

## NONPARAMETRIC SEQUENTIAL PREDICTION FOR STATIONARY PROCESSES

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We study the problem of finding an universal estimation scheme  $h_n : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $n = 1, 2, \dots$  which will satisfy

$$\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=1}^t |h_i(X_0, X_1, \dots, X_{i-1}) - E(X_i | X_0, X_1, \dots, X_{i-1})|^p = 0 \quad \text{a.s.}$$

for all real valued stationary and ergodic processes that are in  $L^p$ . We will construct a single such scheme for all  $1 < p \leq \infty$ , and show that for  $p = 1$  mere integrability does not suffice but  $L \log^+ L$  does.

**1. Introduction.** The problem of sequentially predicting the next value  $X_n$  of a stationary process after observing the initial values  $X_i$  for  $0 \leq i < n$  is one of the central problems in probability and statistics. Usually, one bases the prediction on the conditional expectation  $E(X_n | X_0^{n-1})$  where we write for brevity  $X_0^{n-1} = \{X_0, X_1, \dots, X_{n-1}\}$ . However, when one does not know the distribution of the process one is faced with the problem of estimating the conditional expectation from a single sample of length  $n$ . It was shown long ago by Bailey [5] (cf. also Ryabko [30] and Györfi Morvai and Yakowitz [10]) that even for binary processes no universal scheme  $h_n(X_0^{n-1})$  exists which will almost surely satisfy  $\lim_{n \rightarrow \infty} (h_n(X_0^{n-1}) - E(X_n | X_0^{n-1})) = 0$ . This is in contrast to the backward estimation problem where one is trying to estimate  $E(X_0 | X_{-\infty}^{-1})$  based on the successive observations of  $X_{-\infty}^{-1}$ . Here, it was Ornstein [29] who constructed the first such universal estimator for finite valued processes. This was generalized to bounded processes by Algoet [1], Morvai [16] and Morvai Yakowitz and Györfi [18]. For unbounded processes, several universal estimators were constructed (see Algoet [3] and Györfi et al. [9]).

Returning to our original problem of sequential prediction it was already observed by Bailey that backward schemes could be used for the sequential prediction problem successfully in the sense that that the error tends to zero in the Cesàro

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mean. To establish this, he applied a generalized ergodic theorem which requires some technical hypotheses which were satisfied in his case.

Over the years some authors have extended this work, namely of adapting backward schemes to sequential prediction, but only for bounded processes (see Algoet [1, 3], Morvai [16], Morvai Yakowitz and Györfi [18] and Györfi et al. [9]).

Another approach to the sequential prediction used a weighted average of expert schemes, and with these results were extended to the general unbounded case by Nobel [28] and Ottucsak [12] (see also the survey of Feder and Merhav [8]). However, none of these results were optimal in the sense that moment conditions higher than necessary were assumed. It is our purpose to obtain these optimal conditions and to show why they are necessary. We consider the following problem for  $1 \leq p \leq \infty$ . Does there exist a scheme  $h_n(X_0^{n-1})$  which will satisfy

$$\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=1}^t |h_i(X_0^{i-1}) - E(X_i | X_0^{i-1})|^p = 0 \quad \text{a.s.}$$

for all real valued stationary and ergodic processes that are in  $L^p$ . The only case that has been solved completely is when  $p$  is infinity. Even the recent schemes Nobel [28] and Györfi and Ottucsak [12] put a higher moment condition on the process than is manifestly required. Our main result is that the basic scheme first introduced by the first author in his thesis can be adapted to give a scheme which will answer our problem positively for all  $1 < p$ . For  $p = 1$ , we shall show that stronger hypothesis is necessary, as is usually the case, and we will establish the convergence under the hypothesis that  $X_0 \in L \log^+ L$ .

In the third section, we will show how this hypothesis cannot be weakened to  $X_0 \in L^1$ . Our construction will be based on one of the simplest ergodic transformation, the adding machine, and illustrates the richness of behavior that is possible for processes that are almost periodic (in the sense of Besicovich).

As soon as one knows that the errors converge to zero in Cesáro mean, it follows that there is a set of density one of time moments along which the errors converge to zero. However, in general one does not know what this sequence is. In the framework of estimation, schemes adapted to a sequence of stopping times (see [19–21, 23–26]) one may ask can one find a sequence of stopping times with density one along which the errors of a universal sequential prediction scheme will tend to zero. We have been unable to do this in general and regard it as an important open problem. Finally, we refer the interested reader to some other papers which are relevant to this line of research [2, 11, 17, 27, 34].

Some technical probabilistic results have been relegated to the [Appendix](#), they are of a classical nature and may be known, but we were unable to find references.

**2. The main result.** Let  $\mathbf{X} = \{X_n\}$  denote a real-valued doubly infinite stationary ergodic time series. Let

$$X_i^j = (X_i, X_{i+1}, \dots, X_j)$$

be notation for a data segment, where  $i$  may be minus infinity. Let

$$\mathbf{X}^- = X_{-\infty}^{-1}.$$

Let  $G_k$  denote the quantizer

$$G_k(x) = \begin{cases} 0, & \text{if } -2^{-k} < x < 2^{-k}, \\ -i2^{-k}, & \text{if } -(i+1)2^{-k} < x \leq -i2^{-k} \text{ for some } i = 1, 2, \dots, \\ i2^{-k}, & \text{if } i2^{-k} \leq x < (i+1)2^{-k}. \end{cases}$$

Define the sequences  $\lambda_{k-1}$  and  $\tau_k$  recursively ( $k = 1, 2, \dots$ ). Put  $\lambda_0 = 1$  and let  $\tau_k$  be the time between the occurrence of the pattern

$$B(k) = (G_k(X_{-\lambda_{k-1}}), \dots, G_k(X_{-1})) = G_k(X_{-\lambda_{k-1}}^{-1})$$

at time  $-1$  and the last occurrence of the same pattern prior to time  $-1$ . More precisely, let

$$\tau_k = \min\{t > 0 : G_k(X_{-\lambda_{k-1}-t}^{-1-t}) = G_k(X_{-\lambda_{k-1}}^{-1})\}.$$

Put

$$\lambda_k = \tau_k + \lambda_{k-1}.$$

Define

$$(2.1) \quad R_k = \frac{1}{k} \sum_{1 \leq j \leq k} X_{-\tau_j}.$$

To obtain a fixed sample size  $t > 0$  version, let  $\kappa_t$  be the maximum of integers  $k$  for which  $\lambda_k \leq t$ . For  $t > 0$ , put

$$(2.2) \quad \hat{R}_{-t} = \frac{1}{\kappa_t} \sum_{1 \leq j \leq \kappa_t} X_{-\tau_j}.$$

Motivated by Bailey [5], for  $t > 0$  consider the estimator

$$\hat{R}_t(\omega) = \hat{R}_{-t}(T^t \omega),$$

which is defined in terms of  $(X_0, \dots, X_{t-1})$  in the same way as  $\hat{R}_{-t}(\omega)$  was defined in terms of  $(X_{-t}, \dots, X_{-1})$ . ( $T$  denotes the left shift operator.) The estimator  $\hat{R}_t$  may be viewed as an online predictor of  $X_t$ . This predictor has special significance not only because of potential applications, but additionally because Bailey [5] proved that it is impossible to construct estimators  $\hat{R}_t$  such that always  $\hat{R}_t - E(X_t | X_0^{t-1}) \rightarrow 0$  almost surely.

**THEOREM 1.** *Let  $\{X_n\}$  be stationary and ergodic. Assume that*

$$E(|X_0| \log^+(|X_0|)) < \infty.$$

Then

$$(2.3) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=1}^t |\hat{R}_i - E(X_i | X_0^{i-1})| = 0 \quad a.s.$$

and

$$(2.4) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=1}^t |\hat{R}_i - X_i| = E(|E(X_0 | X_{-\infty}^{-1}) - X_0|) \quad a.s.$$

Furthermore, if for some  $1 < p < \infty$ ,  $E(|X_0|^p) < \infty$ , then

$$(2.5) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=1}^t |\hat{R}_i - E(X_i | X_0^{i-1})|^p = 0 \quad a.s.$$

and

$$(2.6) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=1}^t |\hat{R}_i - X_i|^p = E(|E(X_0 | X_{-\infty}^{-1}) - X_0|^p) \quad a.s.$$

PROOF. The proof will follow the same pattern in all four cases. We will verify that the backward estimator scheme converges almost surely and we will see that the sequence of errors is dominated by an integrable function. This allows us to conclude from the generalized ergodic theorem of Maker (rediscovered by Breiman, cf. Theorem 1 in Maker [15] or Theorem 12 in Algoet [2]) that the forward scheme converges in Cesaro mean. For the first case, we will carry this out in full detail, for the others we will just check the requisite properties for the backward scheme. First, consider

$$\begin{aligned} R_k &= \frac{1}{k} \sum_{1 \leq j \leq k} [X_{-\tau_j} - G_j(X_{-\tau_j})] \\ &\quad + \frac{1}{k} \sum_{1 \leq j \leq k} [G_j(X_{-\tau_j}) - E(G_j(X_{-\tau_j}) | G_{j-1}(X_{-\lambda_{j-1}}^{-1}))] \\ &\quad + \frac{1}{k} \sum_{1 \leq j \leq k} [E(G_j(X_{-\tau_j}) | G_{j-1}(X_{-\lambda_{j-1}}^{-1})) - E(X_{-\tau_j} | G_{j-1}(X_{-\lambda_{j-1}}^{-1}))] \\ &\quad + \frac{1}{k} \sum_{1 \leq j \leq k} [E(X_{-\tau_j} | G_{j-1}(X_{-\lambda_{j-1}}^{-1})) - E(X_0 | G_{j-1}(X_{-\lambda_{j-1}}^{-1}))] \\ &\quad + \frac{1}{k} \sum_{1 \leq j \leq k} E(X_0 | G_{j-1}(X_{-\lambda_{j-1}}^{-1})) \\ &= A_k + B_k + C_k + D_k + E_k. \end{aligned}$$

Obviously,

$$|A_k| + |C_k| \leq \frac{2}{k} \sum_{1 \leq j \leq k} 2^{-j} \leq \frac{2}{k} \rightarrow 0.$$

Now we will deal with  $D_k$ . By Lemma 1, in Morvai, Yakowitz and Györfi [18],

$$P(X_{-\tau_j} \in C | G_{j-1}(X_{-\lambda_{j-1}}^{-1})) = P(X_0 \in C | G_{j-1}(X_{-\lambda_{j-1}}^{-1})).$$

Using this, we get that  $D_k = 0$ .

Assume that  $E(|X_0| \log^+(|X_0|)) < \infty$ . Toward mastering  $B_k$ , one observes that  $\{X_{-\tau_j}\}$  are identically distributed by Lemma 1 in Morvai, Yakowitz and Györfi [18] and  $B_k$  is an average of martingale differences. By Proposition 1 in the Appendix,  $|B_k| \rightarrow 0$  almost surely and  $E(\sup_{1 \leq k} |B_k|) < \infty$ .

Now we deal with the last term  $E_k$ . By assumption,

$$\sigma(G_j(X_{-\lambda_j}^{-1})) \uparrow \sigma(\mathbf{X}^-).$$

Consequently by the a.s. martingale convergence theorem, we have that

$$E(X_0 | G_j(X_{-\lambda_j}^{-1})) \rightarrow E(X_0 | \mathbf{X}^-) \quad \text{a.s.,}$$

and thus

$$E_k \rightarrow E(X_0 | \mathbf{X}^-) \quad \text{a.s.}$$

Furthermore, by Doob's inequality, cf. Theorem 1 on page 464, Section 3, Chapter VII in Shiryaev [32],  $E(\sup_{1 \leq k} |E_k|) \leq E(\sup_{1 \leq j} |E(X_0 | G_j(X_{-\lambda_j}^{-1}))|) < \infty$ .

We have so far proved that

$$R_k \rightarrow E(X_0 | \mathbf{X}^-) \quad \text{almost surely}$$

and

$$E\left(\sup_{1 \leq k} |R_k|\right) < \infty.$$

This in turn implies that

$$\lim_{t \rightarrow \infty} \hat{R}_{-t} = E(X_0 | \mathbf{X}^-) \quad \text{almost surely}$$

and

$$E\left(\sup_{1 \leq t} |\hat{R}_{-t}|\right) < \infty.$$

Now since  $E(X_0 | X_{-t}^{-1}) \rightarrow E(X_0 | \mathbf{X}^-)$  almost surely,

$$\lim_{t \rightarrow \infty} |\hat{R}_{-t} - E(X_0 | X_{-t}^{-1})| = 0 \quad \text{almost surely}$$

and by Doob's inequality,

$$\begin{aligned} E\left(\sup_{1 \leq t} |\hat{R}_{-t} - E(X_0|X_{-t}^{-1})|\right) &\leq E\left(\sup_{1 \leq t} |\hat{R}_{-t}|\right) \\ &\quad + E\left(\sup_{1 \leq t} |E(X_0|X_{-t}^{-1})|\right) \\ &< \infty. \end{aligned}$$

Now, apply the generalized ergodic theorem to conclude that

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=1}^t (|\hat{R}_{-i} - E(X_0|X_{-i}^{-1})|(T^i \omega)) &= \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=1}^t |\hat{R}_i - E(X_i|X_0^{i-1})| \\ &= 0 \quad \text{a.s.} \end{aligned}$$

and the proof of (2.3) is complete. Similarly,

$$\lim_{t \rightarrow \infty} |\hat{R}_{-t} - X_0| = |E(X_0|X_{-\infty}^{-1}) - X_0| \quad \text{almost surely}$$

and

$$E\left(\sup_{1 \leq t} |\hat{R}_{-t} - X_0|\right) \leq E\left(\sup_{1 \leq t} |\hat{R}_{-t}|\right) + E(|X_0|) < \infty$$

and the generalized ergodic theorem gives

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=1}^t (|\hat{R}_{-i} - X_0|(T^i \omega)) &= \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=1}^t |\hat{R}_i - X_i| \\ &= E(|E(X_0|X_{-\infty}^{-1}) - X_0|) \quad \text{a.s.} \end{aligned}$$

and the proof of (2.4) is complete.

Now, we assume that for some  $1 < p < \infty$ ,  $E(|X_0|^p) < \infty$ , and we prove (2.5). Observe that

$$|R_k|^p \leq 3^p \left[ \left(\frac{2}{k}\right)^p + |B_k|^p + |E_k|^p \right]$$

and since by Proposition 2 in the Appendix  $|B_k| \rightarrow 0$  almost surely and  $E(\sup_{1 \leq k} |B_k|^p) < \infty$  and by Doob's inequality,  $E(\sup_{1 \leq k} |E_k|^p) < \infty$  and  $E_k \rightarrow E(X_0|\mathbf{X}^-)$  almost surely (for the same reason as before).

We have so far proved that

$$R_k \rightarrow E(X_0|\mathbf{X}^-) \quad \text{almost surely}$$

and

$$E\left(\sup_{1 \leq k} |R_k|^p\right) < \infty.$$

This in turn implies that

$$\lim_{t \rightarrow \infty} \hat{R}_{-t} = E(X_0 | \mathbf{X}^-) \quad \text{almost surely}$$

and

$$E\left(\sup_{1 \leq t} |\hat{R}_{-t}|^p\right) < \infty.$$

Now since  $E(X_0 | X_{-t}^{-1}) \rightarrow E(X_0 | \mathbf{X}^-)$  almost surely,

$$\lim_{t \rightarrow \infty} |\hat{R}_{-t} - E(X_0 | X_{-t}^{-1})|^p = 0 \quad \text{almost surely}$$

and by Doob's inequality,

$$\begin{aligned} E\left(\sup_{1 \leq t} |\hat{R}_{-t} - E(X_0 | X_{-t}^{-1})|^p\right) &\leq 2^p E\left(\sup_{1 \leq t} |\hat{R}_{-t}|^p\right) \\ &\quad + 2^p E\left(\sup_{1 \leq t} |E(X_0 | X_{-t}^{-1})|^p\right) \\ &< \infty. \end{aligned}$$

By Maker's (or Breiman's) generalized ergodic theorem (cf. Theorem 1 in Maker [15] or Theorem 12 in Algoet [2]) one gets (2.5). Similarly,

$$\lim_{t \rightarrow \infty} |\hat{R}_{-t} - X_0|^p = |E(X_0 | X_{-\infty}^{-1}) - X_0|^p \quad \text{almost surely}$$

and

$$E\left(\sup_{1 \leq t} |\hat{R}_{-t} - X_0|^p\right) \leq 2^p E\left(\sup_{1 \leq t} |\hat{R}_{-t}|^p\right) + 2^p E(|X_0|^p) < \infty.$$

Now, apply Maker's (or Breiman's) generalized ergodic theorem to prove (2.6). The proof of Theorem 1 is complete.  $\square$

REMARK 1. We are indebted to the referee for the following remark. Using the notion of Bochner integrability of strongly measurable functions with values in  $c_0$  and the extension of Birkhoff's ergodic theorem to Banach space valued functions (see Krengel [14], page 167), one can give an easy proof of Maker's theorem. The key condition now becomes the fact that the norm of the sequence  $\{f - f_k\}$  in  $c_0$  is integrable, and then the convergence in the norm of  $c_0$  allows one to deduce the convergence of the diagonal sequence which is what appears in Maker's theorem.

**3. Integrability alone is not enough.** In Theorem 1 for the Cesàro convergence in the  $L^1$  norm, we assumed that  $X_0$  was not merely in  $L^1$  but in  $L^1 \log^+ L$ . In this section, we shall show that some additional condition is really necessary. We will first give an example to show that the maximal function of the conditional expectations  $\sup_{1 \leq n} |E(X_0 | X_{-n}^{-1})|$  may be nonintegrable for an integrable process.

We shall do so in an indirect fashion by showing that the estimate  $E(X_n|X_0^{n-1})$  for  $E(X_n|X_{-\infty}^{n-1})$  does not converge in Cesàro mean to zero. This means that even though we are may be in the distant future the information of the prehistory can make a serious difference. This example serves as a model for the main result of the section where we show that for any estimation scheme for  $E(X_n|X_0^{n-1})$  which converges almost surely in Cesàro mean for all bounded processes there will be some ergodic integrable process where it fails to converge. Indeed the processes that we need to consider are countably valued and in fact are zero entropy and finitarily Markovian (see below for a definition), a generalization of finite order Markov chains.

First, let us fix the notation. Let  $\{X_n\}_{n=-\infty}^{\infty}$  be a stationary and ergodic time series taking values from finite or countable alphabet  $\mathcal{X}$ . (Note that all stationary time series  $\{X_n\}_{n=0}^{\infty}$  can be thought to be a two sided time series, that is,  $\{X_n\}_{n=-\infty}^{\infty}$ .)

**DEFINITION 1.** The stationary time series  $\{X_n\}$  is said to be finitarily Markovian if almost surely the sequence of the conditional distributions  $\mathcal{L}(X_1|X_{-k}^0)$  is constant for large  $k$  (it is random how large  $k$  should be).

This class includes of course all finite order Markov chains but also many other processes such as the finitarily determined processes of Kalikow, Katznelson and Weiss [13], which serve to represent all isomorphism classes of zero entropy processes.

For some concrete examples that are not Markovian, consider the following example.

**EXAMPLE 1.** Let  $\{M_n\}$  be any stationary and ergodic first order Markov chain with finite or countably infinite state space  $S$ . Let  $s \in S$  be an arbitrary state with  $P(M_1 = s) > 0$ . Now let  $X_n = I_{\{M_n=s\}}$ . By Shields [31], Chapter I.2.c.1, the binary time series  $\{X_n\}$  is stationary and ergodic. It is also finitarily Markovian. Indeed, the conditional probability  $P(X_1 = 1|X_{-\infty}^0)$  does not depend on values beyond the first (going backward) occurrence of one in  $X_{-\infty}^0$  which identifies the first (going backward) occurrence of state  $s$  in the Markov chain  $\{M_n\}$ . The resulting time series  $\{X_n\}$  is not a Markov chain of any order in general.

We note that Morvai and Weiss [22] proved that there is no classification rule for discriminating the class of finitarily Markovian processes from other ergodic processes. For more about estimation for finitarily Markovian processes, see Morvai and Weiss [23, 24, 26].

**THEOREM 2.** Let  $\mathcal{X} = \{0, 10^{-k}, \frac{2^k}{3^m}, k = 1, 2, \dots, m = 1, 2, \dots\}$ . There exists a stationary and ergodic finitarily Markovian time series  $\{X_n\}$  taking values from



$\mathcal{X}$  such that  $E|X_0| < \infty$  and

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N |E(X_n|X_0^{n-1}) - E(X_n|X_{-\infty}^{n-1})| = \infty$$

almost surely. Therefore,

$$E\left(\sup_{1 \leq n} |E(X_0|X_n^{-1})|\right) = \infty.$$

PROOF. Let  $\Omega$  be the one sided sequence space over  $\{0, 1\}$ . Let  $\omega = (\omega_1, \omega_2, \dots) \in \Omega$ . Define the transformation  $T : \Omega \rightarrow \Omega$  as follows:

$$(T\omega)_i = \begin{cases} 0, & \text{if } \omega_j = 1 \text{ for all } j \leq i, \\ 1, & \text{if } \omega_i = 0 \text{ and for all } j < i : \omega_j = 1, \\ \omega_i, & \text{otherwise.} \end{cases}$$

Consider the product measure  $P = \prod_{i=1}^{\infty} \{1/2, 1/2\}$  on  $\Omega$  which is preserved by  $T$ . It is well known (cf. Aaronson [4], page 25) that  $(\Omega, P, T)$  is an ergodic process, called the adding machine or dyadic odometer. The process will be defined by a function  $f : \Omega \rightarrow \mathbb{R}$  as  $X_n(\omega) = f(T^n\omega)$ . Let  $l_3 < \dots < l_{k-1} < l_k \rightarrow \infty$ . Define  $a_k = a$  and  $b_k = b$  when  $k = 2^a + b$  where  $1 \leq b \leq 2^a$ . Define

$$C_k = \{\omega : \omega_i = 1 \text{ for } 1 \leq i < l_k, \omega_{l_k} = 0\},$$

clearly  $P(C_k) = 2^{-l_k}$ . Let

$$D_k = \{\omega : \omega_i = 1 \text{ for } 1 \leq i < l_k - a_k, \omega_{l_k - a_k} = 0, \omega_i = 1 \text{ for } l_k - a_k < i < l_k\}$$

and

$$E_k = \bigcup_{i=0}^{2^{l_k - a_k - 1} - 1} T^{-i} D_k.$$

Notice that

$$E_k = \{\omega : \omega_{l_k - a_k} = 0, \omega_j = 1, \text{ for all } l_k - a_k < j < l_k\}.$$

It is clear that if the  $l_k$ 's are chosen large enough so that for all  $k' > k$   $l_k < l_{k'} - 2a_{k'}$ :

- the family  $C_k, D_l$   $k, l \geq 3$  consists of disjoint sets,
- the intervals  $[l_k - a_k, l_k - 1]$  are also disjoint and therefore the sets  $E_k$  are independent.

The signaling function  $u$  is defined by

$$u(\omega) = \sum_{k=3}^{\infty} 10^{-k} I_{D_k}(\omega)$$

and the main contributor to  $f$  will be

$$v(\omega) = \sum_{k=3}^{\infty} \frac{2^{l_k}}{3^{a_k}} I_{C_k}(\omega).$$

Clearly,

$$\begin{aligned} E(v(\omega)) &= \sum_{k=3}^{\infty} \frac{2^{l_k}}{3^{a_k}} P(C_k) = \sum_{k=3}^{\infty} \frac{1}{3^{a_k}} = \sum_{a=1}^{\infty} \sum_{b=1}^{2^a} \frac{1}{3^a} \\ &= \sum_{a=1}^{\infty} \left(\frac{2}{3}\right)^a < \infty. \end{aligned}$$

Define a process by  $f(\omega) = u(\omega) + v(\omega)$  and

$$X_n(\omega) = f(T^n \omega).$$

Notice that  $X_n \in \{0, 10^{-k}, \frac{2^{l_k}}{3^{a_k}}, k = 3, 4, \dots\}$ . Observe that  $P(E_k) = 2^{-a_k}$  and

$$\sum_{k=3}^{\infty} P(E_k) = \sum_{a=1}^{\infty} \sum_{b=1}^{2^a} 2^{-a} = \sum_{a=1}^{\infty} 1 = \infty.$$

By the Borel–Cantelli lemma, a point  $\omega$  belongs to  $E_k$  infinitely often. When  $\omega \in E_k$ ,

$$T^{i_0} \omega \in D_k \quad \text{for some } 0 \leq i_0 \leq 2^{l_k - a_k - 1} - 1.$$

For  $\omega \in E_k$ , we know that  $X_{i_0}(\omega) = 10^{-k}$ . At time  $i_0 + 2^{l_k - a_k - 1} - 1$ ,

$$(T^{i_0 + 2^{l_k - a_k - 1} - 1}(\omega))_j = \begin{cases} 0, & \text{if } j = 1, \\ 1, & \text{if } 1 < j \leq l_k - 1, \\ \omega_j, & \text{otherwise.} \end{cases}$$

Let's compute for a fixed  $i_0$  such that  $T^{i_0} \omega \in D_k$  (i.e.,  $X_{i_0} = 10^{-k}$ )

$$E(X_{i_0 + 2^{l_k - a_k}} | X_0^{i_0 + 2^{l_k - a_k - 1}}).$$

Take  $N = 2^{l_k - a_k}$  and consider

$$\frac{1}{N} \sum_{n=1}^N |E(X_n | X_0^{n-1}) - E(X_n | X_{-\infty}^{n-1})|.$$

For  $\omega \in T^{-i_0} D_k$  (i.e.,  $X_{i_0} = 10^{-k}$ ), we know that

$$(T^{i_0 + 2^{l_k - a_k - 1}} \omega)_j = \begin{cases} 1, & \text{if } 1 \leq j \leq l_k - 1, \\ \omega_j, & \text{otherwise.} \end{cases}$$

Therefore if  $X_{i_0+2^{l_k-a_k-1}} > 0$ , then we must have

$$T^{i_0+2^{l_k-a_k-1}} \omega \in C_k \cup \bigcup_{j>k} (C_j \cup D_j)$$

(because if  $k' < k$  then  $l_{k'} < l_k$  and  $C_{k'}, D_{k'}$  are defined by zero values of  $\omega_i$  with  $i < l_k$ ) and

$$\begin{aligned} & E(X_{i_0+2^{l_k-a_k-1}} | X_0^{i_0+2^{l_k-a_k-1}-1}) \\ &= \frac{2^{l_k} / 3^{a_k} 2^{-l_k} + \sum_{j>k} 2^{l_j} / 3^{a_j} 2^{-l_j} + \sum_{j>k} 10^{-j} 2^{-l_j} + 0}{P(D_k)} \\ &\geq \frac{(2/3)^{a_k+1}}{2 \cdot 2^{-l_k}} \\ &= \frac{1}{2} 2^{l_k} \left(\frac{2}{3}\right)^{a_k+1}. \end{aligned}$$

Similarly,

$$\begin{aligned} & E(X_{i_0+2^{l_k-a_k}} | X_0^{i_0+2^{l_k-a_k}-1}) \\ &= \frac{2^{l_k} / 3^{a_k} 2^{-l_k} + \sum_{j>k} 2^{l_j} / 3^{a_j} 2^{-l_j} + \sum_{j>k} 10^{-j} 2^{-l_j} + 0}{P(D_k)} \\ &\leq \frac{10^{-k-1} + \sum_{i=0}^{\infty} (2/3)^{a_k+i}}{2 \cdot 2^{-l_k}} \\ &= \frac{1}{2} 2^{l_k} \left(10^{-k-1} + \left(\frac{2}{3}\right)^{a_k} 3\right) \\ &\leq 4 \cdot 2^{l_k} \left(\frac{2}{3}\right)^{a_k}. \end{aligned}$$

On the other hand,  $X_{-\infty}^{i_0+2^{l_k-a_k-1}-1}$  determines exactly the value of  $X_{i_0+2^{l_k-a_k-1}}$ . There are four cases. If  $X_{i_0+2^{l_k-a_k-1}}$  is equal with 0,  $\frac{2^{l_k}}{3^{a_k}}$ , or for some  $k < k' : 10^{-k'}$  or  $\frac{2^{l_{k'}}}{3^{a_{k'}}}$ . That is,

$$\begin{aligned} & E(X_{i_0+2^{l_k-a_k-1}} | X_{-\infty}^{i_0+2^{l_k-a_k-1}-1}) \\ &= \begin{cases} \frac{2^{l_k}}{3^{a_k}}, & \text{if } T^{i_0+2^{l_k-a_k-1}} \omega \in C_k, \\ 10^{-k'}, & \text{if } T^{i_0+2^{l_k-a_k-1}} \omega \in D_{k'} \text{ for some } k < k', \\ \frac{2^{l_{k'}}}{3^{a_{k'}}}, & \text{if } T^{i_0+2^{l_k-a_k-1}} \omega \in C_{k'} \text{ for some } k < k', \\ 0, & \text{if otherwise.} \end{cases} \end{aligned}$$

Now

$$|E(X_{i_0+2^{l_k-a_k-1}}|X_0^{i_0+2^{l_k-a_k-1}-1}) - E(X_{i_0+2^{l_k-a_k-1}}|X_{-\infty}^{i_0+2^{l_k-a_k-1}-1})|$$

$$\geq \begin{cases} 0.52^{l_k} \frac{2^{a_k}}{3^{a_k}}, & \text{if } T^{i_0+2^{l_k-a_k-1}} \omega \in C_k, \\ 10^{-k'}, & \text{if } T^{i_0+2^{l_k-a_k-1}} \omega \in D_{k'} \text{ for some } k < k', \\ 2^{l_k}, & \text{if } T^{i_0+2^{l_k-a_k-1}} \omega \in C_{k'} \text{ for some } k < k', \\ 0.52^{l_k} \left(\frac{2}{3}\right)^{a_k+1}, & \text{if otherwise,} \end{cases}$$

where we assumed that  $l_{k'} - 2a_{k'} > l_k$  if  $k' > k$ . Now

$$|E(X_{i_0+2^{l_k-a_k-1}}|X_{-\infty}^{i_0+2^{l_k-a_k-1}-1}) - E(X_{i_0+2^{l_k-a_k-1}}|X_{-\infty}^{i_0+2^{l_k-a_k-1}-1})|$$

$$\geq \frac{1}{4} 2^{l_k} \left(\frac{2}{3}\right)^{a_k+1}.$$

Therefore,

$$\frac{1}{N} \sum_{n=1}^N |E(X_n|X_0^{n-1}) - E(X_n|X_{-\infty}^{n-1})| \geq \frac{1}{N} \frac{1}{4} 2^{l_k} \left(\frac{2}{3}\right)^{a_k+1}$$

$$= 2^{-l_k+a_k} \frac{1}{4} 2^{l_k} \left(\frac{2}{3}\right)^{a_k+1}$$

$$= \frac{1}{6} \left(\frac{4}{3}\right)^{a_k}.$$

Since  $\limsup_{k \rightarrow \infty} a_k = \infty$ , the proof of Theorem 2 will be complete as soon as we verify that the process is ergodic and finitarily Markovian. The first property follows from the fact that  $T$  is an ergodic transformation. To see the second, what we need to do is to show that the values of  $f(T^{-n}\omega)$  will reveal to us more and more of the values of  $\omega_m$  as  $n$  increases. Almost every point is in infinitely many  $T^{2^{l_j-a_j}} E_j$ 's. For any such  $j$ , there is a unique  $i < 2^{l_j-a_j}$  such that  $T^{i-2^{l_j-a_j}} \omega \in D_j$  and this is revealed to us by the value of  $f$  at the point in the negative orbit of  $\omega$ . This information will give us the values of  $\omega_m$  for all  $m$  up to  $l_j - a_j$  and this completes the proof.  $\square$

REMARK 2. The referee pointed out that a simpler and equivalent formulation of the first statement of the theorem above is as follows.

Let  $\mathcal{X} = \{0, 10^{-k}, \frac{2^k}{3^m}, k = 1, 2, \dots, m = 1, 2, \dots\}$ . There exists a stationary and ergodic finitarily Markovian time series  $\{X_n\}$  taking values from  $\mathcal{X}$  such that  $E|X_0| < \infty$  and

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N |E(X_n|X_0^{n-1})| = \infty$$

almost surely.

[This is because  $E(X_n|X_{-\infty}^{n-1})(\omega) = E(X_0|X_{-\infty}^{-1})(T^n\omega)$  and by the ergodic theorem

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N E(X_n|X_{-\infty}^{n-1}) = E(X_0) < \infty$$

almost surely.]

**THEOREM 3.** *Let  $\mathcal{X} = \{0, 10^{-k}, \frac{2^k}{3^m}k = 1, 2, \dots, m = 1, 2, \dots\}$ . Suppose  $h_m : \mathcal{X}^m \rightarrow \mathbb{R}$  is a scheme that for any bounded ergodic finitarily Markovian process  $\{Y_n\}$  taking values from  $\mathcal{X}$ , almost surely satisfies*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N |E(Y_n|Y_0^{n-1}) - h_n(Y_0^{n-1})| = 0.$$

*Then there is an ergodic finitarily Markovian process  $\{X_n\}$  taking values from  $\mathcal{X}$  for which*

$$E|X_0| < \infty$$

and

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N |E(X_n|X_0^{n-1}) - h_n(X_0^{n-1})| = \infty$$

almost surely.

**PROOF.** I. *A Master process.* We shall prepare a master process with many possibilities for constructing a process such as in the earlier example with  $l_k$  in a fashion that will be dictated by the estimation scheme. For  $1 \leq j \leq n$ , define

$$q(n, j) = (n^2 + j)!$$

and sets

$$C_{q(n,j)} = \{\omega : \omega_i = 1 \text{ for } 1 \leq i < q(n, j), \omega_{q(n,j)} = 0\},$$

clearly  $P(C_{q(n,j)}) = 2^{-q(n,j)}$ . Let

$$D_{q(n,j)} = \{\omega : \omega_i = 1 \text{ for } 1 \leq i < q(n, j) - j, \\ \omega_{q(n,j)-j} = 0, \omega_i = 1 \text{ for } q(n, j) - j < i < q(n, j)\}$$

and

$$E_{q(n,j)} = \bigcup_{i=0}^{2^{q(n,j)-j-1}-1} T^{-i} D_{q(n,j)}.$$

Notice that

$$E_{q(n,j)} = \{\omega : \omega_{q(n,j)-j} = 0, \omega_i = 1, \text{ for all } q(n, j) - j < i < q(n, j)\}$$

and it follows that the sets  $\{E_{q(n,j)}, 1 \leq j \leq n, n \in \mathbf{N}\}$  are mutually independent.

Letting

$$u(\omega) = \sum_{n=1}^{\infty} \sum_{j=1}^n 10^{-q(n,j)} I_{D_{q(n,j)}}(\omega)$$

the master process is defined by  $Y_n(\omega) = u(T^n \omega)$ . For later use, observe that the  $D_{q(n,j)}$ 's are disjoint.

We will need the following easy consequence of our assumption on the estimators  $h_n$ , namely that for any bounded process  $Y_n$  defined on  $\Omega$  as in the theorem and for any  $k$  there is an integer  $N_k$  and a set  $H_k \subset \Omega$  with  $P(H_k) \geq 1 - 2^{-k}$  and for all  $\omega \in H_k$  and  $m \geq N_k$  we have:  $|h_m(Y_0, \dots, Y^{(k-1)})| \leq \frac{m}{10}$ .

II. *The construction.* We shall now define a sequence  $l_k, k = 2^{a_k} + b_k, 1 \leq b \leq 2^a$  inductively, together with functions  $v_k$  which are bounded. As  $k$  tends to infinity, the  $v_k$  will converge to  $v$  and we will use  $u + v$  to get our desired process. We may take  $v_2 = 0$  to start the inductive construction.

Assume that we have already defined  $l_3 < l_4 < \dots < l_{k-1}$  a subsequence of the  $q(n, j)$ 's and

$$v_{k-1} = \sum_{i=3}^{k-1} \left( \frac{2^{l_i}}{3^{a_i}} \right) I_{C_{l_i}}$$

we want to define  $l_k$  and  $v_k$ . Recalling the notation  $k - 1 = 2^{a_{k-1}} + b_{k-1}$ , we have that  $b_{k-1} = b_k - 1$  unless  $k - 1 = 2^a$ , in which case  $a_{k-1} = a - 1$  and  $b_{k-1} = 2^{a-1}$ .

Since  $v_{k-1}$  is bounded, the process defined by

$$X_n^{(k-1)} = f_{k-1}(T^n \omega) = u(T^n \omega) + v_{k-1}(T^n \omega)$$

is bounded. Now, by assumption, there is an  $N_k$  and a set  $H_k$  with  $P(H_k) \geq 1 - 2^{-k}$  and for all  $\omega \in H_k$  and  $m \geq N_k$  we know that

$$|h_m(X_0^{(k-1)}, \dots, X_{m-1}^{(k-1)})| \leq \frac{m}{10}.$$

Choose  $n$  large enough so that  $2^{q(n, a_k) - a_k} > 10N_k$  and we make sure that  $q(n, a_k) - a_k > 10l_{k-1}$ . Set

$$l_k = q(n, a_k)$$

and

$$v_k = v_{k-1} + \left( \frac{2^{l_k}}{3^{a_k}} \right) I_{C_{l_k}}.$$

This defines a new process

$$X_n^{(k)}(\omega) = f_k(T^n \omega) = u(T^n \omega) + v_k(T^n \omega).$$

It is important to observe that if for some  $i_0 \leq 2^{l_k - a_k - 1}$  we have  $T^{i_0} \omega \in D_{l_k}$  then for all  $0 \leq j \leq i_0 + 2^{l_k - a_k - 1} - 1$

$$X_j^{(k)}(\omega) = X_j^{(k-1)}(\omega).$$

This is because the way  $C_{l_k}$  is defined, we know that  $T^{i_0 + 2^{l_k - a_k - 1}} \omega$  can be in  $C_{l_k}$  which implies that earlier iterates of  $\omega$  cannot be there. Indeed,

$$C_{q(n,j)} \subset T^{2^{l_k - a_k - 1}} D_{l_k} \quad \text{for all } q(n, j) \geq l_k,$$

which implies that during all the later stages of the construction the values of  $X_i^{(k-1)}$  in this range will not change. So we will have for

$$v = \sum_{k=3}^{\infty} \left( \frac{2^{l_k}}{3^{a_k}} \right) I_{C_{l_k}}$$

and

$$X_n(\omega) = f(T^n \omega) = u(T^n \omega) + v(T^n \omega)$$

that

$$X_j(\omega) = X_j^{(k-1)}(\omega) \quad \text{for all } 0 \leq j \leq i_0 + 2^{l_k - a_k - 1} - 1,$$

if  $T^{i_0} \omega \in D_{l_k}$ .

It is clear that if the  $l_k$ 's are chosen large enough so that for all  $k' > k$   $l_k < l_{k'} - 2a_{k'}$ :

- the sets  $\{C_k, D_k\}_{k=3}^{\infty}$  are disjoint,
- the intervals  $[l_k - a_k, l_k - 1]$  are also disjoint and therefore the sets  $E_{l_k}$  are independent.

The signaling function  $u$  is bounded and the main contributor to  $f$  will be

$$v(\omega) = \sum_{k=3}^{\infty} \frac{2^{l_k}}{3^{a_k}} I_{C_{l_k}}(\omega).$$

Clearly,

$$E(v(\omega)) = \sum_{k=3}^{\infty} \frac{2^{l_k}}{3^{a_k}} P(C_{l_k}) = \sum_{k=3}^{\infty} \frac{1}{3^{a_k}} = \sum_{a=1}^{\infty} \sum_{b=1}^{2^a} \frac{1}{3^a} = \sum_{a=1}^{\infty} \left( \frac{2}{3} \right)^a < \infty.$$

Define a process by  $f(\omega) = u(\omega) + v(\omega)$  and

$$X_n(\omega) = f(T^n \omega).$$

Note that  $X_n \in \mathcal{X}$  as advertised.

III. *Checking the properties.* Observe that  $P(E_{l_k}) = 2^{-a_k}$  and

$$\sum_{k=3}^{\infty} P(E_{l_k}) = \sum_{a=1}^{\infty} \sum_{b=1}^{2^a} 2^{-a} = \sum_{a=1}^{\infty} 1 = \infty.$$

By the Borel–Cantelli lemma, a point  $\omega$  belongs to  $E_{l_k}$  infinitely often. In addition, since  $P(H_k) > 1 - 2^{-k}$ , almost every point will belong to  $H_k$  for all sufficiently large  $k$ . Suppose then that  $\omega \in E_{l_k} \cap H_k$ . When  $\omega \in E_{l_k}$ ,

$$T^{i_0} \omega \in D_k \quad \text{for some } 0 \leq i_0 \leq 2^{l_k - a_k - 1} - 1.$$

For  $\omega \in E_{l_k}$ , we know that  $X_{i_0}(\omega) = 10^{-l_k}$ . At time  $i_0 + 2^{l_k - a_k - 1} - 1$ ,

$$(T^{i_0 + 2^{l_k - a_k - 1} - 1}(\omega))_j = \begin{cases} 0, & \text{if } j = 1, \\ 1, & \text{if } 1 < j \leq l_k - 1, \\ \omega_j, & \text{otherwise.} \end{cases}$$

Let's compute for a fixed  $i_0$  such that  $T^{i_0} \omega \in D_{l_k}$  (i.e.,  $X_{i_0} = 10^{-l_k}$ )

$$E(X_{i_0 + 2^{l_k - a_k}} | X_0^{i_0 + 2^{l_k - a_k - 1}}).$$

For  $\omega \in T^{-i_0} D_{l_k}$  (i.e.,  $X_{i_0} = 10^{-l_k}$ ) we know that

$$(T^{i_0 + 2^{l_k - a_k - 1}} \omega)_j = \begin{cases} 1, & \text{if } 1 \leq j \leq l_k - 1, \\ \omega_j, & \text{otherwise.} \end{cases}$$

Therefore if  $X_{i_0 + 2^{l_k - a_k - 1}} > 0$ , then we must have

$$T^{i_0 + 2^{l_k - a_k - 1}} \omega \in C_{l_k} \cup \bigcup_{m > k} C_m \cup \bigcup_{1 \leq n, 1 \leq j \leq 2^n : q(n, j) > l_k} D_{q(n, j)},$$

because if  $k' < k$  then  $l_{k'} < l_k$  and the  $C_{k'}$ , are defined by zero values of  $\omega_i$  with  $i < l_k$ , and similarly for  $D_{q(n, j)}$  with  $q(n, j) < l_k$ ,

$$\begin{aligned} & E(X_{i_0 + 2^{l_k - a_k - 1}} | X_0^{i_0 + 2^{l_k - a_k - 1}}) \\ &= \frac{2^{l_k} / 3^{a_k} 2^{-l_k} + \sum_{j > k} 2^{l_j} / 3^{a_j} 2^{-l_j} + \sum_{q(n, j) > l_k} 10^{-q(n, j)} 2^{-q(n, j)} + 0}{P(D_{l_k})} \\ &\geq \frac{(2/3)^{a_k + 1}}{2 \cdot 2^{-l_k}} \\ &= \frac{1}{2} 2^{l_k} \left(\frac{2}{3}\right)^{a_k + 1}. \end{aligned}$$



Similarly,

$$\begin{aligned} & E(X_{i_0+2^{l_k-a_k}} | X_0^{i_0+2^{l_k-a_k}-1}) \\ &= \frac{2^{l_k}/3^{a_k}2^{-l_k} + \sum_{j>k} 2^{l_j}/3^{a_j}2^{-l_j} + \sum_{q(n,j)>l_k} 10^{-q(n,j)}2^{-q(n,j)} + 0}{P(D_{l_k})} \\ &\leq \frac{10^{-l_k} + \sum_{i=0}^\infty (2/3)^{a_k+i}}{2 \cdot 2^{-l_k}} \\ &= \frac{1}{2}2^{l_k} \left( 10^{-k-1} + \left(\frac{2}{3}\right)^{a_k} 3 \right) \\ &\leq 4 \cdot 2^{l_k} \left(\frac{2}{3}\right)^{a_k}. \end{aligned}$$

On the other hand, because  $\omega \in H_k$  and our remark about  $X_j = X_j^{(k-1)}$  for  $0 \leq j \leq 2^{l_k-a_k-1} - 1$ , we have that

$$|h_{i_0+2^{l_k-a_k-1}}(X_0^{i_0+2^{l_k-a_k-1}-1})| \leq \frac{i_0 + 2^{l_k-a_k-1} - 1}{10}.$$

Therefore, if we take  $N = 2^{l_k-a_k}$

$$\begin{aligned} \frac{1}{N} \sum_{n=1}^N |E(X_n | X_0^{n-1}) - h_n(X_0^{n-1})| &\geq \frac{1}{N} \frac{1}{4} 2^{l_k} \left(\frac{2}{3}\right)^{a_k+1} \\ &= 2^{-l_k+a_k} \frac{1}{4} 2^{l_k} \left(\frac{2}{3}\right)^{a_k+1} \\ &= \frac{1}{6} \left(\frac{4}{3}\right)^{a_k}. \end{aligned}$$

Since  $\limsup_{k \rightarrow \infty} a_k = \infty$ , the proof of Theorem 3 is complete.  $\square$

### APPENDIX

The next result is a generalization of a result due to Elton; cf. Theorems 2 and 4 in Elton [7].

**PROPOSITION 1.** *For  $n = 0, 1, 2, \dots$ , let  $\mathcal{F}_n$  be an increasing sequence of  $\sigma$ -fields, and  $X_n$  random variables measurable with respect to  $\mathcal{F}_n$ , be identically distributed with  $E(|X_0| \log^+(|X_0|)) < \infty$ . Let  $g_n(X_n)$  be quantizing functions so that for all  $n$ ,  $|g_n(X_n) - X_n| \leq 1$ , and for an increasing sequence of sub  $\sigma$ -fields,  $\mathcal{G}_n \subseteq \mathcal{F}_n$  such that  $g_n(X_n) = Y_n$  is measurable with respect to  $\mathcal{G}_n$ , form the sequence of martingale differences*

$$Z_n = g_n(X_n) - E(g_n(X_n) | \mathcal{G}_{n-1}) = Y_n - E(Y_n | \mathcal{G}_{n-1}).$$

Then

$$(A.1) \quad E \left( \sup_{1 \leq n} \left| \frac{1}{n} \sum_{i=1}^n Z_i \right| \right) < \infty$$

and

$$(A.2) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n Z_i = 0 \quad \text{almost surely.}$$

PROOF. We follow Elton [7], who gave the proof when the martingale differences  $Z_n$  are identically distributed. Write

$$Y_n = Y'_n + Y''_n,$$

where  $|Y'_n| \leq n$  and  $|Y''_n| > n$ . Now

$$Z_n = Y'_n - E(Y'_n | \mathcal{G}_{n-1}) + Y''_n - E(Y''_n | \mathcal{G}_{n-1}).$$

Since for any sequence of real numbers  $\{a_i\}$ ,

$$\sup_{1 \leq n} \left| \frac{1}{n} \sum_{i=1}^n a_i \right| \leq 2 \left( \sup_{1 \leq n} \left| \sum_{i=1}^n \frac{1}{i} a_i \right| \right),$$

(cf. Lemma 7 in Elton [7]), letting

$$d_n = Y'_n - E(Y'_n | \mathcal{G}_{n-1})$$

and

$$e_n = Y''_n - E(Y''_n | \mathcal{G}_{n-1})$$

we get

$$(A.3) \quad E \left( \sup_{1 \leq n} \left| \frac{1}{n} \sum_{i=1}^n Z_i \right| \right)$$

$$(A.4) \quad \leq 2E \left( \sup_{1 \leq n} \left| \sum_{i=1}^n \frac{1}{i} Z_i \right| \right)$$

$$(A.5) \quad \leq 2E \left( \sup_{1 \leq n} \left| \sum_{i=1}^n \frac{1}{i} d_i \right| \right)$$

$$(A.6) \quad + 2E \left( \sup_{1 \leq n} \left| \sum_{i=1}^n \frac{1}{i} e_i \right| \right).$$

For (A.5) by Davis' inequality (valid for all martingale differences cf. e.g., Shiryaev [32], page 470), we get

$$2E \left( \sup_{1 \leq n} \left| \sum_{i=1}^n \frac{1}{i} d_i \right| \right) \leq 2BE \left[ \left( \sum_{i=1}^{\infty} \frac{1}{i^2} (d_i)^2 \right)^{0.5} \right] \leq 2B \left[ E \left( \sum_{i=1}^{\infty} \frac{1}{i^2} (d_i)^2 \right) \right]^{0.5}.$$

Now,  $E((d_i)^2) \leq E((Y_i')^2)$ . But since  $|Y_i - X_i| \leq 1$ , we get

$$E((Y_i')^2) = E((Y_i)^2 I_{\{|Y_i| \leq i\}}) \leq E((X_i + 1)^2 I_{\{|X_i - 1| \leq i\}})$$

and the  $X_i$ 's are identically distributed therefore

$$\begin{aligned} & \sum_{i=1}^{\infty} \frac{1}{i^2} E((X_i + 1)^2 I_{\{|X_i - 1| \leq i\}}) \\ &= \sum_{i=1}^{\infty} \left( E((X_i + 1)^2 I_{\{|X_i - 1| \leq i\}}) \left( \sum_{j=i}^{\infty} \frac{1}{j^2} \right) \right) \\ &\leq K E(|X_0|), \end{aligned}$$

where  $K$  is a suitable constant (cf. the last line of the proof of Lemma 1 in Elton [7]).

For (A.6),

$$E|e_n| \leq 2E|Y_n''| \leq 2E((1 + |X_n|) I_{\{|X_n| > n-1\}})$$

and now  $X_n$ ' are identically distributed. Now since  $E(|X| \log^+(|X|)) < \infty$ , Lemma 2 in Elton [7] implies that

$$\sum_{n=1}^{\infty} \frac{1}{n} E((1 + |X_n|) I_{\{|X_n| > n-1\}}) < \infty$$

and so

$$\begin{aligned} 2E \left( \sup_{1 \leq n} \left| \sum_{i=1}^n \frac{1}{i} e_i \right| \right) &\leq 2 \sum_{i=1}^n \frac{1}{i} E|e_i| \\ &< \infty \end{aligned}$$

and this completes the proof of (A.1).

Now, we prove (A.2). By (A.4),

$$U_n = \sum_{i=1}^n \frac{1}{i} Z_i$$

is a martingale with

$$\sup_{1 \leq n} E(|U_n|) < \infty$$

and by Doob's convergence theorem  $U_n$  converges almost surely. Then by Kroecker's lemma (cf. Shirayev [32], page 365),

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n Z_i = 0$$

almost surely. The proof of Proposition 1 is complete.  $\square$

PROPOSITION 2. *Let  $\{\phi_n, \mathcal{F}_n\}$  be a martingale difference sequence. If, for some  $1 < p < \infty$ ,  $\sup_{1 \leq n} E(|\phi_n|^p) < \infty$  then*

$$(A.7) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \phi_i = 0 \quad \text{almost surely}$$

and

$$(A.8) \quad E \left( \sup_{1 \leq n} \left| \frac{1}{n} \sum_{i=1}^n \phi_i \right|^p \right) < \infty.$$

PROOF. Choose a positive integer  $K$  such that  $K(p - 1) > 1$ . Define

$$f_n = \frac{1}{n} \sum_{i=1}^n \phi_i.$$

Assume first that  $1 < p \leq 2$ . Now by Theorem 2 in von Bahr and Esseen [33],

$$(A.9) \quad E(|f_n|^p) \leq 2 \frac{n}{n^p} \sup_{1 \leq i} E(|\phi_i|^p) = 2 \frac{\sup_{1 \leq i} E(|\phi_i|^p)}{n^{(p-1)}}.$$

Define

$$F = \sum_{n=1}^{\infty} |f_{n^K}|^p.$$

By (A.9), and since by assumption  $\sup_{1 \leq n} E(|\phi_n|^p) < \infty$ ,  $K(p - 1) > 1$ ,

$$(A.10) \quad E(F) = 2 \sum_{n=1}^{\infty} \frac{\sup_{1 \leq i} E(|\phi_i|^p)}{n^{K(p-1)}} < \infty.$$

Define

$$g_n = \max_{1 \leq k < (n+1)^K - n^K} |f_{n^K} - f_{n^K+k}|^p$$

and let

$$G = \sum_{n=1}^{\infty} g_n.$$

To complete the proof of (A.7) and (A.8), it is enough to show that  $E(F + G) < \infty$ . By (A.10), it is enough to show that  $E(G) < \infty$ . Now for some  $m = n^K + k$ ,  $1 \leq k < (n + 1)^K - n^K$ ,

$$f_m = (f_{n^K+k} - f_{n^K}) + f_{n^K}$$

and

$$|f_m|^p \leq 2^p (|f_{n^K+k} - f_{n^K}|^p + |f_{n^K}|^p) \leq 2^p (g_n + |f_{n^K}|^p) \leq 2^p (G + F).$$

Now

$$|f_{n^{K+k}} - f_{n^K}| = \left(\frac{1}{n^K} - \frac{1}{n^{K+k}}\right) \sum_{i=1}^{n^K} \phi_i - \frac{1}{n^{K+k}} \sum_{j=1}^k \phi_{n^{K+j}}$$

and so

$$\begin{aligned} |f_{n^{K+k}} - f_{n^K}|^p &\leq 2^p \left( \left| \frac{k}{n^K(n^K+k)} \sum_{i=1}^{n^K} \phi_i \right|^p + \left| \frac{1}{n^{K+k}} \sum_{j=1}^k \phi_{n^{K+j}} \right|^p \right) \\ &\leq 2^p \left( \left| \frac{(n+1)^K - n^K - 1}{n^K n^K} \sum_{i=1}^{n^K} \phi_i \right|^p + \left| \frac{1}{n^K} \sum_{j=1}^k \phi_{n^{K+j}} \right|^p \right). \end{aligned}$$

Now

$$g_n \leq 2^p \left( \left| \frac{(n+1)^K - n^K}{n^K n^K} \sum_{i=1}^{n^K} \phi_i \right|^p + \left| \frac{1}{n^{Kp}} \max_{1 \leq k < (n+1)^K - n^K} \sum_{j=1}^k \phi_{n^{K+j}} \right|^p \right).$$

Now by von Bahr and Eseen [33] and Doob’s inequality (cf. e.g., Theorem 1, Chapter 3 in Shiriyayev [32]),

$$\begin{aligned} E(g_n) &\leq 2^p \left( \frac{(n+1)^K - n^K}{n^{2K}} \right)^p 2n^K \left( \sup_{1 \leq i} E(|\phi_i|^p) \right) \\ &\quad + \left( \frac{p}{(p-1)} \right)^p \frac{1}{n^K} E \left( \left| \sum_{j=1}^{(n+1)^K - n^K} \phi_{n^{K+j}} \right|^p \right) \\ &\leq 2^p \left( \frac{(n+1)^K - n^K}{n^{2K}} \right)^p 2n^K \left( \sup_{1 \leq i} E(|\phi_i|^p) \right) \\ &\quad + \left( \frac{p}{(p-1)} \right)^p \left( \frac{(n+1)^K - n^K}{n^K} \right)^p \sup_{1 \leq i} E(|\phi_i|^p) \end{aligned}$$

and the right-hand side is summable. We have completed the proof for  $1 < p \leq 2$ .

Now assume  $2 < p < \infty$ . By the theorem of Dharmadhikari, Fabian and Jogdeo [6],

$$E(|f_n|^p) \leq C(p) \frac{\sup_{1 \leq i} E(|\phi_i|^p)}{n^{p/2}}.$$

Applying this one gets that

$$E \left( \sum_{n=1}^{\infty} |f_n|^p \right) \leq \sum_{n=1}^{\infty} C(p) \frac{\sup_{1 \leq i} E(|\phi_i|^p)}{n^{p/2}} < \infty.$$

Thus,

$$\sum_{n=1}^{\infty} |f_n|^p < \infty \quad \text{almost surely}$$

and this yields (A.7) and (A.8). The proof of Proposition 2 is complete.  $\square$

REMARK 3. The referee pointed out that the second statement of the proposition above could be proved in a simpler way as follows. By maximal Doob inequality and Burkholder inequality, we obtain

$$\left( E \sup_n \left| \frac{1}{n} \sum_{i=1}^n \phi_i \right|^p \right)^{1/p} \leq 2p \max \left\{ 1, \frac{1}{(p-1)^2} \right\} \left[ E \left( \sum_{i=1}^{\infty} \left( \frac{\phi_i}{i} \right)^2 \right)^{p/2} \right]^{1/p}.$$

Now if  $p \geq 2$ , then by the triangle inequality in  $L_{p/2}$ ,

$$\begin{aligned} \left\{ \left[ E \left( \sum_{i=1}^{\infty} \left( \frac{\phi_i}{i} \right)^2 \right)^{p/2} \right]^{2/p} \right\}^{1/2} &\leq \left[ \sum_{i=1}^{\infty} \left( E \frac{|\phi_i|^p}{i^p} \right)^{2/p} \right]^{1/2} \\ &\leq \left( \sum_{i=1}^{\infty} \frac{1}{i^p} \right)^{1/2} \sup_i (E |\phi_i|^p)^{1/p}. \end{aligned}$$

If  $p \leq 2$ , then since

$$\left( \sum_i a_i \right)^{p/2} \leq \sum_i (a_i)^{p/2}$$

for all positive numbers  $a_i$  we get

$$\left[ E \left( \sum_{i=1}^{\infty} \left( \frac{\phi_i}{i} \right)^2 \right)^{p/2} \right]^{1/p} \leq \sum_{i=1}^{\infty} \left( E \frac{|\phi_i|^p}{i^p} \right)^{1/p} \leq \left( \sum_{i=1}^{\infty} \frac{1}{i^2} \right)^{1/2} \sup_i (E |\phi_i|^p)^{1/p}.$$

Thus, in each case it is

$$\left( E \sup_n \left| \frac{1}{n} \sum_{i=1}^n \phi_i \right|^p \right)^{1/p} \leq C_p \sup_i (E |\phi_i|^p)^{1/p}.$$

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