables: the total area of pores, the total perimeter of pores and shape. We analyze the shape perimeter by squared (area) variable. Table 10 gives the dataset.

The Newton–Raphson procedure to calculate the MLEs is performed by taking the initial values $a = 55.0$, $b = 98.0$, $\alpha = 0.4$ and $\beta = 0.2$ leading to the following

Table 10 *Shape perimeter by squared (area) from measurements on petroleum rock samples*

|					
0.0903296	0.2036540	0.2043140	0.2808870	0.1976530	0.3286410
0.1486220	0.1623940	0.2627270	0.1794550	0.3266350	0.2300810
0.1833120	0.1509440	0.2000710	0.1918020	0.1541920	0.4641250
0.1170630	0.1481410	0.1448100	0.1330830	0.2760160	0.4204770
0.1224170	0.2285950	0.1138520	0.2252140	0.1769690	0.2007440
0.1670450	0.2316230	0.2910290	0.3412730	0.4387120	0.2626510
0.1896510	0.1725670	0.2400770	0.3116460	0.1635860	0.1824530
0.1641270	0.1534810	0.1618650	0.2760160	0.2538320	0.2004470
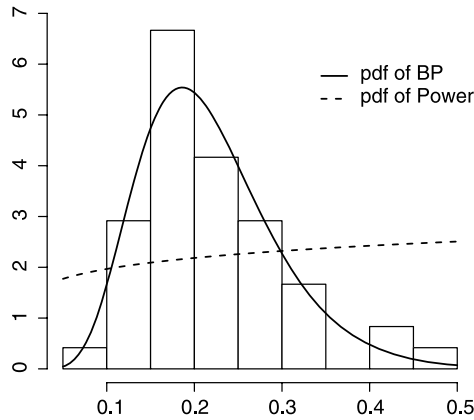
Figure 5   *The histogram of the first dataset and both fitted density functions.*
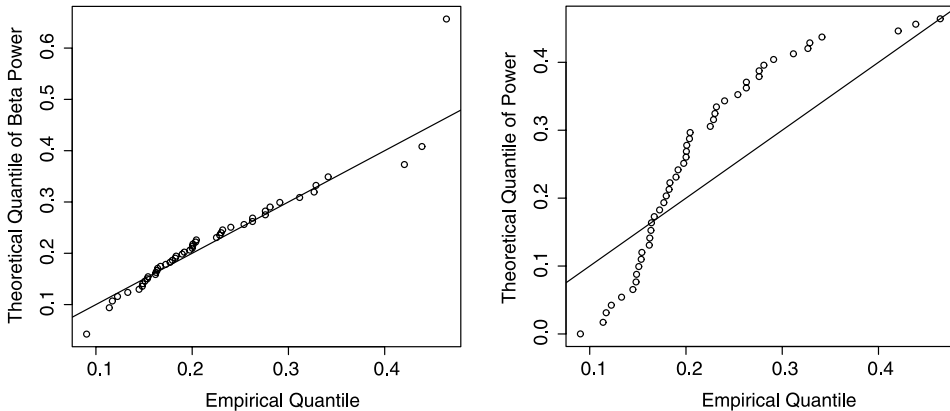

Figure 6   *Plots of the theoretical quantiles versus empirical quantiles for both distributions fitted to the first dataset.*

estimates for the BP distribution: $\hat{a} = 56.0247$, $\hat{b} = 97.8101$, $\hat{\alpha} = 0.2949$ and $\hat{\beta} = 0.1561$. For the power distribution, the MLEs are: $\tilde{\alpha} = 1.1506$ and $\tilde{\beta} = 2.1546$. The LR statistic is equal to $w = 31.2172$ and supports the hypothesis that the BP distribution is a better model. Figure 5 provides the histogram of these data and the fitted BP and power densities.

We note that the BP distribution produces a better fit than the power distribution. Figure 6 plots the theoretical quantiles versus empirical quantiles for both fitted distributions, and again the BP distribution is more appropriate to fit these data because the points are closer to the straight line.

Table 11 *Proportion of total milk production*

0.4365	0.4260	0.5140	0.6907	0.7471	0.2605	0.6196
0.8781	0.4990	0.6058	0.6891	0.5770	0.5394	0.1479
0.2356	0.6012	0.1525	0.5483	0.6927	0.7261	0.3323
0.0671	0.2361	0.4800	0.5707	0.7131	0.5853	0.6768
0.5350	0.4151	0.6789	0.4576	0.3259	0.2303	0.7687
0.4371	0.3383	0.6114	0.3480	0.4564	0.7804	0.3406
0.4823	0.5912	0.5744	0.5481	0.1131	0.7290	0.0168
0.5529	0.4530	0.3891	0.4752	0.3134	0.3175	0.1167
0.6750	0.5113	0.5447	0.4143	0.5627	0.5150	0.0776
0.3945	0.4553	0.4470	0.5285	0.5232	0.6465	0.0650
0.8492	0.8147	0.3627	0.3906	0.4438	0.4612	0.3188
0.2160	0.6707	0.6220	0.5629	0.4675	0.6844	–
0.3413	0.4332	0.0854	0.3821	0.4694	0.3635	–
0.4111	0.5349	0.3751	0.1546	0.4517	0.2681	–
0.4049	0.5553	0.5878	0.4741	0.3598	0.7629	–
0.5941	0.6174	0.6860	0.0609	0.6488	0.2747	–

Second dataset

The BP distribution is now fitted to the data about the total milk production in the first birth of 107 cows from SINDI race. These cows are property of the Carnaúba farm which belongs to the *Agropecuária Manoel Dantas Ltda* (AMDA), located in Taperoá City, Paraíba (Brazil).

The original data is not in the interval $(0, 1)$, and it was necessary to make a transformation given by $x_i = [y_i - \min(y_i)]/[\max(y_i) - \min(y_i)]$, for $i = 1, \ldots, 107$. These data are presented in Table 11 and the values of y_i are given in Table 3.1 of Brito (2009, p. 46).

The initial values for the iterative algorithm are: $a = 0.5$, $b = 42.0$, $\alpha = 4.0$ and $\beta = 0.8$. The MLEs of the parameters of the BP distribution are: $\hat{a} = 0.2704$, $\hat{b} = 42.0228$, $\hat{\alpha} = 6.6402$ and $\hat{\beta} = 0.7756$. For the power distribution, the estimates of the parameters are: $\tilde{\alpha} = 5.0027$ and $\tilde{\beta} = 1.1364$. The LR statistic $w = 333.615$ indicates that the BP distribution is a better model than the power distribution. Figure 7 gives the histogram of the second dataset and the plots of both fitted densities. It is possible to verify the good performance of the BP distribution. The plots of the theoretical quantiles versus empirical quantiles given in Figure 8 suggest that the BP distribution is very suitable for these data.

12 Conclusion

We study the mathematical properties of the beta power (BP) distribution which extends the power distribution defined by Balakrishnan and Nevzorov (2003). The BP distribution is not new since it was defined as the name of the generalized
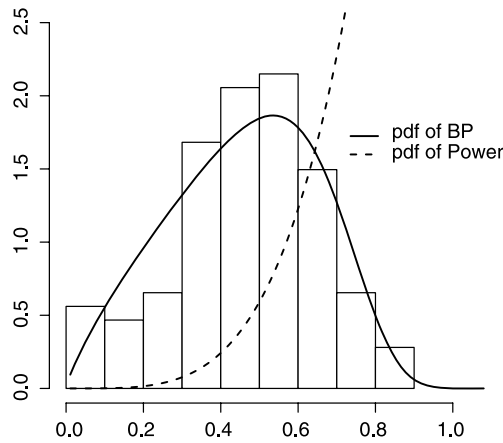
Figure 7 *The histogram of the second dataset and both fitted density functions.*
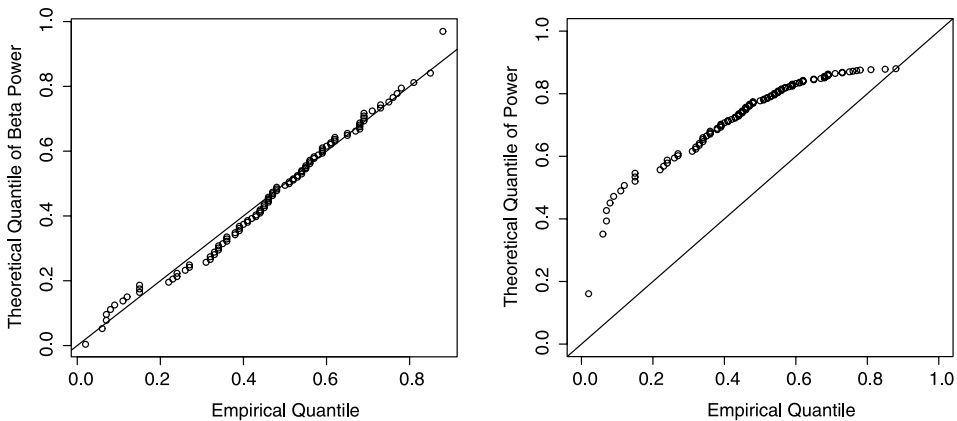

Figure 8 *Plots of theoretical quantiles versus empirical quantiles for both distributions fitted to the second dataset.*

beta distribution of the second kind by McDonald and Richards (1987). However, they do not study its properties. We demonstrate that the BP density is a mixture of power densities. We provide a mathematical treatment for the distribution including closed-form expressions for the moments, probability weighted moments, moment generating function, mean deviations, Bonferroni and Lorenz curves, moments of order statistics, entropy and reliability. We discuss maximum likelihood estimation of the model parameters. We give two applications to real data to show that the BP distribution can be used quite effectively to give better fits than the power model. We hope that the BP model may attract wider applications in statistics.

Appendix A: A Mathematica script to calculate $m_{r,j}$ for the BP distribution

```
Clear[a,b,alpha0,beta0,r,j];

FunX:= Function[{a,b,alpha0,beta0}, (1/(beta0^(r + 1)))*
  ((Gamma[a + b]/Gamma[a])^(j))*
   (Product[ Sum[(((-1)^(Mq))/(Gamma[b - Mq]*(N[Mq!])*
    (a + Mq)))/(r + (alpha0*(j*a + Sum[Mq, {g, 1, j}]))) + 1),
       {Mq, 0, 10000}], {g, 1, j}])]

Clear[mx];

mx = Table[FunX[2.5, 3.5, 1.0, 1.0], {r, 4}, {j, 1, 6}]
```

In this case, the algorithm calculates $m_{r,j}$ for the parameter values $a = 2.5$, $b = 3.5$, $\alpha = 1.0$ and $\beta = 1.0$ with the sums truncated at 10,000.

Appendix B: Proof of the moments

From equation (3.1), we have

$$\mu'_r = \frac{\alpha \beta^{\alpha a}}{B(a,b)} \int_0^{1/\beta} x^{r+\alpha a-1}[1 - (\beta x)^{\alpha}]^{b-1}\, dx.$$

Setting $u = (\beta x)^{\alpha}$, we obtain

$$\mu'_r = \frac{\alpha \beta^{\alpha a}}{B(a,b)} \int_0^1 \left(\frac{u^{1/\alpha}}{\beta}\right)^{r+\alpha a-1} \left[\frac{(1-u)^{b-1}}{\alpha \beta u^{1-1/\alpha}}\right] du$$

$$= \frac{1}{\beta^r B(a,b)} \int_0^1 u^{a+r/\alpha-1}(1-u)^{b-1}\, du$$

and then

$$\mu'_r = \frac{B(a + r/\alpha, b)}{\beta^r B(a,b)}.$$

Appendix C: Proof of the moments of the order statistics

We have, assuming that the moments exist,

$$E(X^k_{i:n}) = k \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i}\binom{n}{j} I_j(k), \qquad (C.1)$$

where

$$I_j(k) = \int_0^{1/\beta} x^{k-1}[1 - F(x)]^j \, dx.$$

Using the binomial expansion for $[1 - F(x)]^j$, it follows

$$[1 - F(x)]^j = \sum_{l=0}^{j}(-1)^l \binom{j}{l} F(x)^l$$

$$= \sum_{l=0}^{j}(-1)^l \binom{j}{l} \left[\frac{\Gamma(a+b)}{\Gamma(a)} \sum_{u=0}^{\infty} \frac{(-1)^u(\beta x)^{\alpha(a+u)}}{\Gamma(b-u)u!(a+u)}\right]^l.$$

For k a positive integer, we also have

$$\left(\sum_{u=0}^{\infty} a_u\right)^m = \sum_{m_1,\dots,m_k=0}^{\infty} a_{m_1}\cdots a_{m_k}.$$

We can rewrite

$$I_j(k) = \int_0^{1/\beta} x^{k-1} \sum_{l=0}^{j}(-1)^l \binom{j}{l} \left[\frac{\Gamma(a+b)}{\Gamma(a)}\right]^l \tag{C.2}$$

$$\times \sum_{m_1,\dots,m_l=0}^{\infty} a_{m_1}\cdots a_{m_l} x^{\alpha \sum_{q=1}^{l}(a+m_q)} \, dx,$$

where the quantities a_{m_q} are given by

$$a_{m_q} = \frac{(-1)^{m_q}\beta^{\alpha(a+m_q)}}{\Gamma(b-m_q)m_q!(a+m_q)}, \qquad q = 1,\dots,l.$$

Plugging (C.2) into (C.1), the moments of the order statistics can be written as

$$E(X_{i:n}^k) = k \sum_{j=n-i+1}^{n} \sum_{l=0}^{j}(-1)^{j-n+i+l-1} \binom{j-1}{n-i}\binom{n}{j}\binom{j}{l}\left[\frac{\Gamma(a+b)}{\Gamma(a)}\right]^l$$

$$\times \sum_{m_1,\dots,m_l=0}^{\infty} a_{m_1}\cdots a_{m_l}\left\{\beta^{k+\alpha\sum_{q=1}^{l}(a+m_q)}\right.$$

$$\left.\times\left[k + \alpha \sum_{q=1}^{l}(a+m_q)\right]^{-1}\right\}.$$

Appendix D: A Mathematica script to calculate the moments of the order statistics for the BP distribution

```
Clear[a,b,alpha0,beta0,k,i,n];
Clear[EmordX];

EMordX:= Function[{a, b, alpha0, beta0, n}, k*
Sum[ Sum[((-1)^(j - n + i + l - 1))*Binomial[j - 1, n - i]*
 Binomial[n, j]*Binomial[j, l]*
 (N[(Gamma[a + b]/Gamma[a])]^l)*(Product[ Sum[(((((-1)^(Mq))*
  (beta0^(alpha0*(a + Mq))))/(N[Gamma[b - Mq]]*(N[Mq!])*(a +
   Mq)))*((beta0^(k + (alpha0*(Sum[(
            a + Mq), {q, 1, l}])))))))*((k + (alpha0*(Sum[(a +
      Mq), {q, 1, l}])))))^(-1)), {Mq, 0, 500}], {s, 1, l}]),
         {l, 0, j}], {j, n - i + 1, n}]]

Clear[Emx]; Clear[i,j,k];
Emx = Table[EMordX[1.5, 1.0, 1.0, 1.0, 10], {i, 1, 10, 3},
                                            {k, 4}]
```

In this case, the algorithm calculates $E(X_{i:n}^k)$ for the parameter values $i = 1, 4, 7$ and 10, $n = 10$, $a = 1.5$, $b = 1.5$, $\alpha = 1.0$ and $\beta = 1.0$ with the sums truncated at 500.

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Departamento de Estatística e Informática
Universidade Federal Rural de Pernambuco
52171-900, Recife, PE
Brazil
E-mail: gausscordeiro@uol.com.br
        janesbrito@gmail.com