

## SCALING LIMIT FOR A CLASS OF GRADIENT FIELDS WITH NONCONVEX POTENTIALS

BY MAREK BISKUP<sup>1</sup> AND HERBERT SPOHN

*University of California at Los Angeles and University of South Bohemia, and  
Technische Universität München*

We consider gradient fields  $(\phi_x : x \in \mathbb{Z}^d)$  whose law takes the Gibbs–Boltzmann form  $Z^{-1} \exp\{-\sum_{\langle x, y \rangle} V(\phi_y - \phi_x)\}$ , where the sum runs over nearest neighbors. We assume that the potential  $V$  admits the representation

$$V(\eta) := -\log \int \varrho(d\kappa) \exp\left[-\frac{1}{2}\kappa\eta^2\right],$$

where  $\varrho$  is a positive measure with compact support in  $(0, \infty)$ . Hence, the potential  $V$  is symmetric, but nonconvex in general. While for strictly convex  $V$ 's, the translation-invariant, ergodic gradient Gibbs measures are completely characterized by their tilt, a nonconvex potential as above may lead to several ergodic gradient Gibbs measures with zero tilt. Still, every ergodic, zero-tilt gradient Gibbs measure for the potential  $V$  above scales to a Gaussian free field.

**1. Introduction.** Gradient fields belong to a class of models that arise in equilibrium statistical mechanics, for example, as approximations of critical systems and as effective interface models. Although their definition is rather simple and, in fact, quite a lot is known (see the reviews by Funaki [15], Velenik [27] or Sheffield [25]), there is still much to be learned. In this note, we study gradient fields on a lattice. Here, the field is a collection of real-valued random variables  $\phi := \{\phi_x : x \in \mathbb{Z}^d\}$  and the distribution of  $\phi$  on  $\mathbb{R}^{\mathbb{Z}^d}$  is given by the formal expression

$$(1.1) \quad \frac{1}{Z} \exp\left\{-\sum_{\langle x, y \rangle} V(\phi_y - \phi_x)\right\} \prod_{x \in \mathbb{Z}^d} d\phi_x,$$

where  $d\phi_x$  is the Lebesgue measure,  $\langle x, y \rangle$  refers to an unordered nearest-neighbor pair on  $\mathbb{Z}^d$  and  $V$  is an even, measurable function, called the *potential*, which is bounded from below and grows superlinearly at  $\pm\infty$ .

Of course, to define the measure (1.1) precisely, we have to restrict the above expression to a finite subset of  $\mathbb{Z}^d$  and fix the  $\phi$ 's on its boundary;  $Z$  is then the normalizing constant. Another way to regularize the expression (1.1) is to consider

---

Received April 2007; revised August 2009.

<sup>1</sup>Supported by NSF Grant DMS-05-05356.

*AMS 2000 subject classifications.* 60K35, 60F05, 82B41.

*Key words and phrases.* Gradient fields, scaling limit, Gaussian free field.

directly measures on all of  $\mathbb{R}^{\mathbb{Z}^d}$  whose conditional probabilities in finite sets take the above form. In our context, this standard definition is hampered by the fact that, due to the unbounded nature of the fields  $\phi_x$ , no such infinite-volume measures may exist at all. However, if one restricts attention to (the  $\sigma$ -algebra generated by) the gradient variables

$$(1.2) \quad \eta_{xy} := \phi_y - \phi_x, \quad |x - y| = 1,$$

then infinite-volume measures exist under the above conditions on  $V$ . Since the measure depends only on gradients, we refer to such measures as *gradient Gibbs measures* (GGM), in accordance with Funaki [15] and Sheffield [25].

Throughout, we will focus on translation-invariant GGMs. An important characteristic is their *tilt*. For a translation-invariant GGM  $\mu$ , there exists a unique tilt vector  $t \in \mathbb{R}^d$  such that

$$(1.3) \quad E_\mu(\eta_b) = t \cdot b$$

for every edge  $b$  of  $\mathbb{Z}^d$ , which we regard as a vector in this formula. Of course, this definition is only really meaningful for the GGMs that are ergodic, that is, trivial on the  $\sigma$ -algebra of events invariant under all lattice translations. Indeed, in the ergodic case,  $t$  represents the average incline of typical configurations.

For the case of quadratic  $V$ , the *massless free field*, the measure (1.1) is Gaussian and so many desired characteristics are amenable to explicit computations. The challenge for mathematicians has been to develop an equivalent level of understanding for nonquadratic  $V$ 's. A good amount of progress in this direction has been made in the last ten to fifteen years: Brydges and Yau [7] (and also earlier works, e.g., by Gawędzki and Kupiainen [12] and Magnen and Sénéor [19]) studied the effect of analytic perturbations of the quadratic potentials and concluded that the large-scale behavior is that of the massless free field. Naddaf and Spencer [23] proved the same nonperturbatively for strictly convex potentials  $V$  and GGMs with zero tilt. The corresponding extension to nonzero tilt was obtained by Giacomin, Olla and Spohn [13]. For the same class of potentials, Funaki and Spohn [16] proved a bijection between the ergodic GGMs and their tilt. Sheffield [25] characterized translation-invariant GGMs by means of a Gibbs variational principle and extended Funaki and Spohn's results to fields taking only a discrete set of values. We refer to the reviews by Funaki [15], Velenik [27] and Sheffield [25] for further results and references.

As a unifying feature, all the (nonperturbative) results mentioned are based on the strict convexity of the potential  $V$ , be it for the use of the Brascamp–Lieb inequality [13, 16, 23], Helffer–Sjöstrand random walk representation [13, 23], coupling to Langevin dynamics [16] and the cluster-swapping algorithm [25]. One would naturally like to have a nonperturbative approach that works even for nonconvex potentials. With this motivation, Biskup and Kotecký [5] recently studied the GGMs for nonconvex  $V$  that are a log-mixture of centered Gaussians,

$$(1.4) \quad V(\eta) := -\log \int \varrho(d\kappa) e^{-1/2\kappa\eta^2},$$

where  $\varrho$  is a positive measure with compact support in  $(0, \infty)$ . Surprisingly, already for the simplest nontrivial case,

$$(1.5) \quad \varrho := p\delta_{\kappa_1} + (1-p)\delta_{\kappa_2}$$

with  $\kappa_1 \gg \kappa_2 > 0$ , it was shown that, in  $d = 2$ , there is a value  $p \in (0, 1)$  at which one can construct *two* distinct, translation-invariant, gradient Gibbs measures of zero tilt.

The relevant conclusion from [5] for the general theory is that the one-to-one correspondence between ergodic GGMs and their tilt breaks down once  $V$  is sufficiently nonconvex. The next question which naturally arises is how to understand what happens to the scaling limit. The purpose of this note is to show that, regardless of the occurrence of phase transitions, for potentials of the form (1.4), every translation-invariant, ergodic GGM with zero tilt scales to a Gaussian free field (GFF).

The proof is based on the fact—utilized already in [5]—that (1.4) allows us to represent every GGM as a mixture over Gaussian gradient measures with a random coupling constant  $\kappa_{xy}$  for each edge  $\langle x, y \rangle$ . Its covariance is simply the inverse of the operator

$$(1.6) \quad (\mathcal{L}_\kappa f)(x) := \sum_{y: |y-x|=1} \kappa_{xy}[f(y) - f(x)],$$

where we take, once and for all,  $\kappa_{xy} = \kappa_{yx}$ . The *fluctuations* in the Gaussian measure can be analyzed by invoking a random walk representation;  $\mathcal{L}_\kappa$  is the generator of a random walk with symmetric random jump rates, known, equivalently, as a *random conductance model*. The name arises naturally from the electrostatic interpretation of this problem (cf. Doyle and Snell [9], in which one views  $\mathbb{Z}^d$  as a resistor network with *conductance*  $\kappa_{xy}$ —or resistivity  $1/\kappa_{xy}$ —assigned to an edge  $\langle x, y \rangle$ ). As it turns out (see Lemma 3.2), if the initial GGM is ergodic, then so is the law of the conductances. This makes homogenization a possible tool.

Much work has been done in the past two decades on the problem of random walks with random conductances. For our purposes, it suffices to invoke two known results: Kipnis and Varadhan's [18] invariance principle (i.e., scaling of the random walk to Brownian motion) and Delmotte and Deuschel's [10] annealed derivative heat kernel bounds. (Note that in the Helffer–Sjöstrand random walk representation, as used in [13, 23], one also has to study a random walk in a random environment. However, this random environment fluctuates in time, while, in our case, it is static.) This takes care of the fluctuations of the field; an important technical issue is thus the control of the mean. This is where the zero-tilt restriction comes in (see Lemma 3.4, Corollary 5.8 and discussion in Section 6).

*Note.* While this paper has been in the process of revision, further developments have occurred in the study of gradient models with nonconvex potentials. Cotar, Deuschel and Müller [8] have shown that for nonconvex perturbations of

potentials  $V$  where the size of the nonconvex region is small compared to typical fluctuations of the field, the conclusions are as in the convex case. (Their precise condition is a bound on the  $L^1$ -norm of the negative part of the second derivative.) This is a high-temperature result; work in progress by Adams, Kotecký and Müller [1] addresses the low-temperature case when nonconvexities are allowed only sufficiently far away from the absolute minimum of  $V$ . Our contribution remains valuable despite these advances as it applies to all potentials of the type (1.4), including those for which phase coexistence occurs.

This paper is organized as follows. In Section 2, we precisely define the concept of the gradient Gibbs measure and state our main theorem. In Section 3, we introduce the extended gradient Gibbs measures and characterize their conditional marginals. This will naturally lead to the aforementioned connections with random walks in reversible random environments. To keep the main line of the argument intact, we first finish proving our main result in Section 4 and only then expound on the random walk connections in Section 5. Section 6 is devoted to the discussion of the limitations to zero tilt and some open questions concerning gradient Gibbs measures.

**2. Main results.**

2.1. *Gradient Gibbs measures.* As mentioned above, infinite-volume measures on the field variables  $(\phi_x)$  may not always exist, particularly in sufficiently low dimensions. To make our statements uniform in dimension, we will focus attention on the gradient variables. However, not even that will be entirely straightforward because the gradient variables satisfy a host of “hard-core” constraints which, in a sense, encapsulate most of the interaction. Since  $\eta$  is gradient, one has

$$(2.1) \quad \eta_{x_1,x_2} + \eta_{x_2,x_3} + \eta_{x_3,x_4} + \eta_{x_4,x_1} = 0,$$

whenever  $(x_1, \dots, x_4)$  are the vertices of a cycle in  $\mathbb{Z}^d$  of length four. We will often write  $\eta_b$  for the *positively oriented* edge  $b$  in  $\mathbb{Z}^d$ . Throughout, we will only work with positively oriented edges and will use  $\mathbb{B}(\Lambda)$  to denote the set of such edges with both endpoints in the set  $\Lambda \subset \mathbb{Z}^d$ .

The constraints (2.1) are implemented at the level of the a priori measure which is defined as follows. Fix a configuration  $\eta \in \mathbb{R}^{\mathbb{B}(\mathbb{Z}^d)}$  that obeys (2.1) and, for  $\Lambda \subset \mathbb{Z}^d$  finite, let  $\nu_\Lambda^{(\eta_{\mathbb{B}(\Lambda)^c})}$  be the Lebesgue measure on the linear subspace of configurations  $(\eta'_b)$  such that  $\eta'_b = \eta_b$  for all  $b \notin \mathbb{B}(\Lambda)$  and that  $\eta'$  obeys the constraints (2.1). Note that if  $\bar{\phi}$  is a configuration such that  $\eta_{xy} = \bar{\phi}_y - \bar{\phi}_x$  for every nearest-neighbor pair  $\langle x, y \rangle$ , then  $\nu_\Lambda^{(\eta_{\mathbb{B}(\Lambda)^c})}$  is, to within a normalization constant, the projection to gradient variables of the Lebesgue measure on  $\{\phi_x : x \in \Lambda\}$  subject to the boundary condition  $\bar{\phi}$ .

Next, we will give a precise definition of the notion of gradient Gibbs measure. For a finite  $\Lambda \subset \mathbb{Z}^d$ , consider the *specification*  $\gamma_\Lambda$ , which is a measure in the

first coordinate and a function of the boundary condition in the second coordinate, defined by

$$(2.2) \quad \begin{aligned} &\gamma_\Lambda(d\eta_{\mathbb{B}(\Lambda)}|\eta_{\mathbb{B}(\Lambda)^c}) \\ &:= \frac{1}{Z_\Lambda(\eta_{\mathbb{B}(\Lambda)^c})} \exp\left\{-\sum_{\substack{(x,y) \\ x \in \Lambda, y \in \Lambda \cup \partial\Lambda}} V(\eta_{xy})\right\} \nu_\Lambda^{(\eta_{\mathbb{B}(\Lambda)^c})}(d\eta_{\mathbb{B}(\Lambda)}). \end{aligned}$$

Here,  $Z_\Lambda(\eta_{\mathbb{B}(\Lambda)^c})$  is the normalizing constant.

DEFINITION 2.1. Let  $\mathcal{E}_{\mathbb{B}(\Lambda)} := \sigma(\{\eta_b : b \in \mathbb{B}(\Lambda)\})$ . We say that a measure  $\mu$  on  $\mathbb{R}^{\mathbb{B}(\mathbb{Z}^d)}$  is a *gradient Gibbs measure* if the regular conditional probability  $\mu(-|\mathcal{E}_{\mathbb{B}(\Lambda)^c})$  in any finite  $\Lambda \subset \mathbb{Z}^d$  satisfies

$$(2.3) \quad \mu(-|\mathcal{E}_{\mathbb{B}(\Lambda)^c})(\eta) = \gamma_\Lambda(-|\eta_{\mathbb{B}(\Lambda)^c})$$

for  $\mu$ -a.e.  $\eta$ .

Most of this paper is restricted to translation-invariant gradient Gibbs measures. To define the required notation, for each  $x \in \mathbb{Z}^d$ , let  $\tau_x : \mathbb{R}^{\mathbb{B}(\mathbb{Z}^d)} \rightarrow \mathbb{R}^{\mathbb{B}(\mathbb{Z}^d)}$  be the “translation by  $x$ ” which acts on configurations  $\eta$  by shifting the origin to position  $x$ ,

$$(2.4) \quad (\tau_x \eta)_{yz} := \eta_{y+x, z+x}, \quad (y, z) \in \mathbb{B}(\mathbb{Z}^d).$$

We say that  $\mu$  is *translation-invariant* if  $\mu \circ \tau_x^{-1} = \mu$  for all  $x \in \mathbb{Z}^d$  and that it is *ergodic* if  $\mu(A) \in \{0, 1\}$  for every event  $A$  such that  $\tau_x^{-1}(A) = A$  for all  $x \in \mathbb{Z}^d$ .

2.2. *Scaling limit.* As is usual for problems involving random fields, we will interpret samples from gradient Gibbs measures as random linear functionals on an appropriate space of functions. Let  $C_0^\infty(\mathbb{R}^d)$  denote the set of all infinitely differentiable functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  with compact support. Given a configuration  $\eta = (\eta_b)$  of gradients satisfying the conditions (2.1), we can find a configuration of the field  $\phi = (\phi_x)$  such that (1.2) holds for every nearest-neighbor pair of sites. The configuration  $\phi$  is determined uniquely once we fix the value at one site, for example,  $\phi_0$ . For any function  $f \in C_0^\infty(\mathbb{R}^d)$ , we introduce the random linear functional

$$(2.5) \quad \phi(f) := \int dx f(x) \phi_{[x]},$$

which, under the condition

$$(2.6) \quad \int dx f(x) = 0,$$

does not depend on the choice of the special value  $\phi_0$ .

The functional  $\phi(f)$  can be naturally extended to a somewhat larger space, defined as follows. Let  $\Delta$  denote the Laplace differential operator in  $\mathbb{R}^d$  and consider the set

$$(2.7) \quad \mathcal{H}_0 := \{\Delta g : g \in C_0^\infty(\mathbb{R}^d)\}.$$

Note that each  $f \in \mathcal{H}_0$  automatically obeys (2.6). The set  $\mathcal{H}_0$  is endowed with a natural quadratic form  $f \mapsto (f, f) + (f, -\Delta^{-1} f)$ , defined as

$$(2.8) \quad (\Delta g, \Delta g) + (\Delta g, -\Delta^{-1} \Delta g) = \int_{\mathbb{R}^d} dx (|\Delta g(x)|^2 + |\nabla g(x)|^2).$$

We thus define the norm

$$(2.9) \quad \|f\|_{\mathcal{H}} := [(f, f) + (f, -\Delta^{-1} f)]^{1/2}$$

and let  $\mathcal{H}$  be the completion of  $\mathcal{H}_0$  in this norm. Note that  $\mathcal{H}$  corresponds to the case  $k = -1/2$  in the family of Sobolev spaces  $W^{k,2}(\mathbb{R}^d)$ . The condition that  $(f, -\Delta^{-1} f) < \infty$  is natural once we realize that this quantity will represent the variance of the limiting Gaussian field.

The extension of  $\phi$  to  $\mathcal{H}$  is implied by the following lemma.

LEMMA 2.2. *Suppose that  $\varrho$  in (1.4) has support bounded away from zero and let  $\mu$  be a translation-invariant, ergodic, zero-tilt gradient Gibbs measure for the potential  $V$ . There then exists a constant  $c < \infty$  such that for each  $f \in \mathcal{H}_0$ ,*

$$(2.10) \quad \|\phi(f)\|_{L^2(\mu)} \leq c \|f\|_{\mathcal{H}}.$$

*In particular,  $\phi$  extends to a linear functional  $\phi : \mathcal{H} \rightarrow \mathbb{R}$ .*

Note that (2.10) means that the map  $f \mapsto \phi(f)$  is continuous in  $L^2$ -norm. If we want to avoid questions about accumulations of null sets, this permits us to work with only a countable number of  $f$ 's at any each time. [In particular, we do not claim that  $f \mapsto \phi(f)$  is continuous in any pointwise sense.] This will not pose any problems because we will content ourselves with the following (weaker) definition of a Gaussian free field based on the standard approach via Gaussian Hilbert spaces (cf. Sheffield [26], Section 2.4).

DEFINITION 2.3. We say that a family  $\{\psi(f) : f \in \mathcal{H}\}$  of random variables on a probability space  $(\Omega, \mathcal{F}, P)$  is a Gaussian free field if the map  $f \mapsto \psi(f)$  is linear a.s. and each  $\psi(f)$  is Gaussian with mean zero and variance

$$(2.11) \quad E(\psi(f)^2) = (f, -\Delta^{-1} f).$$

Our goal is to show that the family of random variables  $\{\phi(f) : f \in \mathcal{H}\}$  has, asymptotically, in the scaling limit, the law of a linear transformation of a Gaussian

free field. To pass to this limit, we have to impose the condition that the test functions are slowly varying, which we take to be on the scale  $\varepsilon^{-1}$ . For  $\varepsilon > 0$  and a function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ , let

$$(2.12) \quad f_\varepsilon(x) := \varepsilon^{(d/2+1)} f(\varepsilon x)$$

and note that the normalization ensures that

$$(2.13) \quad \begin{aligned} \|f_\varepsilon\|_{\mathcal{H}}^2 &= (f_\varepsilon, f_\varepsilon) + (f_\varepsilon, (-\Delta)^{-1} f_\varepsilon) \\ &= \varepsilon^2 (f, f) + (f, (-\Delta)^{-1} f) \leq_{\varepsilon \leq 1} \|f\|_{\mathcal{H}}^2. \end{aligned}$$

Let  $\phi_\varepsilon$  denote the linear functional acting on  $f \in C_0^\infty(\mathbb{R}^d)$  via

$$(2.14) \quad \phi_\varepsilon(f) := \phi(f_\varepsilon) = \int dx f(x) (\varepsilon^{(-d/2+1)} \phi_{[x/\varepsilon]}).$$

The main theorem is the Gaussian scaling limit for  $\phi_\varepsilon(f)$ .

**THEOREM 2.4 (Scaling to GFF).** *Suppose that  $V$  is as in (1.4) with  $\varrho$  compactly supported in  $(0, \infty)$ . Let  $\mu$  be a gradient Gibbs measure for the potential  $V$  which we assume to be ergodic with respect to the translations of  $\mathbb{Z}^d$  and to have zero tilt. Then, for every  $f \in \mathcal{H}$ ,*

$$(2.15) \quad \lim_{\varepsilon \downarrow 0} E_\mu(e^{i\phi_\varepsilon(f)}) = \exp\left\{\frac{1}{2} \int dx f(x) (Q^{-1} f)(x)\right\},$$

where  $Q^{-1}$  is the inverse of the operator

$$(2.16) \quad Qf := \sum_{i,j=1}^d q_{ij} \frac{\partial^2}{\partial x_j \partial x_i} f,$$

with  $(q_{ij})$  denoting some positive semidefinite, nondegenerate,  $d \times d$  matrix. In other words, the law of  $\phi_\varepsilon$  on the linear dual  $\mathcal{E}'$  of any finite-dimensional linear subspace  $\mathcal{E} \subset \mathcal{H}$  converges weakly to that of a Gaussian field with mean zero and covariance  $(-Q)^{-1}$ .

**REMARKS 2.5.** There follow some additional observations and remarks concerning the model under consideration and the results above.

- (1) Since  $(-Q)$  is dominated by a multiple of  $(-\Delta)$  from below, the integral in (2.15), interpreted as the quadratic form  $(f, Q^{-1} f)$ , is well defined for all  $f \in \mathcal{H}$ .
- (2) Note that in (2.14), the individual  $\phi$ 's get scaled by  $\varepsilon^{(-d/2+1)}$ , not  $\varepsilon^{-d/2}$  as one might expect from the conventional central limiting reasoning. This has to do

with the fact that the variables  $(\phi_x)$  are strongly correlated. These correlations are weaker for the gradients  $\eta_{xy} := \phi_y - \phi_x$  which adhere to the “usual” central limit scaling. In  $d = 1$  and for general potentials  $V$ , the increments  $\eta_b$  are in fact i.i.d. and the scaling limit follows from the standard central limit theorem.

- (3) In  $d > 1$ , the matrix  $(q_{ij})$  is not necessarily a multiple of unity since, in general,  $\mu$  is not guaranteed to be invariant under reflections and rotations of  $\mathbb{Z}^d$ . [Nevertheless, we expect that every zero-tilt, translation-invariant, ergodic measure for the isotropic interaction (2.2) will inherit these symmetries.] To get convergence of  $\phi_\varepsilon$  to GFF in the sense of Definition 2.3, one must thus scale the argument of  $\phi$  by the root of the corresponding eigenvalue of  $q$  in each of its principal directions.
- (4) The absence of strict convexity does not permit us to use the general argument of Funaki and Spohn [16] for the *existence* of an ergodic GGM with zero (or any other prescribed) tilt. To show that such GGMs do exist—and that our Theorem 2.4 is not vacuous—we note that, by Lemma 4.8 of Biskup and Kotecký [5], every weak limit of torus measures exhibits exponential concentration of the empirical tilt; one then just needs to choose any ergodic component. Note that this lemma applies only to zero tilt (cf. [5], Remark 4.9).
- (5) The restriction to zero tilt is actually a significant drawback of our analysis. The main reason is our inability to characterize the scaling limit of the so-called corrector for the corresponding random walk problem. See Section 6 for more details.
- (6) In the example studied by Biskup and Kotecký [5] [cf. (1.5)], the two GGMs coexisting at the transitional value  $p_t$  of  $p$  were proven to exhibit different characteristic fluctuations. It follows that the corresponding scaling limits will be distinguished by their stiffness coefficients  $q_{ij}$ .

Moreover, by Theorem 2.5 of [5], for  $\kappa_1 \gg \kappa_2$ , the transition in the  $d = 2$  model with (1.5) lies on a self-dual line, that is,

$$(2.17) \quad \frac{p_t}{1 - p_t} = \left(\frac{\kappa_2}{\kappa_1}\right)^{1/4}.$$

The transition presumably stays on this line even as one slides the ratio  $\kappa_1/\kappa_2$  toward one. However, it disappears before  $\kappa_1/\kappa_2$  hits one because, for  $\kappa_1 \approx \kappa_2$ , the potential  $V$  is convex and so there is only one GGM with zero tilt [16]. At such a point of disappearance, physicists often expect nontrivial critical fluctuations. Notwithstanding, our results show that this is not the case.

- (7) We avoid the context of the “stronger” definition of GFF as a random element in an appropriate Banach space (cf. Gross [14] or Sheffield [26], Section 2.2). This definition is appealing in  $d = 1$ , where the limiting functional  $f \mapsto \psi(f)$  actually admits the integral representation

$$(2.18) \quad \psi(f) = \int_{\mathbb{R}} f(t)\psi_t dt$$

with  $t \mapsto \psi_t$  denoting a continuous diffusion with generator  $Q$ , but in  $d > 1$ , the corresponding field becomes less and less regular with increasing dimension and the appeal is lost. However, this context would be ideal if one wished to discuss the notion of tightness and convergence in law for the limit in Theorem 2.4.

Both Lemma 2.2 and Theorem 2.4 are proved in Section 4.

### 3. Extended gradient Gibbs measures.

3.1. *Coupling to random conductance model.* The key idea underlying the representation (1.4) is that the auxiliary variable  $\kappa$  in the expression for  $V$  may be elevated to a genuine degree of freedom associated with the corresponding edge. Specifically, given a gradient Gibbs measure  $\mu$  with potential (1.4), for each finite  $\Lambda \subset \mathbb{B}(\mathbb{Z}^d)$ , consider the measure  $\tilde{\mu}_\Lambda$  on  $\mathbb{R}^{\mathbb{B}(\mathbb{Z}^d)} \times \mathbb{R}^\Lambda$  defined by

$$(3.1) \quad \tilde{\mu}_\Lambda(\mathcal{A} \times \mathcal{B}) := \int_{\mathcal{B}} \prod_{b \in \Lambda} \varrho(d\kappa_b) E_\mu \left( \mathbf{1}_{\mathcal{A}} \prod_{b \in \Lambda} e^{V(\eta_b) - 1/2\kappa_b \eta_b^2} \right),$$

where  $\mathcal{A} \subset \mathbb{R}^{\mathbb{B}(\mathbb{Z}^d)}$  and  $\mathcal{B} \subset \mathbb{R}^\Lambda$  are Borel sets. The representation (1.4) ensures that  $(\tilde{\mu}_\Lambda)$  is a consistent family of measures; by Kolmogorov’s extension theorem, these are projections from a unique measure  $\tilde{\mu}$  onto configurations  $(\eta_b, \kappa_b) \in \mathbb{R}^{\mathbb{B}(\mathbb{Z}^d)} \times \mathbb{R}^{\mathbb{B}(\mathbb{Z}^d)}$ . The restriction of  $\tilde{\mu}$  to the  $\eta$ ’s gives us back  $\mu$ ; we call  $\tilde{\mu}$  an *extension* of  $\mu$ . The measure  $\tilde{\mu}$  is Gibbs for the Hamiltonian  $\sum_{\langle x,y \rangle} \frac{1}{2} \kappa_{xy} \eta_{xy}^2$ , so we will refer to it as an *extended gradient Gibbs measure* (see Biskup and Kotecký [5] for further facts on extended GGMs).

To ease the notation, whenever  $b$  is an edge between  $x$  and  $y$ , we may interchangeably write  $\kappa_b$  and  $\kappa_{xy}$  for the same quantity. Furthermore, for the same reasons, it will even be convenient to assume that

$$(3.2) \quad \kappa_{xy} = \kappa_{yx}, \quad |x - y| = 1$$

and work with the  $\kappa$ ’s as symmetric objects.

We proceed with a series of lemmas that characterize the properties of  $\tilde{\mu}$ .

LEMMA 3.1. *Let  $\mu$  be a gradient Gibbs measure for the potential  $V$  and let  $\tilde{\mu}$  be its extension to  $\mathbb{R}^{\mathbb{B}(\mathbb{Z}^d)} \times \mathbb{R}^{\mathbb{B}(\mathbb{Z}^d)}$ . Consider the  $\sigma$ -field  $\mathcal{E} := \sigma(\{\eta_b : b \in \mathbb{B}(\mathbb{Z}^d)\})$ . For  $\tilde{\mu}$ -a.e.  $\eta$ , the regular conditional distribution  $\tilde{\mu}(-|\mathcal{E})(\eta)$ , regarded as a measure on the  $\kappa$ ’s, takes the product form*

$$(3.3) \quad \tilde{\mu}(d\kappa|\mathcal{E})(\eta) = \bigotimes_{b \in \mathbb{B}(\mathbb{Z}^d)} [e^{V(\eta_b) - 1/2\kappa_b \eta_b^2} \varrho(d\kappa_b)].$$

PROOF. Recall that  $\mathcal{E}_{\mathbb{B}(\Lambda)} := \sigma(\{\eta_b : b \in \mathbb{B}(\Lambda)\})$ . The identity (3.1) implies that  $\tilde{\mu}_\Lambda$  coincides with  $\tilde{\mu}$  on  $\mathbb{R}^{\mathbb{B}(\Lambda)} \times \mathbb{R}^{\mathbb{B}(\Lambda)}$ . However,  $\mu_\Lambda(-|\mathcal{E}_\Lambda)$  has the desired product form by definition and so the claim follows by standard approximation arguments.  $\square$

LEMMA 3.2. *Let  $\mu$  be a gradient Gibbs measure and let  $\tilde{\mu}$  be its extension to  $\mathbb{R}^{\mathbb{Z}^d} \times \mathbb{R}^{\mathbb{Z}^d}$ . If  $\mu$  is translation-invariant and ergodic, then so is  $\tilde{\mu}$ .*

PROOF. The uniqueness of the extension of measures (3.1) implies that  $\tilde{\mu}$  is translation-invariant if  $\mu$  is translation-invariant and so it remains to prove that ergodicity is also inherited. Let  $A \subset \mathbb{R}^{\mathbb{Z}^d} \times \mathbb{R}^{\mathbb{Z}^d}$  be a translation-invariant event, that is,  $(\eta, \kappa) \in A$  if and only if  $(\tau_x \eta, \tau_x \kappa) \in A$  for all  $x$ . Our task is to show that  $\tilde{\mu}(A) \in \{0, 1\}$ .

First, we invoke the ergodicity of  $\mu$ . Consider the function

$$(3.4) \quad f(\eta) := E_{\tilde{\mu}}(\mathbf{1}_A | \mathcal{E})(\eta).$$

Since  $A$  and  $\tilde{\mu}$  are translation-invariant, we have

$$(3.5) \quad f(\tau_x \eta) = E_{\tilde{\mu}}(\mathbf{1}_A | \mathcal{E})(\tau_x \eta) = E_{\tilde{\mu}}(\mathbf{1}_A \circ \tau_x^{-1} | \mathcal{E})(\eta) = f(\eta), \quad \tilde{\mu}\text{-a.s.}$$

But  $f$  is  $\mathcal{E}$ -measurable and the restriction of  $\tilde{\mu}$  to  $\mathcal{E}$  is  $\mu$ , which we assumed to be ergodic. Hence,  $f$  is constant almost surely. Let  $c$  denote this constant.

We will use a standard approximation argument to show that  $c \in \{0, 1\}$ . Since  $A$  is an event from the product  $\sigma$ -algebra, there exists a sequence of events

$$(3.6) \quad A_n \in \sigma(\{\eta_{xy}, \kappa_{xy} : |x - y| = 1, |x| \leq n\})$$

such that

$$(3.7) \quad \tilde{\mu}(A \Delta A_n) \xrightarrow{n \rightarrow \infty} 0.$$

The bound

$$(3.8) \quad \|\mathbf{1}_{A_n} - \mathbf{1}_A\|_{L^1(\tilde{\mu})} \leq \tilde{\mu}(A \Delta A_n)$$

then shows that  $\mathbf{1}_{A_n} \rightarrow \mathbf{1}_A$  in  $L^1(\tilde{\mu})$ . Since  $A$  is translation-invariant, we have  $\mathbf{1}_A = \mathbf{1}_A \mathbf{1}_{\tau_x^{-1}(A)}$ . Each indicator can be approximated by the indicator of the event  $A_n$ ; a simple bound gives

$$(3.9) \quad \|\mathbf{1}_{A_n} \mathbf{1}_{\tau_x^{-1}(A_n)} - \mathbf{1}_A \mathbf{1}_{\tau_x^{-1}(A)}\|_{L^1(\tilde{\mu})} \leq 2\tilde{\mu}(A \Delta A_n).$$

For  $x$  with  $|x| > 2n + 1$ , the fact that  $\tilde{\mu}(-|\mathcal{E})$  is a product measure (cf. Lemma 3.1) implies that  $A_n$  and  $\tau_x^{-1}(A_n)$  are independent. Hence,

$$(3.10) \quad E_{\tilde{\mu}}(\mathbf{1}_{A_n} \mathbf{1}_{\tau_x^{-1}(A_n)} | \mathcal{E}) = E_{\tilde{\mu}}(\mathbf{1}_{A_n} | \mathcal{E}) E_{\tilde{\mu}}(\mathbf{1}_{\tau_x^{-1}(A_n)} | \mathcal{E}).$$

Rolling the approximations backward, we then conclude that the left-hand side converges to  $c$  in  $L^1(\tilde{\mu})$ , while the right-hand side converges to  $c^2$  (note that all expectations are bounded). It follows that  $c = c^2$ , that is,  $c \in \{0, 1\}$ . As

$$(3.11) \quad \tilde{\mu}(A) = E_{\tilde{\mu}}(f) = c,$$

the proof is finished.  $\square$

3.2. *Random walk connections.* Our next goal will be to characterize also the conditional measure given the  $\kappa$ 's. This will, in turn, require some facts from the theory of random walks with random conductances. We will frequently borrow facts from an associated potential theory which will be expounded in Section 5.

Let us choose a configuration  $\kappa = (\kappa_b)$  with  $\kappa_b \in (0, \infty)$  and recall the formula (1.6) for the generator  $\mathcal{L}_\kappa$  of the random walk among conductances  $\kappa$ . We will focus on the action of  $\mathcal{L}_\kappa$  on functions of both the environment  $\kappa$  and the position  $x$  that satisfy the following *shift covariance* property:

$$(3.12) \quad g(\kappa, x + b) - g(\kappa, x) = g(\tau_x \kappa, b),$$

with  $x \in \mathbb{Z}^d$  and  $b$  a coordinate unit vector in  $\mathbb{R}^d$ , subject to the condition

$$(3.13) \quad g(\kappa, 0) = 0.$$

This makes the function completely determined by its values at the neighbors of the origin. A function of this kind is said to be *harmonic* for the above random walk if

$$(3.14) \quad \mathcal{L}_\kappa g(\kappa, \cdot) = 0$$

for (almost) every  $\kappa$ . As it turns out, harmonic, shift-covariant functions are uniquely determined (a.s.) by their mean with respect to ergodic measures on the conductances.

LEMMA 3.3. *Let  $\nu$  be a translation-invariant, ergodic probability measure on configurations  $\kappa = (\kappa_b) \in \mathbb{R}^{\mathbb{B}(\mathbb{Z}^d)}$  such that  $\nu(\varepsilon \leq \kappa_b \leq 1/\varepsilon) = 1$  for some  $\varepsilon > 0$ . Let  $g : \mathbb{R}^{\mathbb{B}(\mathbb{Z}^d)} \times \mathbb{Z}^d \rightarrow \mathbb{R}$  be a measurable function which is:*

- (1) *harmonic in the sense of (3.14),  $\nu$ -a.s.;*
- (2) *shift-covariant in the sense of (3.12) and (3.13),  $\nu$ -a.s.;*
- (3) *square integrable in the sense that  $E_\nu |g(\cdot, x)|^2 < \infty$  for all  $x$  with  $|x| = 1$ .*

*If  $E_\nu(g(\cdot, x)) = 0$  for all  $x$  with  $|x| = 1$ , then  $g(\cdot, x) = 0$  a.s. for all  $x \in \mathbb{Z}^d$ .*

We defer the proof, and further discussion of the consequences of shift covariance and harmonicity, to Section 5. Returning to the gradient fields, we now characterize the conditional law given the  $\kappa$ 's.

LEMMA 3.4. *Let  $\mu$  be a translation-invariant, ergodic gradient Gibbs measure with zero tilt and let  $\tilde{\mu}$  be its extension to  $\mathbb{R}^{\mathbb{B}(\mathbb{Z}^d)} \times \mathbb{R}^{\mathbb{B}(\mathbb{Z}^d)}$ . Consider the  $\sigma$ -field  $\mathcal{F} := \sigma(\{\kappa_b : b \in \mathbb{B}(\mathbb{Z}^d)\})$ . For  $\tilde{\mu}$ -a.e.  $\kappa$ , the conditional law  $\tilde{\mu}(-|\mathcal{F})(\kappa)$ , regarded as a measure on the set of configurations  $\{(\phi_x) \in \mathbb{R}^{\mathbb{Z}^d} : \phi_0 = 0\}$  with the  $\phi$ 's defined from the  $\eta$ 's via (1.2), is Gaussian with mean zero,*

$$(3.15) \quad E_{\tilde{\mu}}(\phi_x | \mathcal{F})(\kappa) = 0, \quad x \in \mathbb{Z}^d,$$

and covariance given by  $(-\mathcal{L}_\kappa)^{-1}$ . Explicitly, for each  $f : \mathbb{Z}^d \rightarrow \mathbb{R}$  with finite support and  $\sum_x f(x) = 0$ ,

$$(3.16) \quad \text{Var}_{\tilde{\mu}}\left(\sum_x f(x)\phi_x \mid \mathcal{F}\right)(\kappa) = \sum_x f(x)(-\mathcal{L}_\kappa^{-1}f)(x).$$

PROOF. The fact that the conditional measure is a multivariate Gaussian law with covariance  $\mathcal{L}_\kappa^{-1}$  can be checked by direct inspection of (3.1). The only non-trivial task is to identify the mean. First, we note that the loop conditions (2.1) ensure that there exists a function  $u : \mathbb{R}^{\mathbb{B}(\mathbb{Z}^d)} \times \mathbb{Z}^d \rightarrow \mathbb{R}$  such that

$$(3.17) \quad u(\kappa, 0) = 0$$

and

$$(3.18) \quad u(\kappa, x + b) - u(\kappa, x) = E_{\tilde{\mu}}(\eta_{x,x+b} \mid \mathcal{F})(\kappa)$$

for all unit vectors  $b$  in the coordinate directions. We claim that  $u$  is harmonic in the sense of (3.14). Indeed,

$$(3.19) \quad \mathcal{L}_\kappa u(\kappa, x) = E_{\tilde{\mu}}\left(\sum_{y: |y-x|=1} \kappa_{xy}(\phi_y - \phi_x) \mid \mathcal{F}\right)(\kappa),$$

where we write, thanks to the loop conditions,  $\eta_{xy} = \phi_y - \phi_x$ . Using the fact that  $\tilde{\mu}$  is Gibbs, we can now also condition on  $\sigma(\phi_y : y \neq x)$ ; the conditional measure  $\mu_{\{x\}}$  is Gaussian with the explicit form

$$(3.20) \quad \mu_{\{x\}}(d\phi_x) = \frac{1}{Z} \exp\left\{-\frac{1}{2}\phi_x^2 \sum_{y: |y-x|=1} \kappa_{xy} + \phi_x \sum_{y: |x-y|=1} \kappa_{xy}\phi_y\right\} d\phi_x,$$

where  $Z$  is an appropriate normalization constant. It is easy to check that the mean of  $\phi_x \sum_{y: |y-x|=1} \kappa_{xy}$  under  $\mu_{\{x\}}$  is exactly  $\sum_{y: |y-x|=1} \kappa_{xy}\phi_y$ , proving that  $\mathcal{L}_\kappa u(\kappa, x) = 0$ .

Next, we observe that the translation invariance of  $\tilde{\mu}$  implies that

$$(3.21) \quad \begin{aligned} u(\tau_x \kappa, b) - u(\tau_x \kappa, 0) &= E_{\tilde{\mu}}(\eta_{0,b} \mid \mathcal{F})(\tau_x \kappa) \\ &= E_{\tilde{\mu}}(\eta_{x,x+b} \mid \mathcal{F})(\kappa) \\ &= u(\kappa, x + b) - u(\kappa, x) \end{aligned}$$

and so  $u$  is shift-covariant, as defined in (3.12) and (3.13). Finally, the definition of  $u$  and the fact that  $\tilde{\mu}$  has zero tilt imply that

$$(3.22) \quad E_{\tilde{\mu}}(u(\cdot, x)) = E_{\tilde{\mu}}(\eta_{0,x}) = 0, \quad |x| = 1.$$

As  $u$  obeys all conditions of Lemma 3.3, we have  $E_{\tilde{\mu}}(\phi_x \mid \mathcal{F}) = u(\cdot, x) = 0$   $\tilde{\mu}$ -a.s. □

Our reference to the random walk with generator  $\mathcal{L}_\kappa$  is not limited to Lemma 3.3; we will also need to know some specific properties of this random walk. First, we will need to know that the position of the walk satisfies a central limit theorem. Let  $X = (X_t)$  denote the continuous-time random walk with the generator  $\mathcal{L}_\kappa$  and let  $P_\kappa^x$  denote the law of  $X$  subject to the initial condition  $P_\kappa^x(X_0 = x) = 1$ . The following lemma goes back to Kipnis and Varadhan [18].

LEMMA 3.5 (Annealed central limit theorem). *Let  $\mu$  be a measure on  $\mathbb{R}^{\mathbb{B}(\mathbb{Z}^d)}$  which is translation-invariant, ergodic and obeys  $\mu(\varepsilon \leq \omega_b \leq 1/\varepsilon) = 1$  for some  $\varepsilon > 0$ . There then exists a positive semidefinite, nondegenerate,  $d \times d$  matrix  $q$  such that for every  $t > 0$ , the annealed distribution  $E_\mu P_\kappa^0(\varepsilon X_{t\varepsilon^{-2}} \in \cdot)$  converges weakly to the law of the multivariate normal  $\mathcal{N}(0, tq)$ .*

The main result of [18] actually shows that the annealed law of the entire path  $t \mapsto \varepsilon X_{t\varepsilon^{-2}}$  converges to that of (a linear transform of) Brownian motion. However, the above is all that will be needed for the purposes of the present paper.

Apart from a central limit asymptotics, we will also need an estimate on the heat kernel of the above random walk. The following lemma is a consequence of the main result of Delmotte and Deuschel [10].

LEMMA 3.6 (Heat kernel upper bound). *Let  $\mu$  be a law of the conductances satisfying the ellipticity condition  $\mu(\varepsilon < \kappa_b < 1/\varepsilon) = 1$  for some  $\varepsilon > 0$ . There is then a  $c_1 < \infty$  such that*

$$(3.23) \quad E_\mu |\nabla_i \nabla_j P_\kappa^0(X_t = \cdot)| \leq \frac{c_1}{t^{d/2+1}}, \quad 1 \leq i, j \leq d, t > 0.$$

Here,  $\nabla_i$  is the discrete spatial derivative in the  $i$ th coordinate direction, that is,  $\nabla_i f(x) := f(x + \hat{e}_i) - f(x)$ .

PROOF. By formula (1.5b) in [10], Theorem 1.1,

$$(3.24) \quad E_\mu |\nabla_i \nabla_j P_\kappa^0(X_t = x)| \leq c'_1 \frac{P^{c'_2 t}(0, x)}{t},$$

where  $P^{c'_2 t}(0, x)$  is the probability of the continuous-time simple random walk to be at  $x$  at time  $c'_2 t$ . This probability is bounded from above by a constant times  $t^{-d/2}$ .  $\square$

#### 4. Proof of main result.

4.1. *Regularity estimates.* The goal of this section is to prove Theorem 2.4 concerning the scaling limit of  $\phi(f_\varepsilon)$ . We begin by proving  $L^2$ -continuity of the random functional  $f \mapsto \phi(f)$  on  $\mathcal{H}$ , as stated in Lemma 2.2. For convenience of

notation, whenever  $\mathcal{R}$  is an operator on  $\ell^2(\mathbb{Z}^d)$ , we will extend it to an operator on  $L^2(\mathbb{R}^d)$  via the formula

$$(4.1) \quad (f, \mathcal{R}f) := \int dx dy f(x)f(y)\mathcal{R}(\lfloor x \rfloor, \lfloor y \rfloor),$$

where  $\mathcal{R}(x, y)$  is the kernel of  $\mathcal{R}$  in the canonical basis in  $\ell^2(\mathbb{Z}^d)$ .

**PROOF OF LEMMA 2.2.** Let  $\tilde{\mu}$  be the extended gradient Gibbs measure corresponding to  $\mu$ . Recall the notation  $\mathcal{L}_\kappa$  for the generator of the random walk among conductances  $\kappa = (\kappa_b)$  and let  $\mathcal{L}$  denote the generator of the simple random walk (i.e., the special case of  $\mathcal{L}_\kappa$  when all  $\kappa_b = 1$ ). Choose  $f \in \{\Delta g : g \in C_0^\infty(\mathbb{R}^d)\}$ . Lemma 3.4 and the fact that  $\phi(f)$  is linear in the  $\eta$ 's imply that

$$(4.2) \quad \|\phi(f)\|_{L^2(\tilde{\mu})}^2 = E_{\tilde{\mu}}((f, -\mathcal{L}_\kappa^{-1}f)),$$

where  $(f, \mathcal{L}_\kappa^{-1}f)$  is as defined above. By assumption on the support of  $\varrho$ , we know that  $\kappa_b \geq a$   $\tilde{\mu}$ -a.s., by which we immediately have the operator inequalities

$$(4.3) \quad (-\mathcal{L}_\kappa) \geq a(-\mathcal{L}) \quad \text{and} \quad (-\mathcal{L}_\kappa)^{-1} \leq a^{-1}(-\mathcal{L})^{-1}.$$

Therefore, it suffices to bound the quadratic form associated with the (homogeneous) discrete Laplacian  $\mathcal{L}$  by the quadratic form defining the  $\mathcal{H}$ -space:

$$(4.4) \quad (f, (-\mathcal{L})^{-1}f) \leq c\|f\|_{\mathcal{H}}^2$$

for some constant  $c < \infty$  and all  $f$  in a dense subset of  $\mathcal{H}$ .

To this end, we pick  $f \in \mathcal{H}$  in the Schwartz class and let

$$(4.5) \quad \hat{f}(k) := (2\pi)^{-d/2} \int f(x)e^{ik \cdot x} dx$$

be its ( $L^2$ -norm-preserving) Fourier transform. A direct calculation now yields

$$(4.6) \quad (f, (-\mathcal{L})^{-1}f) = \int_{[-\pi, \pi]^d} dk \frac{1}{(-\hat{\mathcal{L}})(k)} \left| \sum_{\substack{k': \exists \ell \in \mathbb{Z}^d \\ k-k'=2\pi\ell}} \hat{f}(k') \prod_{j=1}^d \frac{1 - e^{-ik'_j}}{ik'_j} \right|^2,$$

where

$$(4.7) \quad (-\hat{\mathcal{L}})(k) := 4 \sum_{j=1}^d \sin^2(k_j/2)$$

is the generalized eigenvalue of the lattice Laplacian. Introducing  $-\hat{\Delta}(k) = |k|^2$  to denote the corresponding quantity for the continuum Laplacian, we invoke the Cauchy-Schwarz inequality for the sum over  $k'$  to get

$$(4.8) \quad (f, (-\mathcal{L})^{-1}f) \leq \int_{[-\pi, \pi]^d} dk \left\{ \sum_{\substack{k': \exists \ell \in \mathbb{Z}^d \\ k-k'=2\pi\ell}} |\hat{f}(k')|^2 (1 - \hat{\Delta}(k')^{-1}) \right\} c(k),$$

where

$$(4.9) \quad c(k) := \sum_{\substack{k' : \exists \ell \in \mathbb{Z}^d \\ k-k'=2\pi\ell}} \frac{1}{(-\hat{\mathcal{L}})(k)} \frac{-\hat{\Delta}(k')}{1-\hat{\Delta}(k')} \prod_{j=1}^d \left( \frac{2 \sin(k'_j/2)}{k'_j} \right)^2$$

is well defined on the set  $\mathcal{B} := \{k \in [-\pi, \pi]^d : k_j \neq 0, j = 1, \dots, d\}$  of full Lebesgue measure in  $[-\pi, \pi]^d$ . We claim that

$$(4.10) \quad c := \sup_{k \in \mathcal{B}} c(k) < \infty.$$

We will show this by proving that the summand in (4.9) is bounded by a constant times the product  $\prod_j (|k'_j| + 1)^{-2}$ . Indeed, for the  $k' = k$  term, we use the fact that the ratio  $\hat{\Delta}(k)/\hat{\mathcal{L}}(k)$  is bounded throughout  $\mathcal{B}$ , and the same for the ratios  $k \mapsto \sin(k_j/2)/k_j$ . When  $k' \neq k$ , we set  $i$  to be the first index  $j$  such that  $k'_j \neq k_j$  and bound the  $4 \sin(k'_i/2)^2$  term by  $-\hat{\mathcal{L}}(k)$ . We then bound the ratio of  $\hat{\Delta}(k)$  terms by unity and the  $j$ th term in the product by a constant times  $(|k'_j|^2 + 1)^{-2}$ . [The  $\sin(k'_i/2)$  term is not needed because  $|k'_i| \geq 2\pi$ .]

The product  $\prod_j (|k'_j| + 1)^{-2}$  is summable over  $k' \in k + (2\pi\mathbb{Z})^d$  uniformly in  $k \in \mathcal{B}$  and so (4.10) is proved. Bounding  $c(k)$  by its supremum in (4.8), we can merge the sum and the integral to get  $\|f\|_{\mathcal{H}}^2$ . The desired bound (4.4) then follows. □

REMARK 4.1. The inclusion of  $L^2$ -norm of  $f$  in  $\|f\|_{\mathcal{H}}$  is crucial for the bound (4.4). Indeed, on the basis of (4.6), it is not hard to construct functions for which the ratio  $(f, -\mathcal{L}^{-1}f)/(f, -\Delta^{-1}f)$  is arbitrarily large. This is caused by the fact that the spectrum of  $-\Delta$  extends all the way to infinity, while that of  $-\mathcal{L}$  is bounded.

The continuity established in Lemma 2.2 allows us to work only with smooth and compactly supported test functions. We will nevertheless need one more regularity bound before we can delve into the proof of our main result.

LEMMA 4.2. *Let  $\mu$  be a translation-invariant law on the conductances subject to the ellipticity condition  $\mu(\varepsilon < \kappa_b < 1/\varepsilon) = 1$  for some  $\varepsilon > 0$ . There then exists  $c < \infty$  such that whenever  $f = \Delta g$  for some  $g \in C_0^\infty(\mathbb{R}^d)$ ,*

$$(4.11) \quad E_\mu(f, e^{t\mathcal{L}_x} f) \leq c \|\nabla g\|_\infty^2 \lambda(\text{supp } g)^2 \frac{1}{t^{d/2+1}},$$

where  $\lambda(A)$  is the set function on Borel subsets of  $\mathbb{R}^d$  defined by

$$(4.12) \quad \lambda(A) := \sum_{x \in \mathbb{Z}^d} \sum_{i=1}^d \int_{R_i} dz \mathbf{1}_{\{x+z \in A\}},$$

with  $R_i$  denoting the set of points in  $[0, 1]^d$  with vanishing  $i$ th coordinate.

PROOF. Translation invariance of  $\mu$  and a simple integration by parts tells us that

$$\begin{aligned}
 & E_\mu(\Delta g, e^{t\mathcal{L}_\kappa} \Delta g) \\
 &= \sum_{x,y \in \mathbb{Z}^d} \int_{[0,1]^d} dz \int_{[0,1]^d} dz' \Delta g(x+z) \Delta g(y+z') E_\mu P_\kappa^x(X_t = y) \\
 (4.13) \quad &= \sum_{x,y \in \mathbb{Z}^d} \sum_{i,j=1}^d \int_{R_i} dz \int_{R_j} dz' \partial_i g(x+z) \partial_j g(y+z') \\
 &\quad \times \nabla_i \nabla_j E_\mu P_\kappa^x(X_t = y),
 \end{aligned}$$

where  $\partial_i$  stands for the partial derivative with respect to the  $i$ th coordinate. Restricting the integrations and sums so that the arguments  $x+z$  and  $y+z'$  are in the support of  $g$ , bounding the partial derivatives by  $\|\nabla g\|_\infty$  and applying the estimate (3.23), we obtain the desired bound.  $\square$

The consequence of Lemma 4.2 that will concern us is as follows.

COROLLARY 4.3. For  $\mu$  as in Lemma 4.2 and any  $f \in \{\Delta g : g \in C_0^\infty(\mathbb{R}^d)\}$ ,

$$(4.14) \quad \lim_{M \rightarrow \infty} \sup_{0 < \varepsilon < 1} \int_M^\infty dt E_\mu \varepsilon^{-2} (f_\varepsilon, e^{t\varepsilon^{-2}\mathcal{L}_\kappa} f_\varepsilon) = 0.$$

PROOF. Choose  $f$  of the form  $f = \Delta g$  and note that  $f_\varepsilon = \Delta g^{(\varepsilon)}$ , where  $g^{(\varepsilon)}(x) := \varepsilon^{d/2+1} g(x\varepsilon)$ . First, we observe that

$$(4.15) \quad \|\nabla g^{(\varepsilon)}\|_\infty = \varepsilon^{d/2} \|\nabla g\|_\infty.$$

Next, we note that, since the support of  $g$  is the closure of a nonempty bounded open set, a simple covering argument tells us that

$$(4.16) \quad \varepsilon^d \lambda(\text{supp } g^{(\varepsilon)}) \xrightarrow{\varepsilon \downarrow 0} d |\text{supp } g|,$$

where  $|\text{supp } g|$  is the Lebesgue measure of  $\text{supp } g$ . As a consequence, there exists a constant  $C(g) < \infty$  such that

$$(4.17) \quad \lambda(\text{supp } g^{(\varepsilon)}) \leq C(g) \varepsilon^{-d}, \quad 0 < \varepsilon < 1.$$

Plugging (4.15) and (4.17) into (4.11), we get, for  $\varepsilon \in (0, 1)$ ,

$$(4.18) \quad E_\mu \varepsilon^{-2} (f_\varepsilon, e^{t\varepsilon^{-2}\mathcal{L}_\kappa} f_\varepsilon) \leq c \|\nabla g\|_\infty^2 C(g)^2 \frac{1}{t^{d/2+1}}.$$

The functions on the left (indexed by  $\varepsilon$ ) are uniformly integrable in  $t$  in all  $d \geq 1$ .  $\square$

4.2. *Scaling limit.* Having dispensed with regularity considerations, we can now proceed to establish the principal fact underlying the proof of Theorem 2.4.

PROPOSITION 4.4. *Let  $\mu$  be a translation-invariant, ergodic measure on  $\kappa = (\kappa_b) \in \mathbb{R}^{\mathbb{B}(\mathbb{Z}^d)}$  such that  $\mu(\delta \leq \kappa_b \leq 1/\delta) = 1$  for some  $\delta > 0$ . There then exists a positive semidefinite, nondegenerate  $d \times d$  matrix  $q = (q_{ij})$  such that*

$$(4.19) \quad \lim_{\varepsilon \downarrow 0} (f_\varepsilon, (-\mathcal{L}_\kappa)^{-1} f_\varepsilon) = (f, (-Q)^{-1} f)$$

in  $\mu$ -probability for each  $f \in C_0^\infty(\mathbb{R}^d) \cap \mathcal{H}$ , where  $Q$  is defined from  $q$  by (2.16).

The key to the proof is the following lemma.

LEMMA 4.5. *For every  $t > 0$  and any  $f \in C_0^\infty(\mathbb{R}^d) \cap \mathcal{H}$ ,*

$$(4.20) \quad \Theta_\varepsilon(t) := \varepsilon^{-2} (f_\varepsilon, e^{t\varepsilon^{-2}\mathcal{L}_\kappa} f_\varepsilon) - (f, e^{tQ} f) \xrightarrow{\varepsilon \downarrow 0} 0 \quad \text{in } L^2(\mu).$$

PROOF. Let  $(X_t)_{t \geq 0}$  be the continuous-time random walk with the generator  $\mathcal{L}_\kappa$  and let  $P_\kappa^x$  denote the law of the walk started from  $x$ . By Lemma 3.5, the annealed law of  $\varepsilon X_{t\varepsilon^{-2}}$  tends weakly to that of the multivariate normal

$$(4.21) \quad \mathcal{N}_t := \mathcal{N}(0, tq)$$

for some positive semidefinite, nondegenerate  $d \times d$  matrix  $q = (q_{ij})$ . As a consequence, if  $\mathcal{G} \subset C_0^\infty(\mathbb{R}^d)$  is a family of functions that are uniformly equicontinuous and bounded, then we have

$$(4.22) \quad E_\mu \left( \sup_{g \in \mathcal{G}} |E_\kappa^0(g(\varepsilon X_{t\varepsilon^{-2}})) - E g(\mathcal{N}_t)|^2 \right) \xrightarrow{\varepsilon \downarrow 0} 0.$$

Now, let  $Q$  be the generator of the Brownian motion with mean zero and covariance  $q$ , that is,  $Q$  is the operator in (2.16). We then have

$$(4.23) \quad (f, e^{tQ} f) = \int dy f(y) E(f(y + \mathcal{N}_t)).$$

As  $\mathcal{L}_\kappa$  is the generator of the random walk  $(X_t)$ , we similarly derive

$$(4.24) \quad \begin{aligned} & \varepsilon^{-2} (f_\varepsilon, e^{t\varepsilon^{-2}\mathcal{L}_\kappa} f_\varepsilon) \\ &= \varepsilon^d \int_{[0,1]^d \times [0,1]^d} dz_1 dz_2 \sum_{x \in \mathbb{Z}^d} f(\varepsilon x + \varepsilon z_1) E_\kappa^x(f(\varepsilon z_2 + \varepsilon X_{t\varepsilon^{-2}})). \end{aligned}$$

We thus need to show that the right-hand side of (4.24) tends to that of (4.23). Note that if  $f$  is supported in  $[-M, M]^d$ , then the integral in (4.23) can be restricted to  $y \in [-M, M]^d$  and the sum over  $x$  in (4.24) to, say,  $|x| \leq 2M/\varepsilon$  (once  $\varepsilon \ll 1$ ).

Substituting  $y = \varepsilon x + \varepsilon z$  with  $x \in \mathbb{Z}^d$ ,  $|x| \leq 2M/\varepsilon$  and  $z \in [0, 1]^d$  in (4.23) allows us to put both terms on the same footing. Subtracting (4.23) from (4.24), taking expectation with respect to  $\mu$  and applying the Cauchy–Schwarz inequality, we thus get

$$(4.25) \quad \begin{aligned} E_\mu \Theta_\varepsilon(t)^2 &\leq \|f\|_2 \varepsilon^d \int_{[0,1]^d} dz \sum_{x: |x| \leq 2M/\varepsilon} E_\mu |E_\kappa^x f(\varepsilon z + \varepsilon X_{t\varepsilon^{-2}}) \\ &\quad - Ef(\varepsilon x + \varepsilon z + \mathcal{N}_t)|^2. \end{aligned}$$

Using the translation invariance of  $\mu$ , we may replace  $E_\kappa^x f(\varepsilon z + \varepsilon X_{t\varepsilon^{-2}})$  by the expression  $E_\kappa^0 f(\varepsilon x + \varepsilon z + \varepsilon X_{t\varepsilon^{-2}})$  inside the expectation. For  $f \in C_0^\infty(\mathbb{R}^d)$ ,

$$(4.26) \quad \mathcal{G} := \{f(\varepsilon x + \varepsilon z + \cdot) : |x| \leq 2M/\varepsilon, z \in [0, 1]^d\}$$

is an equicontinuous family of uniformly bounded functions. Then, (4.22) tells us that the right-hand side of (4.25) tends to zero as  $\varepsilon \downarrow 0$ , which proves the desired claim (4.20).  $\square$

**PROOF OF PROPOSITION 4.4.** To extract (4.19) from (4.20), we note that for any  $f \in C_0^\infty(\mathbb{R}^d) \cap \mathcal{H}$ ,

$$(4.27) \quad (f, (-\mathcal{L}_\kappa)^{-1} f) = \int_0^\infty dt (f, e^{t\mathcal{L}_\kappa} f).$$

Replacing  $f$  by  $f_\varepsilon$  and scaling  $t$  by  $\varepsilon^2$ , we find that

$$(4.28) \quad (f_\varepsilon, (-\mathcal{L}_\kappa)^{-1} f_\varepsilon) = \int_0^\infty dt \varepsilon^{-2} (f_\varepsilon, e^{t\varepsilon^{-2}\mathcal{L}_\kappa} f_\varepsilon).$$

By (4.20), the function being integrated tends to  $(f, e^{tQ} f)$  in probability for each  $t$ ; the monotonicity in  $t$  (and continuity of the limit) ensures that the convergence is actually uniform (in probability) on compact intervals. By Corollary 4.3, the integral can be truncated to a finite interval in  $L^1$ -norm and similarly for the integral of the limit, which is finite since  $f$  is finite in the domain of  $Q^{-1}$ . It follows that

$$(4.29) \quad (f_\varepsilon, (-\mathcal{L}_\kappa)^{-1} f_\varepsilon) \xrightarrow{\varepsilon \downarrow 0} \int_0^\infty dt (f, e^{tQ} f) = (f, -Q^{-1} f)$$

in  $\mu$ -probability [and  $L^1(\mu)$ ]. This is the desired conclusion (4.19).  $\square$

**REMARK 4.6.** We note that to control the tail of the integral in (4.27) in  $d \geq 3$ , it suffices to invoke the diagonal heat kernel estimate

$$(4.30) \quad E_\mu P_\kappa^0(X_t = x) \leq \frac{c_1}{t^{d/2}}, \quad x \in \mathbb{Z}^d,$$

which, in the elliptic case, is an immediate consequence of the mixing theory for Markov chains based on isoperimetric inequalities. This is sufficient because the finiteness of the Green function in  $d \geq 3$  permits us to define  $(f, (-\mathcal{L}_\kappa)^{-1} f)$ , even for  $f \geq 0$ . This enables us to reduce the general case to positive  $f$  by decomposing the test function into a positive and a negative part and applying

$$(4.31) \quad E_\nu(f, e^{t\mathcal{L}_\kappa} f) \leq \|f\|_1^2 \frac{c_1}{t^{d/2}},$$

which is uniformly integrable when  $d \geq 3$ . However, to include  $d = 2$ , we cannot disregard the cancellations due to the vanishing of  $\int f(x) dx$  and thus the stronger derivative bound (3.23) is necessary. A similar situation occurred in Giacomini, Olla and Spohn [13] where a stronger Nash continuity estimate was required to include  $d = 2$ .

We are now ready to establish the main result of this paper.

**PROOF OF THEOREM 2.4.** Let  $\mu$  be a translation-invariant, ergodic, gradient Gibbs measure with zero tilt and let  $\tilde{\mu}$  be its extension to  $\mathbb{R}^{\mathbb{B}(\mathbb{Z}^d)} \times \mathbb{R}^{\mathbb{B}(\mathbb{Z}^d)}$ . We want to prove that  $\phi(f_\varepsilon)$  tends weakly to a normal random variable with mean zero and variance  $(f, (-Q)^{-1} f)$ . By Lemma 2.2, it suffices to prove this for  $f \in C_0^\infty(\mathbb{R}^d) \cap \mathcal{H}$ .

By Lemma 3.4, we know that  $\phi(f)$  is Gaussian conditional on  $\kappa$ . The standard formula for any Gaussian random variable  $X$ ,

$$(4.32) \quad E(e^{iX}) = e^{iE(X) - 1/2 \text{Var}(X)},$$

implies, via (3.15) and (3.16), that

$$(4.33) \quad E_{\tilde{\mu}}(e^{i\phi(f)} | \mathcal{F})(\kappa) = e^{-1/2(f, (-\mathcal{L}_\kappa)^{-1} f)}.$$

By Proposition 4.4, we have  $(f_\varepsilon, (-\mathcal{L}_\kappa)^{-1} f_\varepsilon) \rightarrow (f, (-Q)^{-1} f)$  in  $\tilde{\mu}$ -probability. Since the right-hand side of (4.33) is a bounded continuous function of this inner product, (2.15) follows by means of the bounded convergence theorem.  $\square$

**5. Potential theory for random conductance models.** The proof of the key Lemma 3.3 leads us to the study of potential theory for operators depending on a random environment that fall into the class of *random conductance models*. A good deal of what is to follow exists explicitly, or implicitly, in the literature. We have borrowed some of the notation from the paper of Mathieu and Piatnitski [21], although the formalism draws on earlier works in homogenization theory; see, for instance, the book by Jikov, Kozlov and Oleinik [17]. Notwithstanding, the content of Section 5.2 appears to be new.

5.1. *Basic notions.* Consider a translation-invariant  $\nu$  probability measure on  $\Omega := \mathbb{R}_+^{\mathbb{B}(\mathbb{Z}^d)}$  (endowed with the product  $\sigma$ -algebra) satisfying the ellipticity condition

$$(5.1) \quad \exists \varepsilon > 0 : \nu \left( \varepsilon \leq \kappa_b \leq \frac{1}{\varepsilon} \right) = 1, \quad b \in \mathbb{B}(\mathbb{Z}^d).$$

Let  $L^2(\nu)$  denote the closure of the set of all local functions in the topology induced by the inner product

$$(5.2) \quad \langle h, g \rangle := E_\nu(h(\kappa)g(\kappa)).$$

Let  $B := \{\hat{e}_1, \dots, \hat{e}_d\}$  denote the set of coordinate vectors in  $\mathbb{Z}^d$ . The translations by the vectors in  $B$  induce natural unitary maps  $T_1, \dots, T_d$  on  $L^2(\nu)$  defined via

$$(5.3) \quad (T_j h) := h \circ \tau_{\hat{e}_j}, \quad j = 1, \dots, d.$$

Apart from square integrable functions, we will also need to work with vector fields, by which we will generally mean measurable functions  $u : \Omega \times B \rightarrow \mathbb{R}$  or  $\Omega \times B \rightarrow \mathbb{R}^d$ , depending on the context. We will sometimes write  $u_1, \dots, u_d$  for  $u(\cdot, \hat{e}_1), \dots, u(\cdot, \hat{e}_d)$ —note that these may still be vector-valued.

REMARK 5.1. While we index vector fields only by the *positive* coordinate vectors, in certain situations, it is convenient to have them also defined for the negative coordinate directions via

$$(5.4) \quad u(\kappa, -b) := -u(\tau_{-b}\kappa, b), \quad b \in B.$$

As we will see, this definition will automatically ensure that the cycle condition (see Lemma 5.2 below) holds for the trivial cycles crossing only a single edge.

Let  $L^2_{\text{vec}}(\nu)$  be the set of all vector fields with  $(u, u) < \infty$ , where  $(\cdot, \cdot)$  denotes the inner product

$$(5.5) \quad (u, v) := E_\nu \left( \sum_{b \in B} \kappa_b u(\kappa, b) \cdot v(\kappa, b) \right).$$

Examples of such functions are the gradients  $\nabla h$  of local functions  $h \in L^2(\nu)$  defined component-wise via the formula

$$(5.6) \quad (\nabla h)_j := T_j h - h, \quad j = 1, \dots, d.$$

We denote by  $L^2_{\nabla}(\nu)$  the closure of the set of gradients of local functions in the topology induced by the above inner product.

LEMMA 5.2. *Let  $u \in L^2_{\nabla}$ . Then,  $u$  satisfies the cycle condition*

$$(5.7) \quad \sum_{j=0}^n u(\tau_{x_j}\kappa, x_{j+1} - x_j) = 0$$

for any finite (nearest-neighbor) cycle  $(x_0, x_1, \dots, x_n = x_0)$  on  $\mathbb{Z}^d$ . In particular, there exists a shift-covariant function  $\bar{u} : \Omega \times \mathbb{Z}^d \rightarrow \mathbb{R}^d$  such that  $u(\kappa, b) = \bar{u}(\kappa, b)$  for every  $b \in B$ .

PROOF. The cycle condition (5.7) holds trivially for all gradients of local functions. Indeed, if  $u = \nabla h$ , then, in light of (5.4), we have

$$(5.8) \quad u(\tau_{x_j} \kappa, x_{j+1} - x_j) = (\nabla h)_{x_{j+1} - x_j}(\tau_{x_j} \kappa) = h \circ \tau_{x_{j+1}}(\kappa) - h \circ \tau_{x_j}(\kappa).$$

A corresponding limit extends this to all of  $L^2_{\nabla}$ . To define  $\bar{u}(\cdot, x)$ , we integrate properly shifted values of  $u$  along a path from zero to  $x$ ; the cycle condition guarantees that the result is independent of the choice of path and that  $\bar{u}$  is shift-covariant.  $\square$

We will henceforth use the convention of writing  $\bar{u}$  for the extension of a shift-covariant vector field  $u \in L^2_{\text{vec}}$  to a function on  $\mathbb{Z}^d$ . Notice that the shift  $T_j$  extends naturally via

$$(5.9) \quad T_j \bar{u}(\kappa, x) := \bar{u}(\tau_{\hat{e}_j} \kappa, x) = \overline{(T_j u)}(\kappa, x).$$

Next, let us characterize the functions in  $(L^2_{\nabla})^{\perp}$ .

LEMMA 5.3. For  $u \in L^2_{\text{vec}}(\nu)$ , let  $\mathcal{L}u$  be the function in  $L^2(\nu)$  defined by

$$(5.10) \quad (\mathcal{L}u)(\kappa) := \sum_{b \in B} [\kappa_b u(\kappa, b) - (\tau_{-b} \kappa)_b u(\tau_{-b} \kappa, b)],$$

where  $-b$  is the coordinate vector opposite to  $b$ . We then have

$$(5.11) \quad u \in (L^2_{\nabla})^{\perp} \iff \mathcal{L}u = 0, \quad \nu\text{-a.s.}$$

If  $u$  satisfies the cycle condition and  $\bar{u}$  is its extension, then  $\mathcal{L}u(\tau_x \kappa) = \mathcal{L}_{\kappa} \bar{u}(\kappa, x)$ .

PROOF. These are direct consequences of the definitions, the translation invariance of  $\nu$  and a simple calculation.  $\square$

Note that  $\mathcal{L}u$  plays the role of the *divergence*—that is, the total flow out of a given vertex—of vector field  $u$ . However, we prefer to denote it by  $\mathcal{L}$  to emphasize its connection with the operator  $\mathcal{L}_{\kappa}$ .

5.2. *Uniqueness of harmonic embedding.* Clearly, all  $u \in L^2_{\nabla}$  are shift-covariant and have zero mean. A question which naturally arises is whether every shift-covariant zero-mean  $u$  is in  $L^2_{\nabla}$ . (Note that this is analogous to asking whether every closed differential form is exact.) Our answer to this is in the affirmative.

THEOREM 5.4. Suppose  $\nu$  is ergodic. Then, every  $u \in L^2_{\text{vec}}(\nu)$  which obeys the cycle condition (5.7) and  $E_{\nu} u = 0$  satisfies  $u \in L^2_{\nabla}$ .

Again, recall that (5.7) and zero expectation are necessary for  $u \in L^2_{\nabla}$ . The above implies that these conditions are also sufficient. To prove the theorem, we will need the following lemma.

LEMMA 5.5. *Let  $P_j$  denote the orthogonal projection onto  $\text{Ker}(1 - T_j)$  in  $L^2_{\text{vec}}(\nu)$ . If  $\nu$  is ergodic and  $u \in L^2_{\text{vec}}(\nu)$  satisfies (5.7), then  $P_j u = E_{\nu}(P_j u)$ ,  $\nu$ -almost surely.*

PROOF. Fix  $u \in L^2_{\text{vec}}(\nu)$  that obeys (5.7) and let  $\bar{u}$  be the corresponding shift-covariant function. We will prove the claim only for the component  $u_1 = u(\cdot, \hat{e}_1)$ ; the other cases follow analogously. By translation covariance and the  $L^2$  ergodic theorem, we have

$$(5.12) \quad \frac{\bar{u}(\cdot, n\hat{e}_1)}{n} = \frac{1}{n} \sum_{k=0}^{n-1} T_1^k u_1 \xrightarrow[n \rightarrow \infty]{} P_1 u_1 \quad \text{in } L^2(\nu).$$

If  $\nu$  were separately ergodic (i.e., ergodic with respect to  $T_1$  alone), then the claim would immediately follow by the fact that every  $T_1$ -invariant function must be constant. To make up for the potential lack of separate ergodicity, we note that translation covariance of  $u$  and the fact that  $\bar{u}$  obeys the cycle conditions together yield

$$(5.13) \quad \begin{aligned} T_j \bar{u}(\cdot, n\hat{e}_1) &= \bar{u}(\cdot, n\hat{e}_1 + \hat{e}_j) - \bar{u}(\cdot, \hat{e}_j) \\ &= \bar{u}(\cdot, n\hat{e}_1) - u(\cdot, \hat{e}_j) + T_{n\hat{e}_1} u(\cdot, \hat{e}_j). \end{aligned}$$

It follows that  $T_j \frac{1}{n} \bar{u}(\cdot, n\hat{e}_1)$  also converges to  $P_1 u_1$  (in  $L^2$ ) and so, by the continuity of  $T_j$ ,

$$(5.14) \quad T_j P_1 u_1 = P_1 u_1.$$

Hence,  $P_1 u$  is invariant under all shifts and is therefore constant  $\nu$ -a.s.  $\square$

PROOF OF THEOREM 5.4. Suppose  $\nu$  is ergodic and let  $u \in L^2_{\text{vec}}(\nu)$  obey (5.7) and  $E_{\nu} u = 0$ . The boundedness of the  $\kappa_b$ 's away from zero and infinity ensures that  $u \in L^2_{\text{vec}}(\nu)$  if and only if all of its components are in  $L^2(\nu)$ . Our task is to construct functions  $h_{\varepsilon} \in L^2(\nu)$  such that  $\nabla h_{\varepsilon} \rightarrow u$  in  $L^2_{\text{vec}}(\nu)$ . We define

$$(5.15) \quad h_{\varepsilon} := - \sum_{n \geq 0} \frac{T_1^n u_1}{(1 + \varepsilon)^{n+1}}$$

and note that this is the unique solution of the equation  $(1 + \varepsilon - T_1)h_{\varepsilon} = -u_1$ . This observation implies that

$$(5.16) \quad (1 - T_1)h_{\varepsilon} = -u_1 - \varepsilon h_{\varepsilon}$$

and so the first component of  $\nabla h_\varepsilon$  converges to that of  $u$ , provided  $\varepsilon h_\varepsilon \rightarrow 0$  in  $L^2(\nu)$ . To see what happens with the other components of  $\nabla h_\varepsilon$ , we note that the cycle condition (5.7) translates into

$$(5.17) \quad (1 - T_j)u_1 = (1 - T_1)u_j.$$

Applying this to the definition of  $h_\varepsilon$ , we conclude that

$$(5.18) \quad (1 - T_j)h_\varepsilon = -u_j - \varepsilon \tilde{h}_\varepsilon,$$

where  $\tilde{h}_\varepsilon$  is defined as  $h_\varepsilon$ , but with  $u_1$  replaced by  $u_j$ . Again, it suffices to show that  $\varepsilon \tilde{h}_\varepsilon \rightarrow 0$  in  $L^2(\nu)$ , which will boil down to the same argument as for  $j = 1$ .

To prove that  $\varepsilon h_\varepsilon \rightarrow 0$ , we note that, for the inner product in (5.2),

$$(5.19) \quad \langle h_\varepsilon, h_\varepsilon \rangle = \sum_{n \geq 0} \frac{n+1}{(1+\varepsilon)^{n+2}} \langle u_1, T_1^n u_1 \rangle.$$

Introducing the notation  $A_n u$  for the average

$$(5.20) \quad A_n u := \frac{1}{n} \sum_{k=0}^{n-1} T_1^k u,$$

inserting this into the above sum and reordering the terms, we get

$$(5.21) \quad \langle h_\varepsilon, h_\varepsilon \rangle = \frac{\langle u_1, u_1 \rangle}{(1+\varepsilon)^2} + \sum_{n \geq 1} \frac{n(\varepsilon n - 1)}{(1+\varepsilon)^{n+2}} \langle u_1, A_n u_1 \rangle.$$

By Lemma 5.5, the  $L^2$  ergodic theorem and the fact that  $u$  has zero expectation in  $\nu$ , we have

$$(5.22) \quad A_n u_1 \xrightarrow{n \rightarrow \infty} P_1 u_1 = E_\nu u_1 = 0 \quad \text{in } L^2(\nu)$$

and so  $\langle u_1, A_n u_1 \rangle \rightarrow 0$  as  $n \rightarrow \infty$ . A straightforward estimate now shows that the sum in (5.21) is  $o(\varepsilon^{-2})$  and so  $\varepsilon^2 \langle h_\varepsilon, h_\varepsilon \rangle \rightarrow 0$ , as desired.  $\square$

Not every function in  $L^2_{\text{vec}}(\nu)$  necessarily belongs to  $L^2_{\nabla}(\nu)$ . A prime example is the *position* vector field  $x(\kappa, b) = b$ . Indeed, let  $\chi : \mathbb{R}^{\mathbb{B}(\mathbb{Z}^d)} \times B \rightarrow \mathbb{R}^d$  be the projection

$$(5.23) \quad \chi := -\text{proj}_{L^2_{\nabla}(\nu)} x.$$

Since  $\chi \in L^2_{\nabla}(\nu)$ , it satisfies (5.7) and we may extend it to a function  $\bar{\chi}$  mapping  $\mathbb{R}^{\mathbb{B}(\mathbb{Z}^d)} \times \mathbb{Z}^d \rightarrow \mathbb{R}^d$  by setting  $\bar{\chi}(\cdot, 0) = 0$  and integrating the gradients along oriented paths. Lemma 5.3 implies that  $\bar{x} + \bar{\chi}$  is harmonic,  $\mathcal{L}_\kappa(\bar{x} + \bar{\chi}) = 0$ . Moreover, Lemma 5.5 and (5.12) show that  $\bar{\chi}(\cdot, n\hat{e}_j)/n \rightarrow 0$  and so  $x + \chi \neq 0$ . It follows that  $x \notin L^2_{\nabla}$  and so  $L^2_{\nabla} \neq L^2_{\text{vec}}$ .

The function  $\bar{\chi}$  is generally referred to as the *corrector* because it corrects for the nonharmonicity of the position function. The corrector can be defined by appealing to spectral theory (Kipnis and Varadhan [18], also Berger and Biskup [4]); the above “projection” definition is inspired by those in Giacomini, Olla and Spohn [13] and Mathieu and Piatnitski [21]. As a side remark, we note that the function  $\bar{x} + \bar{\chi}$  actually allows us to characterize the space of all square integrable shift-covariant functions.

**COROLLARY 5.6.** *Suppose  $\nu$  is ergodic. Then, every shift-covariant  $\mathbb{R}$ -valued  $u \in (L^2_{\nabla})^\perp$  can be obtained from  $x + \chi$ , where  $x$  is the position function and  $-\chi$  is its orthogonal projection onto  $L^2_{\nabla}$ , by means of the linear transformation*

$$(5.24) \quad u_j(\kappa) = \sum_{k=1}^d [\hat{e}_k \cdot (x_j + \chi_j)] E_\nu u_k, \quad j = 1, \dots, d.$$

Here,  $u_1, \dots, u_d$  stand for  $u(\cdot, \hat{e}_1), \dots, u(\cdot, \hat{e}_d)$  and similarly for ( $\mathbb{R}^d$ -valued objects)  $x_1, \dots, x_k$  and  $\chi_1, \dots, \chi_d$ . In particular, for  $\mathbb{R}$ -valued vector fields, we have

$$(5.25) \quad \{u \in L^2_{\text{vec}}(\nu) : \text{shift-covariant}\} = L^2_{\nabla} \oplus \{\lambda \cdot (x + \chi) : \lambda \in \mathbb{R}^d\}.$$

**REMARK 5.7.** The observation (5.25)—which, in the language borrowed from differential geometry, implies that linear transforms of  $x + \chi$  are the only closed forms that are not exact—was previously discovered in the context of multicolored exclusion processes (Quastel [24], Theorem 9.1). However, while our proof is based only on a soft, Poisson equation-based argument (5.15)–(5.22), that of [24] requires an explicit bound on the spectral gap for the corresponding dynamics. The reason may be the reliance of [24] on the *spatial* ergodic theorem, which, naturally, leads to bounds involving the Poincaré inequality and/or spectral gap.

**PROOF OF COROLLARY 5.6.** This is a simple consequence of Theorem 5.4. Let  $w = (w_1, \dots, w_d)$  be the  $d$ -component vector field whose ( $\mathbb{R}$ -valued) components are defined by

$$(5.26) \quad w_j := \sum_{k=1}^d [\hat{e}_k \cdot (x_j + \chi_j)] E_\nu u_k, \quad j = 1, \dots, d.$$

We will show that  $w = u$ . First, both  $u$  and  $w$  obey the cycle condition, so  $u - w$  also does. As  $\chi \in L^2_{\nabla}(\nu)$  implies  $E_\nu \chi = 0$ , the fact that  $E_\nu x_k = x_k = \hat{e}_k$  shows that  $E_\nu w_j = E_\nu u_j$ , that is,  $E_\nu(u - w) = 0$ . Theorem 5.4 implies that  $u - w \in L^2_{\nabla}$ . On the other hand,  $\mathcal{L}(x + \chi) = 0$  implies that  $\mathcal{L}w = 0$  and so, by Lemma 5.3,  $w \in (L^2_{\nabla})^\perp$ . Thus,  $u - w \in (L^2_{\nabla})^\perp$  also holds. It follows that  $u = w$ , as claimed.

As for (5.25), the argument we just used ensures that a shift-covariant field  $u \in L^2_{\text{vec}}(\nu)$  can be written as  $\lambda \cdot (x + \chi)$ , where  $\lambda$  is a vector with components  $\lambda_j :=$

$E_\nu u_j$ , plus a shift-covariant vector field with zero expectation. By Theorem 5.4, the latter is in  $L^2_\nabla$ .  $\square$

We still have to supply the proof of Lemma 3.3.

**PROOF OF LEMMA 3.3.** Since  $g$  has square integrable components, we have  $g \in L^2_{\text{vec}}(\nu)$ . As  $g$  is shift-covariant and has zero expectation, Theorem 5.4 implies that  $g \in L^2_\nabla$ . However,  $g$  is also harmonic and so, in turn, Lemma 5.3 forces  $g \in (L^2_\nabla)^\perp$ . Thus,  $g = 0$ , as desired.  $\square$

Recall that  $\bar{\chi}$  is the extension of  $\chi$  subject to the condition  $\bar{\chi}(\cdot, 0) = 0$ . With this object at hand, we may even remove the restriction to zero slope in Lemma 3.4.

**COROLLARY 5.8.** *Let  $\mu$  be a translation-invariant, ergodic, gradient Gibbs measure for the potential (1.4) and let  $\tilde{\mu}$  be its extension to  $\mathbb{R}^{\mathbb{B}(\mathbb{Z}^d)} \times \mathbb{R}^{\mathbb{B}(\mathbb{Z}^d)}$ . Let  $t \in \mathbb{R}^d$  be the tilt of  $\mu$  and let  $\mathcal{F} := \sigma(\{\kappa_b : b \in \mathbb{B}(\mathbb{Z}^d)\})$ . Then,  $\tilde{\mu}(-|\mathcal{F})(\kappa)$  is Gaussian with mean*

$$(5.27) \quad E_{\tilde{\mu}}(\phi_x - \phi_0 | \mathcal{F})(\kappa) = t \cdot [x + \bar{\chi}(\kappa, x)]$$

and covariance given by  $(-\mathcal{L}_\kappa)^{-1}$ .

**PROOF.** Let  $\bar{u}$  denote the extension of the shift-covariant vector field  $u(\kappa, b) := E_{\tilde{\mu}}(\eta_{0,b} | \mathcal{F})(\kappa)$ ,  $b \in B$ . Clearly, we have

$$(5.28) \quad \bar{u}(\kappa, x) = E_{\tilde{\mu}}(\phi_x - \phi_0 | \mathcal{F})(\kappa), \quad x \in \mathbb{Z}^d.$$

Inspecting the proof of Lemma 3.4, all formulae carry over, except (3.22), which becomes

$$(5.29) \quad E_{\tilde{\mu}}(\bar{u}(\cdot, x)) = t \cdot x, \quad x \in \mathbb{Z}^d.$$

Thus,  $\bar{u} - t \cdot (\bar{x} + \bar{\chi})$  is harmonic, shift-covariant and of zero mean, so  $\bar{u} = t \cdot (\bar{x} + \bar{\chi})$  by Lemma 5.3 and Theorem 5.4.  $\square$

**6. Discussion and open problems.** The proofs in the present note often rely on the fact that the random walk with generator  $\mathcal{L}_\kappa$  is uniformly elliptic. This enters via the assumption that  $\varrho$ , defining the potential  $V$ , has support bounded away from zero. While we believe that the general picture carries over, even if we let the support extend all the way to zero, a number of steps in the proof become quite subtle. For instance, the pointwise heat kernel asymptotic for this walk may take a radically different form (Berger et al. [3]) and it is not known under what conditions on the environment the walk scales to Brownian motion. Progress in this direction for i.i.d. (or i.i.d. dominated) environments has been made only recently (Mathieu [20], Biskup and Prescott [6], Barlow and Deuschel [2]).

Another interesting open question concerns the annoying restriction to *zero-tilt* gradient Gibbs measures. As is seen from Corollary 5.8, once the tilt is nonzero, (4.33) has to be modified to

$$(6.1) \quad E_{\tilde{\mu}}(e^{i[\phi(f)-E_{\tilde{\mu}}(\phi(f))]}|_{\mathcal{F}})(\kappa) = e^{-i(t \cdot \bar{\chi})(f)-1/2(f, (-\mathcal{L}_{\kappa})^{-1} f)}.$$

When we plug in  $f_{\varepsilon}$  for  $f$ , the second term in the exponent still converges to  $(f, (-\mathcal{Q})^{-1} f)$  in probability as  $\varepsilon \downarrow 0$ . To show convergence of  $\phi_{\varepsilon} - E_{\tilde{\mu}}\phi_{\varepsilon}$  to Gaussian free field, we thus have to show that  $(t \cdot \bar{\chi})(f_{\varepsilon})$  converges in law to a normal random variable. If  $V$  is strictly convex and of the form (1.4), then we know this to be true by the main result of Giacomin, Olla and Spohn [13]. However, the general case seems to be a hard open problem (cf. [4], Conjecture 5). Already in  $d = 1$ , where the corrector can be written explicitly as the sum

$$(6.2) \quad \bar{\chi}(\kappa, x) = \frac{1}{C} \sum_{n=0}^{x-1} \left( \frac{1}{\kappa_{n,n+1}} - C \right),$$

with  $C := E_{\tilde{\mu}}(1/\kappa_b)$ , the desired result requires that the resistivities  $1/\kappa_b$  exhibit a specific correlation decay [11], Theorem 7.6. Very little information concerning this is known in  $d \geq 2$ ; see Mourrat [22] for some recent progress in this direction.

Our final remark concerns a generalization to potentials  $V$ , where (1.4) has been modified to

$$(6.3) \quad V(\eta) := -\log \int \varrho(d\kappa) e^{-W_{\kappa}(\eta)},$$

with  $(W_{\kappa})$  denoting a family of strictly convex, even, measurable functions with uniformly superlinear growth at  $\pm\infty$  and a uniform lower bound. In this case, we may still consider the extended gradient Gibbs measures; however, the conditional law given the  $\kappa$ 's is no longer Gaussian. Notwithstanding, given  $\kappa$ , the Helffer–Sjöstrand representation still applies and leads to a random walk in a dynamic random environment. Its annealed central limit theorem would imply the Gaussian scaling limit for the  $\phi$ -field.

**Acknowledgments.** This project was initiated at the Kavli Institute for Theoretical Physics during the program “Stochastic Geometry and Field Theory: From Growth Phenomena to Disordered Systems.” The KITP program was supported by the NSF Grant PHY99-07949. We wish to thank Jean-Dominique Deuschel for a number of interesting suggestions, and Scott Sheffield and Jeremy Quastel for discussions on related subjects. We are also grateful to the anonymous referees for careful reading of the first version of this paper and for a number of constructive suggestions.

REFERENCES

[1] ADAMS, S., KOTECKÝ, R. and MÜLLER, S. In preparation.  
 [2] BARLOW, M. T. and DEUSCHEL, J. D. (2010). Invariance principle for the random conductance model with unbounded conductances. *Ann. Probab.* **38** 234–276. MR2599199

- [3] BERGER, N., BISKUP, M., HOFFMAN, C. E. and KOZMA, G. (2008). Anomalous heat-kernel decay for random walk among bounded random conductances. *Ann. Inst. H. Poincaré Probab. Statist.* **44** 374–392. [MR2446329](#)
- [4] BERGER, N. and BISKUP, M. (2007). Quenched invariance principle for simple random walk on percolation clusters. *Probab. Theory Related Fields* **137** 83–120. [MR2278453](#)
- [5] BISKUP, M. and KOTECKÝ, R. (2007). Phase coexistence of gradient Gibbs states. *Probab. Theory Related Fields* **139** 1–39. [MR2322690](#)
- [6] BISKUP, M. and PRESCOTT, T. M. (2007). Functional CLT for random walk among bounded random conductances. *Electron. J. Probab.* **12** 1323–1348. [MR2354160](#)
- [7] BRYDGES, D. and YAU, H.-T. (1990). Grad  $\phi$  perturbations of massless Gaussian fields. *Comm. Math. Phys.* **129** 351–392. [MR1048698](#)
- [8] COTAR, C., DEUSCHEL, J.-D. and MÜLLER, S. (2009). Strict convexity of the free energy for a class of non-convex gradient models. *Comm. Math. Phys.* **286** 359–376. [MR2470934](#)
- [9] DOYLE, P. G. and SNELL, J. L. (1984). *Random Walks and Electric Networks*. *Carus Mathematical Monographs* **22**. Math. Assoc. America, Washington, DC. [MR920811](#)
- [10] DELMOTTE, T. and DEUSCHEL, J. D. (2005). On estimating the derivatives of symmetric diffusions in stationary random environment, with applications to  $\nabla\phi$  interface model. *Probab. Theory Related Fields* **133** 358–390. [MR2198017](#)
- [11] DURRETT, R. (2005). *Probability: Theory and Examples*, 3rd ed. Brooks/Cole—Thomson Learning, Belmont, CA.
- [12] GAWĘDZKI, K. and KUPIAINEN, A. (1983). Block spin renormalization group for dipole gas and  $(\nabla\phi)^4$ . *Ann. Physics* **147** 198–243. [MR707525](#)
- [13] GIACOMIN, G., OLLA, S. and SPOHN, H. (2001). Equilibrium fluctuations for  $\nabla\phi$  interface model. *Ann. Probab.* **29** 1138–1172. [MR1872740](#)
- [14] GROSS, L. (1967). Abstract Wiener spaces. In *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), Vol. II: Contributions to Probability Theory, Part 1* 31–42. Univ. California Press, Berkeley, CA. [MR0212152](#)
- [15] FUNAKI, T. (2005). *Stochastic Interface Models* (J. PICARD, ed.). *Lecture Notes in Math.* **1869**. Springer, Berlin. [MR2227242](#)
- [16] FUNAKI, T. and SPOHN, H. (1997). Motion by mean curvature from the Ginzburg–Landau  $\nabla\phi$  interface model. *Comm. Math. Phys.* **185** 1–36. [MR1463032](#)
- [17] JIKOV, V. V., KOZLOV, S. M. and OLEĬNIK, O. A. (1994). *Homogenization of Differential Operators and Integral Functionals*. Springer, Berlin. [MR1329546](#)
- [18] KIPNIS, C. and VARADHAN, S. R. S. (1986). Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Comm. Math. Phys.* **104** 1–19. [MR834478](#)
- [19] MAGNEN, J. and SÉNÉOR, R. (1984). The infrared behaviour of  $(\nabla\Phi)_3^4$ . *Ann. Physics* **152** 130–202. [MR734211](#)
- [20] MATHIEU, P. (2008). Quenched invariance principles for random walks with random conductances. *J. Stat. Phys.* **130** 1025–1046. [MR2384074](#)
- [21] MATHIEU, P. and PIATNITSKI, A. (2007). Quenched invariance principles for random walks on percolation clusters. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **463** 2287–2307. [MR2345229](#)
- [22] MOURRAT, J. C. (2009). Variance decay for functionals of the environment viewed from the particle. Available at [arxiv.org/pdf/0902.0204v1](http://arxiv.org/pdf/0902.0204v1).
- [23] NADDAF, A. and SPENCER, T. (1997). On homogenization and scaling limit of some gradient perturbations of a massless free field. *Comm. Math. Phys.* **183** 55–84. [MR1461951](#)
- [24] QUASTEL, J. (1992). Diffusion of color in the simple exclusion process. *Comm. Pure Appl. Math.* **45** 623–679. [MR1162368](#)
- [25] SHEFFIELD, S. (2005). Random surfaces. *Astérisque* **304** vi+175. [MR2251117](#)

- [26] SHEFFIELD, S. (2007). Gaussian free fields for mathematicians. *Probab. Theory Related Fields* **139** 521–541. [MR2322706](#)
- [27] VELENIK, Y. (2006). Localization and delocalization of random interfaces. *Probab. Surv.* **3** 112–169 (electronic). [MR2216964](#)

DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF CALIFORNIA AT LOS ANGELES  
LOS ANGELES  
CALIFORNIA 90095-1555  
USA  
AND  
SCHOOL OF ECONOMICS  
UNIVERSITY OF SOUTH BOHEMIA  
STUDENTSKÁ 13  
CZ-37005 ČESKÉ BUDĚJOVICE  
CZECH REPUBLIC  
E-MAIL: [biskup@math.ucla.edu](mailto:biskup@math.ucla.edu)  
URL: <http://www.math.ucla.edu/~biskup/>

ZENTRUM MATHEMATIK  
TECHNISCHE UNIVERSITÄT MÜNCHEN  
BOLTZMANNSTRASSE 3  
D-85748 GARCHING BEI MÜNCHEN  
GERMANY  
E-MAIL: [spohn@ma.tum.de](mailto:spohn@ma.tum.de)  
URL: <http://www-m5.ma.tum.de/pers/spohn/>