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A TRIGONOMETRIC APPROACH TO QUATERNARY CODE DESIGNS WITH APPLICATION TO ONE-EIGHTH AND ONE-SIXTEENTH FRACTIONS

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The study of good nonregular fractional factorial designs has received significant attention over the last two decades. Recent research indicates that designs constructed from quaternary codes (QC) are very promising in this regard. The present paper shows how a trigonometric approach can facilitate a systematic understanding of such QC designs and lead to new theoretical results covering hitherto unexplored situations. We focus attention on one-eighth and one-sixteenth fractions of two-level factorials and show that optimal QC designs often have larger generalized resolution and projectivity than comparable regular designs. Moreover, some of these designs are found to have maximum projectivity among all designs.

1. Introduction and preliminaries. Fractional factorial designs play a key role in efficient and economic experimentation with multiple factors and have gained immense popularity in various fields of application such as engineering, agriculture and medicine. These designs are broadly categorized as *regular* and *nonregular* depending on whether or not they can be generated via defining relations among the factors. In regular designs, any two factorial effects are either mutually orthogonal or completely aliased, and the criterion of *maximum resolution* [2] and its refinement, *minimum aberration* (MA) [10], are commonly used in discriminating amongst these designs. We refer to [14, 22] for detailed surveys and extensive references on regular designs.

The last two decades, especially the last ten years, have witnessed a significant spurt in research on nonregular designs. The case of two-level factors has received

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particular attention. The notions of resolution and aberration have been generalized, with statistical justifications, to these designs; see [7, 9, 13, 19, 20, 27, 28]. More recent work on nonregular designs and related topics include [4] giving theoretical results on generalized MA designs, [12] giving a catalog of generalized MA designs, [24] on moment aberration projection designs, [23] on designs obtained from the Nordstrom and Robinson code, and [17] on a complete classification of certain two-level orthogonal arrays. It is well recognized now that although nonregular designs have a complex aliasing structure, they can outperform their regular counterparts with regard to resolution or projectivity, and this is one of the principal motivating forces for the current surge of interest in these designs. For more details, see [25] giving a state-of-the-art review of nonregular designs with a comprehensive list of references.

A recent major development in nonregular two-level designs has been the use of quaternary codes (QC) for their efficient construction. The resulting two-level designs are hereafter called QC designs. While QCs are known to yield good binary codes in coding theory [11], QC designs have been seen to be attractive with regard to resolution, aberration and projectivity. Moreover, as noted in [25], these designs are relatively straightforward to construct and have simple design representation. Xu and Wong [26] pioneered research on QC designs and reported theoretical as well as computational results. Phoa and Xu [16] obtained comprehensive analytical results on quarter fraction QC designs and showed that they often have larger resolution and projectivity than regular designs of the same size.

The present paper aims at extending [16] to more highly fractionated settings. A serious hurdle in this regard is that the inductive proofs in [16] become unmanageable when one attempts to go beyond quarter fractions. A trigonometric representation for QC designs is employed here in order to overcome this difficulty. This approach is found to be quite convenient for one-eighth and one-sixteenth fractions which form our main focus. Earlier, Phoa [15] in his unpublished Ph.D. dissertation reported partial results on one-sixteenth fraction QC designs under a certain assumption. An advantage of our approach is that it involves no such restrictive assumption and enables us to obtain unified and more comprehensive results on these fractions. As discussed in the concluding remarks, the trigonometric formulation holds the promise of being applicable to even more general settings as well.

In Section 2, we introduce the trigonometric approach and present results pertaining to generalized resolution and wordlength pattern (WLP) for the case where the number of runs, N, is an even power of 2. The corresponding results when N equals an odd power of 2 appear in Section 3. Section 4 dwells on projectivity and some directions for future work are indicated in Section 5. Satisfyingly, with one-eighth and one-sixteenth fractions, at least over the range covered by our tables, (a) the same design turns out to be optimal among all QC designs with respect to all the commonly used criteria like resolution, aberration and projectivity, and (b) such an optimal design is often seen to have higher resolution and projectivity than what regular designs can achieve. The point in (b) reinforces the findings in [16] for quarter fractions but that in (a) is in contrast to what they observed in their setup. It is also seen that some of our optimal QC designs have maximum projectivity among all designs.

Before concluding this section, we reproduce some definitions from [16] for ease in reference. A two-level design D with N runs and q factors is represented by an $N \times q$ matrix with entries ± 1, where the rows and columns are identified with the runs and factors, respectively. For any subset $S = \{c_1, \dots, c_k\}$ of k columns of D, define

$$J_k(S; D) = \sum_{s=1}^{N} c_{s1} \cdots c_{sk}, \tag{1.1}$$

where c_{sj} is the sth entry of column c_j. The $J_k(S; D)$ values are called the *J-characteristics* of design D; cf. [9, 20]. Following [8], the *aliasing index* of S is defined as $\rho_k(S; D) = |J_k(S; D)|/N$. Clearly, $0 \le \rho_k(S; D) \le 1$. If $\rho_k(S; D) = 1$, then the columns in S are fully aliased with one another and form a *complete word* of length k and aliasing index 1. If $0 < \rho_k(S; D) < 1$, then these columns are partially aliased with one another and form a *partial word* of length k and aliasing index $\rho_k(S; D)$. Finally, if $\rho_k(S; D) = 0$, then these columns do not form a word.

Let r be the smallest integer such that $\max_{\#S=r} \rho_r(S; D) > 0$, where # denotes cardinality of a set and the maximum is over all subsets S of r columns of D. The *generalized resolution* [9] of D is defined as

$$R(D) = r + 1 - \max_{\#S=r} \rho_r(S; D). \tag{1.2}$$

For $1 \le k \le q$, let

$$A_k(D) = \sum_{\#S=k} \{\rho_k(S; D)\}^2. \tag{1.3}$$

The vector $(A_1(D), \dots, A_q(D))$ is called the *generalized WLP* of D. The *generalized MA* criterion [27], also known as minimum G_2 aberration [20], calls for sequential minimization of $(A_1(D), \dots, A_q(D))$. When restricted to regular designs, generalized resolution, generalized WLP and generalized MA reduce to the traditional resolution, WLP and MA, respectively. For simplicity, we use the terminology resolution, WLP and MA for both regular and nonregular designs.

Following [1], the design D is said to have *projectivity* p if every p-factor projection contains a complete 2^p factorial design, possibly with some points replicated. Evidently, a regular design of resolution R has projectivity $R - 1$. As shown in [9], a design with resolution $R > r$ has projectivity greater than r.

2. Quaternary code designs in 2^{2n} runs.

2.1. *One-sixteenth fractions.* In the spirit of [16], let C be the QC given by the $n \times (n + 2)$ generator matrix $[u \;\; v \;\; I_n]$, where $u = (u_1, \dots, u_n)'$ and

$v = (v_1, \dots, v_n)'$ are $n \times 1$ vectors over $Z_4 = \{0, 1, 2, 3\}$ (mod 4), I_n is the identity matrix of order n over Z_4, and the primes stand for transpose. The code C, consisting of $4^n (=2^{2n})$ codewords, each of size $n + 2$, can be described as

$$C = \{(a'u, a'v, a_1, \dots, a_n) : a_1, \dots, a_n \in Z_4\}, \tag{2.1}$$

where $a = (a_1, \dots, a_n)'$, and $a'u$ and $a'v$ are reduced mod 4. The *Gray map*, which replaces each element of Z_4 with a pair of two symbols, transforms C into a binary code D, called the binary image of C. For convenience, the two symbols are taken as 1 and -1, instead of the more conventional 0 and 1. Then the Gray map is defined as

$$0 \to (1, 1), \qquad 1 \to (1, -1), \qquad 2 \to (-1, -1), \qquad 3 \to (-1, 1). \tag{2.2}$$

With its codewords as rows, D is a $2^{2n} \times (2n+4)$ matrix having entries ± 1. Indeed, with columns and rows identified with factors and runs, respectively, D represents a design involving $2n + 4$ two-level factors and 2^{2n} runs. In this sense, D will be referred to as a $2^{(2n+4)-4}$ QC design.

A representation of D using trigonometric functions facilitates the study of its statistical properties which depend on the choice of u and v. Since the pair $(\sqrt{2}\sin(\frac{\pi}{4} + \frac{\pi}{2}k), \sqrt{2}\cos(\frac{\pi}{4} + \frac{\pi}{2}k))$ equals $(1, 1)$, $(1, -1)$, $(-1, -1)$ and $(-1, 1)$ for $k = 0, 1, 2$ and 3 (mod 4), respectively, by (2.1) and (2.2), the 2^{2n} runs in D can be expressed as

$$\begin{aligned} &\sqrt{2}\Big[\sin\Big(\frac{\pi}{4} + \frac{\pi}{2}a'u\Big), \cos\Big(\frac{\pi}{4} + \frac{\pi}{2}a'u\Big), \sin\Big(\frac{\pi}{4} + \frac{\pi}{2}a'v\Big), \\ &\quad \cos\Big(\frac{\pi}{4} + \frac{\pi}{2}a'v\Big), \sin\Big(\frac{\pi}{4} + \frac{\pi}{2}a_1\Big), \cos\Big(\frac{\pi}{4} + \frac{\pi}{2}a_1\Big), \dots, \\ &\qquad \sin\Big(\frac{\pi}{4} + \frac{\pi}{2}a_n\Big), \cos\Big(\frac{\pi}{4} + \frac{\pi}{2}a_n\Big)\Big], \\ &\qquad\qquad a_1, \dots, a_n \in Z_4. \end{aligned} \tag{2.3}$$

Denote the $2n + 4$ factors in D by $F_1, \dots, F_4, F_{11}, F_{12}, \dots, F_{n1}, F_{n2}$, in conformity with the ordering in (2.3), that is, $\sqrt{2}\sin(\frac{\pi}{4} + \frac{\pi}{2}a'u)$ and $\sqrt{2}\cos(\frac{\pi}{4} + \frac{\pi}{2}a'u)$ are the levels of F_1 and F_2, and so on.

The factors $F_1, \dots, F_4$, with levels dictated by u or v, require special attention. From this perspective, for any nonempty collection of factors (or equivalently, columns of D), let $x_k = 1$ or 0 according as whether F_k is included in the collection or not, $1 \le k \le 4$. With this notation, the collection is said to be of the type $x = x_1x_2x_3x_4$. Thus, for any binary 4-tuple $x = x_1x_2x_3x_4$, a typical collection of type x consists of factors F_k with $x_k = 1$ $(1 \le k \le 4)$, and also factors F_{j1}, F_{j2} $(j \in S_1)$, F_{j2} $(j \in S_2)$ and F_{j1} $(j \in S_3)$, where S_1, S_2, S_3 are any disjoint subsets of $\{1,\dots,n\}$. The total number of factors in the collection is then $m+X$, where $X=x_1+x_2+x_3+x_4$, $m=2n_1+n_2+n_3$ and $n_j=\#S_j$ $(1\le j\le 3)$. Here S_1,S_2,S_3 can be empty sets as well but if $x=0000$ then at least one of them is nonempty, for otherwise, the collection contains no factor at all. From (1.1), (2.3) and the definition of aliasing index, it follows that the aliasing index of a collection of type x $(=x_1x_2x_3x_4)$ as described above is given by $|V(x)|$, where

$$V(x)=\sum_{a_1=0}^{3}\cdots\sum_{a_n=0}^{3}\phi(x;a_1,\dots,a_n) \tag{2.4}$$

with

$$\begin{aligned}\phi(x;a_1,\dots,a_n)&\\ &=2^{(1/2)(m+X)-2n}\sin^{x_1}\Big(\frac{\pi}{4}+\frac{\pi}{2}a'u\Big)\cos^{x_2}\Big(\frac{\pi}{4}+\frac{\pi}{2}a'u\Big)\\ &\quad\times\sin^{x_3}\Big(\frac{\pi}{4}+\frac{\pi}{2}a'v\Big)\cos^{x_4}\Big(\frac{\pi}{4}+\frac{\pi}{2}a'v\Big)\\ &\quad\times\psi(a_1,\dots,a_n)\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\psi(a_1,\dots,a_n)&=\Big[\prod_{j\in S_1}\Big\{\sin\Big(\frac{\pi}{4}+\frac{\pi}{2}a_j\Big)\cos\Big(\frac{\pi}{4}+\frac{\pi}{2}a_j\Big)\Big\}\Big]\\ &\quad\times\Big[\prod_{j\in S_2}\cos\Big(\frac{\pi}{4}+\frac{\pi}{2}a_j\Big)\Big]\Big[\prod_{j\in S_3}\sin\Big(\frac{\pi}{4}+\frac{\pi}{2}a_j\Big)\Big].\end{aligned} \tag{2.6}$$

By (2.4)–(2.6), for any fixed x, the value of $V(x)$ depends on the sets S_1,S_2,S_3 in addition to u and v. Any choice of S_1,S_2,S_3 that makes $V(x)$ nonzero entails a word of length $m+X$ and aliasing index $|V(x)|$. Such a word will be called a word of type x.

We now present Theorem 1 below giving an account of words of all possible types. For $x=0101$, this result has been proved in the Appendix. The proofs for all other x are similar and occasionally simpler. In particular, the case $x=0000$ is evident from the presence of I_n in the generator matrix $[u\;\; v\;\; I_n]$ of C. Some more notations will help. With reference to the vectors u and v, let

$$\begin{aligned}&\Delta_{ks}=\{j:1\le j\le n,u_j=k,v_j=s\},\\ &f_{ks}=\#\Delta_{ks},\qquad 0\le k,s\le 3,\end{aligned} \tag{2.7}$$

$$\begin{aligned}&\lambda_1=f_{10}+f_{30},\qquad \lambda_2=f_{01}+f_{03},\qquad \lambda_3=f_{12}+f_{32},\\ &\lambda_4=f_{21}+f_{23},\qquad \lambda_5=f_{11}+f_{33},\qquad \lambda_6=f_{13}+f_{31},\\ &\lambda_7=f_{02},\qquad \lambda_8=f_{20},\qquad \lambda_9=f_{22},\qquad \lambda_{10}=f_{00},\end{aligned} \tag{2.8}$$

$$
\begin{aligned}
&l_1 = 2(\lambda_4 + \lambda_8 + \lambda_9) + \lambda_1 + \lambda_3 + \lambda_5 + \lambda_6,\\
&l_2 = 2(\lambda_3 + \lambda_7 + \lambda_9) + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_6,\\
&l_3 = 2(\lambda_2 + \lambda_8 + \lambda_9) + \lambda_1 + \lambda_3 + \lambda_5 + \lambda_6,\\
&l_4 = 2(\lambda_1 + \lambda_7 + \lambda_9) + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_6,\\
&l_5 = 2(\lambda_1 + \lambda_3 + \lambda_5 + \lambda_6), \qquad l_6 = 2(\lambda_2 + \lambda_4 + \lambda_5 + \lambda_6),\\
&l_7 = 2(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4), \qquad l_8 = 2(\lambda_7 + \lambda_8) + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4,\\
&l_9 = 2(\lambda_5 + \lambda_7 + \lambda_8) + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4,\\
&l_{10} = 2(\lambda_6 + \lambda_7 + \lambda_8) + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4,
\end{aligned} \tag{2.9}
$$

$$
\begin{aligned}
&\rho_1 = 1/2^{\langle (1/2)(\lambda_1+\lambda_3+\lambda_5+\lambda_6)\rangle}, \qquad \rho_2 = 1/2^{\langle (1/2)(\lambda_2+\lambda_4+\lambda_5+\lambda_6)\rangle},\\
&\xi_1 = 1/2^{\langle (1/2)(\lambda_1+\lambda_3)\rangle}, \qquad \xi_2 = 1/2^{\langle (1/2)(\lambda_2+\lambda_4)\rangle},\\
&\xi = 1/2^{\langle (1/2)(\lambda_1+\lambda_2+\lambda_3+\lambda_4+1)\rangle},
\end{aligned} \tag{2.10}
$$

where $\langle y\rangle$ is the largest integer not exceeding y. By (2.7), f_{ks} equals the frequency with which (ks) occurs as a row of the $n \times 2$ matrix $[u \;\; v]$. Also, each of the quantities introduced in (2.8)–(2.10) is uniquely determined by these frequencies.

THEOREM 1. *With reference to the* $2^{(2n+4)-4}$ *QC design* D, *the following hold*:

(a) *For* $x = 0000$, *there is no word of type* x.
(b) *For each of* $x = 0100$ *and* 1000, *there are* $1/\rho_1^2$ *words of type* x; *every such word has aliasing index* ρ_1 *and length* $l_1 + 1$.
(c) *For each of* $x = 0001$ *and* 0010, *there are* $1/\rho_2^2$ *words of type* x; *every such word has aliasing index* ρ_2 *and length* $l_2 + 1$.
(d) *For each of* $x = 0111$ *and* 1011, *there are* $1/\rho_1^2$ *words of type* x; *every such word has aliasing index* ρ_1 *and length* $l_3 + 3$.
(e) *For each of* $x = 1101$ *and* 1110, *there are* $1/\rho_2^2$ *words of type* x; *every such word has aliasing index* ρ_2 *and length* $l_4 + 3$.
(f) *For* $x = 1100$, *there is one word of type* x, *with aliasing index* 1 *and length* $l_5 + 2$.
(g) *For* $x = 0011$, *there is one word of type* x, *with aliasing index* 1 *and length* $l_6 + 2$.
(h) *For* $x = 1111$, *there is one word of type* x, *with aliasing index* 1 *and length* $l_7 + 4$.
(i) *For each of* $x = 0101$ *and* 1010,
  (i1) *if* $\lambda_5 + \lambda_6 = 0$, *then there are* $1/(\xi_1^2\xi_2^2)$ *words of type* x; *every such word has aliasing index* $\xi_1\xi_2$ *and length* $l_8 + 2$;
  (i2) *if* $\lambda_5 + \lambda_6 > 0$ *and* $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$, *then there is one word of type* x, *with aliasing index* 1 *and length* $l_{10} + 2$;

(i3) *if* $\lambda_5 + \lambda_6 > 0$ *and* $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 > 0$, *then there are* $1/\xi^2$ *words of type* x, *each with aliasing index* ξ; *half of these words have length* $l_9 + 2$ *and the rest have length* $l_{10} + 2$.

(j) *For each of* $x = 0110$ *and* 1001,

(j1) *if* $\lambda_5 + \lambda_6 = 0$, *then the same conclusion as in* (i1) *holds*;

(j2) *if* $\lambda_5 + \lambda_6 > 0$ *and* $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$, *then there is one word of type* x, *with aliasing index* 1 *and length* $l_9 + 2$;

(j3) *if* $\lambda_5 + \lambda_6 > 0$ *and* $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 > 0$, *then the same conclusion as in* (i3) *holds*.

Since $\xi = 1$ when $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$, one can merge (i2), (i3), (j2) and (j3) above when words of all types are considered together. Summarizing Theorem 1, we thus get the next result.

THEOREM 2. *With reference to the* $2^{(2n+4)-4}$ *QC design* D, *the following hold*:

(a) *There are* $4/\rho_1^2$ *words each with aliasing index* ρ_1, *half of these words have length* $l_1 + 1$ *and the rest have length* $l_3 + 3$.

(b) *There are* $4/\rho_2^2$ *words each with aliasing index* ρ_2; *half of these words have length* $l_2 + 1$ *and the rest have length* $l_4 + 3$.

(c) *There are three words each with aliasing index* 1; *these have lengths* $l_5 + 2$, $l_6 + 2$ *and* $l_7 + 4$.

(d) *In addition*,

(d1) *if* $\lambda_5 + \lambda_6 = 0$, *then there are* $4/(\xi_1^2 \xi_2^2)$ *words each with aliasing index* $\xi_1\xi_2$ *and length* $l_8 + 2$;

(d2) *if* $\lambda_5 + \lambda_6 > 0$, *then there are* $4/\xi^2$ *words each with aliasing index* ξ; *half of these words have length* $l_9 + 2$ *and the rest have length* $l_{10} + 2$.

Theorem 2, in conjunction with (2.9) and (2.10), shows that the resolution and WLP of the design D depend on u and v only through $\lambda_1, \ldots, \lambda_{10}$. Indeed, for any given u and v, Theorem 2 readily yields these features of D. This is illustrated below.

EXAMPLE 1. With $n = 3$, let $u = (2, 1, 1)'$ and $v = (1, 1, 3)'$. Then $f_{21} = f_{11} = f_{13} = 1$ and all other f's equal 0, so that by (2.8), $\lambda_4 = \lambda_5 = \lambda_6 = 1$ and all other λ's are zeros. Hence by (2.9) and (2.10),

$$l_1 = 4, \quad l_2 = 3, \quad l_3 = 2, \quad l_4 = 3, \quad l_5 = 4, \quad l_6 = 6,$$
$$l_7 = 2, \quad l_8 = 1, \quad l_9 = l_{10} = 3, \quad \rho_1 = \rho_2 = \xi = \tfrac{1}{2}.$$

As a result, parts (a), (b) and (d2) of Theorem 2 entail 48 words each with aliasing index $\frac{1}{2}$; of these, 8 have length four, 32 have length five and 8 have length 6. Similarly, part (c) of Theorem 2 entails three words having lengths six, eight, and six, and each with aliasing index 1. Hence by (1.2) and (1.3), in this case the QC design D, which is a 2^{10-4} design, has resolution 4.5 and WLP

$(0, 0, 0, 2, 8, 4, 0, 1, 0, 0)$. As seen later in Table 3, this design has maximum resolution and MA among all 2^{10-4} QC designs. Also, it has the same WLP but higher resolution than the regular 2^{10-4} MA design.

Even though the trigonometric formulation keeps our derivation tractable, Theorem 2 is considerably more involved than its counterpart, namely, Theorem 1 of [16], for quarter fractions. Consequently, analytical expressions for the optimal choice of u and v, or equivalently, of $\lambda_1, \dots, \lambda_{10}$, maximizing the resolution or minimizing the aberration of D do not exist in easily comprehensible forms. On the other hand, as Example 1 demonstrates, for any given $\lambda_1, \dots, \lambda_{10}$, the resolution and WLP of D can be obtained immediately from Theorem 2. Hence, we find the best choice of the λ's, with regard to resolution and aberration, by complete enumeration of all possible nonnegative integer-valued $\lambda_1, \dots, \lambda_{10}$ subject to $\lambda_1 + \cdots + \lambda_{10} = n$, a condition which is evident from (2.7) and (2.8). Because of the substantial reduction of the problem as achieved in Theorem 2, such complete enumeration can be done instantaneously, for example by MATLAB, for reasonable values of n. The results are summarized in Table 3 and discussed in the next section along with their counterparts for QC designs in 2^{2n+1} runs.

2.2. *One-eighth fractions.* Deletion of any one of the first four columns of the matrix D in Section 2.1 leads to a QC design involving $2n + 3$ two-level factors and 2^{2n} runs, that is, a $2^{(2n+3)-3}$ QC design. The competing class of QC designs, corresponding to all possible choices of u and v, remains the same up to isomorphism whichever of these four columns is deleted. This follows by interchanging the roles of u and v and noting that

$$\cos\left(\frac{\pi}{4} + \frac{\pi}{2}a'u\right) = \sin\left(\frac{\pi}{4} + \frac{\pi}{2}a'(3u)\right),$$
$$\cos\left(\frac{\pi}{4} + \frac{\pi}{2}a'v\right) = \sin\left(\frac{\pi}{4} + \frac{\pi}{2}a'(3v)\right)$$

as $a'u$ and $a'v$ are integers. Therefore, without loss of generality, we consider the deletion of the first column of D. Let $D^{(1)}$ denote the resulting design. Since the deletion of the first column of D amounts to dropping the factor F_1, continuing with the notation of Section 2.1, only collections of factors of types $x = 0000, 0001, 0010, 0011, 0100, 0101, 0110$ or 0111 can arise now. For all such x, Theorem 1 again describes the numbers of words of type x as well as the aliasing indices and lengths of these words. Analogously to Theorem 2, this can be summarized as follows.

THEOREM 3. *With reference to the $2^{(2n+3)-3}$ QC design $D^{(1)}$, the following hold*:

(a) *There are $2/\rho_1^2$ words each with aliasing index ρ_1; half of these words have length $l_1 + 1$ and the rest have length $l_3 + 3$.*
(b) *There are $2/\rho_2^2$ words each with aliasing index ρ_2 and length $l_2 + 1$.*

(c) *There is one word with aliasing index* 1 *and length* $l_6 + 2$.

(d) *In addition*,

(d1) *if* $\lambda_5 + \lambda_6 = 0$, *then there are* $2/(\xi_1^2\xi_2^2)$ *words each with aliasing index* $\xi_1\xi_2$ *and length* $l_8 + 2$;

(d2) *if* $\lambda_5 + \lambda_6 > 0$, *then there are* $2/\xi^2$ *words each with aliasing index* ξ; *half of these words have length* $l_9 + 2$ *and the rest have length* $l_{10} + 2$.

Theorem 3 shows that the resolution and WLP of $D^{(1)}$ depend on u and v only through $\lambda_1, \dots, \lambda_{10}$ and greatly simplifies the task of finding, by complete enumeration, the optimal λ's maximizing the resolution or minimizing the aberration of $D^{(1)}$. The results are summarized in Table 4 and discussed in the next section.

3. Quaternary code designs in 2^{2n+1} runs. First, consider one-sixteenth fraction QC designs in 2^{2n+1} runs as obtained by a branching technique studied in [16] for quarter fractions. In the present context, this technique can be conveniently described as follows. Let $\tilde{u} = (u_0, u_1, \dots, u_n)'$, $\tilde{v} = (v_0, v_1, \dots, v_n)'$ be $(n + 1) \times 1$ vectors and I_{n+1} be the identity matrix of order $n + 1$ over Z_4. Consider the QC given by the generator matrix $[\tilde{u}\ \ \tilde{v}\ \ I_{n+1}]$, and let $\tilde{C}$ be a collection of 2^{2n+1} codewords thereof, each of size $n + 3$, as given by

$$\tilde{C} = \{(\tilde{a}'\tilde{u}, \tilde{a}'\tilde{v}, a_0, a_1, \dots, a_n) : a_0 = 0, 1; a_1, \dots, a_n \in Z_4\}, \tag{3.1}$$

where $\tilde{a} = (a_0, a_1, \dots, a_n)'$, and $\tilde{a}'\tilde{u}$ and $\tilde{a}'\tilde{v}$ are reduced mod 4. Apply the Gray map (2.2) to $\tilde{C}$ to get a $2^{2n+1} \times (2n + 6)$ matrix $\tilde{D}$ having entries ± 1. By (3.1), the entries in the fifth and sixth columns of $\tilde{D}$ correspond to the third entry a_0 in the codewords of $\tilde{C}$. Since $a_0 = 0$ or 1, it is evident from (2.2) that every entry in the fifth column of $\tilde{D}$ is 1, while in the sixth column of $\tilde{D}$ half of the entries equal 1 and the remaining half -1. Delete the fifth column of $\tilde{D}$ to get the final design matrix D_0, of order $2^{2n+1} \times (2n + 5)$ and having entries ± 1. With its columns and rows identified with factors and runs, respectively, D_0 represents a design involving $2n + 5$ two-level factors and 2^{2n+1} runs. In this sense, D_0 will be called a $2^{(2n+5)-4}$ QC design. Evidently, the role of u_0 and v_0 in this construction is different from that of $u_1, \dots, u_n$ and $v_1, \dots, v_n$ and this will be reflected in the statistical properties of D_0. Let $u = (u_1, \dots, u_n)'$ and $v = (v_1, \dots, v_n)'$.

We consider a trigonometric representation for the runs in D_0. Since D_0 is obtained by deleting the fifth column of $\tilde{D}$, by (3.1) and analogously to (2.3), D_0 has 2^{2n} runs

$$\begin{aligned} &\sqrt{2}\Big[\sin\Big(\frac{\pi}{4} + \frac{\pi}{2}a'u\Big), \cos\Big(\frac{\pi}{4} + \frac{\pi}{2}a'u\Big), \sin\Big(\frac{\pi}{4} + \frac{\pi}{2}a'v\Big), \\ &\quad \cos\Big(\frac{\pi}{4} + \frac{\pi}{2}a'v\Big), \frac{1}{\sqrt{2}}, \sin\Big(\frac{\pi}{4} + \frac{\pi}{2}a_1\Big), \\ &\quad \cos\Big(\frac{\pi}{4} + \frac{\pi}{2}a_1\Big), \dots, \sin\Big(\frac{\pi}{4} + \frac{\pi}{2}a_n\Big), \cos\Big(\frac{\pi}{4} + \frac{\pi}{2}a_n\Big)\Big], \\ &\qquad\qquad a_1, \dots, a_n \in Z_4, \end{aligned} \tag{3.2}$$

which correspond to $a_0 = 0$, and another 2^{2n} runs

$$\begin{aligned}
\sqrt{2}\Big[&\sin\Big\{\frac{\pi}{4}+\frac{\pi}{2}(u_0+a'u)\Big\}, \cos\Big\{\frac{\pi}{4}+\frac{\pi}{2}(u_0+a'u)\Big\},\\
&\sin\Big\{\frac{\pi}{4}+\frac{\pi}{2}(v_0+a'v)\Big\}, \cos\Big\{\frac{\pi}{4}+\frac{\pi}{2}(v_0+a'v)\Big\}, -\frac{1}{\sqrt{2}},\\
&\sin\Big(\frac{\pi}{4}+\frac{\pi}{2}a_1\Big), \cos\Big(\frac{\pi}{4}+\frac{\pi}{2}a_1\Big), \dots, \sin\Big(\frac{\pi}{4}+\frac{\pi}{2}a_n\Big), \cos\Big(\frac{\pi}{4}+\frac{\pi}{2}a_n\Big)\Big],\\
&\qquad a_1, \dots, a_n \in Z_4,
\end{aligned} \tag{3.3}$$

which correspond to $a_0 = 1$. Here $a = (a_1, \dots, a_n)'$. Denote the $2n + 5$ factors in D_0 by $F_1, \dots, F_4, F_5, F_{11}, F_{12}, \dots, F_{n1}, F_{n2}$ in conformity with the ordering in (3.2) or (3.3).

In the spirit of Section 2, for any nonempty collection of factors, let $x_k = 1$ if F_k occurs in the collection, and 0 otherwise, $1 \le k \le 5$. The collection is then said to be of the type (x, x_5), where $x = x_1x_2x_3x_4$. Thus, a typical collection of type (x, x_5) consists of factors F_k with $x_k = 1$ $(1 \le k \le 5)$, and also factors F_{j1}, F_{j2} $(j \in S_1)$, F_{j2} $(j \in S_2)$ and F_{j1} $(j \in S_3)$, where S_1, S_2, S_3 are any disjoint and possibly empty subsets of $\{1, \dots, n\}$. Such a collection has $m + X + x_5$ factors, where X and m are as in Section 2, and by (1.1), (3.2) and (3.3), its aliasing index equals $|G(x) + (-1)^{x_5}H(x)|$, where

$$G(x) = \frac{1}{2}\sum_{a_1=0}^{3}\cdots\sum_{a_n=0}^{3}\phi(x; a_1, \dots, a_n),$$

$$H(x) = \frac{1}{2}\sum_{a_1=0}^{3}\cdots\sum_{a_n=0}^{3}\phi^*(x; a_1, \dots, a_n)$$

with $\phi(x; a_1, \dots, a_n)$ defined by (2.5) and $\phi^*(x; a_1, \dots, a_n)$ defined similarly replacing $a'u$ and $a'v$ in (2.5) by $u_0 + a'u$ and $v_0 + a'v$. Any choice of S_1, S_2, S_3 making $G(x) + (-1)^{x_5}H(x)$ nonzero entails a word of length $m + X + x_5$ and aliasing index $|G(x) + (-1)^{x_5}H(x)|$. Such a word is called a word of type (x, x_5).

Steps similar to but more elaborate than those in the Appendix may now be employed to develop an analog of Theorem 1 giving an account of words of all possible types for the $2^{(2n+5)-4}$ QC design D_0. As hinted above, this has to be done separately for each possible pair u_0v_0. One can, thereafter, summarize the findings to get a counterpart of Theorem 2. However, given the multitude of possibilities for the pair u_0v_0, a tabular representation of these summary results is easier to comprehend than a statement in the form of a theorem. For any u_0v_0 and any combination of the wordlength (wl) and aliasing index (ai), denote the corresponding number of words in D_0 by $N(u_0v_0, wl, ai)/(ai)^2$. Table 1 lists all

TABLE 1
Values of $N(u_0v_0, wl, ai)$ for the $2^{(2n+5)-4}$ QC design D_0

	u_0v_0									
wl*, *ai	**00**	**01/03**	**02**	**10/30**	**11/33**	**12/32**	**13/31**	**20**	**21/23**	**22**
l_1+1, θ_1	2	2	2	1	1	1	1	0	0	0
l_1+2, θ_1	0	0	0	1	1	1	1	2	2	2
l_2+1, θ_2	2	1	0	2	1	0	1	2	1	0
l_2+2, θ_2	0	1	2	0	1	2	1	0	1	2
l_3+3, θ_1	2	0	2	1	1	1	1	0	2	0
l_3+4, θ_1	0	2	0	1	1	1	1	2	0	2
l_4+3, θ_2	2	1	0	0	1	2	1	2	1	0
l_4+4, θ_2	0	1	2	2	1	0	1	0	1	2
$l_5+2, 1$	1	1	1	0	0	0	0	1	1	1
$l_5+3, 1$	0	0	0	1	1	1	1	0	0	0
$l_6+2, 1$	1	0	1	1	0	1	0	1	0	1
$l_6+3, 1$	0	1	0	0	1	0	1	0	1	0
$l_7+4, 1$	1	0	1	0	1	0	1	1	0	1
$l_7+5, 1$	0	1	0	1	0	1	0	0	1	0
l_8+2, ω_0*	4	2	0	2	0	2	0	0	2	4
l_8+3, ω_0*	0	2	4	2	0	2	0	4	2	0
l_9+2, ω#	2	1	0	1	0	1	2	0	1	2
l_9+3, ω#	0	1	2	1	2	1	0	2	1	0
$l_{10}+2, \omega$#	2	1	0	1	2	1	0	0	1	2
$l_{10}+3, \omega$#	0	1	2	1	0	1	2	2	1	0

*Entries in these rows, except the ones for $u_0v_0 = 11, 13, 31, 33$, arise if and only if $\lambda_5 + \lambda_6 = 0$;
#Entries of these rows, except the ones for $u_0v_0 = 11, 13, 31, 33$, arise if and only if $\lambda_5 + \lambda_6 > 0$.

possible (wl, ai) and, for any such (wl, ai), shows $N(u_0v_0, wl, ai)$ for every u_0v_0. The derivation underlying this table is omitted to save space. In Table 1, $l_1, \dots, l_{10}$ are as in (2.9), where $\lambda_1, \dots, \lambda_{10}$ continue to be given by (2.8) with reference to $u = (u_1, \dots, u_n)'$ and $v = (v_1, \dots, v_n)'$. Also,

$$\theta_1 = 1/2^{\langle (1/2)(\lambda_1+\lambda_3+\lambda_5+\lambda_6+\delta_1)\rangle}, \qquad \theta_2 = 1/2^{\langle (1/2)(\lambda_2+\lambda_4+\lambda_5+\lambda_6+\delta_2)\rangle},$$
$$\omega_0 = \omega_1\omega_2, \qquad \omega_1 = 1/2^{\langle (1/2)(\lambda_1+\lambda_3+\varepsilon_1)\rangle}, \qquad \omega_2 = 1/2^{\langle (1/2)(\lambda_2+\lambda_4+\varepsilon_2)\rangle}, \tag{3.4}$$
$$\omega = 1/2^{\langle (1/2)(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\varepsilon+1)\rangle},$$

where $\delta_1 = I(u_0 = 1 \text{ or } 3)$, $\delta_2 = I(v_0 = 1 \text{ or } 3)$, $\varepsilon_1 = I(u_0v_0 = 10, 12, 30 \text{ or } 32)$, $\varepsilon_2 = I(u_0v_0 = 01, 03, 21 \text{ or } 23)$, $\varepsilon = \varepsilon_1 + \varepsilon_2$, and $I(\cdot)$ is the indicator function.

EXAMPLE 2. With $n = 2$, let $u = (1, 2)'$, $v = (2, 1)'$ and $u_0v_0 = 11$. Then $f_{12} = f_{21} = 1$ and all other f's equal 0, so that by (2.8), $\lambda_3 = \lambda_4 = 1$ and all other

λ's are zeros. Hence, by (2.9) and (3.4),

$$l_1 = l_2 = 3, \qquad l_3 = l_4 = 1, \qquad l_5 = l_6 = 2, \qquad l_7 = 4, \qquad l_8 = l_9 = l_{10} = 2,$$
$$\theta_1 = \theta_2 = \tfrac{1}{2}, \qquad \omega_0 = 1, \qquad \omega = \tfrac{1}{2}.$$

As a result, Table 1 shows that there are 48 words each with aliasing index $\frac{1}{2}$; of these, 24 have length four and 24 have length five. In addition, there are three words having lengths five, five and eight, and each with aliasing index 1. Hence by (1.2) and (1.3), in this case the QC design D_0, which is a 2^{9-4} design, has resolution 4.5 and WLP $(0, 0, 0, 6, 8, 0, 0, 1, 0, 0)$. As seen later in Table 3, this design has maximum resolution and MA among all 2^{9-4} QC designs. Also, it has the same WLP but higher resolution than the regular 2^{9-4} MA design.

We next turn to one-eighth fraction QC designs in 2^{2n+1} runs. Deletion of any one of the first four columns of D_0 introduced earlier leads to a design involving $2n + 4$ two-level factors and 2^{2n+1} runs, that is, a $2^{(2n+4)-3}$ QC design. Using the same logic as in Section 2.2, without loss of generality, suppose the first column of D_0 is deleted. Let $D_0^{(1)}$ denote the resulting design. Table 2 lists all possible combinations (wl, ai) of the wordlength and aliasing index in $D_0^{(1)}$ and, for any such

TABLE 2
Values of $N(u_0v_0, wl, ai)$ for the $2^{(2n+4)-3}$ quaternary code design $D_0^{(1)}$

	u_0v_0													
wl, ai	**00**	**01/03**	**02**	**10**	**11**	**12**	**13**	**20**	**21/23**	**22**	**30**	**31**	**32**	**33**
l_1+1, θ_1	1	1	1	k_{11}	k_{11}	k_{11}	k_{11}	0	0	0	k_{12}	k_{12}	k_{12}	k_{12}
l_1+2, θ_1	0	0	0	k_{12}	k_{12}	k_{12}	k_{12}	1	1	1	k_{11}	k_{11}	k_{11}	k_{11}
l_2+1, θ_2	2	1	0	2	1	0	1	2	1	0	2	1	0	1
l_2+2, θ_2	0	1	2	0	1	2	1	0	1	2	0	1	2	1
l_3+3, θ_1	1	0	1	k_{11}	k_{12}	k_{11}	k_{12}	0	1	0	k_{12}	k_{11}	k_{12}	k_{11}
l_3+4, θ_1	0	1	0	k_{12}	k_{11}	k_{12}	k_{11}	1	0	1	k_{11}	k_{12}	k_{11}	k_{12}
$l_6+2, 1$	1	0	1	1	0	1	0	1	0	1	1	0	1	0
$l_6+3, 1$	0	1	0	0	1	0	1	0	1	0	0	1	0	1
$l_8+2, \omega_0{}^*$	2	1	0	k_{21}	0	k_{22}	0	0	1	2	k_{22}	0	k_{21}	0
$l_8+3, \omega_0{}^*$	0	1	2	k_{22}	0	k_{21}	0	2	1	0	k_{21}	0	k_{22}	0
$l_9+2, \omega^{\#}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$	0
$l_9+3, \omega^{\#}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1
$l_{10}+2, \omega^{\#}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1
$l_{10}+3, \omega^{\#}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0

*Entries in these rows, except the ones for $u_0v_0 = 11, 13, 31, 33$, arise if and only if $\lambda_5 + \lambda_6 = 0$;
#Entries of these rows, except the ones for $u_0v_0 = 11, 13, 31, 33$, arise if and only if $\lambda_5 + \lambda_6 > 0$.

TABLE 3
One-sixteenth fraction QC designs with maximum resolution and MA

Design	**QC design with maximum resolution and MA**	**Regular MA design**
2^{8-4}	$\lambda = 0011000000$, $R = 4$, $A = (14, 0, 0, 0, 1)$	$R = 4$, A same
2^{9-4}	$\lambda = 0011000000$, $u_0v_0 = 11$, $R = 4.5$, $A = (6, 8, 0, 0, 1, 0)$	$R = 4$, A same
2^{10-4}	$\lambda = 0001110000$, $R = 4.5$, $A = (2, 8, 4, 0, 1, 0, 0)$	$R = 4$, A same
2^{11-4}	$\lambda = 0001110000$, $u_0v_0 = 12$, $R = 5.5$, $A = (0, 6, 6, 2, 1, 0, 0, 0)$	$R = 5$, A same
2^{12-4}	$\lambda = 0011110000$, $R = 6.5$, $A = (0, 0, 12, 0, 3, 0, 0, 0, 0)$	$R = 6$, A same
2^{13-4}	$\lambda = 0011110000$, $u_0v_0 = 22$, $R = 6.5$, $A = (0, 0, 4, 8, 3, 0, 0, 0, 0, 0)$	$R = 6$, A same
2^{14-4}	$\lambda = 1011110000$, $R = 6.5$, $A = (0, 0, 2, 8, 3, 0, 2, 0, 0, 0, 0)$	$R = 7$, A better

(wl, ai), shows $N(u_0v_0, wl, ai)$ for every u_0v_0, where $N(u_0v_0, wl, ai)$ is defined as above but now refers to $D_0^{(1)}$. In Table 2, l_1, l_2 etc. are as in (2.9), $\theta_1, \theta_2, \omega_0$ and ω are as in (3.4), and

$$k_{11} = \tfrac{1}{2} I(\lambda_1 + \lambda_3 > 0), \qquad k_{12} = 1 - k_{11},$$
$$k_{21} = I(\lambda_1 + \lambda_3 + \lambda_5 + \lambda_6 > 0), \qquad k_{22} = 2 - k_{21}.$$

As illustrated in Example 2, Tables 1 and 2 readily yield, in their respective contexts, the resolution and WLP of a QC design for any given $\lambda_1, \dots, \lambda_{10}$ and u_0v_0. Hence, we find the best choice of the λ's and u_0v_0, with regard to resolution and aberration, by complete enumeration of all possibilities. Again, because of the significant reduction achieved in Tables 1 and 2 by theoretical means, such complete enumeration can be done instantaneously, for example by MATLAB, for reasonable values of n. The results are summarized in Tables 3 and 4 for one-sixteenth and one-eighth fractions, respectively.

TABLE 4
One-eighth fraction QC designs with maximum resolution and MA

Design	**QC design with maximum resolution and MA**	**Regular MA design**
2^{7-3}	$\lambda = 0011000000$, $R = 4$, $A = (7, 0, 0, 0)$	$R = 4$, A same
2^{8-3}	$\lambda = 0011000000$, $u_0v_0 = 11$, $R = 4.5$, $A = (3, 4, 0, 0, 0)$	$R = 4$, A same
2^{9-3}	$\lambda = 0010110000$, $R = 4.5$, $A = (1, 4, 2, 0, 0, 0)$	$R = 4$, A same
2^{10-3}	$\lambda = 0010110000$, $u_0v_0 = 21$, $R = 5.5$, $A = (0, 3, 3, 1, 0, 0, 0)$	$R = 5$, A same
2^{11-3}	$\lambda = 0011110000$, $R = 6.5$, $A = (0, 0, 6, 0, 1, 0, 0, 0)$	$R = 6$, A same
2^{12-3}	$\lambda = 0011110000$, $u_0v_0 = 12$, $R = 6.75$, $A = (0, 0, 2, 4, 1, 0, 0, 0, 0)$	$R = 6$, A same
2^{13-3}	$\lambda = 0021110000$, $R = 7.75$, $A = (0, 0, 0, 4, 3, 0, 0, 0, 0, 0)$	$R = 7$, A same

A brief discussion on Tables 3 and 4, showing one-sixteenth and one-eighth fraction QC designs with maximum resolution and MA is in order. The 14 QC designs shown in these tables are optimal, among all comparable QC designs, under both criteria. All these 14 designs have resolution four or higher, and for each, the resolution R and $A = (A_4, A_5, \ldots)$ are shown. For ease in comparison, we also show R and comment on A for the corresponding regular MA designs, as obtained by Chen and Wu [6]. Out of the 14 optimal QC designs in Tables 3 and 4, there are two, that is, the first design in either table, which have the same R and A as the corresponding regular MA designs. It can be seen that these two designs involve only full words and hence are themselves regular. A comparison with Table 6 of Sun, Li and Ye [18] shows that these two designs have maximum resolution and MA in their sense among all designs of the same size. Eleven of the remaining twelve optimal QC designs in our Tables 3 and 4 have the same WLP but higher resolution than the corresponding regular MA designs. Only the 2^{14-4} optimal QC design turns out to be worse than the regular MA design. In both Tables 3 and 4, λ stands for the 10-tuple $\lambda_1\lambda_2\cdots\lambda_{10}$.

Indeed, the theoretical results reported in Theorems 2, 3 and Tables 1, 2 readily allow extension of Tables 3 and 4 beyond the ranges considered here, if the situation so demands. For instance, from Table 2, one can check that the 2^{16-3} QC design with maximum resolution and MA is given by $\lambda = 0020220000$, $u_0v_0 = 20$. This design has $R = 8.875$ and $A = (0, 0, 0, 0, 1, 4, 2, 0, 0, 0, 0, 0, 0)$, while the corresponding regular MA design has the same A but $R = 8$.

4. Results on projectivity. The following results give upper bounds on the projectivity of the one-sixteenth fraction QC designs D and D_0 introduced in Sections 2 and 3, respectively.

THEOREM 4. *The projectivity p of the $2^{(2n+4)-4}$ QC design D satisfies* (i) $p \le \frac{4}{3}n + 1$, *if* $n = 0 \bmod 3$, (ii) $p \le \frac{4}{3}(n - j) + 3$, *if* $n = j \bmod 3$, *with* $j = 1$ *or* 2.

PROOF. We prove only (ii). The proof of (i) is similar. Let $n = 3t + j$, where t is an integer and $j = 1$ or 2. We need to show that $p \le 4t + 3$. If $p \ge 4t + 4$, then all full words in D have length at least $4t + 5$, so that by Theorem 2(c) and (2.9),

$$l_5 + 2 = 2(\lambda_1 + \lambda_3 + \lambda_5 + \lambda_6) + 2 \ge 4t + 5,$$

$$l_6 + 2 = 2(\lambda_2 + \lambda_4 + \lambda_5 + \lambda_6) + 2 \ge 4t + 5,$$

$$l_7 + 4 = 2(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) + 4 \ge 4t + 5,$$

that is, invoking the integrality of $\lambda_1, \ldots, \lambda_6$,

$$\lambda_1 + \lambda_3 + \lambda_5 + \lambda_6 \ge 2t + 2, \qquad \lambda_2 + \lambda_4 + \lambda_5 + \lambda_6 \ge 2t + 2,$$

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \ge 2t + 1.$$

Adding the last three inequalities, $2(\lambda_1 + \cdots + \lambda_6) \geq 6t + 5$, that is, $n \geq \lambda_1 + \cdots + \lambda_6 \geq 3t + 3$, again using the integrality of $\lambda_1, \ldots, \lambda_6$, and we reach a contradiction. □

From Table 1, now observe that the $2^{(2n+5)-4}$ QC design D_0 has at least three full words. These have lengths (a) $l_5 + 2$, $l_6 + 2$, $l_7 + 4$ if $u_0 v_0 = 00, 02, 20, 22$, (b) $l_5 + 2$, $l_6 + 3$, $l_7 + 5$ if $u_0 v_0 = 01, 03, 21, 23$, (c) $l_5 + 3$, $l_6 + 2$, $l_7 + 5$ if $u_0 v_0 = 10, 12, 30, 32$ and (d) $l_5 + 3$, $l_6 + 3$, $l_7 + 4$ if $u_0 v_0 = 11, 13, 31, 33$. Hence, arguments similar to but more elaborate than those in Theorem 4 lead to the following result.

THEOREM 5. *The projectivity p of the $2^{(2n+5)-4}$ QC design D_0 satisfies* (i) $p \leq \frac{4}{3}n + 2$ *if* $n = 0 \bmod 3$, (ii) $p \leq \frac{4}{3}(n-1) + 3$, *if* $n = 1 \bmod 3$, (iii) $p \leq \frac{4}{3}(n-2) + 4$, *if* $n = 2 \bmod 3$.

Table 5 shows the projectivities of the one-sixteenth fraction QC designs reported in Table 3. It is easily seen that these designs attain the upper bounds on projectivity as shown in Theorems 4 or 5. Hence, in addition to having maximum resolution and MA, they have maximum projectivity among all comparable QC designs. Indeed, for $8 \leq q \leq 12$, the 2^{q-4} QC designs in Table 3 have projectivity $q - 5$ which is the highest among all designs of the same size. This holds because, otherwise, one would get a 2^{q-4} design with projectivity $q - 4$, that is, an orthogonal array OA $(2^{q-4}, q, 2, q - 4)$ of index unity, which is nonexistent; see [3]. Table 5 also shows that the use of QC designs leads to gain in projectivity over regular MA designs for $9 \leq q \leq 14$.

We next consider one-eighth fractions and show in Table 6 the projectivities of the QC designs reported in Table 4. For $7 \leq q \leq 11$, the 2^{q-3} QC designs in Table 4 are seen to have projectivity $q - 4$ which is the highest among all designs of the same size. This follows as in the last paragraph using a nonexistence result in [3] on orthogonal arrays of index unity. Also, for $q = 12$ and 13, the QC designs in Table 4 were computationally verified to have maximum projectivity at least among all comparable QC designs. Incidentally, for one-eighth fraction QC designs, it is hard to develop analogs of Theorems 4 and 5 as there is only

TABLE 5
Projectivities of the one-sixteenth fraction QC designs in Table 3

Design	2^{8-4}	2^{9-4}	2^{10-4}	2^{11-4}	2^{12-4}	2^{13-4}	2^{14-4}
Projectivity of the QC design in Table 3	3	4	5	6	7	7	7
Projectivity of the regular MA design	3	3	3	4	5	5	6

TABLE 6
Projectivities of the one-eighth fraction QC designs in Table 4

Design	2^{7-3}	2^{8-3}	2^{9-3}	2^{10-3}	2^{11-3}	2^{12-3}	2^{13-3}
Projectivity of the QC design in Table 4	3	4	5	6	7	7	7
Projectivity of the regular MA design	3	3	3	4	5	5	6

one guaranteed full word (cf. Theorem 3 and Table 2) but the computational study of projectivity remains manageable with a moderate number of factors. Table 6 also shows the projectivity of regular MA designs and the gains via the use of QC designs, for $8 \le q \le 13$, are evident.

The foregoing discussion reveals that, unlike what often happens with quarter fraction QC designs [16], maximum projectivity is not in conflict with maximum resolution or MA in our setup at least over the range covered by Tables 3–6, where the same QC design turns out to be optimal, among all comparable QC designs, with regard to all the three criteria.

5. Summary and future work. In the present paper, a trigonometric approach was developed to obtain theoretical results on QC designs with focus on one-eighth and one-sixteenth fractions of two-level factorials. It was seen that optimal QC designs often have larger resolution and projectivity than comparable regular designs. In addition, some of these designs were found to have maximum projectivity among all designs.

Before concluding, we indicate a few open issues. It should be possible to use the trigonometric approach to obtain further theoretical results on the projectivity of QC designs. This calls for examining the existence of solutions to certain trigonometric equations. For instance, by (2.3), the first four columns of the $2^{(2n+4)-4}$ QC design D in Section 2 contain a full 2^4 factorial if and only if for every $y_1, \dots, y_4$ in $\{-1, 1\}$, the equations

$$\sqrt{2}\sin\left(\frac{\pi}{4}+\frac{\pi}{2}a'u\right)=y_1, \qquad \sqrt{2}\cos\left(\frac{\pi}{4}+\frac{\pi}{2}a'u\right)=y_2,$$
$$\sqrt{2}\sin\left(\frac{\pi}{4}+\frac{\pi}{2}a'v\right)=y_3, \qquad \sqrt{2}\cos\left(\frac{\pi}{4}+\frac{\pi}{2}a'v\right)=y_4,$$

admit a solution for $a = (a_1, \dots, a_n)'$ in Z_4. A study of equations of this kind, however, branches out into too many cases, depending on u and v, compared to the derivation of results on wordlength and aliasing index as done here.

It would also be of interest to investigate how the trigonometric approach can be implemented for even more highly fractionated QC designs. The trigonometric formulation as well as the mathematical tools are expected to be essentially same as the ones here and the main difficulty will lie in handling the multitude of cases that such an effort will involve. A related issue concerns the development of a complementary design theory for QC designs in the spirit of [5, 21] and with respect to an appropriately defined reference set.

Some kind of symbolic computation may help in addressing the open problems mentioned above. We hope that the present endeavor will generate further interest in these and related issues.

APPENDIX: PROOF OF THEOREM 1 FOR $x = 0101$

The proof will be worked out through a sequence of lemmas. We concentrate on $V(x)$ and begin by giving an expression for $\psi(a_1, \dots, a_n)$ in (2.6). Recall that in (2.6), S_1, S_2, S_3 are disjoint subsets of $\{1, \dots, n\}$ and that $m = 2n_1 + n_2 + n_3$ with $n_k = \#S_k$. For $1 \le k \le 3$, let Σ_k denote the sum over 2^{n_k} terms corresponding to the 2^{n_k} subsets W_k of S_k, and for any such subset W_k, write $\bar{W}_k = S_k \setminus W_k$, $w_k = \#W_k$, $\bar{w}_k = \#\bar{W}_k$. Thus if $S_1 = \{2, 3\}$, then Σ_1 denotes the sum over 2^2 terms corresponding to $W_1 =$ empty set, $\{2\}$, $\{3\}$ and $\{2, 3\}$. For any given subsets W_1, W_2, W_3 of S_1, S_2, S_3, we also write $\Sigma^{(1)}$, $\bar{\Sigma}^{(1)}$, $\Sigma^{(23)}$ and $\bar{\Sigma}^{(23)}$ to denote sums over $j \in W_1$, $j \in \bar{W}_1$, $j \in W_2 \cup W_3$ and $j \in \bar{W}_2 \cup \bar{W}_3$, respectively. Similarly, $\Sigma^{(4)}$ denotes sum over $j \in S_4$, where $S_4 = \{1, \dots, n\} \setminus (S_1 \cup S_2 \cup S_3)$. Let $i = \sqrt{-1}$. Then the following lemma is not hard to obtain using elementary facts such as $\sin y \cos y = \frac{1}{2} \sin 2y$, $\cos y = \frac{1}{2}(e^{iy} + e^{-iy})$, $\sin y = \frac{1}{2i}(e^{iy} - e^{-iy})$.

LEMMA A.1.

$$\psi(a_1, \dots, a_n) = \frac{1}{2^m i^{n_1+n_3}} \Sigma_1 \Sigma_2 \Sigma_3 M(\bar{w}_1, \bar{w}_2, \bar{w}_3) \times \exp\left\{\frac{i\pi}{2}\left(2\Sigma^{(1)} a_j - 2\bar{\Sigma}^{(1)} a_j + \Sigma^{(23)} a_j - \bar{\Sigma}^{(23)} a_j\right)\right\},$$

where $M(\bar{w}_1, \bar{w}_2, \bar{w}_3) = (-1)^{\bar{w}_1 + \bar{w}_3} \exp\{\frac{i\pi}{4}(m - 4\bar{w}_1 - 2\bar{w}_2 - 2\bar{w}_3)\}$.

Let $g = (g_1, \dots, g_n)'$, $h = (h_1, \dots, h_n)'$, where

$$g_j = u_j + v_j \pmod 4, \qquad h_j = u_j - v_j \pmod 4, \qquad 1 \le j \le n. \tag{A.1}$$

Using the same elementary facts that led to Lemma A.1, we now note that

$$\begin{aligned} &\cos\left(\frac{\pi}{4} + \frac{\pi}{2} a' u\right) \cos\left(\frac{\pi}{4} + \frac{\pi}{2} a' v\right) \\ &\quad = \frac{1}{4}\left[\exp\left\{\frac{i\pi}{2}(1 + a'g)\right\} + \exp\left\{\frac{i\pi}{2} a' h\right\}\right. \\ &\qquad \left. + \exp\left\{-\frac{i\pi}{2} a' h\right\} + \exp\left\{-\frac{i\pi}{2}(1 + a'g)\right\}\right]. \end{aligned} \tag{A.2}$$

Throughout the rest of the Appendix, including the lemmas below, we consider $x = 0101$. Then $X = x_1 + x_2 + x_3 + x_4 = 2$, and hence (2.5), (A.2) and Lemma A.1 yield

$$\begin{aligned}\phi(x; a_1, \ldots, a_n) &= \frac{1}{2^{(1/2)(m+2)+2n} i^{n_1+n_3}} \Sigma_1 \Sigma_2 \Sigma_3 M(\bar{w}_1, \bar{w}_2, \bar{w}_3) \\ &\quad \times \sum_{k=1}^{4} \phi_k^W(x; a_1, \ldots, a_n),\end{aligned} \tag{A.3}$$

where, with $g_0 = 1$,

$$\begin{aligned}&\phi_1^W(x; a_1, \ldots, a_n) \\ &\quad = \exp\biggl[\frac{i\pi}{2}\{g_0 + \Sigma^{(1)}(g_j + 2)a_j + \bar{\Sigma}^{(1)}(g_j - 2)a_j \\ &\qquad + \Sigma^{(23)}(g_j + 1)a_j + \bar{\Sigma}^{(23)}(g_j - 1)a_j + \Sigma^{(4)} g_j a_j\}\biggr],\end{aligned} \tag{A.4}$$

and, for $k = 2, 3, 4$, $\phi_k^W(x; a_1, \ldots, a_n)$ are analogous to $\phi_1^W(x; a_1, \ldots, a_n)$, with $(g_0, g_1, \ldots, g_n)$ in the latter replaced by $(0, h_1, \ldots, h_n)$, $(0, -h_1, \ldots, -h_n)$ and $(-g_0, -g_1, \ldots, -g_n)$, respectively. The superscript W here indicates the dependence of each $\phi_k^W(x; a_1, \ldots, a_n)$ on W_1, W_2, W_3 via the sums $\Sigma^{(1)}$, $\bar{\Sigma}^{(1)}$, $\Sigma^{(23)}$, $\bar{\Sigma}^{(23)}$. Writing

$$V_k(x) = \frac{1}{2^{(1/2)(m+2)+2n} i^{n_1+n_3}} \Sigma_1 \Sigma_2 \Sigma_3 M(\bar{w}_1, \bar{w}_2, \bar{w}_3) V_k^W(x), \tag{A.5}$$

where

$$V_k^W(x) = \sum_{a_1=0}^{3} \cdots \sum_{a_n=0}^{3} \phi_k^W(x; a_1, \ldots, a_n), \qquad 1 \le k \le 4, \tag{A.6}$$

the following lemma is immediate from (2.4) and (A.3).

LEMMA A.2. $V(x) = V_1(x) + V_2(x) + V_3(x) + V_4(x)$.

Some more notation will help in presenting the subsequent lemmas. With reference to the sets Δ_{ks} in (2.7) and the g_j and h_j in (A.1), let

$$\begin{aligned}&\Delta^{(1)} = \Delta_{10} \cup \Delta_{12} \cup \Delta_{30} \cup \Delta_{32}, \qquad \Delta^{(2)} = \Delta_{01} \cup \Delta_{03} \cup \Delta_{21} \cup \Delta_{23}, \\ &\quad \Delta = \Delta^{(1)} \cup \Delta^{(2)},\end{aligned} \tag{A.7}$$

$$\begin{aligned}&\Delta_k^g = \{j : 1 \le j \le n, g_j = k\}, \qquad \Delta_k^h = \{j : 1 \le j \le n, h_j = k\}, \\ &f_k^g = \#\Delta_k^g, \qquad f_k^h = \#\Delta_k^h, \qquad 0 \le k \le 3.\end{aligned} \tag{A.8}$$

Then by (2.7), (2.8) and (A.1),

$$\begin{aligned}\beta_1 &= \#\Delta^{(1)} = f_{10} + f_{12} + f_{30} + f_{32} = \lambda_1 + \lambda_3,\\ \beta_2 &= \#\Delta^{(2)} = f_{01} + f_{03} + f_{21} + f_{23} = \lambda_2 + \lambda_4,\\ \beta &= \#\Delta = \beta_1 + \beta_2 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4,\end{aligned} \tag{A.9}$$

$$\begin{aligned}&\Delta_0^g = \Delta_{00} \cup \Delta_{13} \cup \Delta_{22} \cup \Delta_{31}, \qquad \Delta_0^h = \Delta_{00} \cup \Delta_{11} \cup \Delta_{22} \cup \Delta_{33},\\ &\Delta_2^g = \Delta_{02} \cup \Delta_{11} \cup \Delta_{20} \cup \Delta_{33}, \qquad \Delta_2^h = \Delta_{02} \cup \Delta_{13} \cup \Delta_{20} \cup \Delta_{31},\\ &\Delta_1^g \cup \Delta_3^g = \Delta_1^h \cup \Delta_3^h = \Delta,\end{aligned} \tag{A.10}$$

$$\begin{aligned}&f_2^g = \lambda_5 + \lambda_7 + \lambda_8,\\ &f_2^h = \lambda_6 + \lambda_7 + \lambda_8, \qquad f_1^g + f_3^g = f_1^h + f_3^h = \beta.\end{aligned} \tag{A.11}$$

We also write

$$\beta_1^* = f_{12} + f_{30} - f_{10} - f_{32}, \qquad \beta_2^* = f_{03} + f_{21} - f_{01} - f_{23}, \tag{A.12}$$

$$\mu_1 = \#(S_3 \cap \Delta^{(1)}), \qquad \mu_2 = \#(S_3 \cap \Delta^{(2)}), \tag{A.13}$$

$$n(g,k) = \#(S_3 \cap \Delta_k^g), \qquad n(h,k) = \#(S_3 \cap \Delta_k^h), \qquad 0 \le k \le 3. \tag{A.14}$$

LEMMA A.3. (i) $V_1(x) = V_4(x) = 0$, *unless*

$$S_1 = \Delta_2^g, \qquad S_2 \cup S_3 = \Delta_1^g \cup \Delta_3^g, \qquad S_4 = \Delta_0^g. \tag{A.15}$$

(ii) $V_2(x) = V_3(x) = 0$, *unless*

$$S_1 = \Delta_2^h, \qquad S_2 \cup S_3 = \Delta_1^h \cup \Delta_3^h, \qquad S_4 = \Delta_0^h. \tag{A.16}$$

PROOF. We prove (i). The proof of (ii) is similar. Since for any integer k,

$$\begin{aligned}\sum_{s=0}^{3} \exp\left(\frac{i\pi}{2}ks\right) &= 4 \qquad \text{if } k = 0 \bmod 4,\\ &= 0 \qquad \text{otherwise},\end{aligned} \tag{A.17}$$

it follows from (A.4) and (A.6) that $V_1^W(x)$ vanishes, for every W_1 ($\subset S_1$), W_2 ($\subset S_2$) and W_3 ($\subset S_3$), and hence by (A.5) $V_1(x) = 0$, if either (a) S_1 is nonempty and $g_j \neq 2$ for some $j \in S_1$, or (b) $S_2 \cup S_3$ is nonempty and $g_j \neq 1, 3$ for some $j \in S_2 \cup S_3$, or (c) S_4 is nonempty and $g_j \neq 0$ for some $j \in S_4$. Thus $V_1(x) = 0$, unless $S_1 \subset \Delta_2^g$, $S_2 \cup S_3 \subset \Delta_1^g \cup \Delta_3^g$ and $S_4 \subset \Delta_0^g$. These conditions are equivalent to those in (A.15) because S_1, S_2, S_3, S_4 form a partition of $\{1, \dots, n\}$ as $\Delta_0^g, \Delta_1^g, \Delta_2^g, \Delta_3^g$ do. The same arguments apply to $V_4(x)$. Hence, (i) follows. □

LEMMA A.4. (i) *If* (A.15) *holds then*

$$
\begin{aligned}
&V_1(x) + V_4(x) \\
&\quad = \frac{(-1)^{n(g,1)}}{2^{(1/2)\beta+1} i^{n_3}} \left[\exp\left\{ \frac{i\pi}{4}(2 + f_3^g - f_1^g) \right\} \right. \\
&\qquad\qquad \left. + (-1)^{n_3} \exp\left\{ -\frac{i\pi}{4}(2 + f_3^g - f_1^g) \right\} \right].
\end{aligned}
$$

(ii) *If* (A.16) *holds*, *then*

$$
\begin{aligned}
V_2(x) + V_3(x) = \frac{(-1)^{n(h,1)}}{2^{(1/2)\beta+1} i^{n_3}} &\left[\exp\left\{ \frac{i\pi}{4}(f_3^h - f_1^h) \right\} \right. \\
&\left. + (-1)^{n_3} \exp\left\{ -\frac{i\pi}{4}(f_3^h - f_1^h) \right\} \right].
\end{aligned}
$$

PROOF. We prove (i), the proof of (ii) being similar. Let (A.15) hold. Then by (A.4), noting that $g_0 = 1$,

$$
\begin{aligned}
&\phi_1^W(x; a_1, \ldots, a_n) \\
&\quad = \exp\left[\frac{i\pi}{2} \{ 1 + \Sigma^{(23)}(g_j + 1)a_j + \bar{\Sigma}^{(23)}(g_j - 1)a_j \} \right] \qquad \text{(A.18)} \\
&\quad = \exp\left\{ \frac{i\pi}{2}(1 + 2\Sigma^* a_j + 2\Sigma^{**} a_j) \right\},
\end{aligned}
$$

Σ^* and Σ^{**} being sums over $j \in (W_2 \cup W_3) \cap \Delta_1^g$ and $j \in (\bar{W}_2 \cup \bar{W}_3) \cap \Delta_3^g$, respectively. By (A.6), (A.17) and (A.18), $V_1^W(x) = 0$ unless the ranges of Σ^* and Σ^{**} are both empty. From this, invoking the second equation in (A.15), a little reflection shows that $V_1^W(x) = 0$ unless

$$
\begin{aligned}
W_2 &= S_2 \cap \Delta_3^g, & W_3 &= S_3 \cap \Delta_3^g, \\
\bar{W}_2 &= S_2 \cap \Delta_1^g, & \bar{W}_3 &= S_3 \cap \Delta_1^g.
\end{aligned}
\qquad \text{(A.19)}
$$

Given S_2, S_3, (A.19) determines W_2, W_3, $\bar{W}_2$ and $\bar{W}_3$ uniquely. Moreover, if (A.19) holds then $\bar{w}_3 = n(g, 1)$ by (A.14), $V_1^W(x) = 4^n \exp(\frac{i\pi}{2})$ by (A.6) and (A.18), and

$$
\begin{aligned}
&m - 4\bar{w}_1 - 2\bar{w}_2 - 2\bar{w}_3 \\
&\quad = 2n_1 - 4\bar{w}_1 + w_2 + w_3 - \bar{w}_2 - \bar{w}_3 \\
&\quad = 2n_1 - 4\bar{w}_1 + \#\{(S_2 \cup S_3) \cap \Delta_3^g\} - \#\{(S_2 \cup S_3) \cap \Delta_1^g\} \\
&\quad = 2n_1 - 4\bar{w}_1 + \#\Delta_3^g - \#\Delta_1^g = 2n_1 - 4\bar{w}_1 + f_3^g - f_1^g
\end{aligned}
$$

by (A.8), (A.15) and the facts that $m = 2n_1 + n_2 + n_3, n_k = w_k + \bar{w}_k$ $(k = 2, 3)$. If we summarize the above and recall the definition of $M(\bar{w}_1, \bar{w}_2, \bar{w}_3)$ from Lemma A.1, then from (A.5), we get

$$\begin{aligned} V_1(x) &= \frac{4^n \exp(i\pi/2)}{2^{(1/2)(m+2)+2n} i^{n_1+n_3}} \Sigma_1 (-1)^{\bar{w}_1 + n(g,1)} \\ &\quad \times \exp\left\{\frac{i\pi}{4}(2n_1 - 4\bar{w}_1 + f_3^g - f_1^g)\right\}. \end{aligned} \tag{A.20}$$

Since $m + 2 = 2n_1 + f_1^g + f_3^g + 2 = 2n_1 + \beta + 2$ by (A.11) and (A.15), and

$$\Sigma_1 (-1)^{\bar{w}_1} \exp\left\{\frac{i\pi}{4}(2n_1 - 4\bar{w}_1)\right\} = (2i)^{n_1}$$

as one can verify after a little algebra, (A.20) yields

$$V_1(x) = \frac{(-1)^{n(g,1)}}{2^{(1/2)\beta+1} i^{n_3}} \exp\left\{\frac{i\pi}{4}(2 + f_3^g - f_1^g)\right\}. \tag{A.21}$$

Similarly, it can be shown that under (A.15),

$$V_4(x) = \frac{(-1)^{n(g,3)}}{2^{(1/2)\beta+1} i^{n_3}} \exp\left\{-\frac{i\pi}{4}(2 + f_3^g - f_1^g)\right\}. \tag{A.22}$$

By (A.14) and (A.15), $n_3 = n(g, 1) + n(g, 3)$. Hence, (i) follows from (A.21) and (A.22). □

LEMMA A.5. (i) $(-1)^{n(h,1)-n(g,1)} = (-1)^{\mu_2}$. (ii) *If either* (A.15) *or* (A.16) *holds then* $n_3 = \mu_1 + \mu_2$.

PROOF. (i) Let $\mu_0 = \#\{S_3 \cap (\Delta_{01} \cup \Delta_{23})\}$. From (2.7), (A.1) and (A.8),

$$\Delta_1^g = \Delta_{01} \cup \Delta_{10} \cup \Delta_{23} \cup \Delta_{32}, \qquad \Delta_1^h = \Delta_{03} \cup \Delta_{10} \cup \Delta_{21} \cup \Delta_{32}.$$

Hence by (A.7), (A.13) and (A.14), $n(h, 1) - n(g, 1) = \mu_2 - 2\mu_0$, and (i) follows.

(ii) If either (A.15) or (A.16) holds, then by (A.7) and (A.10), $S_3 \subset \Delta$ $[=\Delta^{(1)} \cup \Delta^{(2)}]$. Now (ii) is immediate from (A.13). □

LEMMA A.6. (i) *If* $\lambda_5 + \lambda_6 = 0$, *then* (A.15) *and* (A.16) *become identical and both reduce to*

$$S_1 = \Delta_{02} \cup \Delta_{20}, \qquad S_2 \cup S_3 = \Delta, \qquad S_4 = \Delta_{00} \cup \Delta_{22}. \tag{A.23}$$

(ii) *If* $\lambda_5 + \lambda_6 > 0$, *then* (A.15) *and* (A.16) *cannot hold simultaneously.*

PROOF. If $\lambda_5 + \lambda_6 = 0$, then by (2.7) and (2.8), the sets $\Delta_{11}, \Delta_{13}, \Delta_{31}$ and Δ_{33} are empty, so that by (A.10) , $\Delta_2^g = \Delta_2^h = \Delta_{02} \cup \Delta_{20}$ and $\Delta_0^g = \Delta_0^h = \Delta_{00} \cup \Delta_{22}$. Hence, (i) follows recalling the last identity in (A.10). On the other hand, if $\lambda_5 + \lambda_6 > 0$, then at least one of $\Delta_{11}, \Delta_{13}, \Delta_{31}$ and Δ_{33} is nonempty. Therefore, $\Delta_2^g \neq \Delta_2^h$ and $\Delta_0^g \neq \Delta_0^h$ and (ii) follows. □

LEMMA A.7. *Let* $\lambda_5 + \lambda_6 = 0$. *Then*:

(i) $V(x) = 0$, *if* (A.23) *does not hold*.
(ii) $V(x) = (-1)^{n(g,1)} Q_1(x) Q_2(x)$, *if* (A.23) *holds*, *where for* $k = 1, 2$,

$$Q_k(x) = \frac{1}{2^{(1/2)(\beta_k+1)} i^{\mu_k}} \biggl[\exp\biggl\{\frac{i\pi}{4}(1+\beta_k^*)\biggr\} + (-1)^{\mu_k} \exp\biggl\{-\frac{i\pi}{4}(1+\beta_k^*)\biggr\}\biggr]. \tag{A.24}$$

PROOF. Part (i) is evident from Lemmas A.2, A.3 and A.6(i). To prove (ii), let (A.23) hold and write $\alpha_1 = \frac{\pi}{4}(2 + f_3^g - f_1^g)$, $\alpha_2 = \frac{\pi}{4}(f_3^h - f_1^h)$. Now, by Lemmas A.2, A.4, A.5 and A.6(i),

$$\begin{aligned} V(x) &= \frac{(-1)^{n(g,1)}}{2^{(1/2)\beta+1} i^{n_3}} \{e^{i\alpha_1} + (-1)^{\mu_1+\mu_2} e^{-i\alpha_1} + (-1)^{\mu_2} e^{i\alpha_2} + (-1)^{\mu_1} e^{-i\alpha_2}\} \\ &= \frac{(-1)^{n(g,1)}}{2^{(1/2)\beta+1} i^{n_3}} \{e^{i(\alpha_1+\alpha_2)/2} + (-1)^{\mu_1} e^{-i(\alpha_1+\alpha_2)/2}\} \\ &\quad \times \{e^{i(\alpha_1-\alpha_2)/2} + (-1)^{\mu_2} e^{-i(\alpha_1-\alpha_2)/2}\}. \end{aligned} \tag{A.25}$$

But $\frac{1}{2}(\alpha_1 + \alpha_2) = \frac{\pi}{8}(2 + f_3^g - f_1^g + f_3^h - f_1^h) = \frac{\pi}{4}(1 + \beta_1^*)$, as one can verify from (2.7), (A.1), (A.8) and (A.12), on simplification. Similarly, $\frac{1}{2}(\alpha_1 - \alpha_2) = \frac{\pi}{4}(1 + \beta_2^*)$. Hence, (ii) follows from (A.25) using Lemma A.5(ii) and the fact that $\beta = \beta_1 + \beta_2$; cf. (A.9). □

LEMMA A.8. (i) *If both* β_1 *and* β_2 *are odd*, *then the number of choices*, *say* σ, *of* S_1, S_2, S_3 *which meet* (A.23) *and keep both* μ_1 *and* μ_2 *odd equals* $2^{\beta-2}$.

(ii) *The number of choices of* S_1, S_2, S_3 *meeting* (A.16) *is* 2^β.

(iii) *If* β (>0) *is even*, *then* $\sigma_0 = \sigma_1 = 2^{\beta-1}$, *where* σ_0 *and* σ_1 *denote the numbers of choices of* S_1, S_2, S_3 *which meet* (A.16) *and keep* n_3 *even and odd*, *respectively.*

PROOF. We prove only (i). Proofs of (ii), (iii) are similar. For a given QC design, (A.23) determines S_1 uniquely and fixes $S_2 \cup S_3$ at Δ [$=\Delta^{(1)} \cup \Delta^{(2)}$, by (A.7)]. Thus, by (A.13), σ equals the number of ways in which one can choose an odd number of elements from each of $\Delta^{(1)}$ and $\Delta^{(2)}$. Hence, $\sigma = (2^{\beta_1-1})(2^{\beta_2-1}) = 2^{\beta-2}$, as $\beta_1(=\#\Delta^{(1)})$ and $\beta_2(=\#\Delta^{(2)})$ are both odd and hence positive. □

PROOF OF THEOREM 1 FOR $x = 0101$. The proof is given separately for three cases corresponding to (i1), (i2) and (i3) of Theorem 1. Recall that any choice of S_1, S_2, S_3 which makes $V(x)$ nonzero entails a word of type x, length $m + X$ and aliasing index $|V(x)|$.

Case 1. Let $\lambda_5 + \lambda_6 = 0$. Then Lemma A.7 is applicable. Thus if $V(x) \neq 0$ then (A.23) holds, so that $m + X = 2(f_{02} + f_{20}) + \#\Delta + 2 = l_8 + 2$, by (2.7)–(2.9) and

(A.9). Hence, each word of type x has length $l_8 + 2$. It remains to show that there are $1/(\xi_1^2\xi_2^2)$ words of type x, each with aliasing index $\xi_1\xi_2$. To that effect, from (A.24) note that for any S_1, S_2, S_3 meeting (A.23),

$$|V(x)| = |Q_1(x)||Q_2(x)|, \tag{A.26}$$

where for $k = 1, 2$,

$$|Q_k(x)| = 2^{-(1/2)(\beta_k-1)}\left|\cos\left\{\frac{\pi}{4}(1+\beta_k^*)\right\}\right| \qquad \text{if } \mu_k \text{ is even}$$
$$= 2^{-(1/2)(\beta_k-1)}\left|\sin\left\{\frac{\pi}{4}(1+\beta_k^*)\right\}\right| \qquad \text{if } \mu_k \text{ is odd.}$$

First, suppose both β_1 and β_2 are odd. Then by (A.9) and (A.12), β_1^* and β_2^* are also both odd. For $k = 1, 2$, if $\beta_k^* = 1$ or $5 \pmod 8$, then $|Q_k(x)|$ equals 0 or $2^{-(1/2)(\beta_k-1)}$ depending on whether μ_k is even or odd, while if $\beta_k^* = 3$ or 7 (mod 8), then the roles of even and odd μ_k are switched. Hence if both β_1^* and β_2^* equal 1 or 5 (mod 8), then by (A.26), a choice of S_1, S_2, S_3 meeting (A.23) yields a nonzero $V(x)$ and hence leads to a word if and only if μ_1 and μ_2 are both odd. By (2.10), (A.9) and Lemma A.8(i), the number of such choices is $2^{\beta-2}$ [$=1/(\xi_1^2\xi_2^2)$] and, for any such choice, $|V(x)| = 2^{-(1/2)(\beta_1+\beta_2-2)}$ [$=\xi_1\xi_2$]. It is easily seen that the same holds if both β_1^* and β_2^* equal 3 or 7 (mod 8) or if one of them equals 1 or 5 (mod 8) and the other 3 or 7 (mod 8). This settles Case 1 when β_1 and β_2 are both odd. Similar arguments work when they are both even or one of them is odd and the other even.

Case 2. Let $\lambda_5 + \lambda_6 > 0$ and $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$. By (A.9)–(A.11), here $\beta = 0$, $f_1^g = f_3^g = f_1^h = f_3^h = 0$, and $\Delta_1^g \cup \Delta_3^g$ and $\Delta_1^h \cup \Delta_3^h$ are empty sets. Hence, by (A.14), $n_3 = n(g, 1) = n(h, 1) = 0$, for any S_1, S_2, S_3 meeting (A.15) or (A.16). Therefore, Lemma A.4 yields $V_1(x) + V_4(x) = 0$ under (A.15), and $V_2(x) + V_3(x) = 1$ under (A.16). Since Lemma A.6(ii) rules out simultaneous occurrence of (A.15) and (A.16), Lemmas A.2 and A.3 show that $V(x)$ equals 1 if (A.16) holds, and 0 otherwise. Moreover, as $\Delta_1^h \cup \Delta_3^h$ is empty, (A.16) determines S_1, S_2, S_3 uniquely and, under (A.16), $m + X = 2f_2^h + 2 = l_{10} + 2$, using (2.9), (A.11). Thus, in this case, there is one word of type x, with aliasing index 1 and length $l_{10} + 2$.

Case 3. Let $\lambda_5 + \lambda_6 > 0$ and $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 > 0$. Again, Lemma A.6(ii) precludes coincidence of (A.15) and (A.16). Thus, by Lemmas A.2 and A.3, words of type x can arise in two mutually exclusive and exhaustive ways: (a) from S_1, S_2, S_3 meeting (A.15), (b) from S_1, S_2, S_3 meeting (A.16). First, consider (b). For any S_1, S_2, S_3 meeting (A.16), by Lemmas A.2, A.3 and A.4(ii),

$$|V(x)| = 2^{-(1/2)\beta}\left|\cos\left\{\frac{\pi}{4}(f_3^h - f_1^h)\right\}\right| \qquad \text{if } n_3 \text{ is even}$$
$$= 2^{-(1/2)\beta}\left|\sin\left\{\frac{\pi}{4}(f_3^h - f_1^h)\right\}\right| \qquad \text{if } n_3 \text{ is odd.}$$

Thus for odd $f_3^h - f_1^h$, irrespective of whether n_3 is even or odd, $|V(x)| = 2^{-(1/2)(\beta+1)}$. On the other hand, if $f_3^h - f_1^h = 0$ or 4 (mod 8), then $|V(x)|$ equals $2^{-(1/2)\beta}$ or 0 according as whether n_3 is even or odd, while if $f_3^h - f_1^h = 2$ or 6 (mod 8), then the roles of even and odd n_3 are switched. Since β [$=f_1^h + f_3^h$, by (A.11)] and $f_3^h - f_1^h$ are either both even or both odd, from (2.10), (A.9) and Lemma A.8(ii), (iii), it follows that, irrespective of whether $f_3^h - f_1^h$ is odd or even, (b) yields $1/(2\xi^2)$ words of type x, each with aliasing index ξ. By (2.9), (A.9), (A.11) and (A.16), each such word has length $m + X = 2f_2^h + f_1^h + f_3^h + 2 = l_{10} + 2$. Similarly, (a) yields another $1/(2\xi^2)$ words of type x, each with aliasing index ξ and length $l_9 + 2$. Hence, the conclusion of Theorem 1 for $x = 0101$ follows in this case. □

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