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CURRENT FLUCTUATIONS FOR TASEP: A PROOF OF THE PRÄHOFER–SPOHN CONJECTURE

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We consider the family of two-sided Bernoulli initial conditions for TASEP which, as the left and right densities (ρ_-, ρ_+) are varied, give rise to shock waves and rarefaction fans—the two phenomena which are typical to TASEP. We provide a proof of Conjecture 7.1 of [*Progr. Probab.* **51** (2002) 185–204] which characterizes the order of and scaling functions for the fluctuations of the height function of two-sided TASEP in terms of the two densities ρ_-, ρ_+ and the speed y around which the height is observed.

In proving this theorem for TASEP, we also prove a fluctuation theorem for a class of corner growth processes with external sources, or equivalently for the last passage time in a directed last passage percolation model with two-sided boundary conditions: ρ_- and $1-\rho_+$. We provide a complete characterization of the order of and the scaling functions for the fluctuations of this model's last passage time $L(N, M)$ as a function of three parameters: the two boundary/source rates ρ_- and $1-\rho_+$, and the scaling ratio $\gamma^2 = M/N$. The proof of this theorem draws on the results of [*Comm. Math. Phys.* **265** (2006) 1–44] and extensively on the work of [*Ann. Probab.* **33** (2005) 1643–1697] on finite rank perturbations of Wishart ensembles in random matrix theory.

1. Introduction and results. We study the fluctuations of the height function for the Totally Asymmetric Simple Exclusion Process (TASEP)—a stochastic process of great interest due to its wide applicability and mathematical accessibility. Under hydrodynamic scaling, this height function is the integrated solution to the deterministic Burgers equation [17]. This hydrodynamic limit is sensitive to the initial conditions of TASEP. It is of great interest to determine how the initial conditions of TASEP affect the random fluctuations of the height function. Ultimately, one would like to have a dictionary between initial conditions of TASEP and the resulting orders of the fluctuations of the height function, along with the scaling functions and correlation structures. This paper serves to lay some groundwork for understanding the phenomena which figure into this dictionary. The two phenomena which must be considered in TASEP are shocks and rarefaction fans. We

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study the simplest family of initial conditions which give rise to both of these phenomena. These initial conditions are simply Bernoulli independent at each site x, with density ρ_- for $x \le 0$ and ρ_+ for $x > 0$. We study the fluctuations of the height function or equivalently the current for these two-sided initial conditions. We solve an important conjecture of Prähofer and Spohn [22] (see also [14]). Understanding the fluctuation theory for two-sided TASEP provides the logical link between the well-developed theory for equilibrium initial conditions ($\rho_- = \rho_+$) [10, 14] and step initial conditions ($\rho_- = 1$, $\rho_+ = 0$) [16]. Two-sided TASEP interpolates between systems which are in equilibrium and systems which are entirely out of equilibrium. Our analysis shows how this interpolation occurs. The main result, Theorem 1.1, was first conjectured in [22] based on a scaling theory and analogous results for the PNG model and discrete TASEP [3]. Figure 1 illustrates the

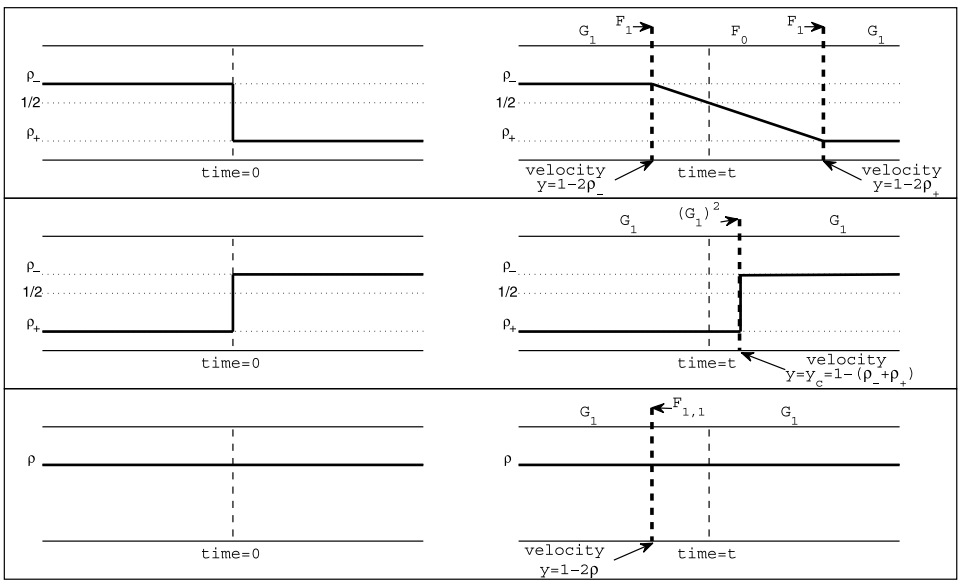

FIG. 1. *Depiction of three types of TASEP* (*particles move right*) *time evolution and identification of different regions of fluctuations for corresponding height functions. The top diagram depicts the phenomena of a rarefaction fan. The height function fluctuations seen by an observer moving at a speed so as to be*: *outside of the fan* (*outside the dashed lines*) *will be of order* $t^{1/2}$ *and Gaussian* (*denoted* G_1); *inside of the fan* (*inside the dashed lines*) *will be of order* $t^{1/3}$ *and Tracy–Widom* GUE (*denoted* F_0); *on the edge of the fan* (*on the dashed lines*) *will be of order* $t^{1/3}$ *and Tracy–Widom* GOE^2 (*denoted* F_1). *Likewise, the middle diagram depicts the phenomena of a moving shock. The height function fluctuations seen by an observer moving at a speed so as to be*: *on the shock* (*on the dashed line*) *will be of order* $t^{1/2}$ *and "Gaussian squared"* [*denoted* $(G_1)^2$]; *away from the shock* (*off the dashed line*) *will be of order* $t^{1/2}$ *and Gaussian* (*denoted* G_1). *The bottom diagram depicts equilibrium initial conditions. The height function fluctuations seen by an observer moving*: *at the critical speed* $y_c = 1 - 2\rho$ (*on the dashed line*) *will be of order* $t^{1/3}$ (*denoted* $F_{1,1}$ *and corresponding to what* [14] *call* F_0); *at all other speeds* (*off the dashed line*) *will be of order* $t^{1/2}$ *and Gaussian* (*denoted* G_1).

main result of this paper—it shows how the order of and scaling functions for the fluctuations of the height function for TASEP depend on the observation location with respect to shocks and rarefaction fans.

The proof of our main results makes use of the aforementioned result of [14] for the critical point (equilibrium $\rho_- = \rho_+ = \rho$ and $y = 1 - 2\rho$). For every other set of initial conditions, the proof relies on the main result of [2], a paper about the largest eigenvalue of finite rank perturbations of Wishart (sample covariance) random matrix ensembles. The connection between these two, seemingly disparate mathematical construction (TASEP and Wishart ensembles) is due to [16] and is facilitated through an intermediate random process known as directed last passage percolation (LPP). Our results about fluctuations of currents (or height functions) for TASEP follow from equivalent results for fluctuations of the last passage time for a directed last passage percolation model with two-sided boundary conditions (given in Theorem 1.3).

Returning to the model, TASEP is a Markov process η_t with state space $\eta_t \in \{0, 1\}^{\mathbb{Z}}$. For a given $t \in \mathbb{R}^+$ (time) and $x \in \mathbb{Z}$ (site location), we say that site x is occupied at time t if $\eta_t(x) = 1$ and it is empty if $\eta_t(x) = 0$. Given an initial configuration η_0 of particles, the TASEP evolves in continuous time as follows: each particle waits independent exponentially distributed times and then attempts to jump one site to its right; if there already exists another particle in the destination site, the particle does not move and its waiting time resets (see [17, 18] for a rigorous construction of this process). In equilibrium (or stationary) initial conditions (parametrized by a number $\rho \in [0, 1]$), the $\eta_0(x)$ are independent Bernoulli random variables with $P(\eta_0(x) = 1) = \rho$. In step initial conditions, $\eta_0(x) = 1$ for all $x \le 0$ and zero otherwise. Finally, in two-sided initial conditions (parametrized by a left density ρ_- and a right density ρ_+) $\eta_0(x)$ are independent Bernoulli random variables with $P(\eta_0(x) = 1) = \rho_-$ for $x \le 0$ and $P(\eta_0(x) = 1) = \rho_+$ for $x > 0$.

A natural and important quantity to study in TASEP is the current of particles past an observer moving with speed y. It is defined as $J_{yt,t}$ = number of particles to the left of the origin at time zero and to the right of yt at time t minus number of particles to the right of the origin at time zero and to the left of yt at time t. The current encodes the same information as the height function $h_t(j)$ [which we will define in (6)]:

$$J_{j,t} = \frac{h_t(j) - j}{2}. \tag{1}$$

For equilibrium TASEP with density ρ, the law of large numbers and central limit theorem [10] states that

$$\lim_{t\to\infty} \frac{J_{yt,t}}{t} = \rho(1-\rho) - y\rho \qquad \text{almost surely,} \tag{2}$$

$$\lim_{t\to\infty} \frac{J_{yt,t} - E(J_{yt,t})}{\sqrt{t}} = N(0, D_J), \tag{3}$$

where $N(0, D_J)$ is a normal with variance

$$D_J = \rho(1-\rho)|(1-2\rho) - y|. \tag{4}$$

For every velocity aside from $y = 1 - 2\rho$, current (and height function) fluctuations are Gaussian of order $t^{1/2}$. However, for a single critical velocity the central limit theorem of [10] is degenerate as the fluctuations are of a lower order than $t^{1/2}$. In terms of the hydrodynamic limit, this velocity corresponds to the slope of the characteristic line for Burgers equation. Heuristically this is the speed at which the initial condition fluctuations travel. Therefore, at any other speed, the current will depend on more initial conditions than just that localized to the origin—it is this that ensures the $t^{1/2}$ fluctuations and Gaussian scaling function for other velocities. At the critical speed, the initial environment's fluctuations are of lesser order, and only the dynamic fluctuations (those due to the actual TASEP process) are felt. These dynamic fluctuations are of central importance to understanding KPZ universality. At the critical speed $y = 1 - 2\rho$ the fluctuations are of order $t^{1/3}$ and converge, under suitable centering and scaling to a distribution function related to the Tracy–Widom GUE distribution [14, 22]. Rewriting expression (1.14) of [14] in terms of the current $J_{yt,t}$, with $y = 1 - 2\rho$, their $w = 0$, and $\chi = \rho(1-\rho)$, the result shows that

$$\lim_{t\to\infty} P\left(\frac{J_{yt,t} - \rho^2 t}{\chi^{2/3} t^{1/3}} \le x\right) = F_{1,1}(x; 0; 0), \tag{5}$$

where $F_{1,1}(x; 0; 0) = \frac{\partial}{\partial x}(F_0(s) g(x, 0))$. The $g(x, 0)$ is a scaling function given in their equation (1.18). See Section 1.2 for an overview of how our notation translates into the notation used in [14, 22].

In TASEP starting with step initial conditions, there are no fluctuations in the initial environment and consequently for every velocity $y \in (-1, 1)$ the current has fluctuations from the dynamics of order $t^{1/3}$ and with scaling function which corresponds to the Tracy–Widom GUE distribution [16] (which we write as F_0 so as to be in line with the notation of [2]). In terms of the hydrodynamic limit, the range of speeds $y \in (-1, 1)$ corresponds to the entire rarefaction fan, and the fluctuations are entirely due to the dynamics of TASEP. Ranges of speed bounded away from the fan correspond to regions which are, in the allotted time, unchanged by the dynamics of TASEP.

Drawing on the heuristics about the fluctuations along flat and fanned regions in the hydrodynamic limit, as well as based on a scaling theory and previous work of [3] for the PNG model, Prähofer and Spohn [22] conjectured that these two fluctuation theorems (for equilibrium and step initial conditions) arise as cases of a complete fluctuation theory for two-sided TASEP (see Figure 1). In their Conjecture 7.1, Prähofer and Spohn claimed that the critical point in [10] of $t^{1/3}$ fluctuations for equilibrium TASEP becomes a critical window (representing the region of the rarefaction fan) as ρ_- is increased and ρ_+ decreased. Ultimately, as $\rho_- = 1$

and $\rho_+ = 0$ the critical window of velocities equals the interval $(-1, 1)$ as showed in [16]. Likewise, they conjectured Gaussian behavior outside of this window, as well as in the case where $\rho_- < \rho_+$.

Previous to this paper, part of the Prähofer–Spohn conjecture had been proved via random matrix techniques in both the papers of Nagao and Sasamoto [20] and Baik, Ben Arous and Péché [2]. Both papers essentially dealt with the case of $\rho_+ = 0$ and any $\rho_- \in [0, 1]$. Our results are dependent on coupling arguments which allow us to bootstrap these boundary cases into every type of two-sided initial condition except for the critical equilibrium case (which is dealt with via the result of [14]). The methods of [25, 29] prove the part of the conjecture corresponding to the shock ($\rho_- < \rho_+$) by means of a microscopic Hopf–Lax–Oleinik formula. In these papers, the entire one time fluctuation process is characterized in the case of the shock. The scaling conjectured for the rarefaction fan was proved in [4] (in terms of the corresponding corner growth/LPP model discussed below), though the scaling functions were not addressed therein.

Beyond giving a complete proof of the Prähofer–Spohn conjecture, we believe that our coupling methods are very natural and provide a highly intuitive explanation for the transition between Gaussian and Tracy–Widom scalings. These methods are also useful in studying last passage percolation models with more general weights and more general boundary conditions. In proving the Prähofer–Spohn conjecture, one may alternatively follow the approach of [14] which is necessary in the critical case $\rho_- = \rho_+ = \rho$ and $y = 1 - 2\rho$. That argument is very strong and widely applicable. It is based on the idea of the Schur measure and involves a shift argument and a necessary analytic continuation argument. Coupling completely avoids these technical issues and replaces the complex analysis and asymptotic analysis with simple and intuitive probability. It also seems to be applicable in certain cases where the Schur measure argument cannot be applied.

Much effort has been devoted to understanding the analogous picture for ASEP, where particles may move to either the left of the right but are still subject to the exclusion rule. Progress in this direction was made in [9–13] in the early 1990s. The work of Baik and Rains [3], Prähofer and Spohn [23] and Imamura and Sasamoto [15] in the context of the closely related PNG model and last passage percolation with geometric weights was very important in formulating and understanding the theory of fluctuations. Very recently, due to the efforts of Tracy and Widom [30–34]), Derrida and Gerschenfeld [8], Balázs and Seppäläinen [5, 6], Quastel and Valkó [24], Mountford and Guiol [19] significant progress has been made in answering this question in the general ASEP. In particular, in [34], Tracy and Widom extend their step initial condition integrable system approach to ASEP with one-sided Bernoulli initial conditions. In that case, they observe the exact same fluctuation regimes as for TASEP. At present, the Prähofer–Spohn conjecture has not been proved for ASEP. It is tempting to try to use coupling methods

to extend the one-sided picture for ASEP to the two-sided initial condition case. It is not clear if this is possible, as ASEP is not related to a last passage percolation model and the coupling occurs at the level of such a model.

1.1. *Results*. The main result of this paper is a complete proof of [22] Conjecture 7.1—our Theorem 1.1.

Following [22], assign to a TASEP configuration $\eta_t(j)$ the height function

$$h_t(j) = \begin{cases} 2J_{0,t} + \sum_{i=1}^{j}(1-2\eta_t(i)), & j \geq 1, \\ 2J_{0,t}, & j = 0, \\ 2J_{0,t} - \sum_{i=j+1}^{0}(1-2\eta_t(i)), & j \leq -1. \end{cases} \tag{6}$$

Recall that $J_{0,t}$ is defined as the number of particles which have crossed the bond $(0,1)$ up to time t. For $|y| < 1$ denote

$$\lim_{t\to\infty} \frac{1}{t} h_t([yt]) = \bar{h}(y), \tag{7}$$

which exists almost surely due to the law of large numbers established via the hydrodynamic theory [26, 28]. This limit $\bar{h}$ depends not just on y, but also on ρ_- and ρ_+ as follows.

If $\rho_- < \rho_+$, then

$$\bar{h}(y) = \begin{cases} (1-2\rho_-)y + 2\rho_-(1-\rho_-), & \text{for } y \leq y_c, \\ (1-2\rho_+)y + 2\rho_+(1-\rho_+), & \text{for } y > y_c, \end{cases} \tag{8}$$

with $y_c = (\rho_+(1-\rho_+) - \rho_-(1-\rho_-))/(\rho_+ - \rho_-) = 1 - (\rho_- + \rho_+)$.

If $\rho_- > \rho_+$, then

$$\bar{h}(y) = \begin{cases} (1-2\rho_-)y + 2\rho_-(1-\rho_-), & \text{for } y \leq 1-2\rho_-, \\ \frac{1}{2}(y^2+1), & \text{for } 1-2\rho_- < y \leq 1-2\rho_+, \\ (1-2\rho_+)y + 2\rho_+(1-\rho_+), & \text{for } 1-2\rho_+ < y. \end{cases} \tag{9}$$

We prove the following (note that in parenthesis we record the distribution names as used in [22]). For an illustration of the results below, see Figure 1.

THEOREM 1.1 ([22], Conjecture 7.1).

(F_G). *Let either $\rho_- < \rho_+$, $y > y_c$ and $y < 1-\rho_+$, or $\rho_- > \rho_+$, $y > 1-2\rho_+$ and $y < 1-\rho_+$. Then*

$$\begin{aligned} &\lim_{t\to\infty} P\big(t\bar{h}(y) - h_t([yt]) \leq \big(4\rho_+(1-\rho_+)(y-1+2\rho_+)t\big)^{1/2}x\big) \\ &\qquad = G_1(x). \end{aligned} \tag{10}$$

Let either $\rho_- < \rho_+$, $y < y_c$ *and* $-\rho_- < y$, *or* $\rho_- > \rho_+$, $y < 1 - 2\rho_-$ *and* $-\rho_- < y$. *Then*

$$\begin{aligned}\lim_{t\to\infty} P\big(t\bar{h}(y) - h_t([yt]) \le \big(4\rho_-(1-\rho_-)(-y+1-2\rho_-)t\big)^{1/2}x\big) \\ = G_1(x).\end{aligned} \tag{11}$$

(F_G^2). *Let* $\rho_- < \rho_+$ *and* $y = y_c$, *then*

$$\begin{aligned}\lim_{t\to\infty} P\big(t\bar{h}(y) - h_t([yt]) \le \big((\rho_+ - \rho_-)t\big)^{1/2}x\big) \\ = G_1\big(\big(4\rho_+(1-\rho_+)\big)^{-1/2}x\big)G_1\big(\big(4\rho_-(1-\rho_-)\big)^{-1/2}x\big).\end{aligned} \tag{12}$$

(F_{GUE}). *Let* $\rho_- > \rho_+$ *and* $1 - 2\rho_- < y < 1 - 2\rho_+$. *Then*

$$\lim_{t\to\infty} P\big(t\bar{h}(y) - h_t([yt]) \le 2^{-1/3}(1-y^2)^{2/3}t^{1/3}x\big) = F_0(x). \tag{13}$$

(F_{GOE}^2). *Let* $\rho_- > \rho_+$ *and either* $y = 1 - 2\rho_-$ *or* $y = 1 - 2\rho_+$. *Then*

$$\lim_{t\to\infty} P\big(t\bar{h}(y) - h_t([yt]) \le 2^{-1/3}(1-y^2)^{2/3}t^{1/3}x\big) = F_1(x;0). \tag{14}$$

(F_0). *Let* $\rho_- = \rho = \rho_+$ *and* $y = 1 - 2\rho$. *Then*

$$\lim_{t\to\infty} P\big(t\bar{h}(y) - h_t([yt]) \le 2^{-1/3}(1-y^2)^{2/3}t^{1/3}x\big) = F_{1,1}(x;0;0). \tag{15}$$

REMARK 1.2. There is one difference between what we prove in Theorem 1.1 and what is stated in Conjecture 7.1 of [22] which is that for the case of the Gaussian scaling limit, by virtue of the fact that our proof goes by way of a mapping with directed last passage percolation, there are certain parts of the Gaussian region (with respect to y, ρ_-, ρ_+) for which our methods do not apply.

In this model, random weights $w_{i,j}$ are associated to each site (i, j) in the upper-right corner of $\mathbb{Z}^2$ (with $i \ge 0$ and $j \ge 0$). The weights are usually independent, and often exponential random variables, or geometric random variables. Every directed (up/right only) path π from $(0,0)$ to (N, M) then has weight $T(\pi) = \sum_{(i,j)\in\pi} w_{i,j}$, the sum of all weights along the path. The last passage time from $(0,0)$ to (N, M) is the maximum path weight over all directed paths:

$$L(N, M) = \max_{\pi:(0,0)\to(N,M)} T(\pi). \tag{16}$$

The statistics of $L(N, M)$ are dependent on the choice of distribution for the random weights, and in certain cases related to eigenvalue statistics for random matrices.

The current fluctuations for TASEP with step initial conditions were determined by identifying the height function with a corner growth model whose growth times

correspond with the last passage times for a specific LPP model with independent rate one exponential random weights $w_{i,j}$ for $i, j > 0$ and boundary weights $w_{i,j} = 0$, for $i = 0$ or $j = 0$ [16]. Theorem 1.6 of [16] shows that as $M, N \to \infty$ such that M/N is in a compact subset of $(0, \infty)$, $L(N, M)$ (as defined by the above weights) is approximated in distribution by

$$(\sqrt{M} + \sqrt{N})^2 + \frac{(\sqrt{M} + \sqrt{N})^{4/3}}{(MN)^{1/6}} \chi_0, \tag{17}$$

where χ_0 is distributed as a Tracy–Widom GUE distribution. The first term gives the asymptotic average for $L(N, M)$ and the second term shows that the fluctuations scale like $M^{1/3}$ and have a well understood scaling function. Via the height functions mapping, these results translate back into the current fluctuations for TASEP with step initial conditions. The corner growth model height function is exactly the random interface bounding the growth region.

This theorem was proved by using the tools of generalized permutations, the RSK correspondence and Young Tableaux, to relate the distribution of the last passage time to the distribution of the largest eigenvalue of a Wishart ensemble, whose statistics are known to follow the Tracy–Widom GUE distribution [16].

By analogy, our method of proof is to first relate two-sided TASEP to a LPP model, which we appropriately call *LPP with two-sided boundary conditions* [see (18) for a definition], and then to relate the statistics of the last passage time for that model to the statistics of eigenvalues of already studied random matrices. The first mapping is already found in [22] and relies on Burke's theorem (we review this mapping in Section 3). LPP with two-sided boundary conditions is not directly connected to a random matrix ensemble, however we can realize its last passage time as the maximum of last passage time for a pair of coupled *LPP with one-sided boundary conditions* [see (26) for a definition]. The last passage time in such one-sided LPP models is related (see Section 6 of [2]) to the largest eigenvalue of Wishart matrices with finite rank perturbations. In fact, the phase transitions, with respect to the magnitude of the finite perturbation, which are discussed in [2] correspond exactly to the transitions between different orders of and scaling functions for the height function of two-sided TASEP. By a set of coupling arguments, and using the results of [2] and [14] we provide a proof of Theorem 1.1. In proving Theorem 1.1, we reprove the fluctuation results for step initial conditions as well as for equilibrium initial conditions (except at the critical point). We also show that these two results arise from a much more complete picture (see Figure 1 for an illustration of this).

As noted above, the proof of this theorem relies on understanding the fluctuations of the last passage time in a LPP model with two-sided boundary conditions. The specific *LPP with two-sided boundary conditions* which we will devote much of this paper to studying has three different types of independent exponential

weights $w_{i,j}$:

$$w_{i,j} = \begin{cases} \text{exponential of rate } \pi, & \text{if } i > 0,\ j = 0; \\ \text{exponential of rate } \eta, & \text{if } i = 0,\ j > 0; \\ \text{exponential of rate } 1, & \text{if } i > 0,\ j > 0; \\ \text{zero}, & \text{if } i = 0,\ j = 0. \end{cases} \tag{18}$$

In the later part of Section 2, we will allow for more general boundary condition where a finite number of columns and rows can have different (though uniform within the column or row) rates. The LPP with one-sided boundary conditions is defined similarly using the weights in (26). At this point, it is worth remarking that changing the distribution of a finite number of weights does not have any affect on the asymptotic fluctuations of the last passage time (see Lemma 3.1).

The statistics considered in this paper are the last passage times $L_2(N, M)$ (we use a subscript 2 to denote two-sided), from $(0, 0)$ to (N, M), as N and M go to infinity together such that $M/N = \gamma^2$. Note that N denotes the number of columns and M the number of rows. Such statistics can be parametrized in terms of the two boundary condition rates π and η, as well as the scaling parameter γ. It is worth keeping in mind that the boundary rates π and η correspond with the TASEP densities ρ_- and $1 - \rho_+$, and the scaling parameter γ corresponds (in a slightly more complicated way) with the TASEP velocity y.

With this connection in mind, we completely characterize both the order and the scaling functions for the fluctuations of the last passage time of LPP with two-sided boundary conditions in terms of the three parameters π, η and γ. As noted before, the main result we appeal to in this paper is from [2] (extended to the case $\gamma < 1$ in [21]) which classifies the fluctuations of the largest eigenvalue of complex Wishart ensembles with finite rank perturbations. There is a single critical point which does not yield to our method of argument, but this corresponds exactly with the critical point considered in [14]. Using these two results and coupling arguments, we prove our LPP with two-sided boundary conditions classification theorem (see Figures 2–4).

THEOREM 1.3.

(1) *For $\gamma \in (0, \infty)$ and $M/N \to \gamma^2$, then for π, η such that $\pi > \frac{1}{1+\gamma}$ and $\eta > \frac{\gamma}{1+\gamma}$ (the* GUE *region)*

$$P\left(L_2(N, M) \le (1 + \gamma^{-1})^2 M + \frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} x\right) \to F_0(x), \tag{19}$$

where $F_0(x)$ is the Tracy–Widom GUE *distribution function.*

(2) *For $\gamma \in (0, \infty)$ and $M/N \to \gamma^2$, then for π, η such that $\pi > \frac{1}{1+\gamma}$ and $\eta = \frac{\gamma}{1+\gamma}$ or $\pi = \frac{1}{1+\gamma}$ and $\eta > \frac{\gamma}{1+\gamma}$ (the* GOE^2 *region),*

$$P\left(L_2(N, M) \le (1 + \gamma^{-1})^2 M + \frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} x\right) \to F_1(x), \tag{20}$$

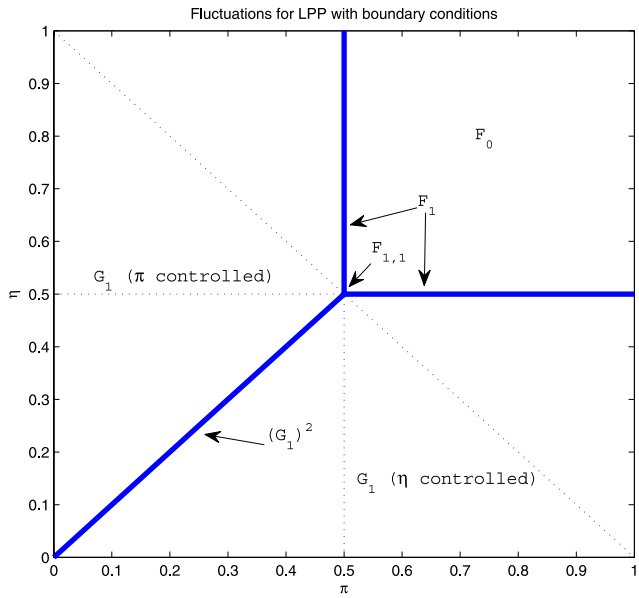

FIG. 2. *Fluctuation diagram for $\gamma = 1$. Note that all G_1 and $(G_1)^2$ regions have $M^{1/2}$ order fluctuations while all other regions have $M^{1/3}$ order fluctuations.*

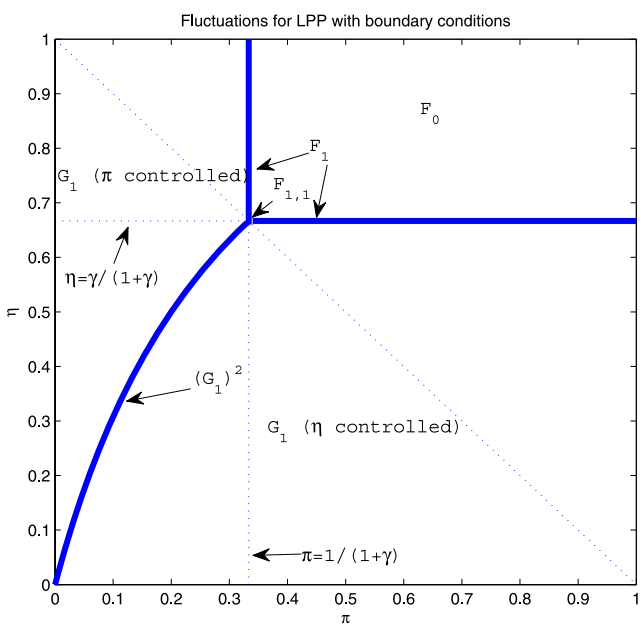

FIG. 3. *Fluctuation diagram for $\gamma = 2$. Compared with Figure 2, the effect of changing γ is that the region of $M^{1/3}$ fluctuations has shifted up and to the left along the anti-diagonal.*

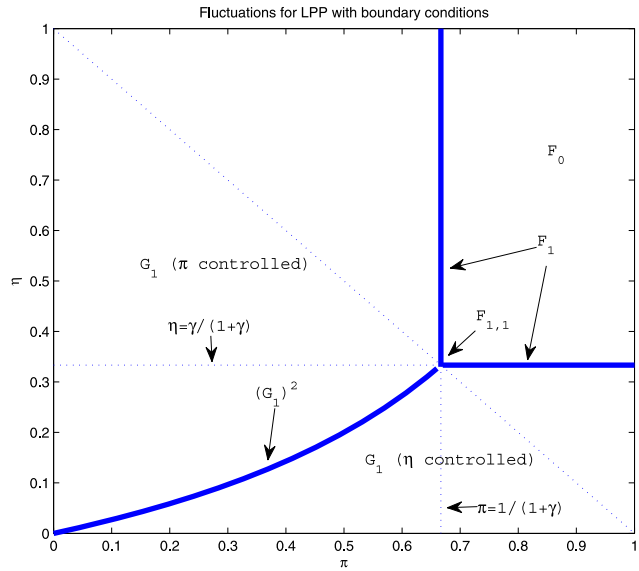

FIG. 4. *Fluctuation diagram for $\gamma = 0.5$. Compared with Figure 2, the effect of changing γ is that the region of $M^{1/3}$ fluctuations has shifted down and to the right along the anti-diagonal.*

where $F_1(x)$ is the square of the Tracy–Widom GOE *distribution function.*

(3) *For $\gamma \in (0, \infty)$ and $M/N \to \gamma^2$, then for $\pi = 1/(1+\gamma)$ and $\eta = \gamma/(1+\gamma)$,*

$$P\left(L_2(N, M) \le (1+\gamma^{-1})^2 M + \frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} x\right) \to F_{1,1}(x; 0; 0), \tag{21}$$

where $F_{1,1}(x; 0; 0)$ is the same distribution as what [14] *refer to as $F_0(x)$.*

(4) *For $\gamma \in (0, \infty)$ and $M/N \to \gamma^2$, then for π, η such that $\pi < 1/(1+\gamma)$ and $\eta > \frac{\pi}{\pi(1-\gamma^{-2})+\gamma^{-2}}$ [the G (π controlled) region],*

$$\begin{aligned} &P\left(L_2(N, M) \le \left(\pi^{-1} + \frac{\pi^{-1}\gamma^2}{\pi^{-1}-1}\right) N + \left(\pi^{-2} - \frac{\pi^{-2}\gamma^2}{(\pi^{-1}-1)^2}\right)^{1/2} N^{1/2} x\right) \\ &\qquad \to G_1(x), \end{aligned} \tag{22}$$

where $G_1(x) = \operatorname{erf}(x)$.

Likewise for π, η such that $\eta < \gamma/(1+\gamma)$ and $\eta < \frac{\pi}{\pi(1-\gamma^{-2})+\gamma^{-2}}$ [the G (η controlled) region],

$$\begin{aligned} &P\left(L_2(N, M) \le \left(\eta^{-1} + \frac{\eta^{-1}\gamma^{-2}}{\eta^{-1}-1}\right) M + \left(\eta^{-2} - \frac{\eta^{-2}\gamma^{-2}}{(\eta^{-1}-1)^2}\right)^{1/2} M^{1/2} x\right) \\ &\qquad \to G_1(x), \end{aligned} \tag{23}$$

where $G_1(x) = \operatorname{erf}(x)$.

(5) *For* $\gamma \in (0, \infty)$ *and* $M/N \to \gamma^2$, *then for* π, η *such that* $\pi + \eta < 1$ *and* $\eta = \frac{\pi}{\pi(1-\gamma^{-2})+\gamma^{-2}}$ *(the* G^2 *line)*,

$$P\Big(L_2(N, M) \le \Big(\pi^{-1} + \frac{\pi^{-1}\gamma^2}{\pi^{-1} - 1}\Big)N + \Big(\frac{(1-\pi+\pi\gamma^2)((1-\pi)^2 - \pi^2\gamma^2)}{\gamma^2\pi^2(1-\pi)^2}\Big)^{1/2} N^{1/2}x\Big) \tag{24}$$

$$\to G_1\Big(x\frac{\gamma}{\sqrt{1-\pi+\gamma^2\pi}}\Big)G_1\Big(x\frac{\sqrt{1-\pi+\gamma^2\pi}}{\gamma}\Big). \tag{25}$$

It is worth noting that there are many ways to write the expressions above, and our choices are to facilitate the greatest ease in our proofs.

This type of fluctuation classification picture has been previously discussed in [3] and [27]. In fact, in [3] Baik and Rains provide a proof of an analogous fluctuation classification result for two closely related particle system models: LPP with geometric weights, and the polynuclear growth model. Recently, [7] studied two-speed (different though related to two-sided initial conditions, half flat and half Bernoulli) TASEP and proved a fluctuation classification theorem for that model. As noted before, [4] previously provided the order of fluctuations for LPP models with two-sided boundary condition corresponding to the rarefaction fan. With an even more general type of boundary condition, the paper establishes $t^{1/3}$ scaling for the fluctuations of the last passage time.

1.2. *Notation*. In a paper such as this which connects two different lines of thought, it is easy to become lost in the disparity between notations. We will adopt notation in the style of [2] throughout, and when making connections with distributions as found in papers such as [3, 14, 22], we will take care to make note of the alternative notation used in those contexts. In this section, we define all of the distributions which we will encounter herein and provide references for their previous use and definition.

(1) $G_k(x)$ is a family of distributions defined in [2], Definition 1.2 and Lemma 1.1. It represents the distribution of the largest eigenvalue of a $k \times k$ GUE. From this representation, it is clear that $G_1(x) = \mathrm{erf}(x)$, the standard Gaussian distribution function.
(2) $F_J(x; x_1, \dots, x_J)$ is a family of distributions defined in [2], Definition 1.3. In the case when the $x_j = 0$ for all j, these distributions coincide with those from [2], Definition 1.1. Of note is $F_0(x)$ which is often written as F_{GUE}, the GUE Tracy–Widom distribution function, and $F_1(x; 0)$ which is often written as $F_{\mathrm{GOE}}(x)^2$, where F_{GOE} is the GOE Tracy–Widom distribution function.

(3) $F_{J,I}(x; x_1, \dots, x_J; y_1, \dots, y_I)$ is a family of distributions which we conjecture come up in LPP with thick two-sided boundary conditions. The only member of this family for which we know the correct definition is $F_{1,1}(x; 0; 0)$ which corresponds to the distribution denoted by F_0 in [14]. As of yet, we do not know how the other distributions should be defined.

1.3. *Outline*. The main theorems (Theorems 1.1 and 1.3) have already been recorded above in this section. Section 2 provides an intuitive sketch of the proof for Theorem 1.3. Section 3 explains the connection between the LPP with two-sided boundary conditions and the TASEP with two-sided initial conditions as well as briefly sketches how to translate the result of Theorem 1.3 into a proof of Theorem 1.1. Section 4 gives the full proof of the two main theorems, complete with the necessary technical lemmas for the coupling arguments.

2. Fluctuations in last passage percolation with boundary conditions. We start this section by reviewing the result of [2] which relates directed last passage percolation with boundary conditions to finite rank perturbations of Wishart ensembles. We then apply these results to prove Theorem 1.3 which fully characterizes the fluctuations of last passage times in terms of boundary conditions and the ratio $M/N = \gamma^2$. Using coupling arguments, supplemented in one case by the result of [14], we provide both the order and the scaling function for these fluctuations. Using the exact same arguments but fully taking advantage of the scope of the results of [2], we prove almost all of the cases in Partial Theorem 2.1. In this section, we will only sketch our proofs, which can be found in entirety in Section 4.

2.1. *LPP with one-sided boundary conditions*. Consider a directed last passage percolation model with one-sided boundary conditions defined as follows:

$$w_{i,j} = \begin{cases} \text{exponential of rate } \eta, & \text{if } i = 0, j > 0, \\ \text{exponential of rate } 1, & \text{if } i > 0, j > 0, \\ \text{zero}, & \text{if } i \geq 0, j = 0. \end{cases} \tag{26}$$

Let $L_1(N, M)$ denote the last passage time from $(0, 0)$ to (N, M) (for this LPP model with one-sided boundary conditions, but also for any LPP model with thick one-sided boundary conditions). Then the distribution of $L_1(N, M)$ is related to the distribution of the largest eigenvalue of the normalized covariance matrix $\frac{1}{M} XX'$ where X is $N \times M$ and each column is drawn (independent of other columns) from a complex N-dimensional Gaussian distribution with covariance matrix Σ. The matrix Σ has eigenvalues all equal to one aside from a single one, which is $l_1 = \eta^{-1}$. Depending on the value of η^{-1}, $L_1(N, M)$ behaves differently.

The following theorem is adapted from Theorem 1.1 of [2] and the extension to all $\gamma \in (0, \infty)$ given in [21], as applied to the one-sided boundary condition LPP. The connection between the largest eigenvalue and the LPP with one-sided boundary conditions given above is explained in Section 6 of [2] and is briefly rehashed in Remark 2.2.

PROPOSITION 2.1. *With $L_1(N, M)$ defined as above, as $M, N \to \infty$ while $M/N = \gamma^2$ is in a compact subset of $(0, \infty)$, the following hold for any real x in a compact set.*

(1) *When $\eta > \frac{\gamma}{1+\gamma}$,*

$$P\left(L_1(N, M) \le (1+\gamma^{-1})^2 M + \frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} x\right) \to F_0(x); \tag{27}$$

(2) *When $\eta = \frac{\gamma}{1+\gamma}$,*

$$P\left(L_1(N, M) \le (1+\gamma^{-1})^2 M + \frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} x\right) \to F_1(x); \tag{28}$$

(3) *When $\eta < \frac{\gamma}{1+\gamma}$,*

$$\begin{aligned} &P\left(L_1(N, M) \le \left(\eta^{-1} + \frac{\eta^{-1}\gamma^{-2}}{\eta^{-1}-1}\right) M + \left(\eta^{-2} - \frac{\eta^{-2}\gamma^{-2}}{(\eta^{-1}-1)^2}\right)^{1/2} M^{1/2} x\right) \\ &\qquad \to G_1(x). \end{aligned} \tag{29}$$

REMARK 2.2. The connection between last passage time in LPP with one-sided boundary conditions and the largest eigenvalue of the spiked Wishart ensemble was observed in [2]. The connection is not via an exact map but rather an equality of distributions. Proposition 6.1 of [2] records this fact and explains how a modification of the argument in [16] can be used to prove this.

An intuitive explanation for the cutoff of $\eta^{-1} = 1 + \gamma^{-1}$ in terms of a simple calculus problem of maximizing the law of large numbers for LPP paths forced to travel a specific fraction of the way along the left column can be found in Section 6 of [2].

2.2. *LPP with two-sided boundary conditions.* Presently, we turn our attention to the LPP with two-sided boundary conditions as defined in (18): on the left-most column there are exponential weights of rate η, on the bottom-most row there are exponential weights of rate π at the origin there is a weight of zero, and for all strictly positive lattice points the weight is of rate one. Define, respectively, $X(N, M)$ and $Y(N, M)$ as the coupled last passage times of paths which have taken the first step to the right and the first step up (resp.). One should be careful to note that we are not conditioning on the location of the optimal path, but rather, for each configuration of weights, defining X to be the length of the optimal path which first goes right, and Y the length of the optimal path which first goes up. It is clear then that X and Y are coupled, dependent and that

$$L_2(N, M) = \max(X(N, M), Y(N, M)). \tag{30}$$

Consider now the marginals of X and Y and observe that each of these marginals is of the type of the last passage time for a LPP model with one-sided boundary conditions. The boundary conditions for Y are exactly as above (η weights and an N by M region). However, for X, the boundary conditions are π weights and an M by N region (note that the region has been flipped in order to conform with the setup for Proposition 2.1). From this observation, we can apply Proposition 2.1 to completely characterize the marginals of the joint distribution for the pair (X, Y). Note that while X and Y are not exactly of the form of a last passage time for a LPP with one-sided boundary conditions, they only differ by a finite number of weights and therefore have the exact same asymptotic statistics via Lemma 3.1.

PROPOSITION 2.3. *With $X(N, M)$ defined as above, as $M, N \to \infty$ while $M/N = \gamma^2$ is in a compact subset of $(0, \infty)$, the following hold for any real x in a compact set*:

(1) *when $\pi > \frac{1}{1+\gamma}$,*

$$P\left(X(N, M) \le (1+\gamma)^2 N + \frac{(1+\gamma^{-1})^{4/3}}{\gamma^{-1}} N^{1/3} x\right) \to F_0(x); \tag{31}$$

(2) *when $\pi = \frac{1}{1+\gamma}$,*

$$P\left(X(N, M) \le (1+\gamma)^2 N + \frac{(1+\gamma^{-1})^{4/3}}{\gamma^{-1}} N^{1/3} x\right) \to F_1(x); \tag{32}$$

(3) *when $\pi < \frac{1}{1+\gamma}$,*

$$\begin{aligned} &P\left(X(N, M) \le \left(\pi^{-1} + \frac{\pi^{-1}\gamma^2}{\pi^{-1}-1}\right) N + \left(\pi^{-2} - \frac{\pi^{-2}\gamma^2}{(\pi^{-1}-1)^2}\right)^{1/2} N^{1/2} x\right) \\ &\qquad \to G_1(x). \end{aligned} \tag{33}$$

PROPOSITION 2.4. *With $Y(N, M)$ defined as above, as $M, N \to \infty$ while $M/N = \gamma^2$ is in a compact subset of $(0, \infty)$, the following hold for any real x in a compact set*:

(1) *when $\eta > \frac{\gamma}{1+\gamma}$,*

$$P\left(Y(N, M) \le (1+\gamma^{-1})^2 M + \frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} x\right) \to F_0(x); \tag{34}$$

(2) *when $\eta = \frac{\gamma}{1+\gamma}$,*

$$P\left(Y(N, M) \le (1+\gamma^{-1})^2 M + \frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} x\right) \to F_1(x); \tag{35}$$

(3) *when* $\eta < \frac{\gamma}{1+\gamma}$,

$$P\Big(Y(N,M) \le \Big(\eta^{-1} + \frac{\eta^{-1}\gamma^{-2}}{\eta^{-1}-1}\Big)M + \Big(\eta^{-2} - \frac{\eta^{-2}\gamma^{-2}}{(\eta^{-1}-1)^2}\Big)^{1/2} M^{1/2}x\Big) \to G_1(x). \tag{36}$$

We assume that both η and π are between zero and one. In fact, it is clear from our proofs that the order and fluctuation of our two-sided last passage time $L_2(N, M)$ for parameters η and π is that same as that for parameters $\eta \wedge 1, \pi \wedge 1$. Thus, it suffices to consider only $\eta, \pi \in [0, 1]^2$.

For each of X and Y, there are two regions of different fluctuation orders, and one critical point which has $1/3$ order fluctuations. We call the point $(\pi, \eta) = (1/(1+\gamma), \gamma/(1+\gamma))$ the critical point for the pair π, η. If $\pi < \frac{1}{1+\gamma}$, then the X fluctuations are of order $1/2$, and likewise if $\eta < \frac{\gamma}{1+\gamma}$ then the Y fluctuations are of order $1/2$, whereas in the complementary cases, the fluctuations are of order $1/3$. By comparing leading (law of large number) terms in Propositions 2.3 and 2.4 we see that if either of these two inequalities hold, then the fluctuations must be of order $1/2$. In this case, either the leading term for X or Y clearly wins, in which case $L_2(N, M)$ has the leading order behavior and Gaussian fluctuations of the winner random variable, or the two random variables have the same leading terms. The second case, or equal leading terms, occurs when

$$\Big(\pi^{-1} + \frac{\pi^{-1}\gamma^2}{\pi^{-1}-1}\Big)N = \Big(\eta^{-1} + \frac{\eta^{-1}\gamma^{-2}}{\eta^{-1}-1}\Big)M. \tag{37}$$

In this case, the fluctuations will remain of order $1/2$, but will behave as the fluctuations of two independent normal random variables (what we call G^2).

If $\gamma = 1$, there are two solutions to (37). One is $\eta = \pi$ and the other is $\eta = 1 - \pi$. Since we are only considering the Gaussian region, the anti-diagonal solution is of no interest, and we find that we have G^2 density for our fluctuations if $\eta = \pi$ and $\eta < 1/2$.

For $\gamma \neq 1$, the solution set is a little harder. Recall $M = \gamma^2 N$ and using this we can factor out N from both sides giving

$$\Big(\pi^{-1} + \frac{\pi^{-1}\gamma^2}{\pi^{-1}-1}\Big) = \Big(\gamma^2\eta^{-1} + \frac{\eta^{-1}}{\eta^{-1}-1}\Big). \tag{38}$$

Applying the change of variable $\pi \to 1 - \pi$, we find that it suffices to solve

$$\Big(\gamma^2\pi^{-1} + \frac{\pi^{-1}}{\pi^{-1}-1}\Big) = \Big(\gamma^2\eta^{-1} + \frac{\eta^{-1}}{\eta^{-1}-1}\Big) \tag{39}$$

and change the solution back to our original variables.

This again has the solution $\eta = \pi$. Solving for the other solution and then changing variables back we get

$$\eta = \frac{\pi}{\pi(1-\gamma^{-2}) + \gamma^{-2}}. \tag{40}$$

By plugging in the critical point $(1/(1+\gamma), \gamma/(1+\gamma))$ it is easy to see that this G^2 curve is continuous between the origin and the critical point, though only linear for $\gamma = 1$.

We have, so far, only accounted for the regions where $\pi < \frac{1}{1+\gamma}$ or $\eta < \frac{\gamma}{1+\gamma}$. There are four other regions to consider which correspond to replacing the or with an and, and the less than sign with either equality, or a greater than sign. In each of these cases, the leading term is independent of π and η and equals $(1+\gamma^{-1})^2 M$. The fluctuations of X and Y are both or order $1/3$, so those of $L_2(N, M)$ are as well. To determine the scaling functions, finer coupling arguments are necessary. For instance, when $\pi > \frac{1}{1+\gamma}$ and $\eta > \frac{\gamma}{1+\gamma}$, the last passage path for X and for Y can be compared to the analogous random variables $\tilde{X}$ and $\tilde{Y}$, for last passage paths for a coupled LPP model with two-sided boundary conditions all of rate one (i.e., $\pi, \eta = 1$) such that pointwise $X \geq \tilde{X}$ and likewise $Y \leq \tilde{Y}$. The maximum of $\tilde{X}$ and $\tilde{Y}$ equals $\tilde{L}(N, M)$, the last passage time, and we show that X and $\tilde{X}$, and likewise for Y and $\tilde{Y}$, under appropriate centering and scaling converge to the same distribution, respectively. This implies, via our Lemma 4.2, that the scaled and centered random variables in fact converge in probability and hence that their maximums converge in probability. This means that the maximum of X and Y behaves just like a regular last passage time which is known to have GUE scaling function. A similar coupling shows that the scaling when either $\pi = \frac{1}{1+\gamma}$ and $\eta > \frac{\gamma}{1+\gamma}$, or $\pi > \frac{1}{1+\gamma}$ and $\eta = \frac{\gamma}{1+\gamma}$ the scaling function behaves like that of a single critical last passage time for a LPP model with one-sided boundary conditions.

Determining the scaling function at the critical point is a harder problem. One may identify it as the maximum of two F_1 distributions, coupled as X and Y are coupled. This characterization, a priori, yields a tight family of random variables. However, how to prove that it converges on more than just a subsequence is not immediately clear, and more over it is not clear to what it convergences. A posteriori, this characterization is justified since the result of [14] can readily be translated into a proof that the scaling function at the critical point is $F_{1,1}(x; 0; 0)$ (what they call F_0).

The results from [2] used in the proof of Theorem 1.3 yield, in fact, a much more general result via essentially the same argument. We now define what we call the *LPP model with thick two-sided boundary conditions* in terms of boundary row and column thickness integer parameters $J, I \geq 0$; two vectors of row and column weight rates $\pi = (\pi_1, \ldots, \pi_J)$, $\eta = (\eta_1, \ldots, \eta_I)$; two vectors of row and column convergence rates $X = (x_1, \ldots, x_J)$, $Y = (y_1, \ldots, y_I)$. With these parameters, our

model is defined in terms of the following LPP weights (which implicitly depend on M and N):

$$w_{i,j} = \begin{cases} \text{exponential of rate } \pi_j + \dfrac{x_j}{M^{1/3}}, & \text{if } i > I,\ j \le J; \\ \text{exponential of rate } \eta_i + \dfrac{y_i}{M^{1/3}}, & \text{if } i \le I,\ j > J; \\ \text{exponential of rate } 1, & \text{if } i > I,\ j > J; \\ \text{zero}, & \text{if } i \le I,\ j \le J. \end{cases} \tag{41}$$

To see that this model is a broad generalization of our previously considered two-sided boundary condition model, take $J, I = 1$, $\pi_1 = \pi$, $\eta_1 = \eta$ and x_1, $y_1 = 0$. Corresponding to this model, we now provide a complete characterization of its asymptotic fluctuations. A number of distributions not previously discussed are introduced in this theorem. A full discussion of these distributions can be found in Section 1.2.

The coupling arguments given to prove Theorem 1.3 can be easily adopted to this new setting. Given the above parameters define, again, two coupled random variables $X(N, M)$ and $Y(N, M)$ as follows. $X(N, M)$ is the last passage time from $(0, 0)$ to (N, M) of the set of up/right paths which cross through at least one vertex from the set $\{(i, j) : i = I, j \in \{0, \dots, J-1\}\}$. Likewise $Y(N, M)$ is the last passage time from $(0, 0)$ to (N, M) of the set of up/right paths which cross through at least one vertex from the set $\{(i, j) : i \in \{0, \dots, I-1\}, j = J\}$. Clearly, any up/right path from $(0, 0)$ to (N, M) must go through one and only one of these two regions. Furthermore, by virtue of the definition of the last passage time, the maximizing path for $X(N, M)$ and $Y(N, M)$ will necessarily go through the points $(0, J)$ and $(I, 0)$ (resp.). Thus we may refine our definitions of $X(N, M)$ and $Y(N, M)$ to require passing through these two points. Again, we see that $L_2(N, M) = \max(X(N, M), Y(N, M))$ and just as before [2] provides an immediate proof of the following.

PROPOSITION 2.5. *For the vector π, fix the set $K_1 \subset \{1, \dots, J\}$ by*

$$K_1 = \left\{ j \in \{1, \dots, J\} : \pi_j = \frac{1}{1+\gamma} \right\} \tag{42}$$

an define X_{K_1} as the elements of X which correspond to indices in K_1. Further, define $\tilde{\pi} = \min_{j \in \{1, \dots, J\}}(\pi_j)$ and let k_1 be the number of π_j which attain the value $\tilde{\pi}$.

Then with $X(N, M)$ defined as above, as $M, N \to \infty$ while $M/N - \gamma^2$ is in a compact subset of $(0, \infty)$, the following holds for any real x in a compact set:

(1) *when $\tilde{\pi} > \frac{1}{1+\gamma}$,*

$$P\left(X(N, M) \le (1+\gamma)^2 N + \frac{(1+\gamma^{-1})^{4/3}}{\gamma^{-1}} N^{1/3} x \right) \to F_0(x); \tag{43}$$

(2) *when* $\tilde{\pi} = \frac{1}{1+\gamma}$,

$$P\left(X(N,M) \le (1+\gamma)^2 N + \frac{(1+\gamma^{-1})^{4/3}}{\gamma^{-1}} N^{1/3} x\right) \to F_{|K_1|}(x; X_{K_1}); \tag{44}$$

(3) *when* $\tilde{\pi} < \frac{1}{1+\gamma}$,

$$\begin{aligned} &P\left(X(N,M) \le \left(\tilde{\pi}^{-1} + \frac{\tilde{\pi}^{-1}\gamma^2}{\tilde{\pi}^{-1}-1}\right)N + \left(\tilde{\pi}^{-2} - \frac{\tilde{\pi}^{-2}\gamma^2}{(\tilde{\pi}^{-1}-1)^2}\right)^{1/2} N^{1/2} x\right) \\ &\qquad \to G_{k_1}(x). \end{aligned} \tag{45}$$

A similar proposition exists for $Y(N,M)$. Using these two results, the same types of coupling arguments then apply and give both the orders and the scaling functions for $L_2(N,M)$. As before, these coupling arguments break down when both boundary conditions are critical. With single width boundary conditions, we appealed to [14], however in this case no existing argument provides a characterization of the behavior in this case. The following partial theorem therefore contains a single conjectured equation (51) whose study seems very difficult.

PARTIAL THEOREM 2.1.

(1) *For vectors* π, η *fix the sets* $K_1 \subset \{1,\dots,J\}$ *and* $K_2 \subset \{1,\dots,I\}$ *by*

$$K_1 = \left\{ j \in \{1,\dots,J\} : \pi_j = \frac{1}{1+\gamma} \right\}, \tag{46}$$

$$K_2 = \left\{ i \in \{1,\dots,I\} : \eta_i = \frac{\gamma}{1+\gamma} \right\}. \tag{47}$$

Then define X_{K_1} *and* Y_{K_2} *as the elements of* X *and* Y *which correspond to indices in* K_1 *and* K_2, *respectively.*

For $\gamma \in (0,\infty)$ *and* $M/N \to \gamma^2$, *then for vectors* π, η *such that* $\pi_j \ge \frac{1}{1+\gamma}$ *for all* $i \in \{1,\dots,J\}$ *and* $\eta_i \ge \frac{\gamma}{1+\gamma}$ *for all* $i \in \{1,\dots,I\}$ *then if*:

(a) $|K_1| = 0$, $|K_2| = 0$,

$$P\left(L_2(N,M) \le (1+\gamma^{-1})^2 M + \frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} x\right) \to F_0(x); \tag{48}$$

(b) $|K_1| > 0$, $|K_2| = 0$,

$$P\left(L_2(N,M) \le (1+\gamma^{-1})^2 M + \frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} x\right) \to F_{|K_1|}(x; X_{K_1}); \tag{49}$$

(c) $|K_1| = 0$, $|K_2| > 0$,

$$P\left(L_2(N,M) \le (1+\gamma^{-1})^2 M + \frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} x\right) \to F_{|K_2|}(x; Y_{K_2}); \tag{50}$$

(d) $|K_1| > 0$, $|K_2| > 0$,

$$P\Big(L_2(N,M) \le (1+\gamma^{-1})^2 M + \frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} x\Big) \to F_{|K_1|,|K_2|}(x; X_{K_1}, Y_{K_1}). \tag{51}$$

(2) *Define* $\tilde\pi = \min_{j\in\{1,\dots,J\}}(\pi_j)$ *and* $\tilde\eta = \min_{i\in\{1,\dots,I\}}(\eta_i)$, *and let* k_1 *be the number of* π_j *which attain the value* $\tilde\pi$ *and likewise* k_2 *be the number of* η_i *which attain the value* $\tilde\eta$.

For $\gamma \in (0,\infty)$ *and* $M/N \to \gamma^2$, *then for* π, η *such that*:

(a) $\tilde\pi < 1/(1+\gamma)$ *and* $\tilde\eta > \frac{\tilde\pi}{\tilde\pi(1-\gamma^{-2})+\gamma^{-2}}$ [*the* G (π *controlled*) *region*],

$$P\Big(L_2(N,M) \le \Big(\tilde\pi^{-1} + \frac{\tilde\pi^{-1}\gamma^2}{\tilde\pi^{-1}-1}\Big)N + \Big(\tilde\pi^{-2} - \frac{\tilde\pi^{-2}\gamma^2}{(\tilde\pi^{-1}-1)^2}\Big)^{1/2} N^{1/2} x\Big) \to G_{k_1}(x); \tag{52}$$

(b) $\tilde\eta < \gamma/(1+\gamma)$ *and* $\tilde\eta < \frac{\tilde\pi}{\tilde\pi(1-\gamma^{-2})+\gamma^{-2}}$ [*the* G ($\tilde\eta$ *controlled*) *region*],

$$P\Big(L_2(N,M) \le \Big(\tilde\eta^{-1} + \frac{\tilde\eta^{-1}\gamma^{-2}}{\tilde\eta^{-1}-1}\Big)M + \Big(\tilde\eta^{-2} - \frac{\tilde\eta^{-2}\gamma^{-2}}{(\tilde\eta^{-1}-1)^2}\Big)^{1/2} M^{1/2} x\Big) \to G_{k_2}(x); \tag{53}$$

(c) $\tilde\pi + \tilde\eta < 1$ *and* $\tilde\eta = \frac{\tilde\pi}{\tilde\pi(1-\gamma^{-2})+\gamma^{-2}}$ (*the* G^2 *line*),

$$P\Big(L_2(N,M) \le \Big(\tilde\pi^{-1} + \frac{\tilde\pi^{-1}\gamma^2}{\tilde\pi^{-1}-1}\Big)N + \Big(\frac{(1-\tilde\pi+\tilde\pi\gamma^2)((1-\tilde\pi)^2-\tilde\pi^2\gamma^2)}{\gamma^2\tilde\pi^2(1-\tilde\pi)^2}\Big)^{1/2} N^{1/2} x\Big) \tag{54}$$

$$\to G_{k_1}\Big(x\frac{\gamma}{\sqrt{1-\tilde\pi+\gamma^2\tilde\pi}}\Big) G_{k_2}\Big(x\frac{\sqrt{1-\tilde\pi+\gamma^2\tilde\pi}}{\gamma}\Big). \tag{55}$$

Finally, let us note two applications of LPP with thick one-sided boundary conditions which can be found in [1]. The first application deals with what Baik called *traffic of slow start from stop* in which particles start in the step initial condition of TASEP and have a start-up profile—that is, every particle moves slower for its first few jumps, and then returns to jumping at rate one. The second application is dual

to the first one and is called *traffic with a few slow cars* in which particles always move at a slower rate. In both cases, Baik identifies the fluctuation scaling limits by using the [2] type results which we have made use of herein.

In the next section, we will give an important application for the LPP with two-sided boundary conditions model to two-sided TASEP. It is unclear whether the thick two-sided boundary conditions model has any similar application to TASEP or related models.

3. Mapping TASEP to last passage percolation with boundary conditions. In this section, we explain the connections between the last passage time in LPP with two-sided boundary conditions and the fluctuations of the height function for the two-sided TASEP model. Making use of this mapping, we explain how the results of Theorem 1.3 imply the results of [10] stated in the Introduction. Furthermore, we briefly explain how this theorem translates into a proof of Conjecture 7.1 of [22] (full proof is given in Section 4).

We start with a lemma which states that finite perturbations of our LPP model, have no affect on the asymptotic behavior of the last passage time.

LEMMA 3.1. *Fix some LPP model with weights $w_{i,j}$ (independent but not necessarily identically distributed) such that*

$$P\left(\frac{L(N,M)-a_N}{b_N} \le x\right) \to F(x) \tag{56}$$

for $M/N \to \gamma^2 \in (0,\infty)$, and for F a nondegenerate probability distribution. Randomly, independent of the values of $w_{i,j}$, change a set of these weights to a new set of weights $w'_{i,j}$ and let $L'(N,M)$ denote the last passage time with respect to the original weight with the newly updated weights. Call A the set of changed indices (i,j) and $W_A = \sum_{(i,j)\in A} w_{i,j} + w'_{i,j}$. Then if $E[W_A] < \infty$ and if $b_N \to \infty$, we also have

$$P\left(\frac{L'(N,M)-a_N}{b_N} \le x\right) \to F(x). \tag{57}$$

Below is an outline of the proof. The full level of details is suppressed since a similar style of proof is given for Lemma 4.1 in full detail.

PROOF OF LEMMA 3.1. Since the total effect of the change of weights corresponding to A has finite expectations, the Markov inequality shows that for any ε we can find l large enough so that $P(W_A \ge l) \le \varepsilon$. If we restrict ourselves to this region of our statespace, then since the b_N goes to infinity, the effect of the change of weights is negligible in the limit. Since this is true on all but an ε region of the state space, we have that the distribution functions are within ε of each other in the limit, but taking ε to zero gives equality. $\square$

Recall our definition of the two-sided TASEP model given by the initial conditions of Bernoulli with parameter ρ_- on the left of zero and with parameter ρ_+ on the right. Corresponding to the TASEP process started with this random initial condition, we consider the height function $h_t(j)$ defined in (6). Theorem 2.1 of [22] relates the joint distributions for this height function to those of the height function for a particular growth model associated with a variant on the LPP with two-sided boundary conditions. The weights for this variant LPP are defined with respect to two independent geometric random variables ζ_+ and ζ_-. Let ζ_+ be geometric with parameter $1-\rho_+$ [i.e., $P(\zeta_+ = n) = \rho_+(1-\rho_+)^n$] and ζ_- be geometric with parameter ρ_- [i.e., $P(\zeta_- = n) = (1-\rho_-)\rho_+^n$]. The weights are then defined as independent random variables with:

$$w_{i,j} = \begin{cases} \text{exponential of rate } 1, & \text{if } i, j \geq 1; \\ \text{zero}, & \text{if } i = j = 0; \\ \text{zero}, & \text{if } 0 \leq i \leq \zeta_+ \text{ and } j = 0; \\ \text{exponential of rate } 1-\rho_+, & \text{if } i > \zeta_+ \text{ and } j = 0; \\ \text{zero}, & \text{if } 0 \leq j \leq \zeta_- \text{ and } i = 0; \\ \text{exponential of rate } \rho_-, & \text{if } j > \zeta_- \text{ and } i = 0. \end{cases} \tag{58}$$

With respect to these random weights, define the last passage time $\tilde{L}(N, M)$. This family of random variables is nondecreasing in both N and M. Therefore, one can associate to this a growth process on the upper-corner and likewise a height process over the number line. Let $A_t = \{(N, M) | N, M \geq 1, \tilde{L}(N, M)\}$ be the growth process and let $\tilde{h}_t$ be defined so as to satisfy $A_t = \{(N, M) | 2 \leq N + M \leq \tilde{h}_t(N - M)\}$. To describe this in words, imagine rotating counter-clockwise, the upper corner in which LPP occurs by $\pi/4$. To each lattice point (labeled by $j \in \mathbb{Z}$) now on the horizontal associate a height $\tilde{h}_t(j)$ equal to two times the number of $\tilde{L}(N, M)$ vertically above j which are less than or equal to t. Then we have the following.

THEOREM 3.2 (Theorem 2.1 of [22]). *In the sense of joint distributions, we have*

$$h_t(j) = \tilde{h}_t(j) \qquad \textit{for } |j| \leq h_t(j). \tag{59}$$

This theorem essentially says that for the height profile which lies above the boundary of the rotated upper corner, the two profiles have the same joint distribution. It is worthwhile to recall that there is a similar map between the TASEP height function for TASEP with step initial conditions and the height function for standard (no boundary condition) LPP [16]. The proof of Theorem 3.2 can be found in [22] and essentially amounts to a study of the dynamics of the right most particle to the left of the origin, as well as the dynamics of the left most hole to the right of the origin. Tagging this particle and this hole, we observe that their initial location is geometric and using Burke's theorem we find that their waiting

times between successive moves is exponential with rate relating to the densities ρ_- and ρ_+. Between the tagged particle on the left and the tagged hole on the right, particles move according to normal TASEP rules, and hence the two height functions evolve with the same dynamics. The dynamics of the boundary of the part of the TASEP height function lying in the rotated upper corner is matched by the effect of the LPP boundary conditions, and the theorem follows.

From this theorem, we see that the following equality:

$$P_{\rho_-,\rho_+}\big(h_t(N-M) \geq N+M\big) = P\big(\tilde{L}(N,M) \leq t\big). \tag{60}$$

This equality is not of much use to use, however, because in order to use the results of Theorems 1.3, we must deal with a slightly different LPP model than corresponding to $\tilde{L}$. However, this model and our LPP with two-sided boundary conditions only differ in expectation by a finite number of weights (the geometric number of zeros from ζ_- and ζ_+). Therefore, while it is true that there is not exact equality then with $P(L_2(N,M) \leq t)$, from Lemma 3.1, we see that for any sort of central limit fluctuation statement with a nontrivial limiting distribution, we have equality in the limit. We will abuse notation in the remainder of this section and the next during the proof of Conjecture 7.1, and write equality between the TASEP height function probability and the probability for the last passage time in the LPP with two-sided boundary conditions model. To sum up, we have the following.

REMARK 3.3. While the boundary conditions (18) differ from those (58) used by Prähofer and Spohn, they are much simpler and also describe the TASEP with two-sided initial conditions, as described in [4].

It is important to note that Lemma 3.1 only applies if both $\rho_- < 1$ and $\rho_+ > 0$. If either of these inequalities is violated, then the geometric number of zeros on the boundary will in fact, almost surely be infinite. However, in any of these cases, the classification of one-sided LPP then readily applies.

The boundary conditions for $\tilde{L}$ corresponded to having exponentials of rate ρ_- on the left boundary and $1-\rho_+$ on the right. Therefore, in terms of π and η, we have $\pi = 1-\rho_+$ and $\eta = \rho_-$. The critical point for π, η is $\frac{1}{1+\gamma}$ and $\frac{\gamma}{1+\gamma}$, therefore we see that the critical point for ρ_-, ρ_+ is

$$\rho_- = \rho_+ = \frac{\gamma}{1+\gamma}. \tag{61}$$

This corresponds to an equilibrium measure on TASEP with density $\frac{\gamma}{1+\gamma}$.

In the next section, we will show how Theorem 1.3 implies an almost complete (all but a few regions of the claimed Gaussian region are fully proved) proof of [22], Conjecture 7.1 (our Theorem 1.1). From this result, we may easily deduce the results of [10] stated in the Introduction.

For simplicity assume $r \in [0, 1]$, as the case $r \in [-1, 0]$ follows similarly. We wish to prove that

$$P\left(\frac{J_{rt,t} - (\rho(1-\rho) - r\rho)t}{t^{1/2}\sqrt{\rho(1-\rho)|(1-2\rho) - r|}} \geq x\right) \to G_1(x), \tag{62}$$

where as defined before $G_1(x)$ is the standard Gaussian distribution function.

In the case of $\rho_- = \rho_+ = \rho$ and $r \geq 0$, we can conclude from (10) that

$$P\big(t\bar{h}(r) - h_t([ry]) \leq \big(4\rho(1-\rho)(r-1+2\rho)t\big)^{1/2}x\big) \to G_1(x), \tag{63}$$

where $\bar{h}(r) = (1-2\rho)r + 2\rho(1-\rho)$. Substituting the relationship in (1) and rearranging terms, we arrive at the exact result of [10] desired.

We now briefly explain the approach to proving Conjecture 7.1 from our Theorem 1.3. The conjecture deals with height functions. We have provided above the relationship between height function distributions and LPP distributions. From (60), we see that if one is to consider $P(h_t(j) \geq x)$ in terms of LPP, you must solve for $N = \frac{x+j}{2}$ and $M = \frac{x-j}{2}$. The variables j and x both are functions of time t and a speed y. If $M/N = \frac{x-j}{x+j}$ has a limit, we call that γ^2. This allows us to asymptotically write M (or N) just as a function of time (thus the y dependence goes into γ). In the cases we consider, we can invert the expression for M in terms of t and get an expression for t in terms of M, thus putting us in the form of the limit theorems we proved in Theorem 1.3.

4. Proof of fluctuation theorems. In this section, we provide a proof of Theorem 1.3 (which easily generalizes to prove Partial Theorem 2.1) and a proof of Theorem 1.1.

4.1. *Proof of Theorem* 1.3 (*fluctuations of LPP with two-sided boundary conditions*). The following two technical lemmas provide the basis for the coupling arguments necessary in our proof of Theorem 1.3.

LEMMA 4.1. *If $X_n \geq \tilde{X}_n$ and $X_n \Rightarrow D$ as well as $\tilde{X}_n \Rightarrow D$, then $X_n - \tilde{X}_n$ converges to zero in probability. Conversely if $X_n \geq \tilde{X}_n$ and $\tilde{X}_n \Rightarrow D$ and $X_n - \tilde{X}_n$ converges to zero in probability, then $X_n \Rightarrow D$ as well.*

LEMMA 4.2. *Assume $X_n \geq \tilde{X}_n$ and $X_n \Rightarrow D_1$ as well as $\tilde{X}_n \Rightarrow D_1$; and similarly $Y_n \geq \tilde{Y}_n$ and $Y_n \Rightarrow D_2$ as well as $\tilde{Y}_n \Rightarrow D_2$. Let $Z_n = \max(X_n, Y_n)$ and $\tilde{Z}_n = \max(\tilde{X}_n, \tilde{Y}_n)$. Then if $\tilde{Z}_n \Rightarrow D_3$, we also have $Z_n \Rightarrow D_3$.*

PROOF OF LEMMA 4.1. While it is likely that this lemma is known in the literature, we do not know where and hence produce a proof. We prove the first assertion. Fix $\varepsilon > 0$ and, from the point of contradiction, assume that $P(X_n - \tilde{X}_n > \varepsilon) > \delta > 0$ for an infinite subsequence of n's. By restricting to this subsequence

and noting that all of the hypothesis of the lemma hold under this restriction, we may equivalently assume that $P(X_n - \tilde{X}_n > \varepsilon) > \delta > 0$ for all n large. Since X_n and $\tilde{X}_n$ converge weakly, each sequence of random variables is tight. This implies that there exists an $M(\varepsilon)$ and $N(\varepsilon)$ such that for all $n > N$, $P(|X_n| > M) < \delta/2$ and likewise $P(|\tilde{X}_n| > M) < \delta/2$. Thus $P(|\tilde{X}_n| > M \cap \{X_n - \tilde{X}_n > \varepsilon\}) < \delta/2$, therefore

$$P(|\tilde{X}_n| < M \cap \{X_n - \tilde{X}_n > \varepsilon\}) > \delta/2. \tag{64}$$

Call this event $A = |\tilde{X}_n| < M \cap \{X_n - \tilde{X}_n > \varepsilon\}$, then conditioned on A, $X_n > \tilde{X}_n + \varepsilon$. For large enough n,

$$P(X_n \le t|A) \le P(\tilde{X}_n \le t|A) - P(\tilde{X}_n \in [t-\varepsilon, t]|A). \tag{65}$$

We now partition the interval $[-M, M]$ into ε size blocks and define deterministic numbers $a_j(n)$ for $j \in \{1, \dots, \lceil \frac{2M}{\varepsilon} \rceil\}$ by

$$a_j(n) = P\big(\tilde{X}_n \in [-M + \varepsilon(j-1), -M + \varepsilon j]|A\big). \tag{66}$$

Observe that $\sum_j a_j(n) = 1$ since having conditioned on A, we know $\tilde{X}_n \in [-M, M]$. Therefore, for each n, there exists at least one $j = j(n)$ for which $a_j(n) \ge \frac{1}{2M/\varepsilon+1} = \frac{\varepsilon}{2M+\varepsilon}$ [if there is more than one j for which $a_j(n)$ is as desired, pick the smallest value of j]. Since j is restricted to a finite set of values, there must be some infinite subsequence of n's which have the same value of $j(n)$. Restricting to that subsequence so every $j(n)$ equals a fixed j, if we set $t = -M + \varepsilon j$ we have

$$P(\tilde{X}_n \in [t-\varepsilon, t]|A) \ge \frac{\varepsilon}{2M+\varepsilon}. \tag{67}$$

Therefore,

$$P(X_n \le t|A) \le P(\tilde{X}_n \le t|A) - \frac{\varepsilon}{2M+\varepsilon}. \tag{68}$$

Multiplying both sides by $P(A)$ and rewriting without conditioning gives

$$P(X_n \le t \cap A) \le P(\tilde{X}_n \le t \cap A) - \frac{P(A)\varepsilon}{2M+\varepsilon}. \tag{69}$$

That $X_n \ge \tilde{X}_n$ also implies that

$$P(X_n \le t \cap A^c) \le P(\tilde{X}_n \le t \cap A^c). \tag{70}$$

Adding these two inequalities and using the fact that $P(A) > \delta/2$ gives, for all n large enough

$$P(X_n \le t) \le P(\tilde{X}_n \le t) - \frac{\delta\varepsilon}{2(2M+\varepsilon)}. \tag{71}$$

This inequality implies, however, that X_n and $\tilde{X}_n$ cannot converge in distribution to the same object. This is a contradiction to our hypothesis, so our assumption must be false. That is, $P(X_n - \tilde{X}_n > \varepsilon)$ must go to zero as n goes to infinity.

The second assertion is easier. For all ε, we can find N such that for $n > N$, $P(X_n - \tilde{X}_n > \varepsilon) \le \varepsilon$. Set inclusion and partitioning implies that

$$\begin{aligned} P(\tilde{X}_n \le t - \varepsilon) &= P(\tilde{X}_n \le t - \varepsilon \cap X_n - \tilde{X}_n < \varepsilon) \\ &\quad + P(\tilde{X}_n \le t - \varepsilon \cap X_n - \tilde{X}_n \ge \varepsilon) \\ &\le P(X_n \le t) + P(X_n - \tilde{X}_n \ge \varepsilon) \\ &\le P(X_n \le t) + \varepsilon. \end{aligned} \tag{72}$$

Since $P(X_n \le t) \le P(\tilde{X}_n \le t)$, we find that

$$P(\tilde{X}_n \le t - \varepsilon) - \varepsilon \le P(X_n \le t) \le P(\tilde{X}_n \le t). \tag{73}$$

If t is any continuity point for D, then we can take ε to zero and we find that

$$D(t) \ge P(X_n \le t) \ge D(t), \tag{74}$$

and hence X_n weakly converges to D. $\square$

PROOF OF LEMMA 4.2. Applying Lemma 4.1 to both X_N and $\tilde{X}_N$, as well as Y_N and $\tilde{Y}_N$ we find that for any ε, large enough N, $P(A_X) < \varepsilon$ and likewise $P(A_Y) < \varepsilon$ where $A_X = \{X_N - \tilde{X}_N > \varepsilon\}$ and $A_Y = \{Y_N - \tilde{Y}_N > \varepsilon\}$. From this, it follows that

$$\begin{aligned} &P(\tilde{Z}_n \le t - \varepsilon) \\ &\quad = P(\tilde{Z}_n \le t - \varepsilon \cap A_X^c \cap A_Y^c) + P(\tilde{Z}_n \le t - \varepsilon \cap A_X^c \cap A_Y) \end{aligned} \tag{75}$$

$$+ P(\tilde{Z}_n \le t - \varepsilon \cap A_X \cap A_Y^c) + P(\tilde{Z}_n \le t - \varepsilon \cap A_X \cap A_Y). \tag{76}$$

The first probability is less than or equal to $P(Z_n \le t)$ while the last three are each trivially bounded by ε. Therefore, noting that $P(Z_n \le t) \le P(\tilde{Z}_n \le t)$ we find

$$P(\tilde{Z}_n \le t - \varepsilon) - 3\varepsilon \le P(Z_n \le t) \le P(\tilde{Z}_n \le t). \tag{77}$$

Taking t to be a continuity point for the limiting distribution for $\tilde{Z}_n$ (for D_3) and taking ε to zero we get that $\lim_{n\to\infty} P(Z_n \le t) = F_{D_3}(t)$, and hence Z_n converges in distribution to D_3. $\square$

PROOF OF THEOREM 1.3, (F_0). This result follows from a coupling argument between the $X(N, M)$, $Y(N, M)$ variables as well as a second set of last passage times $\tilde{X}(N, M)$, $\tilde{Y}(N, M)$. $X(N, M)$ and $Y(N, M)$ are coupled as previously described (they are the last passage times if forced to go right (or up) on the first move). Now to define the tilde versions of $X(N, M)$ and $Y(N, M)$,

for a given realization of weights, divide the boundary weights by their means. This creates a new set of weights coupled and pointwise dominated by the original set of weights. For this new set of weights, define $\tilde{X}(N, M)$ and $\tilde{Y}(N, M)$ as the last passage time if forced right (or up) initially. From this pointwise domination of the new weights by the original weights, we see that $X(N, M) \geq \tilde{X}(N, M)$ and $Y(N, M) \geq \tilde{Y}(N, M)$ pointwise. The advantage of the tilde variables is that $\tilde{Z}(N, M) = \max(\tilde{X}(N, M), \tilde{Y}(N, M))$ is the standard (without boundary conditions) last passage time. We also know that, asymptotically $X(N, M)$ and $\tilde{X}(N, M)$, as well as $Y(N, M)$ and $\tilde{Y}(N, M)$ have the same distribution. We wish to use this information to conclude that $Z(N, M)$ and $\tilde{Z}(N, M)$ have the same distribution as well.

Let us redefine our variables by properly shifting and scaling them, so that they have a limiting distribution. Our new $X(N, M)$ is

$$\frac{X(N, M) - (1 + \gamma^2)N}{\gamma(1 + \gamma^{-1})^{4/3} N^{1/3}}, \tag{78}$$

and similarly we define $\tilde{X}(N, M), Y(N, M), \tilde{Y}(N, M)$ and $\tilde{Z}(N, M)$ and $Z(N, M)$ is redefined in terms of the newly defined variables. Now we have the following setup: $X(N, M) \geq \tilde{X}(N, M)$ and $X(N, M) \Rightarrow F_0$ as well as $\tilde{X}(N, M) \Rightarrow F_0$; similarly $Y(N, M) \geq \tilde{Y}(N, M)$ and $Y(N, M) \Rightarrow F_0$ as well as $\tilde{Y}(N, M) \Rightarrow F_0$. Applying Lemma 4.2, we get that $Z(N, M)$ converges in distribution to F_0. $\square$

PROOF OF THEOREM 1.3, (F_1). There are two cases which yield to the same argument. Thus, we prove the case of $\pi > 1/(1 + \gamma)$ and $\eta = \gamma/(1 + \gamma)$ only. As in the last proof, let $X(N, M)$, $\tilde{X}(N, M)$, $Y(N, M)$ denote the suitably centered and scaled random variables. For the sake of applying Lemma 4.2 we define $\tilde{Y}(N, M) = Y(N, M)$. Again we know that $X(N, M) \geq \tilde{X}(N, M)$ and $X(N, M) \Rightarrow F_0$ as well as $\tilde{X}(N, M) \Rightarrow F_0$ and clearly the same holds for the $Y(N, M)$ and $\tilde{Y}(N, M)$. So by Lemma 4.2 since we know that $\max(\tilde{X}(N, M), \tilde{Y}(N, M))$ converges weakly to F_1, it follows that $\max(X(N, M), Y(N, M)) \Rightarrow F_1$. $\square$

PROOF OF THEOREM 1.3, $(F_{1,1})$. This follows immediately from the main result of [14]. $\square$

PROOF OF THEOREM 1.3, (G). We prove the first case, when $\pi < 1/(1 + \gamma)$ and $\eta > \frac{\pi}{\pi(1-\gamma^{-2})+\gamma^{-2}}$, as the other case has the same proof. In this region of π, η, the leading order term on the expression for $X(N, M)$ is larger than that of the expression for $Y(N, M)$. If we renormalize both $X(N, M)$ and $Y(N, M)$ by the leading order term for $X(N, M)$ and divide by its fluctuation term, we find that $X(N, M)$ converges to a standard normal. On the other hand, since the leading order term for $X(N, M)$ exceeds that for $Y(N, M)$, the renormalized $Y(N, M)$ converges to negative infinity. This implies that $\max(X(N, M), Y(N, M))$ converges in distribution to a standard normal, just like $X(N, M)$. $\square$

PROOF OF THEOREM 1.3, (G^2). We couple $X(N, M)$ with a random variable $\tilde{X}(N, M)$ where $\tilde{X}(N, M)$ is the last passage time when forced to stay along the bottom edge for at least a specific, deterministic fraction of the path [we likewise define $\tilde{Y}(N, M)$]. Specifically we define $\tilde{X}(N, M)$ to be the last passage time when the path is pinned to the bottom edge until the point

$$\left(1 - \frac{\gamma^2}{(\pi^{-1} - 1)^2}\right)N, \tag{79}$$

after which point is forced into the bulk and allowed to follow a last passage path therein. The weights accrued along the bottom edge respect a simple central limit (as they are the sums of a deterministic number of i.i.d. random variables) and the weights accrued after the path is forced into the bulk follows the fluctuations theorem for standard exponential last passage times. As these two random variables are independent by construction, their means add since their fluctuations are of different order ($N^{1/2}$ for the bottom and $N^{1/3}$ for the bulk) the bottom fluctuations win out. Following this idea, we find that the mean of $\tilde{X}(N, M)$ is $(\pi^{-1} + \frac{\pi^{-1}\gamma^2}{\pi^{-1}-1})N$ and the fluctuations are normal with variance

$$\pi^{-2}\left(1 - \frac{\gamma^2}{(\pi^{-1} - 1)^2}\right)N. \tag{80}$$

To see this, observe that if we center $\tilde{X}$ by the mean and divide by the square root of the above variance, we are left with a random variable of the form $\tilde{X}_1(N, M) + N^{-1/6}\tilde{X}_2(N, M)$, where $\tilde{X}_1(N, M)$ converges to a normal, and $\tilde{X}_2(N, M)$ converges to a GUE. Since the second term has a prefactor which goes to zero, we see that $P(\tilde{X}_1(N, M) + N^{-1/6}\tilde{X}_2(N, M) \le l)$ can be partitioned into a region of size ε where $|\tilde{X}_2(N, M)| \ge R$ and a region of size $1 - \varepsilon$ where $|\tilde{X}_2(N, M)| < R$. On the second region, we can replace $\tilde{X}_2(N, M)$ by R and find asymptotically that the probability differs from $P(\tilde{X}(N, M) \le l)$ by only ε. Taking ε to zero gives the desired convergence in distribution.

Therefore if we center $X(N, M)$ and $\tilde{X}(N, M)$ by the same amount and renormalize by the same amount we get two random variables which converge to the same distribution, despite the first one being almost always larger than the second one. This is one of the pieces we will need to apply Lemma 4.2.

We can likewise define $\tilde{Y}(N, M)$ as the last passage time when the path is pinned to the left edge until the point

$$\left(1 - \frac{\gamma^{-2}}{(\eta^{-1} - 1)^2}\right)M, \tag{81}$$

after which point is forced into the bulk and allowed to follow a last passage path therein. As in the prior we see that centered and renormalizing $Y(N, M)$ and $\tilde{Y}(N, M)$ by the same amounts, gives to random variables which converge to the

same distribution, despite the first one being almost always larger than the second one.

From the relationship between π and η we know that the leading order terms for both $\tilde{X}(N, M)$ and $\tilde{Y}(N, M)$ coincide. Therefore, we can write

$$\tilde{X}(N, M) = AN + B\pi N^{1/2}\tilde{X}_1(N, M) + C_\pi N^{1/3}\tilde{X}_2(N, M), \tag{82}$$

$$\tilde{Y}(N, M) = AN + B\eta N^{1/2}\tilde{Y}_1(N, M) + C_\eta N^{1/3}\tilde{Y}_2(N, M), \tag{83}$$

where

$$A = \left(\pi^{-1} + \frac{\pi^{-1}\gamma^2}{\pi^{-1} - 1}\right), \tag{84}$$

$$B_\pi = \left(\pi^{-2} - \frac{\pi^{-2}\gamma^2}{(\pi^{-1} - 1)^2}\right)^{1/2}, \tag{85}$$

$$B_\eta = \left(\eta^{-2} - \frac{\eta^{-2}\gamma^{-2}}{(\eta^{-1} - 1)^2}\right)^{1/2}, \tag{86}$$

and C_π and C_η are constants (which will play no role here). If we consider now $\tilde{Z}(N, M) = \max(\tilde{X}(N, M), \tilde{Y}(N, M))$, we find that

$$P\left(\frac{\tilde{Z}(N, M) - AN}{N^{1/2}\sqrt{B_\pi B_\eta}} \le x\right) = P(E_1 \text{ and } E_2),$$

where E_1 and E_2 are, respectively, the events

$$\sqrt{\frac{B_\pi}{B_\eta}}\tilde{X}_1(N, M) + C'_\pi N^{-1/6}\tilde{X}_2(N, M) \le x, \tag{87}$$

$$\sqrt{\frac{B_\eta}{B_\pi}}\tilde{Y}_1(N, M) + C'_\eta N^{-1/6}\tilde{Y}_2(N, M) \le x. \tag{88}$$

As before, because of the $N^{-1/6}$ prefactor to the $\tilde{X}_2(N, M)$ and $\tilde{Y}_2(N, M)$ terms, we can condition on these terms being bounded by some large number R, and only cost ourselves ε of the sample space. Once we have conditioned on these random variables being bounded by R, we can conclude that their joint probability is bounded between the product

$$\begin{aligned} &P\left(\tilde{X}_1(N, M) \le \sqrt{\frac{B_\eta}{B_\pi}}x - C'_\pi N^{-1/6}R\right) \\ &\quad \times P\left(\tilde{Y}_1(N, M) \le \sqrt{\frac{B_\pi}{B_\eta}}x - C'_\eta N^{-1/6}R\right) \end{aligned} \tag{89}$$

and

$$P\left(\tilde{X}_1(N,M) \le \sqrt{\frac{B_\eta}{B_\pi}}x + C'_\pi N^{-1/6}R\right) \times P\left(\tilde{Y}_1(N,M) \le \sqrt{\frac{B_\pi}{B_\eta}}x + C'_\eta N^{-1/6}R\right). \tag{90}$$

Taking N to infinity gives $P(\tilde{X}_1(N,M) \le x)P(\tilde{Y}_1(N,M) \le x)$ and taking ε to zero, and using the central limit theorem to show that $\tilde{X}_1(N,M)$ is standard normal, we find that

$$P\left(\frac{\tilde{Z}(N,M) - AN}{N^{1/2}\sqrt{B_\pi B_\eta}} \le x\right) = G_1\left(x\sqrt{\frac{B_\eta}{B_\pi}}\right)G_1\left(x\sqrt{\frac{B_\pi}{B_\eta}}\right). \tag{91}$$

Therefore, using the observations made at the beginning of the proof, we can apply Lemma 4.2 to conclude that the probability distribution of $Z(N,M)$ centered and normalized as above, converges to the same G^2 distribution [i.e., equation (91) holds with $Z(N,M)$ in place of $\tilde{Z}(N,M)$]. Working out the coefficients B_π and B_η using the relationship between π and η, we have our desired result. □

4.2. *Proof of Theorem* 1.1 (*Conjecture* 7.1 *from* [22]). The following elementary lemma will find repeated use in what follows.

LEMMA 4.3. *If $M = at + bt^{1/2}$ then for large t,*

$$t = a^{-1}M - a^{-3/2}bM^{1/2} + o(M^{1/2}). \tag{92}$$

Likewise if $M = at + bt^{1/3}$ then for large t,

$$t = a^{-1}M - a^{-4/3}bM^{1/3} + o(M^{1/3}). \tag{93}$$

PROOF OF THEOREM 1.1. We provide proofs of only the G^2 and F_0 cases of this theorem as the G case is analogous to G^2 and the F_1 case is analogous to F_0. The $F_{1,1}$ case of the theorem already is proved in [14]. The proofs are based on the fact that if the height at a given time value exceeds a point (N,M), then the last passage time of that point is less than the above time value. Then Theorem 1.3 applies and gives asymptotic height distribution results. As noted before, the mapping between the TASEP height function and the last passage time for our model of LPP with two-sided boundary conditions is not exact (as the exact LPP model has geometric numbers of boundary zeros) however Lemma 3.1 ensures that asymptotically all results in our LPP model correspond to results for the two-sided TASEP height function.

G^2 *case*: recall that in LPP with two-sided boundary conditions $\pi = 1 - \rho_+$ and $\eta = \rho_-$. We presently assume that $\eta + \pi < 1$ ($\rho_- < \rho_+$) and $y = y_c$. Recalling

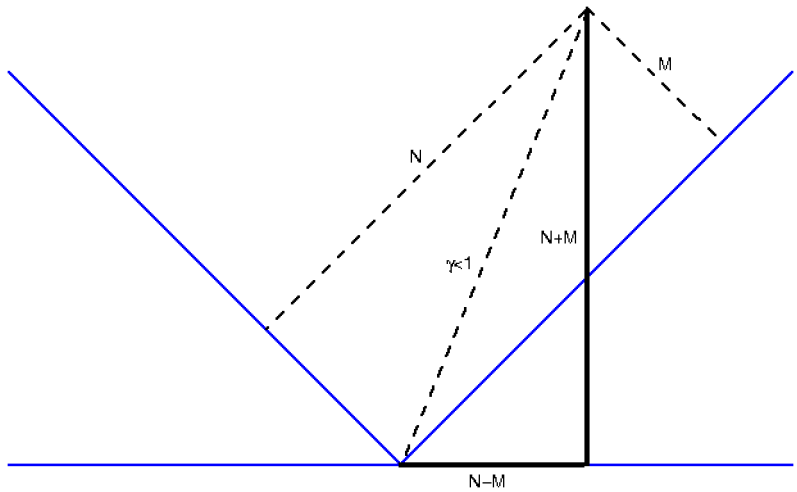

FIG. 5. *The event that the last passage time $L(N, M) \le t$ corresponds to the event that the height function $h_t(N - M) \ge N + M$.*

that $\bar{h}(y) = (1 - 2\eta)y + 2\eta(1 - \eta)$, we wish to determine the asymptotic (large t) value of

$$P\big(h_t(yt) \ge \big((1 - 2\eta)y + 2\eta(1 - \eta)\big)t - \sqrt{(1 - \pi - \eta)}t^{1/2}x\big). \tag{94}$$

We will reduce this probability to a probability in the related LPP with two-sided boundary conditions, and then use Theorem 1.3 to conclude that this probability is the correct product of Gaussian probability functions.

The first step in translating to a LPP problem is to relate the speed to γ. We may solve for the asymptotic value of γ as a function of the speed y. As shown in Figure 5, the height function event corresponds to the LPP event where $N - M = yt$ and $N + M = ((1 - 2\eta)y + 2\eta(1 - \eta))t - \sqrt{(1 - \pi - \eta)}t^{1/2}x)$. From that, we find that

$$M = \eta(1 - y - \eta)t - \tfrac{1}{2}\sqrt{(1 - \pi - \eta)}t^{1/2}x, \tag{95}$$

$$N = \big(y - \eta y + \eta(1 - \eta)\big)t - \tfrac{1}{2}\sqrt{(1 - \pi - \eta)}t^{1/2}x. \tag{96}$$

Therefore,

$$\gamma^2 = \lim_{t\to\infty} M/N = \frac{\eta(1 - y - \eta)}{y - \eta y + \eta(1 - \eta)}. \tag{97}$$

From this equation, we can solve for y as a function of γ:

$$y = \frac{(\gamma^2 - 1)\eta(1 - \eta)}{\gamma^2(\eta - 1) - \eta}. \tag{98}$$

Since we have assume that $y = y_c$, we may use these two expressions for y to relate η, π and γ to find that

$$\eta = \frac{\pi}{\pi(1-\gamma^{-2}) + \gamma^{-2}}. \tag{99}$$

This is exactly the curve along which the G^2 part of Theorem 1.3 applies.

Finally, we may use Lemma 4.3 to invert our expression for N in terms of t. Asymptotically

$$t = \frac{1+\pi(\gamma^2-1)}{\pi(1\pi)} N + \frac{(1-\pi-\eta)^{1/2}(1+\pi(\gamma^2-1))^{3/2}}{2(\pi(1-\pi))^{3/2}} x N^{1/2}. \tag{100}$$

This allows us then to express

$$\begin{aligned} &P\big(h_t(yt) \geq \big((1-2\eta)y + 2\eta(1-\eta)\big)t - \sqrt{(1-\pi-\eta)}t^{1/2}x\big) \\ &\quad = P\big(L_2(N,M) \leq t\big), \end{aligned} \tag{101}$$

where t is as above. It then follows after a little algebra that Theorem 1.3 applies and gives that these probabilities asymptotically equal

$$F_G\left(\frac{x}{2\sqrt{\pi(1-\pi)}}\right) F_G\left(\frac{x}{2\sqrt{\eta(1-\eta)}}\right), \tag{102}$$

as desired to prove this part of Theorem 1.1.

F_0 case: similarly to the previous case, we rehash the height function event in terms of the LPP with two-sided boundary conditions event and show that the desired asymptotic probabilities arise from Theorem 1.3. The region of ρ_-, ρ_+ for which we wish to prove F_0 fluctuations corresponds to $\pi + \eta > 1$. We wish to prove F_0 fluctuations for all y such that $1 - 2\rho_- < y < 1 - 2\rho_+$. Translating this region into η, π, γ variables exactly corresponds to the region in which Theorem 1.3 implies F_0 fluctuations.

We wish to compute the asymptotic formula for

$$P\big(h_t(yt) \geq \tfrac{1}{2}(y^2+1)t - 2^{-1/3}(1-y^2)^{2/3}t^{1/3}x\big), \tag{103}$$

where we have used the fact that $\bar{h}(y) = (y^2+1)/2$ in the region of y, ρ_-, ρ_+ which we are considering. Without loss of generality, let us assume that $y \geq 0$ (the other case follows similarly). As before, set

$$N - M = yt, \tag{104}$$

$$N + M = \tfrac{1}{2}(y^2+1)t - 2^{-1/3}(1-y^2)^{2/3}t^{1/3}x. \tag{105}$$

The height function event we are considering has the same probability as

$$P\big(L_2(N,M) \leq t\big). \tag{106}$$

Using the equations for $N - M$ and $N + M$, we can solve for

$$N = \tfrac{1}{4}(1+y)^2 t - 2^{-4/3}(1-y^2)^{2/3} t^{1/3} x, \tag{107}$$

$$M = \tfrac{1}{4}(1-y)^2 t - 2^{-4/3}(1-y^2)^{2/3} t^{1/3} x. \tag{108}$$

These expressions allow us to express γ asymptotically as

$$\begin{aligned} \gamma^2 = \left(\frac{1-y}{1+y}\right)^2 &\left[1 + 2^{2/3} x t^{-2/3} (1-y^2)^{2/3} \left(\frac{1}{(1+y)^2} - \frac{1}{(1-y)^2}\right)\right] \\ &+ o(t^{-2/3}). \end{aligned} \tag{109}$$

We may use Lemma 4.3 to invert our expression for M in terms of t. Asymptotically

$$t = \frac{4M}{(1-y)^2} + 2^{4/3} \frac{(1+y)^{2/3}}{(1-y)^2} x M^{1/3} + o(M^{1/3}). \tag{110}$$

This implies that $t^{-2/3} = \frac{4^{-2/3}}{(1-y)^{-4/3}} M^{-2/3} + o(M^{-2/3})$. This can be plugged into our expression for γ^2 and gives a new expression for γ^2 in term of M now:

$$\gamma^2 = \left(\frac{1-y}{1+y}\right)^2 \left[1 - 2^{4/3} x M^{-2/3} \frac{y}{(1+y)^{4/3}}\right] + o(M^{-2/3}). \tag{111}$$

From this, we may find that

$$y = \frac{1-\gamma}{1+\gamma} - \frac{(1-\gamma)\gamma}{(1+\gamma)^{5/3}} x M^{-2/3} + o(M^{-2/3}). \tag{112}$$

This can then be substituted into (110) which gives

$$t = (1+\gamma^{-1})^2 M + \frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} x + o(M^{1/3}). \tag{113}$$

Plugging this into $P(L_2(N, M) \le t)$, we find that our height function probability is asymptotically equal to

$$P\left(L_2(N, M) \le (1+\gamma^{-1})M + \frac{(1+\gamma)^{4/3}}{\gamma} M^{1/3} x + o(M^{1/3})\right). \tag{114}$$

As it was already noted, the y, ρ_-, ρ_+ which we are considered maps exactly onto the range of η, π, γ which the LPP probability above is asymptotically equal to $F_0(x)$ and hence the same holds for the the height function probability. $\square$

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