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SOME STOCHASTIC PROCESS WITHOUT BIRTH, LINKED TO THE MEAN CURVATURE FLOW

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Using Huisken's results about the mean curvature flow on a strictly convex hypersurface and Kendall–Cranston's coupling, we will build a stochastic process without birth and show that there exists a unique law of such a process. This process has many similarities with the circular Brownian motion studied by Émery and Schachermayer, and Arnaudon. In general this process is not a stationary process; it is linked to some differential equation without initial condition. We will show that this differential equation has a unique solution up to a multiplicative constant.

1. Tools and first properties. Let M be a compact Riemannian manifold of dimension n without boundary, which is smoothly embedded in $\mathbb{R}^{n+1}$ for $n \geq 2$. We write F_0 the embedding function

$$F_0 : M \hookrightarrow \mathbb{R}^{n+1}.$$

Consider the flow defined by

$$\begin{cases} \partial_t F(t,x) = -H_\nu(t,x)\vec{\nu}(t,x), \\ F(0,x) = F_0(x). \end{cases} \tag{1.1}$$

Let $M_t = F(t, M)$. We identify M with M_0 and F_0 with Id. In (1.1), $\vec{\nu}(t,x)$ is the outer unit normal at $F(t,x)$ on M_t, and $H_\nu(t,x)$ is the mean curvature at $F(t,x)$ on M_t in the direction $\vec{\nu}(t,x)$, that is, $H_\nu(x) = \operatorname{trace}(S_\nu(x))$ where S_ν is the second fundamental form (see [20] for the definition).

REMARK 1.1. In this paper we take this point of view of mean curvature flow (see [14] for existence, and related results). Many other authors give a different point of view for this equation. The viscosity solution (see [7–11]) generalizes the solution after the explosion time and gives a unique solution which is contained in the Brakke family of solutions and passes the singularity. In the sequel we shall only consider smooth solutions until explosion time.

As usual we call M_t the motion by mean curvature. To be self-contained, we include a proof of the next lemma, although it is well known.

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LEMMA 1.2. *Let (M, g) be a Riemannian manifold isometrically embedded in $\mathbb{R}^{n+1}$. We call ι the isometry*

$$(M, g) \overset{\iota}{\hookrightarrow} \mathbb{R}^{n+1}.$$

Then

$$\forall x \in M,\ \Delta\iota(x) = -H_\nu(x)\vec{\nu}(x), \tag{1.2}$$

where Δ is the Laplace–Beltrami operator associated to the metric g.

PROOF. By the flatness of the target manifold, we have

$$\Delta\iota(x) = \begin{pmatrix} \Delta\iota^1(x) \\ \vdots \\ \Delta\iota^{n+1}(x) \end{pmatrix}$$

and

$$\Delta\iota^j(x) = \sum_{i=1}^n \frac{d}{dt^2}\Big|_{t=0} \iota^j(\gamma_i(t)),$$

where $\gamma_i(t)$ is a geodesic in M such that $\gamma_i(0) = x$ and $\dot{\gamma}_i(0) = A_i$, and A_i is an orthogonal basis of T_xM. By definition of a geodesic we obtain

$$\Delta\iota(x) \perp T_{\iota(x)}(\iota(M)),$$

so there exists a function β such that $\Delta\iota(x) = \beta(x)\vec{\nu}(x)$. We compute β as follows:

$$\begin{aligned}
\beta(x) &= \langle \Delta\iota(x), \vec{\nu}(x)\rangle \\
&= \sum_{i=1}^n \Big\langle \frac{d}{dt^2}\Big|_{t=0} \iota(\gamma_i(t)), \vec{\nu}(x)\Big\rangle \\
&= \sum_{i=1}^n \langle \nabla^{\mathbb{R}^n}_{\iota(\dot{\gamma_i}(t))} \iota(\dot{\gamma_i}(t))|_{t=0}, \vec{\nu}(x)\rangle \\
&= \sum_{i=1}^n -\langle \iota(\dot{\gamma_i}(t)), \nabla^{\mathbb{R}^n}_{\iota(\dot{\gamma_i}(t))} \vec{\nu}\rangle|_{t=0}, \text{ metric connection} \\
&= \sum_{i=1}^n -\langle \iota(\dot{\gamma_i}(t)), (\nabla^{\mathbb{R}^n}_{\iota(\dot{\gamma_i}(t))} \vec{\nu})^\top\rangle|_{t=0} \\
&= -\operatorname{trace}(S_\nu(x)).
\end{aligned}$$

$\square$

To give a parabolic interpretation of (1.1), let us define a family of metrics $g(t)$ on M which is the pull-back by $F(t, \cdot)$ of the induced metric on M_t, that is,

$$g(t) := F(t, \cdot)^*(\langle \cdot, \cdot \rangle_{\mathbb{R}^{n+1}})_{|M_t}.$$

Using the previous lemma we rewrite the equation as in [14]

$$\begin{cases} \partial_t F(t,x) = \Delta_t F(t,x), \\ F(0,x) = F_0(x), \end{cases}$$

where Δ_t is the Laplace–Beltrami operator associated to the metric $g(t)$.

REMARK 1.3. Sometimes we follow the probabilistic convention of putting $1/2$ in front of the Laplacian (which just changes the time and makes computations more concise); sometimes we use a geometric convention.

We call T_c the explosion time of the mean curvature flow. Let $T < T_c$, and $g(t)$ be the family of metrics defined as above. Let $(W^i)_{1 \le i \le n}$ be a $\mathbb{R}^n$-valued Brownian motion. Recall from [4] the definition of the $g(t)$-Brownian motion in M started at x which we call $g(t)$-BM(x).

DEFINITION 1.4. Let us take a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\ge 0}, \mathcal{F}, \mathbb{P})$ and a $C^{1,2}$-family $g(t)_{t\in[0,T[}$ of metrics over M. An M-valued process $X(x)$ defined on $\Omega \times [0,T[$ is called a $g(t)$ Brownian motion in M started at $x \in M$ if $X(x)$ is continuous, adapted and for every smooth function f,

$$f(X_s(x)) - f(x) - \frac{1}{2}\int_0^s \Delta_t f(X_t(x))\,dt$$

is a local martingale vanishing at 0.

We give a proposition which yields a characterization of mean curvature flow by the $g(t)$ Brownian motion.

PROPOSITION 1.5. *Let M be an n-dimensional manifold isometrically embedded in $\mathbb{R}^{n+1}$. Consider the application*

$$F : [0,T[\times M \to \mathbb{R}^{n+1}$$

such that $F(t,\cdot)$ are diffeomorphisms and the family of metrics $g(t)$ on M, which is the pull-back by $F(t,\cdot)$ of the induced metric on $M_t = F(t,M)$, that is,

$$g(t) := F(t,\cdot)^*(\langle\cdot,\cdot\rangle_{\mathbb{R}^{n+1}})_{|M_t}.$$

Then the following assertions are equivalent:

(i) *$F(t,\cdot)$ is a solution of mean curvature flow*;
(ii) *$\forall x_0 \in M, \forall T \in [0,T_c[$, let $\tilde g_t^T = \frac{1}{2} g_{T-t}$ and $X^T(x_0)$ be a $(\tilde g_t^T)_{t\in[0,T]}$*-BM($x_0$), *then*

$$Y_t^T = F\big(T-t, X_t^T(x_0)\big)$$

is a local martingale in $\mathbb{R}^{n+1}$.

PROOF. By definition we have a sequence of isometries

$$F(t,\cdot):(M,g_t)\tilde{\to} M_t \hookrightarrow \mathbb{R}^{n+1}.$$

Let $x_0 \in M$ and $T \in [0, T_c[$ and $X^T(x_0)$ a $(\tilde{g}_t^T)_{t\in[0,T]}$-BM($x_0$). We compute the Itô differential of

$$Y_t^{T,i} = F^i\big(T-t, X_t^T(x_0)\big),$$

that is to say

$$\begin{aligned}
d(Y_t^{T,i}) &= -\frac{\partial}{\partial t}F^i\big(T-t, X_t^T(x_0)\big)\,dt + d(F^i_{T-t}(X_t^T(x_0))\\
&\underset{d\mathcal{M}}{\equiv} -\frac{\partial}{\partial t}F^i\big(T-t, X_t^T(x_0)\big)\,dt + \frac{1}{2}\Delta_{\tilde{g}_t}F^i_{T-t}(X_t^T(x_0))\,dt\\
&\underset{d\mathcal{M}}{\equiv} -\frac{\partial}{\partial t}F^i\big(T-t, X_t^T(x_0)\big)\,dt + \Delta_{g_{T-t}}F^i_{T-t}(X_t^T(x_0))\,dt\\
&\underset{d\mathcal{M}}{\equiv} 0.
\end{aligned}$$

Therefore Y_t^T is a local martingale.

Let us show the converse. Let $x_0 \in M$ and $T \in [0, T_c[$ and let $X^T(x_0)$ be a $(\tilde{g}_t^T)_{t\in[0,T]}$-BM($x_0$). Then $Y_t^{T,i}$ is a local martingale since, almost surely, for all $t \in [0, T]$

$$-\frac{\partial}{\partial t}F^i\big(T-t, X_t^T(x_0)\big)\,dt + \Delta_{g_{T-t}}F^i_{T-t}(X_t^T(x_0))\,dt = 0.$$

For any $s \in [0, T]$, we get by integrating

$$\int_0^s -\frac{\partial}{\partial t}F^i\big(T-t, X_t^T(x_0)\big)\,dt + \Delta_{g_{T-t}}F^i_{T-t}(X_t^T(x_0))\,dt = 0.$$

Continuity of $g(t)$-Brownian motions then yields

$$-\frac{\partial}{\partial t}F^i(T, x_0) + \Delta_{g_T}F^i_T(x_0) = 0. \qquad \square$$

In order to apply this proposition, we give an estimation of the explosion time. This is also a consequence of a maximum principle explicitly contained in the $g(t)$-Brownian motion.

The quadratic covariation of Y_t^T is given by

PROPOSITION 1.6. *Let Y_t^T be defined as before; then the quadratic covariation of Y_t^T for the usual scalar product in $\mathbb{R}^{n+1}$ is*

$$\langle dY_t^T, dY_t^T\rangle = 2n\mathbb{1}_{[0,T]}(t)\,dt.$$

PROOF. Let $//^T_{0,t}$ be the parallel transport above X^T_t. It is shown in [4] that this is an isometry:

$$//^T_{0,t} : (T_{X_0}M, \tilde{g}(0)) \longmapsto (T_{X_t}M, \tilde{g}(t)).$$

Let $(e_i)_{1\leq i\leq n}$ be a orthonormal basis of $(T_{X_0}M, \tilde{g}(0))$, and $(W^i)_{1\leq i\leq n}$ be the $\mathbb{R}^n$-valued Brownian motion such that (e.g., [2, 4])

$$*dW_t = //^{T,-1}_{0,t} * dX^T_t,$$

and in the Itô's sense

$$dX^T_t = //^T_{0,t} e_i \, dW^i_t.$$

Hence

$$\begin{aligned}
\langle dY^T_t, dY^T_t \rangle &= \langle d(F_{T-t}(X^T_t(x_0))), d(F_{T-t}(X^T_t(x_0)))\rangle \\
&= \langle d(X^T_t(x_0)), d(X^T_t(x_0))\rangle_{g_{T-t}} \\
&= \langle d(X^T_t(x_0)), d(X^T_t(x_0))\rangle_{2\tilde{g}_t} \\
&= \left\langle \sum_{i=1}^n //^T_{0,t} e_i \, dW^i, \sum_{j=1}^n //^T_{0,t} e_j \, dW^j \right\rangle_{2\tilde{g}_t} \\
&= \sum_{i=1}^n \langle //^T_{0,t} e_i, //^T_{0,t} e_i \rangle_{2\tilde{g}_t} \, dt = \sum_{i=1}^n 2\, dt = 2n\, dt.
\end{aligned}$$

To pass from the first to the second line, we used the fact that F_{T-t} is an isometry, for the last step we used the isometry of the parallel transport. $\square$

REMARK 1.7. Up to convention we recover the same martingale as in [21].

An immediate corollary of Proposition 1.6 is the following result, which appears in [10] and [14].

COROLLARY 1.8. *Let M be a compact Riemannian n-manifold and T_c the explosion time of the mean curvature flow*; *then*

$$T_c \leq \frac{\operatorname{diam}(M_0)^2}{2n}.$$

PROOF. Recall that the mean curvature flow stays in a compact region, like the smallest ball which contains M_0. This result is clear in the case of a strictly convex starting manifold and can be proved in the general setting using P. L. Lions viscosity solution (e.g., Theorem 7.1 in [10]).

For all $T \in [0, T_c[$ take the previous notation. By the above recall that

$$\|Y^T_t\| \leq \operatorname{diam}(M_0);$$

then Y_t^T is a true martingale, and

$$\|Y_t^T\|^2 - \langle Y^T, Y^T\rangle_t$$

is also a true martingale. Hence

$$\mathbb{E}[\|Y_0^T\|^2] + 2nT \le \operatorname{diam}(M_0)^2,$$

and we obtain

$$T \le \frac{\operatorname{diam}(M_0)^2}{2n}. \qquad \square$$

2. Tightness and first example on the sphere. We now define $(\tilde{g}^{T_c})_{t\in]0,T_c]}$-BM in a general setting. When the initial manifold M_0 is a sphere we use conformality of the metric to show that after a deterministic change of time such a process is a $]-\infty, T_c]$ Brownian motion on the sphere (for existence and definition see [1] and [6]). In the next section, we shall give a general uniqueness result when the initial manifold M_0 is strictly convex.

DEFINITION 2.1. Let M be an n-dimensional strictly convex manifold (i.e., with a strictly positive definite second fundamental form), $F(t,\cdot)$ the smooth solution of the mean curvature flow, $(M, g(t))$ the family of metrics constructed by pull-back (as in Proposition 1.5) and T_c the explosion time. We define a family of processes as follows: $\forall \varepsilon \in]0, T_c]$

$$X_t^{\varepsilon}(x_0) = \begin{cases} x_0, & \text{if } 0 < t \le \varepsilon, \\ \mathrm{BM}(\varepsilon, x_0)_t, & \text{if } \varepsilon \le t \le T_c, \end{cases}$$

where $\mathrm{BM}(\varepsilon, x_0)_t$ is a $\frac{1}{2}g(T_c - t)$ Brownian motion that starts at x_0 at time ε, and

$$Y_t^{\varepsilon}(x_0) = \begin{cases} F(T_c - \varepsilon, x_0), & \text{if } 0 \le t \le \varepsilon, \\ F\big(T_c - t, X_t^{\varepsilon}(x_0)\big), & \text{if } \varepsilon \le t \le T_c. \end{cases}$$

REMARK 2.2. We proceed as before because at time T_c, there is not any metric. Huisken shows in [14] that in this case

$$\exists \mathcal{D} \in \mathbb{R}^{n+1} \qquad \text{such that } \forall x_0 \in M, \qquad \lim_{s\to T_c} F(s, x_0) = \mathcal{D}.$$

PROPOSITION 2.3. *With the same notation as in the above definition there exists at least one martingale Y^1 in the adherence (for the weak convergence) of $(Y_\cdot^{\varepsilon}(x_0))_{\varepsilon>0}$ when ε goes to* 0. *Also, every adherence point is a martingale.*

PROOF. We have

$$dY_t^{\varepsilon}(x_0) = \begin{cases} 0, & \text{if } t \le \varepsilon, \\ d\mathcal{M}, & \text{if } t \ge \varepsilon, \end{cases}$$

where $d\mathcal{M}$ is an Itô differential of some martingale. This defines a family of martingales. With the same computation as in Proposition 1.6, we get

$$\langle dY_t^\varepsilon, dY_t^\varepsilon\rangle_{\mathbb{R}^{n+1}} = 2n1_{]\varepsilon,T_c]}(t)\,dt \le 2n\,dt.$$

Also by the above remark Y_0^ε is tight, hence $(Y_\cdot^\varepsilon(x_0))_{\varepsilon>0}$ is tight. As usual, Prokhorov's theorem implies that there exists an adherence point. We also use Huisken [14] (for the strictly convex manifold) to show

$$\|Y^\varepsilon\| \le \operatorname{diam}(M_0). \tag{2.1}$$

By Proposition 1-1 in [16], page 481, and the fact that (Y^ε) are martingales, we conclude that all adherence points of (Y^ε) are martingales with respect to the filtration that they generate. □

REMARK 2.4. The above proposition is also valid for arbitrary M which are isometrically embedded in $\mathbb{R}^{n+1}$ just because the bound (2.1) is also a consequence of Theorem 7.1 in [10].

We will now derive tightness of X_t^ε from those of (Y^ε). This purpose will be completed by the subsequent Lemma 2.6.

Recall some results of [14]: if M_0 is a strictly convex manifold, then M_t is also strictly convex and $\forall 0 \le t_1 < t_2 < T_c$, $M_{t_2} \subset \operatorname{int}(M_{t_1})$, where int is the interior of the bounded connected component of the complementary. Hence there is a foliation of $\overline{\operatorname{int}}(M_0)$

$$\bigsqcup_{t\in[0,T_c[} M_t,$$

where $\bigsqcup$ stand for the disjoint union.

DEFINITION 2.5. We denote

$$\mathcal{C}^f(]0,T_c],\mathbb{R}^{n+1}) = \{\gamma \in \mathcal{C}(]0,T_c],\mathbb{R}^{n+1}) \text{ such that } \gamma(t) \in M_{T_c-t}\}.$$

Note that $\mathcal{C}^f(]0,T_c],\mathbb{R}^n)$ is a closed set of $\mathcal{C}(]0,T_c],\mathbb{R}^n)$ for the Skorokhod topology.

LEMMA 2.6. *Let M be an n-dimensional strictly convex manifold, $F(t,\cdot)$ the smooth solution of the mean curvature flow and T_c the explosion time. Then*

$$F : [0,T_c[\times M \longrightarrow \bigsqcup_{t\in[0,T_c[} M_t$$

is a diffeomorphism in the sense of manifolds with boundary, and

$$\begin{aligned}\Psi : \mathcal{C}^f(]0,T_c],\mathbb{R}^n) &\longrightarrow \mathcal{C}(]0,T_c],M),\\ \gamma &\longmapsto t \mapsto F^{-1}(T_c-t,\gamma(t))\end{aligned}$$

is continuous for the different Skorokhod topologies. To define the Skorokhod topology in $\mathcal{C}(]0,T_c],M)$ we could use the initial metric $g(0)$ on M.

PROOF. It is clear that F is smooth as a solution of a parabolic equation [14], and this result has been used above. Its differential is given at each point by

$$\forall (t,x) \in [0, T_c[\times M, \forall v \in T_x M$$
$$DF(t,x)\left(\frac{\partial}{\partial_t}, v\right) = \frac{\partial}{\partial_t} F(t,x) \oplus DF_t(x)(v),$$

where $\frac{\partial}{\partial_t} F(t,x) = -H(t,x)\vec{\nu}(t,x)$; here $\oplus$ stands for $+$ and means that we cannot cancel the sum without cancelling each term. Since there is no ambiguity we write $H(t,x)$ for $H_\nu(t,x)$. Recall that $H(t,x) > 0$.

For the second part of the lemma, we remark that for $0 \le \delta < T_c$

$$F^{-1} : \bigsqcup_{t \in [0,\delta]} M_t \longrightarrow [0,\delta] \times M$$

is Lipschitz (use the bound of the differential on a compact set).

Recall that a sequence converges to a continuous function in the Skorokhod topology if and only if it converges to this function locally uniformly. We will now show the continuity of Ψ. Take a sequence α_m in $\mathcal{C}^f(]0, T_c], \mathbb{R}^{n+1})$ and $\alpha \in \mathcal{C}^f(]0, T], \mathbb{R}^{n+1})$ such that $\alpha_m \to \alpha$ for the Skorokhod topology. Then for all compact sets A in $]0, T_c]$,

$$\|\alpha_m - \alpha\|_A \longrightarrow 0,$$

where $\|f\|_A = \sup_{t \in A} \|f(t)\|$. Let A be a compact set in $]0, T_c]$; then there exists a Lipschitz constant C_A of F^{-1} in $\bigsqcup_{t \in A} M_t$, such that for all t in A,

$$d_{g(o)}(F^{-1}(\alpha_m(t)), F^{-1}(\alpha(t))) \le C_A \|\alpha_m(t) - \alpha(t)\|,$$

where $d_{g(o)}(x,y)$ is the distance in M between x and y for the metric $g(0)$. We also define

$$d_{g(o),A}(f,g) = \sup_{t \in A} d_{g(o)}(f(t), g(t)),$$

where f and g are M-valued function. We get

$$d_{g(o),A}(\Psi(\alpha_m), \Psi(\alpha)) \le C_A \|\alpha_m - \alpha\|_A.$$

So $\Psi(\alpha_m) \longrightarrow \Psi(\alpha)$ uniformly in all compact, so for the Skorokhod topology in $\mathcal{C}(]0, T_c], M)$. $\square$

Let

$$\tilde{Y}_t^\varepsilon = (Y_t^\varepsilon - Y_0^\varepsilon) + \bigl(Y_0^\varepsilon \mathbb{1}_{[\varepsilon, T_c]}(t) + \mathbb{1}_{[0,\varepsilon]}(t) F(T_c - t, x_o)\bigr).$$

Proposition 2.3 gives the tightness of $Y_t^\varepsilon - Y_0^\varepsilon$, and

$$Y_0^\varepsilon \mathbb{1}_{[\varepsilon, T_c]}(t) + \mathbb{1}_{[0,\varepsilon]}(t) F(T_c - t, x_o)$$

is a nonrandom sequence of functions that converges uniformly; hence $\tilde{Y}^\varepsilon$ is tight. For strictly positive time t,

$$X_t^\varepsilon = F^{-1}(T_c - t, \tilde{Y}_t^\varepsilon).$$

The previous Lemma 2.6 yields the tightness of X^ε. Hence we have shown that

$$\forall \varphi = (\varepsilon_k)_k \to 0, \exists X_{]0,T_c]}^\varphi, \qquad X_{]0,T_c]}^{\varepsilon_k} \stackrel{\mathcal{L}}{\to} X_{]0,T_c]}^\varphi \qquad \text{for a subsequence.}$$

PROPOSITION 2.7. *Let $\varphi = (\varepsilon_k)_k \to 0$ and $X_{]0,T_c]}^\varphi$ such that $X_{]0,T_c]}^{\varepsilon_k} \stackrel{\mathcal{L}}{\to} X_{]0,T_c]}^\varphi$. Then $X_{]0,T_c]}^\varphi$ is a $\frac{1}{2}g(T_c - t)$-BM in the following sense*:

$$\forall \varepsilon > 0 \qquad X_{[\varepsilon,T_c]}^\varphi \stackrel{\mathcal{L}}{=} \mathrm{BM}(\varepsilon, X_\varepsilon^\varphi).$$

PROOF. Let $\varepsilon > 0$; then for large k

$$\begin{cases} X^{\varepsilon_k} \text{ is a } \mathrm{BM}(\varepsilon, X_\varepsilon^{\varepsilon_k}) \text{ after time } \varepsilon, \text{ by the Markov property,} \\ \text{and let } X \text{ be a } \mathrm{BM}(\varepsilon, X_\varepsilon^\varphi) \text{ after time } \varepsilon. \end{cases}$$

We want to show that $X = X^\varphi$ after ε. To sketch the proof

$$X^{\varepsilon_k} \xrightarrow[k\to\infty]{\mathcal{L}} X^\varphi,$$

and hence

$$X_\varepsilon^{\varepsilon_k} \xrightarrow[k\to\infty]{\mathcal{L}} X_\varepsilon^\varphi.$$

We use the Skorokhod theorem, to have a L_2-convergence in a larger probability space

$$X_\varepsilon^{\prime\varepsilon_k} \xrightarrow[k\to\infty]{L_2,\ \text{a.s.}} X_\varepsilon^{\prime\varphi},$$

with $X_\varepsilon^{\prime\varepsilon_k} \stackrel{\mathcal{L}}{=} X_\varepsilon^{\varepsilon_k}$ and $X_\varepsilon^{\prime\varphi} \stackrel{\mathcal{L}}{=} X_\varepsilon^\varphi$. We use convergence of solutions of SDEs with initial conditions converging in L_2 (see Stroock and Varadhan [22]), to get

$$\begin{aligned} \mathrm{BM}(\varepsilon, X_\varepsilon^{\prime\varepsilon_k}) &\xrightarrow[k\to\infty]{\mathcal{L}} \mathrm{BM}(\varepsilon, X_\varepsilon^{\prime\varphi}), \\ \mathrm{BM}(\varepsilon, X_\varepsilon^{\prime\varepsilon_k}) &\stackrel{\mathcal{L}}{=} X_{[\varepsilon,T_c]}^{\varepsilon_k}, \\ \mathrm{BM}(X_\varepsilon^{\prime\varphi}) &\stackrel{\mathcal{L}}{=} \mathrm{BM}(\varepsilon, X_\varepsilon^\varphi). \end{aligned}$$

We use that

$$X^{\varepsilon_k} \xrightarrow[k\to\infty]{\mathcal{L}} X^\varphi$$

to conclude, after identification of the limit,

$$X = \mathrm{BM}(\varepsilon, X_\varepsilon^\varphi) \stackrel{\mathcal{L}}{=} X_{[\varepsilon,T_c]}^\varphi.$$

Hence the process X^{φ} is a $\frac{1}{2}g(T_c - u)_{u\in]0,T_c]}$-BM in the above sense, we call "without birth." $\square$

We now show that in the sphere case the $\frac{1}{2}g(T_c - u)_{u\in]0,T_c]}$-BM is, after a change of time, nothing else than a $\mathrm{BM}(g(0))_{]-\infty,0]}$. This will give uniqueness in law of the process.

PROPOSITION 2.8. *Let $g(t)$ be a family of metrics which arises from a mean curvature flow on the sphere. Then the $\tilde{g}(u) = \frac{1}{2}g(T_c - u)_{u\in]0,T_c]}$-* BM *is unique in law.*

PROOF. Let R_0 be the radius of the $g(0)$-sphere. Then $T_c = \frac{R_0^2}{2n}$, and by direct computation we obtain

$$F(t,x) = \frac{\sqrt{R_0^2 - 2nt}}{R_0}x.$$

Let X be a $\frac{1}{2}g(T_c - u)_{u\in]0,T_c]}$-BM. By Proposition 1.5 we know that the diffusion $Z_t := F(T_c - t, X_t)$ is a local martingale in $\mathbb{R}^{n+1}$. By construction we know that Z_t belongs to the sphere $M_{T_c - t}$, and $X_t = \frac{R_0}{\sqrt{2nt}} Z_t$. By invariance under the orthogonal group $O(n+1)$, the generator of X must have the form $k(t)\Delta_{g(0)}$, where $\Delta_{g(0)}$ is the generator of the spherical Brownian motion; consequently for some deterministic time-change φ, $X_{\varphi(\cdot)}$ is a spherical Brownian motion. To identify φ it suffices to compute the quadratic variation of X in $\mathbb{R}^{n+1}$. Proposition 1.6 gives $\langle dZ_t, dZ_t\rangle = 2n\,dt$, wherefrom

$$\langle dX_t, dX_t\rangle = \left(\frac{R_0}{\sqrt{2nt}}\right)^2 \langle dZ_t, dZ_t\rangle = \frac{R_0^2}{t}\,dt$$

and

$$\langle dX_{\varphi(t)}, dX_{\varphi(t)}\rangle = \frac{R_0^2 \varphi'(t)}{\varphi(t)}\,dt;$$

identifying this with the quadratic variation $n\,dt$ of spherical Brownian motion gives the time-change φ with the initial condition $\varphi(0) = T_c$, that is, the function

$$\varphi(t) = T_c \exp\left(\frac{t}{2T_c}\right).$$

We get that $X_{\varphi(t)} = (\mathrm{BM}_{g(0)})_t$, according to the usual characterization of a Brownian motion. Hence by this deterministic change of time, and by the uniqueness in law of a $(\mathrm{BM}_{g(0)})_{]-\infty,0]}$ on the sphere, we get uniqueness in law of a $\frac{1}{2}g(T_c - u)_{u\in]0,T_c]}$-BM on a sphere. $\square$

REMARK 2.9. By invariance of Z_t under the orthogonal group $O(n+1)$ and using the fact that the norm of Z_t is deterministic [i.e., $\|Z_t\| = f(t)$] we deduce that the generator of Z at a point $z \in \mathbb{R}^{n+1} \setminus \{0\}$ must have the form $c(t)\Delta_{z^\perp}$ [where $c(t)$ depends on $f(t)$, i.e., $2nc(t) = (f^2(t))'$, and $\Delta_{z^\perp}$ denotes the Laplacian in the hyperplanar direction $z^\perp$], just by computing the generator in good coordinates.

In the above proof we essentially made use of conformality of the family of metrics. In the general case of a strictly convex initial manifold the family of metrics may be not conform. But we shall see in the sequel that for any strictly convex initial manifold we can prove the uniqueness in law of the $\frac{1}{2}g(T_c - u)_{u\in]0,T_c]}$-BM, without the assumption of conformality and by using different strategies.

3. Kendall–Cranston coupling. In this section the manifold M is compact and strictly convex. The goal is to prove uniqueness in law of the $g(T_c - t)$-BM. This section will be cut into two parts: in the first one we will give a geometric result inspired by the work of Huisken; the second one will be an adaptation of the Kendall–Cranston coupling. We will, by a deterministic change of time, transform a $g(T_c - t)$-BM (the existence of which comes from Proposition 2.7) into a $\tilde{g}(t)_{]-\infty,0]}$-BM which has good geometric properties.

REMARK 3.1. In the two last sections in [14], Huisken considers, like Hamilton for the Ricci flow, the normalized mean curvature flow. It consists of dilating the manifolds M_t by a coefficient to obtain manifolds of constant volume. He obtains a positive coefficient of dilation $\psi(t)$ that satisfies the following property:

THEOREM 3.2 (Huisken [14]). *For all $t \in [0, T_c[$, define $\tilde{F}(t,\cdot) = \psi(t)F(t,\cdot)$ such that $\int_{\tilde{M}_t} d\tilde{\mu}_t = |M_0|$ and $\tilde{t}(t) = \int_0^t \psi^2(\tau)\,d\tau$, then there exist positive constants δ and C such that*:

(i) $\tilde{T}_c = \infty$;
(ii) $\tilde{H}_{\max}(\tilde{t}) - \tilde{H}_{\min}(\tilde{t}) \le Ce^{-\delta\tilde{t}}$;
(iii) $|\frac{\partial}{\partial\tilde{t}}\tilde{g}_{ij}(\tilde{t})| \le Ce^{-\delta\tilde{t}}$;
(iv) $\tilde{g}_{ij}(\tilde{t}) \to \tilde{g}_{ij}(\infty)$ *when $\tilde{t} \to \infty$ uniformly, for the C^∞-topology, and the convergence is exponentially fast*;
(v) $\tilde{g}(\infty)$ *is a metric such that $(M, \tilde{g}(\infty))$ is a sphere.*

We will now give the change of time propositions.

PROPOSITION 3.3. *Let $\psi : [0, T_c[\to]0,\infty[$ be as above, $\tilde{t}$ defined by*

$$\tilde{t} : [0, T_c[\longrightarrow [0,\infty[, \qquad t \longmapsto \int_0^t \psi^2(\tau)\,d\tau, \tag{3.1}$$

for all $t \in [0, \infty[$, *define*

$$\tilde{g}(t) = \psi^2(\tilde{t}^{-1}(t)) g(\tilde{t}^{-1}(t)),$$

where $g(t)$ *is the family of metrics coming from a mean curvature flow, and* X_t *is a* $g(t)$-BM. *Then*

$$t \mapsto X_{\tilde{t}^{-1}(t)} \text{ is a } \tilde{g}(t)\text{-BM defined on } [0, \infty[.$$

PROOF. Let $f \in \mathcal{C}^\infty(M)$

$$\begin{aligned} f\big(X_{\tilde{t}^{-1}(t)}\big) &\overset{\mathcal{M}}{\equiv} \frac{1}{2}\int_0^{\tilde{t}^{-1}(t)} \Delta_{g(s)} f(X_s)\, ds \\ &\overset{\mathcal{M}}{\equiv} \frac{1}{2}\int_0^t \Delta_{g(\tilde{t}^{-1}(s))} f\big(X_{\tilde{t}^{-1}(s)}\big)(\tilde{t}^{-1})'(s)\, ds \\ &\overset{\mathcal{M}}{\equiv} \frac{1}{2}\int_0^t \Delta_{1/((\tilde{t}^{-1})'(s)) g(\tilde{t}^{-1}(s))} f\big(X_{\tilde{t}^{-1}(s)}\big)\, ds. \end{aligned}$$

Using

$$\psi^2(\tilde{t}^{-1}(s))(\tilde{t}^{-1})'(s) = 1,$$

we obtain

$$\frac{1}{(\tilde{t}^{-1})'(s)} g(\tilde{t}^{-1}(s)) = \tilde{g}(s). \qquad \square$$

PROPOSITION 3.4. *Let* $X_t^{T_c}$, *with* $t \in]0, T_c]$, *be a* $g(T_c - t)$-BM. *Let* τ *be defined by*

$$\begin{aligned} \tau :]0, T_c] &\longrightarrow]-\infty, 0], \\ t &\longmapsto -\tilde{t}(T - t). \end{aligned}$$

Let $\tilde{g}(t)$ *be defined by*

$$\tilde{g}(t) = \psi^2\big(T_c - \tau^{-1}(t)\big) g\big(T_c - \tau^{-1}(t)\big) \qquad \forall t \in]-\infty, 0].$$

Then

$$t \mapsto X^{T_c}_{\tau^{-1}(t)} \text{ is a } \tilde{g}(t)\text{-BM}.$$

PROOF. Let $f \in \mathcal{C}^\infty(M)$ and $s < t$,

$$\begin{aligned} f\big(X^{T_c}_{\tau^{-1}(t)}\big) - f\big(X^{T_c}_{\tau^{-1}(s)}\big) &\overset{\mathcal{M}}{\equiv} \frac{1}{2}\int_{\tau^{-1}(s)}^{\tau^{-1}(t)} \Delta_{g(T_c - u)} f(X_u^{T_c})\, du \\ &\overset{\mathcal{M}}{\equiv} \frac{1}{2}\int_s^t \Delta_{g(T_c - \tau^{-1}(u))} f\big(X^{T_c}_{\tau^{-1}(u)}\big)(\tau^{-1}(u))'(s)\, du \\ &\overset{\mathcal{M}}{\equiv} \frac{1}{2}\int_s^t \Delta_{1/(\tau^{-1})'(u) g(T_c - \tau^{-1}(u))} f\big(X^{T_c}_{\tau^{-1}(u)}\big)\, du. \end{aligned}$$

We have $-\tilde{t}(T_c - \tau^{-1}(u)) = u$, and

$$(\tau^{-1})'(u)\psi^2(T_c - \tau^{-1}(u)) = 1.$$

We obtain

$$f(X^{T_c}_{\tau^{-1}(t)}) - f(X^{T_c}_{\tau^{-1}(s)}) \overset{\mathcal{M}}{\equiv} \frac{1}{2}\int_s^t \Delta_{\psi^2(T_c-\tau^{-1}(u))g(T_c-\tau^{-1}(u))} f(X^{T_c}_{\tau^{-1}(u)})\,du,$$

that is,

$$f(X^{T_c}_{\tau^{-1}(t)}) - f(X^{T_c}_{\tau^{-1}(s)}) \overset{\mathcal{M}}{\equiv} \frac{1}{2}\int_s^t \Delta_{\tilde{g}(u)} f(X^{T_c}_{\tau^{-1}(u)})\,du. \qquad \square$$

REMARK 3.5. By Theorem 3.2, we know that $\tilde{g}(t)$ tends to a sphere metric as t goes to $-\infty$. The above proposition transforms "two" $g(T_c - t)$-BM into "two" $\tilde{g}$-BM. Thus we shall use the regularization of a metric into the sphere metric as well as the large time interval to perform the coupling.

Let τ_x be a plane in T_xM and $g(t)$ be a metric on M. We write $K(t, \tau_x)$ for the sectional curvature of the plane τ_x according to the metric $g(t)$. We will now give a few geometric lemmas that will be used later. For simplicity we will take positive times.

LEMMA 3.6. *Let $g(t)$ be a family of metrics on a manifold M, and $g(\infty)$ a metric that makes M into a sphere. Suppose that*:

(i) *$g(t) \longrightarrow g(\infty)$ uniformly, when $t \longrightarrow \infty$ for the C^∞-topology exponentially fast, that is, $\forall n \in \mathbb{N}, \forall$ multi-indices $(i_1, \dots, i_k)$ such that $\sum i_k = n$, $\exists C_n, \delta_n > 0$, such that*

$$\left|\frac{\partial^n}{\partial X_{i_1}\cdots X_{i_k}} g_{ij}(t) - \frac{\partial^n}{\partial X_{i_1}\cdots X_{i_k}} g_{ij}(\infty)\right| \le C_n e^{-\delta_n t};$$

(ii) *$\exists \delta, C^1 > 0$ such that $|\frac{\partial}{\partial t} g_{ij}(t)| \le C^1 e^{-\delta t}$*;

(iii) $\mathrm{vol}_{g(t)}(M) = \mathrm{vol}_{g(0)}(M)$.

Then, for all $\varepsilon > 0$, there exists $T \in [0, \infty[$, $\exists C$, cst, $\mathrm{cst}_1 \in \mathbb{R}^+$ and $c_n(\mathrm{cst}, V) > 0$ such that, $\forall t \in [T, \infty[$ the following conditions are satisfied:

(i) *for all x in M and for all planes $\tau_x \subset T_xM$, $|K(t, \tau_x) - \mathrm{cst}| \le \varepsilon$*;

(ii) $|\rho_t - \rho_\infty|_{M\times M} \le \mathrm{cst}_1 e^{-\delta t}$;

(iii) *$\rho'_t(x, y) := \frac{d}{dt}\rho_t(x, y) \le C$ in a compact CC of $M \times M$*,

where the constant cst *comes from the radius of M with respect to $g(\infty)$, $\rho_t(x, y)$ is the distance between x and y for the metric $g(t)$, and*

$$CC = \left\{(x, y) \in M \times M : \rho_t(x, y) \le \min\left(\frac{\pi}{2\sqrt{(\mathrm{cst} + \varepsilon)}}, \frac{c_n(\mathrm{cst}, V)}{2}\right), \forall t > T\right\}.$$

PROOF. Let us prove (i).

Curvatures are functions of second-order derivatives of the metric tensor. We give the definitions of curvatures tensors, to make this point clear. Conventions are as in [17, 18, 20], and in particular, we use Einstein's summation convention. For a metric connection without torsion (Levi–Cività connection), we recall the following standard definitions:

- the Christoffel symbols,

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}\left(\frac{\partial}{\partial x_i} g_{jl} + \frac{\partial}{\partial x_j} g_{il} - \frac{\partial}{\partial x_l} g_{ij}\right);$$

- the (3, 1) Riemann tensor,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z;$$

- the (4, 0) curvature tensor,

$$R_m(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle;$$

- the sectional curvature,

$$K(X,Y) = \frac{R_m(X,Y,Y,X)}{|X|^2|Y|^2 - \langle X,Y\rangle^2}.$$

We see that the sectional curvature depends on the metric and its derivatives up to order two, so that $\forall x \in M$, and for all planes $\tau_x \subset T_x M$,

$$\lim_{t\to\infty} K(t,\tau_x) = \mathrm{cst}.$$

Also, for all $\varepsilon > 0$, there exists T such that for all $t > T$, for all x in M and for all planes $\tau_x \subset T_x M$,

$$|K(t,\tau_x) - \mathrm{cst}| \le \varepsilon.$$

For the third point (iii): for $(x, y) \in CC$, where CC is defined above, we will show that we have uniqueness of minimal $g(t)$-geodesic from x to y, for all time $t > T$, because we have the well-known Klingenberg's result (e.g., [12], page 158) about the injectivity radius of a compact manifold whose sectional curvature is bounded above. To use Klingenberg's lemma, we have to bound the shortest length of a closed geodesic. We will use Cheeger's theorem ([3], page 96). By the convergence of the metric, we have the convergence of the Ricci curvature, and thus we obtain that they are bounded by the same constant. We obtain, using Myers's theorem that all diameters are bounded from above. The volumes are constant so bounded from below, all sectional curvatures of M are bounded in absolute value from above. By Cheeger's theorem there exists a constant $c_n(\mathrm{cst}, V) > 0$ that bounds the length of smooth closed geodesics. Hence, for large time, using Klingenberg's lemma, we get a bound from below, uniform in time for large time, of the injectivity radius $\min(\frac{\pi}{2\sqrt{(\mathrm{cst}+\varepsilon)}}, \frac{c_n(\mathrm{cst},V)}{2})$.

Hence for all $t > T$, there exists only one $g(t)$-geodesic between x and y, and we denote it by γ^t. Let $E(\gamma^t) = \int_0^1 \langle \dot{\gamma}^t(s), \dot{\gamma}^t(s)\rangle_{g(t)}\, ds$ be the energy of the geodesic where $\dot{\gamma}^t(s) = \frac{\partial}{\partial s}\gamma^t(s)$, $\rho_t^2(x, y) = E(\gamma^t)$. We compute

$$\begin{aligned}
2\biggl(\frac{\partial}{\partial t}\bigg|_{t=t_0} &\rho_t(x, y)\biggr)(\rho_t(x, y)) \\
&= \frac{\partial}{\partial t}\bigg|_{t=t_0} E(\gamma^t) \\
&= \int_0^1 \langle \dot{\gamma}^{t_0}(s), \dot{\gamma}^{t_0}(s)\rangle_{\partial/\partial t|_{t=t_0} g(t)}\, ds \\
&\quad + 2\int_0^1 \biggl\langle D_t|_{t=t_0} \frac{\partial}{\partial s}\gamma^t(s), \frac{\partial}{\partial s}\gamma^{t_0}(s)\biggr\rangle_{g(t_0)} ds \\
&= \int_0^1 \langle \dot{\gamma}^{t_0}(s), \dot{\gamma}^{t_0}(s)\rangle_{\partial/\partial t|_{t=t_0} g(t)}\, ds \\
&\quad + 2\int_0^1 \biggl\langle D_s \frac{\partial}{\partial t}\bigg|_{t=t_0} \gamma^t(s), \frac{\partial}{\partial s}\gamma^{t_0}(s)\biggr\rangle_{g(t_0)} ds.
\end{aligned}$$

Let $X = \frac{\partial}{\partial t}|_{t=t_0}\gamma^t(s)$ be a vector field such that $X(x) = 0_{T_xM}, X(y) = 0_{T_yM}$ because we do not change the starting and terminal point. The covariant derivative is computed with the Levi–Cività connection associated to $g(t_0)$. Hence we obtain

$$\int_0^1 \biggl\langle D_s \frac{\partial}{\partial t}\bigg|_{t=t_0} \gamma^t(s), \frac{\partial}{\partial s}\gamma^{t_0}(s)\biggr\rangle_{g(t_0)} ds = \int_0^1 \biggl\langle \nabla_{\dot{\gamma}^{t_0}(s)} X, \frac{\partial}{\partial s}\gamma^{t_0}(s)\biggr\rangle_{g(t_0)} ds,$$

and also

$$\biggl\langle \nabla_{\dot{\gamma}^{t_0}(s)} X, \frac{\partial}{\partial s}\gamma^{t_0}(s)\biggr\rangle_{g(t_0)} = \frac{\partial}{\partial s}\biggl\langle X, \frac{\partial}{\partial s}\gamma^{t_0}(s)\biggr\rangle_{g(t_0)},$$

because the connection is metric, and γ^{t_0} is a $g(t_0)$-geodesic. Hence

$$\int_0^1 \frac{\partial}{\partial s}\biggl\langle X, \frac{\partial}{\partial s}\gamma^{t_0}(s)\biggr\rangle_{g(t_0)} ds = \biggl[\biggl\langle X, \frac{\partial}{\partial s}\gamma^{t_0}(s)\biggr\rangle_{g(t_0)}\biggr]_0^1 = 0.$$

Finally, we obtain

$$\frac{\partial}{\partial t}\bigg|_{t=t_0} \rho_t(x, y) = \frac{1}{2\rho_{t_0}(x, y)} \int_0^1 \langle \dot{\gamma}^{t_0}(s), \dot{\gamma}^{t_0}(s)\rangle_{\partial/\partial t|_{t=t_0} g(t)}\, ds. \tag{3.2}$$

We will now control the second term in the previous equation. By the exponential convergence of the metric we can assume that the time is in the compact interval [0, 1]. The manifold is compact, so we have a finite family of charts (indeed, we may assume that we have two charts because the manifold has a metric which turns it into a sphere). The support of this chart could be taken to be relatively compact, and in this chart we can take the Euclidean metric, that is, $\langle \partial_i, \partial_j \rangle_E = \delta_i^j$. In general this is not a metric on M. For the sake of simplicity, after taking the minimum over all charts, we may assume that we just have one chart. Let S_1 be a sphere in $\mathbb{R}^n$ with the Euclidean metric. The functional

$$[0,1] \times S_1 \times M \longrightarrow \mathbb{R}, \qquad (t, v, x) \longmapsto g_{ij}(t,x) v_i v_j,$$

reaches its minimum $C > 0$. Hence

$$\|T\|_E \le C^{-1} \|T\|_{g(t)} \qquad \forall t \in [0,1], \forall T \in TM.$$

Hence for (3.2) we get the estimate

$$\begin{aligned}
\left| \frac{\partial}{\partial t}\bigg|_{t=t_0} \rho_t(x,y) \right| &\le \frac{1}{2\rho_{t_0}(x,y)} C^1 e^{-\delta t_0} \int_0^1 |\langle \dot\gamma^{t_0}(s), \dot\gamma^{t_0}(s) \rangle_E| \, ds \\
&\le \frac{1}{2\rho_{t_0}(x,y)} C^1 (C)^{-1} e^{-\delta t_0} \int_0^1 |\langle \dot\gamma^{t_0}(s), \dot\gamma^{t_0}(s) \rangle_{g(t_0)}| \, ds \\
&\le \frac{1}{2} C^1 (C)^{-1} e^{-\delta t_0}.
\end{aligned}$$

This expression is clearly bounded.

For the second point (ii), let $x, y \in M$ take γ_∞ be a $g(\infty)$-geodesic that joins x to y. Then we have, on the one hand,

$$\begin{aligned}
\rho_t^2(x,y) - \rho_\infty^2(x,y) &\le \int_0^1 \langle \dot\gamma_\infty(s), \dot\gamma_\infty(s) \rangle_{g(t)-g(\infty)} \, ds \\
&\le \mathrm{Cst}\, e^{-\delta t} \int_0^1 \|\dot\gamma_\infty(s)\|_{g(\infty)}^2 \, ds \\
&\le \mathrm{Cst}\, e^{-\delta t} \operatorname{diam}_{g(\infty)}^2(M),
\end{aligned}$$

where the constant changes and depends on the previous constant.

On the other hand, we have

$$\begin{aligned}
\rho_\infty^2(x,y) - \rho_t^2(x,y) &\le \int_0^1 \langle \dot\gamma^t(s), \dot\gamma^t(s) \rangle_{g(\infty)-g(t)} \, ds \\
&\le \mathrm{Cst}\, e^{-\delta t} \int_0^1 \|\dot\gamma^t(s)\|_{g(t)}^2 \, ds \\
&\le \mathrm{Cst}\, e^{-\delta t} \operatorname{diam}_{g(t)}^2(M) \\
&\le \mathrm{cst}_1\, e^{-\delta t},
\end{aligned}$$

for some constant cst_1. We use Myers's theorem for the last inequality to get a uniform upper bound of the diameter (since all Ricci curvatures are uniformly bounded). We get exponential convergence of the length. $\square$

We will now show uniqueness in law of a $g(T_c - t)$-BM. By Proposition 3.4, this uniqueness is equivalent to uniqueness in law of a $\tilde{g}(t)_{]-\infty,0]}$-BM. This family of metrics, $\tilde{g}(t)$, satisfies

$$\tilde{g}(t) \longrightarrow \tilde{g}(-\infty) \qquad \text{for the } C^{\infty}\text{-topology.}$$

Let Z^1, Z^2 be two $\tilde{g}$-$\mathrm{BM}_{]-\infty,0]}$ and $N \ll T$ where T is the time of the Lemma 3.6, that is, the time up to which all bounds of the lemma are under control. The geometry before this time is similar to the geometry of the sphere. So the result of uniqueness in law for Brownian motion defined in a product probability space, indexed by $\mathbb{R}$ in a compact manifold (e.g., [1, 6]) could give the heuristics to our results. As we can see in [4] the $g(t)$-stochastic development and the $g(t)$-horizontal lift of a $g(t)$-BM is well defined.

We shall consider a new process $Z^3_{N,t}$ equal in law to Z^2 after N and equal to Z^2 before. In the sequel we denote Z^3_t for $Z^3_{N,t}$. The construction, after time N, will be given by localization in a stochastic interval.

Let $T_0^N = N$, and for all $t \leq N$, $Z^3_{N,t} = Z^2_t$.

(1) Let Z^3_t evolve independently of Z^1_t, that is, Z^3_t is a $g(T_0^N + \cdot)$-BM which starts at $Z^3_{T_0^N}$ and the $\mathbb{R}^n$-valued Brownian motion that drives Z^3_t will be independent of the one that drives Z^1_t.

Let $T_1^N = (N + \frac{1}{2}) \wedge \inf\{t > T_0^N, \rho_t(Z^1_t, Z^3_t) \leq \frac{1}{4}(\frac{\pi}{\sqrt{\mathrm{cst}+\varepsilon}} \wedge c_n(\mathrm{cst}, V))\} \wedge T$. The constant ε is just taken to be small enough.

Let $C_N = \inf\{t > N, Z^1_t = Z^3_t\}$.

(2) At time T_1^N:

- if $\rho_{T_1^N}(Z^1_{T_1^N}, Z^3_{T_1^N}) \leq \frac{1}{4}(\frac{\pi}{\sqrt{\mathrm{cst}+\varepsilon}} \wedge c_n(\mathrm{cst}, V))$, these two points ($Z^3_{T_1^N}$ and $Z^1_{T_1^N}$) are close enough to make mirror coupling possible. The distance between these two points is strictly less than the injectivity radius $i_{g(t)}(M)$, and hence we have uniqueness of the geodesic that joins these two points. After T_1^N and before C_N, we build Z^3_t as the $g(T_1^N + \cdot)$-BM that starts at $Z^3_{T_1^N}$, and solves

$$*dZ^3_t = U^3_t * d((U^3_t)^{-1} m^t_{Z^1_t, Z^3_t} U^1_t e_i \, dW^i_t)$$

and after C_N,

$$Z^3_t = Z^1_t, \qquad C_N \leq t,$$

where U^3_t is the horizontal lift of Z^3_t. To be correct we have to write down a system of stochastic differential equations as in Kendall [19]: let U^1_t be the horizontal lift of Z^1_t and dW^i_t be the Brownian motions that drive Z^1_t. Then the mirror map $m^t_{x,y}$ consists of transporting a vector along the unique minimal $g(t)$-geodesic that joins x to y and then reflecting it about the hyperplane of $(T_y M, g(t))$ which is perpendicular to the incoming geodesic.

By the isometry property of the horizontal lift of the $g(t)$-BM (see [4]), we have that

$$(U_t^3)^{-1} m^t_{Z_t^1, Z_t^3} U_t^1 \, dW_t^i$$

is an $\mathbb{R}^n$-valued Brownian motion. Let

$$T_2^N = \left(T_1^N + \frac{1}{2}\right) \wedge \inf\left\{t > T_1^N, \rho_t(Z_t^1, Z_t^3) > \frac{\pi/\sqrt{\mathrm{cst} + \varepsilon} \wedge c_n(\mathrm{cst}, V)}{2}\right\}$$
$$\wedge T \wedge C_N.$$

- If $\rho_{T_1^N}(Z^1_{T_1^N}, Z^3_{T_1^N}) > \frac{1}{4}(\frac{\pi}{\sqrt{\mathrm{cst}+\varepsilon}} \wedge c_n(\mathrm{cst}, V))$, then $T_2^N = T_1^N$.

Iterate step 1 and 2 successively (changing T_0^N by T_2^N and T_1^N by T_3^N in step 1, changing T_1^N by T_3^N and T_2^N by T_4^N in step 2, ..., after time T if we have no coupling, we let Z^3 evolve independently of Z_t^1 until the end), we build by induction the process Z_t^3 and a sequence of stopping times. We sketch it as:

- if $C_N < T$,

$$T_0^N \xrightarrow{\text{independent}} T_1^N \xrightarrow{\text{coupling}} T_2^N \xrightarrow{\text{independent}} T_3^N \xrightarrow{\text{coupling}} T_4^N \cdots C_N \xrightarrow{Z_t^3 = Z_t^1} 0;$$

- if $C_N > T$,

$$T_0^N \xrightarrow{\text{independent}} T_1^N \xrightarrow{\text{coupling}} T_2^N \xrightarrow{\text{independent}} T_3^N \xrightarrow{\text{coupling}} T_4^N \cdots T \xrightarrow{\text{independent}} 0.$$

PROPOSITION 3.7. *The two processes Z^3 and Z^2 are equal in law.*

PROOF. It is clear that before N the two processes are equal, so they are equal in law. After N we argue as following:
$Z_N^3 = Z_N^2$.

$$\begin{cases} *dZ_t^3 = \sum_i U_t^3 e_i * dB^i, & \text{when } t \in [T_{2k}^N, T_{2k+1}^N \wedge C_N], \\ *dZ_t^3 = \sum_i U_t^3 * d((U_t^3)^{-1} m^t_{Z_t^1, Z_t^3} U_t^1) e_i \, dW_t^i, & \\ & \text{when } t \in [T_{2k+1}^N, T_{2k+2}^N \wedge C_N], \\ Z_t^3 = Z_t^1, & C_N \le t. \end{cases}$$

We write

$$*dZ_t^3 = \sum_{k=0}^{\infty} \mathbb{1}_{[T_k^N, T_{k+1}^N]}(t) * dZ_t^3 = \sum_{k:\,\mathrm{even}} \cdots + \sum_{k:\,\mathrm{odd}} \cdots.$$

Let $f \in C^\infty(M)$ then we have:

- for even k:

$$df\left(\mathbb{1}_{[T_k^N,T_{k+1}^N]}(t)*dZ_t^3\right)\overset{d\mathcal{M}}{\equiv}\tfrac{1}{2}\mathbb{1}_{[T_k^N,T_{k+1}^N]}(t)\Delta_{\tilde g(t)}f(Z_t^3)\,dt;$$

- for odd k:

$$df\left(\mathbb{1}_{[T_k^N,T_{k+1}^N]}(t)*dZ_t^3\right)\overset{d\mathcal{M}}{\equiv}\tfrac{1}{2}\mathbb{1}_{[T_k^N,T_{k+1}^N]}\Delta_{\tilde g(t)}f(Z_t^3)\,dt.$$

Hence Z^3 and Z^2 are two diffusions with the same starting distribution and the same generator; hence they are equal in law. For the gluing with Z^1 after C_N this is just the strong Markov property for (t,Z). $\square$

PROPOSITION 3.8. *There exists $\alpha>0$ such that*

$$\mathbb{P}\left(T_1^N-N<\tfrac{1}{2}\right)>\alpha.$$

PROOF. By the C^∞-convergence of the metric we get

$$\forall t<T\qquad \left|\Delta_{\tilde g(t)}f-\Delta_{\tilde g(-\infty)}f\right|\le \tilde C e^{\delta t},$$

where the constant comes from Theorem 3.2, and the derivative of f up to order two. We also obtain, by Lemma 3.6, for a constant ε_2 that will be fixed below:

$$|\rho_t-\rho_{-\infty}|\le\varepsilon_2.$$

Over the sphere $(M,\tilde g(-\infty))$, we have by the usual comparison theorem

$$\Delta_{\tilde g(-\infty)}\rho_{-\infty}(x)\le n\cot(\rho_{-\infty}(x)).$$

We can suppose after normalization that the radius of the sphere $(M,\tilde g(-\infty))$ is one, $\mathrm{Radius}_{-\infty}(M)=1$ (i.e., $\mathrm{cst}=1$) in Lemma 3.6. We deduce from above that

$$\Delta_{\tilde g(t)}\rho_{-\infty}(x)\le n\cot(\rho_{-\infty}(x))+\tilde C e^{\delta t}.$$

In $[N,T_1^N[$, we have $\rho_t(Z_t^1,Z_t^3)>\frac{1}{4}(\frac{\pi}{\sqrt{1+\varepsilon}}\wedge c_n(\mathrm{cst},V))$, so

$$\frac{1}{4}\left(\frac{\pi}{\sqrt{1+\varepsilon}}\wedge c_n(\mathrm{cst},V)\right)-\varepsilon_2\le\rho_t(Z_t^1,Z_t^3)-\varepsilon_2\le\rho_{-\infty}(Z_t^1,Z_t^3)\le\pi.$$

We can choose $\varepsilon,\varepsilon_2$ such that $\frac{1}{4}(\frac{\pi}{\sqrt{1+\varepsilon}}\wedge c_n(\mathrm{cst},V))-\varepsilon_2\ge\beta>0$. We obtain

$$\cot(\rho_{-\infty}(Z_t^1,Z_t^3))\le\cot(\beta)$$

and

$$\Delta_{\tilde g(t)}\rho_{-\infty}(Z_t^1,\cdot)(Z_t^3)\le n\cot(\beta)+\tilde C e^{\delta T},$$

(recall that $T\ll 0$). The increments of Z^3 and Z^1 are independent on $[N,T_1^N]$. Hence

$$(Z_t^1,Z_t^3)\text{ is a diffusion with generator }\tfrac{1}{2}\left(\Delta_{\tilde g(t),1}+\Delta_{\tilde g(t),2}\right),$$

that is,

$$d\rho_{-\infty}(Z_t^1, Z_t^3) = dM_t + \tfrac{1}{2}\big(\Delta_{\tilde{g}(t)}\rho_{-\infty}(Z_t^1, \cdot)(Z_t^3) + \Delta_{\tilde{g}(t)}\rho_{-\infty}(\cdot, Z_t^3)(Z_t^1)\big)\, dt,$$

where M_t is a local martingale, so

$$d\rho_{-\infty}(Z_t^1, Z_t^3) \le dM_t + \left(\cot\left(\frac{\pi}{8}\right) + \tilde{C}e^{\delta T}\right) dt.$$

Let us compute the quadratic variation of this local martingale,

$$d\langle M, M\rangle_t = d\rho_{-\infty}(Z_t^1, Z_t^3)\, d\rho_{-\infty}(Z_t^1, Z_t^3)$$

with

$$d\rho_{-\infty}(Z_t^1, Z_t^3) = d\rho_{-\infty}(Z_t^1, \cdot) * dZ_t^3 + d\rho_{-\infty}(\cdot, Z_t^3) * dZ_t^1. \tag{3.3}$$

Let $\gamma_{-\infty}(Z_t^3, Z_t^1)(s)$ be the minimal $\tilde{g}(-\infty)$-geodesic between Z_t^3 and Z_t^1 that exists and is unique almost everywhere because $\mathrm{Cut}_{-\infty}(M)$ is a null measure subspace. We write

$$v_t^1 = \frac{\dot{\gamma}_{-\infty}(Z_t^3, Z_t^1)(0)}{\|\dot{\gamma}_{-\infty}(Z_t^3, Z_t^1)(0)\|_{\tilde{g}(-\infty)}}.$$

We complete v_t^1 with v_t^j to get a $\tilde{g}(-\infty)$-orthonormal basis. We rewrite $*dZ_t^3$ as

$$*dZ_t^3 = \sum U_t^3 e_i * dB^i = \sum_{i,j} \langle U_t^3 e_i, v_t^j\rangle_{\tilde{g}(-\infty)} v_t^j * dB^i.$$

Hence by the Gauss lemma, we obtain

$$\begin{aligned} d\rho_{-\infty}(Z_t^1, \cdot) * dZ_t^3 &= \sum d\rho_{-\infty}(Z_t^1, \cdot) U_t^3 e_i * dB^i \\ &= \sum_{i,j} d\rho_{-\infty}(Z_t^1, \cdot)\langle U_t^3 e_i, v_t^j\rangle_{\tilde{g}(-\infty)} v_t^j * dB^i \\ &= \sum_{i} d\rho_{-\infty}(Z_t^1, \cdot)\langle U_t^3 e_i, v_t^1\rangle_{\tilde{g}(-\infty)} v_t^1 * dB^i \\ &= \sum_{i} \langle U_t^3 e_i, v_t^1\rangle_{\tilde{g}(-\infty)} * dB^i. \end{aligned}$$

It follows that

$$\big(d\rho_{-\infty}(Z_t^1, \cdot) * dZ_t^3\big)\big(d\rho_{-\infty}(Z_t^1, \cdot) * dZ_t^3\big) = \sum_i \langle U_t^3 e_i, v_t^1\rangle^2_{\tilde{g}(-\infty)}\, dt.$$

By the exponential convergence of the metric,

$$\langle U_t^3 e_i, v_t^1\rangle_{\tilde{g}(-\infty)} \ge \langle U_t^3 e_i, v_t^1\rangle_{\tilde{g}(t)} - \tilde{C}e^{\delta T},$$

hence

$$
\begin{aligned}
&\sum_i \langle U_t e_i, v_t^1\rangle_{\tilde g(-\infty)}^2 \\
&\quad \ge \sum_i \langle U_t e_i, v_t^1\rangle_{\tilde g(t)}^2 - 2\tilde C e^{\delta T} \sum_i \langle U_t e_i, v_t^1\rangle_{\tilde g(t)} + n(\tilde C e^{\delta T})^2 \\
&\quad = \|v_t^1\|_{\tilde g(t)}^2 - 2\tilde C e^{\delta T} \sum_i \langle U_t e_i, v_t^1\rangle_{\tilde g(t)} + n(\tilde C e^{\delta T})^2 \\
&\quad \ge \|v_t^1\|_{\tilde g(t)}^2 - 2\tilde C e^{\delta T} n \|v_t^1\|_{\tilde g(t)} + n(\tilde C e^{\delta T})^2 \qquad \text{Schwarz} \\
&\quad \ge \bigl(\|v_t^1\|_{\tilde g(-\infty)} - \tilde C e^{\delta T}\bigr)^2 - 2\tilde C e^{\delta T} n\bigl(\|v_t^1\|_{\tilde g(-\infty)} + \tilde C e^{\delta T}\bigr) \\
&\qquad + n(\tilde C e^{\delta T})^2 \\
&\quad \ge 1 - \tilde C e^{\delta T}\bigl(2 - \tilde C e^{\delta T} + 2(n + n\tilde C e^{\delta T}) - n\tilde C e^{\delta T}\bigr) \\
&\quad \ge \frac{1}{2} \qquad \text{for a small enough } T.
\end{aligned}
$$

The independence of Z_t^1 and Z_t^3 gives

$$
\begin{aligned}
d\langle M_t, M_t\rangle = {} & \bigl(d\rho_{-\infty}(Z_t^1, \cdot) * dZ_t^3\bigr)\bigl(d\rho_{-\infty}(Z_t^1, \cdot) * dZ_t^3\bigr) \\
& + \bigl(d\rho_{-\infty}(\cdot, Z_t^3) * dZ_t^1\bigr)\bigl(d\rho_{-\infty}(\cdot, Z_t^3) * dZ_t^1\bigr),
\end{aligned}
$$

and hence

$$
d\langle M_t, M_t\rangle \ge 1\, dt.
$$

For simplicity we write $\theta = \frac{1}{4}(\frac{\pi}{\sqrt{1+\varepsilon}} \wedge c_n(\mathrm{cst}, V))$. It follows from (3.3) that

$$
\begin{aligned}
&\mathbb{P}(T_1^N - N < 1/2) \\
&\quad = \mathbb{P}\bigl(\exists t \in [N, N+1/2] \text{ s.t. } \rho_t(Z_t^1, Z_t^3) \le \theta\bigr) \\
&\quad \ge \mathbb{P}\bigl(\exists t \in [N, N+1/2] \text{ s.t. } \rho_{-\infty}(Z_t^1, Z_t^3) \le \theta - \varepsilon_2\bigr) \\
&\quad \ge \mathbb{P}\bigl(\exists t \in [N, N+1/2] \text{ s.t. } \pi + M_t \\
&\qquad\qquad + \bigl(\cot(\beta) + \tilde C e^{\delta T}\bigr)(t - N) \le \theta - \varepsilon_2\bigr) \\
&\quad \ge \alpha > 0.
\end{aligned}
$$

For the last step, we use the usual comparison theorem for stochastic processes (e.g., Ikeda and Watanabe [15]). □

We will now show that the coupling can occur between $[T_1^N, T_2^N]$ in a time smaller than $1/2$.

PROPOSITION 3.9. *There exists* $\tilde{\alpha} > 0$ *such that*

$$\mathbb{P}\big(C_N < \big(T_1^N + \tfrac{1}{2}\big) \wedge T_2^N\big) > \tilde{\alpha}.$$

PROOF. Between the two times T_1^N and T_2^N, we have mirror coupling between Z_t^1 and Z_t^3. As in [5, 19] we have

$$d\rho_t(Z_t^1, Z_t^3) = \rho_t'(Z_t^1, Z_t^3)\,dt + 2\,d\beta_t + \frac{1}{2}\sum_{i=2}^{n} I^t(J_i^t, J_i^t)\,dt,$$

$$dZ_t^3 = U_t^3 * d((U_t^3)^{-1} m^t_{Z_t^1, Z_t^3} U_t^1 e_i\,dW_t^i),$$

where:

- β_t is a standard real Brownian motion;
- $\gamma_t(Z_t^1, Z_t^3)(s)$ the minimal $\tilde{g}(t)$ geodesic between Z_t^1 and Z_t^3;
- $(\dot{\gamma}(Z_t^1, Z_t^3)(0), e_i(t))$ a $\tilde{g}(t)$-orthonormal basis of $T_{Z_t^1}M$;
- $J_i^t(s)$ the Jacobi field along γ_t for the metric $\tilde{g}(t)$, with initial condition $J_i^t(0) = e_i(t)$ and $J_i^t(\rho_t(Z_t^1, Z_t^3)) = //^{t,\gamma_t}_{\rho_t(Z_t^1, Z_t^3)} e_i(t)$, that is, the parallel transport for the metric $\tilde{g}(t)$ along γ_t, which is an orthogonal Jacobi field;
- I^t is the index bilinear form for the metric $\tilde{g}(t)$.

Between the times T_1^N and T_2^N, we have

$$\rho_t(Z_t^1, Z_t^3) \le \frac{\pi/\sqrt{\mathrm{cst} + \varepsilon} \wedge c_n(\mathrm{cst}, V)}{2}.$$

Hence by Lemma 3.6, there exists a constant C such that

$$\rho_t'(x, y) \le C.$$

We have to show that between the times T_1^N and T_2^N,

$$\sum_{i=2}^{n} I^t(J_i^t, J_i^t)$$

is bounded from above. We denote $r = \rho_t(Z_t^1, Z_t^3)$, and γ for γ^t. Let $G(s)$ be a real-valued function and K_i^t be the orthogonal vector field over γ defined by

$$K_i^t(s) = G(s)(//_t^{\gamma_t} e_i(t))(s),$$

where $G(0) = G(r) = 1$. We have

$$\|\nabla^t_{\partial/\partial s} K_i^t(s)\|^2_{\tilde{g}(t)} = (\dot{G})^2.$$

By the index lemma (e.g., [20]), we deduce

$$I^t(J_i^t, J_i^t) \le I^t(K_i^t, K_i^t)$$

and

$$I^t(K_i^t, K_i^t) = \int_0^r \langle D_s K_i^t, D_s K_i^t\rangle_{\tilde g(t)} - R_{m,\tilde g(t)}(K_i^t, \dot\gamma, \dot\gamma, K_i^t)\, dt,$$

where $R_{m,\tilde g(t)}$ is the $(4, 0)$ curvature tensor associated to the metric $\tilde g(t)$. Hence

$$\begin{aligned}
\sum_{i=2}^n I^t(K_i^t, K_i^t) &= \sum_{i=2}^n \int_0^r \langle D_s K_i^t, D_s K_i^t\rangle_{\tilde g(t)} - R_{m,\tilde g(t)}(K_i^t, \dot\gamma, \dot\gamma, K_i^t)\, ds \\
&= \sum_{i=2}^n \int_0^r \|\nabla^t_{\partial/\partial s} K_i(s)\|^2_{\tilde g(t)} - R_{m,\tilde g(t)}(K_i^t, \dot\gamma, \dot\gamma, K_i^t)\, ds \\
&= \int_0^r (n-1)(\dot G)^2 - (G)^2 \operatorname{Ric}_{\tilde g(t)}(\dot\gamma, \dot\gamma)\, ds \\
&\le (n-1) \int_0^r \left((\dot G)^2 - (G)^2\left(\frac{1-\varepsilon}{n-1}\right)\right) ds.
\end{aligned}$$

For performing the computation, we impose on G to satisfy the ODE

$$\begin{cases} G(0) = G(r) = 1, \\ \ddot G + \left(\dfrac{1-\varepsilon}{n-1}\right) G = 0. \end{cases}$$

We notice that

$$(\dot G)^2 - (G)^2\left(\frac{1-\varepsilon}{n-1}\right) = (G\dot G)',$$

and the solution of this ODE is given by the function

$$G(s) = \cos\left(\sqrt{\frac{1-\varepsilon}{n-1}} s\right) + \frac{1 - \cos(\sqrt{(1-\varepsilon)/(n-1)}r)}{\sin(\sqrt{(1-\varepsilon)/(n-1)}r)} \sin\left(\sqrt{\frac{1-\varepsilon}{n-1}} s\right).$$

This function does not explode for r in $[0, \frac{\pi}{2\sqrt{(1-\varepsilon)/(n-1)}}]$, and

$$(\dot G)(r) - (\dot G)(0) = -2\sqrt{\frac{1-\varepsilon}{n-1}} \tan\left(\sqrt{\frac{1-\varepsilon}{n-1}} r/2\right).$$

Hence

$$\sum_{i=2}^n I^t(J_i^t, J_i^t) \le -2(n-1)\sqrt{\frac{1-\varepsilon}{n-1}} \tan\left(\sqrt{\frac{1-\varepsilon}{n-1}} r/2\right) \le 0.$$

We get

$$d\rho_t(Z_t^1, Z_t^3) \le C\, dt + 2\, d\beta_t.$$

After conditioning by $\mathcal{F}_{T_1^N}$ we get the following computation

$$\begin{aligned}
&\mathbb{P}\Big(C_N < \Big(T_1^N + \frac{1}{2}\Big) \wedge T_2^N\Big) \\
&\quad = \mathbb{P}\Big(\exists t \in \Big[T_1^N, \Big(T_1^N + \frac{1}{2}\Big) \wedge T_2^N\Big] \text{ such that } \rho_t(Z_t^1, Z_t^3) = 0\Big) \\
&\quad \geq \mathbb{P}\Big(\exists t \in \Big[0, \frac{1}{2}\Big] \text{ such that } Ct + 2\beta_t + \frac{\pi/\sqrt{1+\varepsilon} \wedge c_n(\mathrm{cst}, V)}{4} = 0 \\
&\qquad\qquad \text{and } \sup_{0 \le s \le t}\Big(Cs + 2\beta_s + \frac{\pi/\sqrt{1+\varepsilon} \wedge c_n(\mathrm{cst}, V)}{4}\Big) \\
&\qquad\qquad\qquad < \frac{\pi/\sqrt{1+\varepsilon} \wedge c_n(\mathrm{cst}, V)}{2}\Big) \\
&\quad \geq \tilde{\alpha} > 0.
\end{aligned}$$

□

REMARK 3.10. A better $\tilde{\alpha}$ could be found with a martingale of the type $e^{a\beta_t - a^2 t/2}$.

THEOREM 3.11. *Let* (M, g) *be a compact, strictly convex hypersurface isometrically embedded in* $\mathbb{R}^{n+1}$, $n \geq 2$, *and* $(M, g(t))$ *the family of metrics constructed by the mean curvature flow* (*as in Proposition* 1.5). *There exists a unique* $g(T_c - t)$-BM *in law.*

PROOF. Let X_t^1 and X_t^2 be two $g(T_c - t)$-BM, and by a deterministic change of time we get two $\tilde{g}(t)$-BM which we denote Z_t^1 and Z_t^2. Let $N \leq T \ll 0$. As above we build $Z_{N,t}^3$ and obtain $Z_{N,t}^3 = Z_t^2$ in law. Let $\tilde{k} = E(T - N)$ where $E(t)$ is the integer part of t. We have by construction

$$\mathbb{P}(\exists t \in [N, T],\ \text{s.t. } Z_{N,t}^3 = Z_t^1) \geq \mathbb{P}(\exists t \in [T_0^N, T_{2\tilde{k}}^N],\ \text{s.t. } Z_{N,t}^3 = Z_t^1).$$

Let $\mathcal{F}$ be the natural filtration generated by the two processes. By Propositions 3.8 and 3.9, along with the strong Markov property, we obtain

$$\begin{aligned}
&\mathbb{P}(\exists t \in [N, T_2^N] \text{ such that } Z_{N,t}^3 = Z_t^1) \\
&\quad \geq \mathbb{P}\big(T_1^N < \tfrac{1}{2} + N; C_N < \big(T_1^N + \tfrac{1}{2}\big) \wedge T_2^N\big) \\
&\quad = \mathbb{E}\big[\mathbb{P}\big(C_N \leq \big(T_1^N + \tfrac{1}{2}\big) \wedge T_2^N | \mathcal{F}_{T_1^N}\big)\mathbb{1}_{T_1^N \leq 1/2 + N}\big] \\
&\quad \geq \tilde{\alpha}\mathbb{E}[\mathbb{1}_{T_1^N \leq 1/2+N}] \\
&\quad \geq \alpha\tilde{\alpha} > 0.
\end{aligned}$$

By successive conditioning (by $\mathcal{F}_{T_{2\tilde{k}-2}}, \ldots$) we get

$$\mathbb{P}(\nexists t \in [T_0^N, T_{2\tilde{k}}^N] \text{ such that } Z_{N,t}^3 = Z_t^1) \leq (1 - \alpha\tilde{\alpha})^{\tilde{k}}.$$

Let $f_1, \dots, f_m \in \mathcal{B}_b(M)$ (bounded Borel functions) and $t < t_1 < \cdots < t_m \le 0$,

$$\begin{aligned}
&|\mathbb{E}[f_1(Z^1_{t_1}) \cdots f_m(Z^1_{t_m}) - f_1(Z^2_{t_1}) \cdots f_m(Z^2_{t_m})]| \\
&\quad = |\mathbb{E}[f_1(Z^1_{t_1}) \cdots f_m(Z^1_{t_m}) - f_1(Z^3_{N,t_1}) \cdots f_m(Z^3_{N,t_m})]| \\
&\quad \le \mathbb{E}[|f_1(Z^1_{t_1}) \cdots f_m(Z^1_{t_m}) - f_1(Z^3_{N,t_1}) \cdots f_m(Z^3_{N,t_m})| \mathbb{1}_{Z^1_t \neq Z^3_{N,t}}] \\
&\quad \le 2\|f_1\|_\infty \cdots \|f_m\|_\infty \mathbb{P}(Z^1_t \neq Z^3_{N,t}) \\
&\quad = 2\|f_1\|_\infty \cdots \|f_m\|_\infty \mathbb{P}(\nexists u \in [N,t] \text{ such that } Z^1_u = Z^3_{N,u}) \\
&\quad \le 2\|f\|_\infty \cdots \|f_m\|_\infty (1 - \alpha\tilde{\alpha})^{E(t-N)}.
\end{aligned}$$

We get the result by sending N to $-\infty$. $\square$

REMARK 3.12. We could use Hamilton's results in [13] as well as the same strategies developed before to show the uniqueness in law of a $g(T_c - t)$ Brownian motion, when the family of metrics $g(t)$ comes from a three-dimensional Ricci flow and under the assumption of positive Ricci curvature for the starting manifold.

As application we give uniqueness of a solution of a differential equation without initial condition.

COROLLARY 3.13. *Let (M, g) be a compact, strictly convex hypersurface isometrically embedded in $\mathbb{R}^{n+1}$, $n \ge 2$, and $(M, g(t))$ the family of metrics constructed by the mean curvature flow (as in Proposition 1.5). Then the following equation has a unique solution in $]0, T_c]$, where T_c is the explosion time of the mean curvature flow*:

$$\begin{cases} \dfrac{\partial}{\partial t} h(t,y) + H^2(T_c - t, y) h(t,y) = \dfrac{1}{2} \Delta_{g(T_c - t)} h(t,y), \\ \displaystyle\int_M h(T_c, y)\, d\mu_0 = 1. \end{cases} \tag{3.4}$$

PROOF. Existence: let $X^{T_c}_{]0,T_c]}$ be a $g(T_c - t)$-BM with law $h(t,y)\, d\mu_{T_c - t}$ at time t. Then the law satisfies (3.4); this is a consequence of a Green formula (compare with the similar computation for the Ricci flow in [4], Section 2).

Uniqueness: let $\tilde{h}$ be a solution of (3.4) and ν_k be a nonincreasing sequence in $]0, T_c]$ such that $\lim_{k\to\infty} \nu_k = 0$. Take an M-valued random variable $\tilde{X}^{\nu_k} \sim \tilde{h}_{\nu_k}\, d\mu_{T_c - \nu_k}$, and define the process

$$\overline{X}^{\nu_k}_t = \begin{cases} \tilde{X}^{\nu_k}, & \text{for } t \in]0, \nu_k], \\ g(T_c - t)\text{-BM}(\tilde{X}^{\nu_k}), & \text{for } t \in [\nu_k, T_c]. \end{cases}$$

By a similar argument as in Section 2, we deduce the tightness of the sequence $\overline{X}^{\nu_k}$; let $\overline{X}$ be a limit of a extracted sequence (also denoted by ν_k). It is easy to see (by the uniqueness of solutions of SDE, resp., PDE with initial function) that $\overline{X}^{\nu_{k'}}_{(\cdot)} \overset{\mathcal{L}}{=} \overline{X}^{\nu_k}_{(\cdot)}$ for times greater than ν_k and $k' \geq k$. Sending k' to infinity we obtain $\overline{X}_{(\cdot)} \overset{\mathcal{L}}{=} \overline{X}^{\nu_k}_{(\cdot)}$ for times greater than ν_k. Note also that for $t \geq \nu_k$,

$$\overline{X}^{\nu_k}_{(\cdot)} \overset{\mathcal{L}}{=} g(T_c - \cdot)\text{-BM}(\overline{X}^{\nu_k}_t) \overset{\mathcal{L}}{=} g(T_c - \cdot)\text{-BM}(\overline{X}_t).$$

Hence $\overline{X}$ is a $g(T_c - t)_{]0,T_c]}$ Brownian motion. For $t \geq \nu_k$ we have

$$\overline{X}_t \overset{\mathcal{L}}{=} \overline{X}^{\nu_k}_t \sim \tilde{h}_t \, d\mu_{T_c - t}.$$

By uniqueness in law of such processes we get uniqueness of the solution, hence $h = \tilde{h}$. $\square$

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REFERENCES

[1] ARNAUDON, M. (1999). Appendix to the preceding paper: "A remark on Tsirelson's stochastic differential equation" [in *Séminaire de Probabilités, XXXIII*, 291–303, Lecture Notes in Math., 1709, Springer, Berlin, 1999; MR1768002 (2001e:60111)] by M. Émery and W. Schachermayer. Natural filtration of Brownian motion indexed by **R** in a compact manifold. In *Séminaire de Probabilités, XXXIII*. *Lecture Notes in Math*. **1709** 304–314. Springer, Berlin. MR1768003

[2] ARNAUDON, M., COULIBALY, K. A. and THALMAIER, A. (2008). Brownian motion with respect to a metric depending on time: Definition, existence and applications to Ricci flow. *C. R. Math. Acad. Sci. Paris* **346** 773–778. MR2427080

[3] CHEEGER, J. and EBIN, D. G. (1975). *Comparison Theorems in Riemannian Geometry*. *North-Holland Mathematical Library* **9**. North-Holland, Amsterdam. MR0458335

[4] COULIBALY-PASQUIER, K. A. (2010). Brownian motion with respect to time-changing Riemannian metrics, applications to Ricci flow. *Ann. Inst. H. Poincaré* (*B*). To appear.

[5] CRANSTON, M. (1991). Gradient estimates on manifolds using coupling. *J. Funct. Anal.* **99** 110–124. MR1120916

[6] ÉMERY, M. and SCHACHERMAYER, W. (1999). Brownian filtrations are not stable under equivalent time-changes. In *Séminaire de Probabilités*, *XXXIII*. *Lecture Notes in Math*. **1709** 267–276. Springer, Berlin. MR1768000

[7] EVANS, L. C., SONER, H. M. and SOUGANIDIS, P. E. (1992). Phase transitions and generalized motion by mean curvature. *Comm. Pure Appl. Math.* **45** 1097–1123. MR1177477

[8] EVANS, L. C. and SPRUCK, J. (1992). Motion of level sets by mean curvature. II. *Trans. Amer. Math. Soc.* **330** 321–332. MR1068927

[9] EVANS, L. C. and SPRUCK, J. (1992). Motion of level sets by mean curvature. III. *J. Geom. Anal.* **2** 121–150. MR1151756

[10] EVANS, L. C. and SPRUCK, J. (1999). Motion of level sets by mean curvature. I [MR1100206 (92h:35097)]. In *Fundamental Contributions to the Continuum Theory of Evolving Phase Interfaces in Solids* 328–374. Springer, Berlin. MR1770903

[11] EVANS, L. C. and SPRUCK, J. (1995). Motion of level sets by mean curvature. IV. *J. Geom. Anal.* **5** 77–114. MR1315658

[12] GALLOT, S., HULIN, D. and LAFONTAINE, J. (2004). *Riemannian Geometry*, 3rd ed. Springer, Berlin. MR2088027

[13] HAMILTON, R. S. (1982). Three-manifolds with positive Ricci curvature. *J. Differential Geom.* **17** 255–306. MR664497

[14] HUISKEN, G. (1984). Flow by mean curvature of convex surfaces into spheres. *J. Differential Geom.* **20** 237–266. MR772132

[15] IKEDA, N. and WATANABE, S. (1989). *Stochastic Differential Equations and Diffusion Processes*, 2nd ed. *North-Holland Mathematical Library* **24**. North-Holland, Amsterdam. MR1011252

[16] JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin. MR1943877

[17] JOST, J. (1984). *Harmonic Mappings Between Riemannian Manifolds*. *Proceedings of the Centre for Mathematical Analysis*, *Australian National University* **4**. Australian National Univ. Centre for Mathematical Analysis, Canberra. MR756629

[18] JOST, J. (2005). *Riemannian Geometry and Geometric Analysis*, 4th ed. Springer, Berlin. MR2165400

[19] KENDALL, W. S. (1986). Nonnegative Ricci curvature and the Brownian coupling property. *Stochastics* **19** 111–129. MR864339

[20] LEE, J. M. (1997). *Riemannian Manifolds*: *An Introduction to Curvature*. *Graduate Texts in Mathematics* **176**. Springer, New York. MR1468735

[21] SONER, H. M. and TOUZI, N. (2003). A stochastic representation for mean curvature type geometric flows. *Ann. Probab.* **31** 1145–1165. MR1988466

[22] STROOCK, D. W. and VARADHAN, S. R. S. (2006). *Multidimensional Diffusion Processes*. Springer, Berlin. MR2190038

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