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FEYNMAN–KAC FORMULA FOR HEAT EQUATION DRIVEN BY FRACTIONAL WHITE NOISE

BY YAOZHONG HU, DAVID NUALART AND JIAN SONG

University of Kansas

We establish a version of the Feynman–Kac formula for the multidimensional stochastic heat equation with a multiplicative fractional Brownian sheet. We use the techniques of Malliavin calculus to prove that the process defined by the Feynman–Kac formula is a weak solution of the stochastic heat equation. From the Feynman–Kac formula, we establish the smoothness of the density of the solution and the Hölder regularity in the space and time variables. We also derive a Feynman–Kac formula for the stochastic heat equation in the Skorokhod sense and we obtain the Wiener chaos expansion of the solution.

1. Introduction. Consider the following heat equation on $\mathbb{R}^d$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\Delta u + c(t,x)u, \\ u(0,x) = f(x), \end{cases} \tag{1.1}$$

where f is a bounded measurable function. If $c(t,x)$ is a continuous function of $(t,x) \in [0,\infty) \times \mathbb{R}^d$, then we have the well-known Feynman–Kac formula (see [2]) for the solution of above equation

$$u(t,x) = E\left[f(B_t^x)\exp\left(\int_0^t c(t-s, B_s^x)\,ds\right)\right],$$

where $B_t^x = B_t + x$ is a d-dimensional Brownian motion starting from the point x.

In this paper, we shall extend the above Feynman–Kac formula to the heat equation with fractional noise

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\Delta u + u\,\dfrac{\partial^{d+1} W}{\partial t\,\partial x_1 \cdots \partial x_d}, \\ u(0,x) = f(x), \end{cases} \tag{1.2}$$

where $W(t,x)$ is a fractional Brownian sheet with Hurst parameters H_0 in time and $(H_1, \ldots, H_d)$ in space, respectively. The difference between (1.1) and (1.2) is that

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$\frac{\partial^{d+1} W}{\partial t\,\partial x_1 \cdots \partial x_d}$ is no longer a function of t and x, but a generalized (random) function. For this equation, we can still formally write down the Feynman–Kac formula

$$u(t,x) = E^B\left[f(B_t^x)\exp\left(\int_0^t \int_{\mathbb{R}^d} \delta(B_{t-r}^x - y) W(dr,dy)\right)\right], \tag{1.3}$$

where E^B denotes the expectation with respect to the Brownian motion B_t^x and δ denotes the Dirac delta function.

The aim of this paper is to justify the above formula (1.3), to show that the process $u(t,x)$ is a weak solution of (1.2) and to establish some properties of this process. First, we shall show that the stochastic Feynman–Kac functional $V_{t,x} := \int_0^t \int_{\mathbb{R}^d} \delta(B_{t-r}^x - y) W(dr,dy)$ is a well-defined random variable. This will be done in Section 2 using a suitable approximation of the Dirac delta function, assuming that the Hurst parameters satisfy $2H_0 + \sum_{i=1}^d H_i > d+1$, $H_0 \ge \frac{1}{2}$ and $H_i > \frac{1}{2}$ for $1 \le i \le d$.

After the definition of the random variable $V_{t,x}$, the next problem is to show its exponential integrability. With the use of the covariance structure of the fractional Brownian sheet $W(t,x)$, we show that $u(t,x)$ has exponential moments provided that

$$E\exp\left[\lambda \int_0^1 \int_0^1 |r-s|^{2H_0-2} \prod_{i=1}^d |B_r^i - B_s^i|^{2H_i-2}\, dr\, ds\right] < \infty \tag{1.4}$$

for any $\lambda \in \mathbb{R}$. To show that (1.4) is true, we use a method introduced by Le Gall in [8] to derive the exponential integrability of the renormalized self-intersection local time of the planar Brownian motion, together with the self-similarity of the fractional Brownian sheet and several other techniques. This is done in Section 3.

Another major aim of this paper is to show that $u(t,x)$ defined by (1.3) is a weak solution of (1.2). Instead of following the classical approach based on Itô's formula, which seems complicated in our situation, we again use the approximation technique, together with Malliavin calculus. The main ingredient is to express the Stratonovich integral as the sum of a Skorokhod integral plus a correction term involving Malliavin derivatives. This is a new methodology which is developed in Section 4.

The Feynman–Kac formula gives an explicit form of a weak solution of equation (1.2) which turns out to be very useful for obtaining regularity properties. Several consequences of this expression are derived in Section 5. First, we derive the Hölder continuity of the solution $u(t,x)$ with respect to t and x, and, afterward, we establish the smoothness of the density of the probability law of $u(t,x)$ (with respect to the Lebesgue measure) using techniques of Malliavin calculus.

In the above (1.2), the solution and the noise are multiplied using the ordinary product. This gives rise to the Stratonovich integral when we interpret the equation in its integral form. There are several papers where the Wick product between the solution and the noise is used, this corresponding to the Skorokhod integral. The

Stratonovich integral is more difficult to handle, but it is the right choice if we want to represent a physical model. Applying a Wiener chaos technique pioneered by Dawson and Salehi in [1] and used in several other papers (see, e.g., the work [5] on the relation between moments of the solution and self-intersection local times), one can show that there exists a unique mild solution to the Skorokhod-type equation. We discuss this result in Section 7 and, using Wiener chaos expansions, we obtain a Feynman–Kac formula for this solution.

The above techniques work for $H_i > 1/2$, $i = 1, 2, \ldots, d$. From the condition $2H_0 + \sum_{i=1}^d H_i > d + 1$, it follows that H_0 must be greater than $1/2$ and we cannot allow more than one of the $H_1, \ldots, H_d$ to be less than or equal to $1/2$. Thus, if we want to remove the condition $H_i > 1/2$, $i = 1, 2, \ldots, d$, we need $d = 1$. We show in Section 7 that if $d = 1$, $H_1 = \frac{1}{2}$ and $H_0 > \frac{3}{4}$, then all previous results hold. When $d = 1$, we can also handle the case $H_0 < 1/2$, assuming that the process has a regular spatial covariance. This has been done in the companion paper [4], using different techniques. Finally, the Appendix contains some technical results which are used in the paper.

We would like to close this introduction with some remarks about the motivation of our work and its connection with other related results. The existence of a Feynman–Kac formula like the one we have derived here was mentioned as a conjecture in a paper by Mocioalca and Viens (see [9]), although this problem was circulating long before that. In the lectures by Walsh in Saint Flour (see [13]) it was stated that the one-dimensional equation in the Itô sense driven by a space–time white noise cannot have a Feynman–Kac formula because the Itô–Stratonovich correction term is infinite. In a previous work [5], two of the present authors considered a Skorokhod-type equation assuming $H_i = \frac{1}{2}$ for $i = 1, \ldots, d$. In this case, there exists a unique mild solution obtained by means of the Wiener chaos method if $d = 1$ or $d = 2$, $H_0 > \frac{1}{2}$ and t is small enough, although the Feynman–Kac formula is not available unless $d = 1$ and $H_0 > \frac{3}{4}$ (see Section 7).

A process similar to (1.3) was studied by Viens and Zhang in [12], although it does not have a relation with a stochastic heat equation and, most likely, the asymptotic results obtained in [12] can be extended to the process (1.3).

Recently, Hinz obtained in [3] a Feynman–Kac formula for the stochastic heat equation with a Gaussian multiplicative noise of the form $\frac{\partial W}{\partial t}(t, x)$, where W is a fractional Brownian sheet with Hurst parameter $H > \frac{1}{2}$ in time and $K \in (0, 1)$ in space, and he used this formula to solve a stochastic Burgers equation by means of the Hopf–Cole transformation. In this paper, the noise is more regular in space and this allows the techniques of classical fractional calculus to be used, together with curvilinear integrals.

2. Preliminaries. Fix a vector of Hurst parameters $H = (H_0, H_1, \ldots, H_d)$, where $H_i \in (\frac{1}{2}, 1)$. Suppose that $W = \{W(t, x), t \geq 0, x \in \mathbb{R}^d\}$ is a zero-mean

Gaussian random field with the covariance function

$$E(W(t,x)W(s,y)) = R_{H_0}(s,t)\prod_{i=1}^{d} R_{H_i}(x_i,y_i),$$

where, for any $H \in (0,1)$, we denote by $R_H(s,t)$ the covariance function of the fractional Brownian motion with Hurst parameter H, that is,

$$R_H(s,t) = \tfrac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

In other words, W is a fractional Brownian sheet with Hurst parameters H_0 in the time variable and H_i in the space variables, $i = 1,\dots,d$.

Denote by $\mathcal{E}$ the linear span of the indicator functions of rectangles of the form $(s,t]\times(x,y]$ in $\mathbb{R}_+\times\mathbb{R}^d$. Consider, in $\mathcal{E}$, the inner product defined by

$$\langle I_{(0,s]\times(0,x]}, I_{(0,t]\times(0,y]}\rangle_{\mathcal{H}} = R_{H_0}(s,t)\prod_{i=1}^{d} R_{H_i}(x_i,y_i).$$

In the above formula, if $x_i < 0$, then we assume, by convention, that $I_{(0,x_i]} = -I_{(-x_i,0]}$. We denote by $\mathcal{H}$ the closure of $\mathcal{E}$ with respect to this inner product. The mapping $W: I_{(0,t]\times(0,x]} \to W(t,x)$ extends to a linear isometry between $\mathcal{H}$ and the Gaussian space spanned by W. We will denote this isometry by

$$W(\phi) = \int_0^\infty \int_{\mathbb{R}^d} \phi(t,x)W(dt,dx)$$

if $\phi \in \mathcal{H}$. Notice that if ϕ and ψ are functions in $\mathcal{E}$, then

$$\begin{aligned} E(W(\phi)W(\psi)) &= \langle \phi,\psi\rangle_{\mathcal{H}} \\ &= \alpha_H \int_{\mathbb{R}_+^2\times\mathbb{R}^{2d}} \phi(s,x)\psi(t,y)|s-t|^{2H_0-2} \\ &\qquad \times \prod_{i=1}^{d}|x_i-y_i|^{2H_i-2}\,ds\,dt\,dx\,dy, \end{aligned} \tag{2.1}$$

where $\alpha_H = \prod_{i=0}^{d} H_i(2H_i-1)$. Furthermore, $\mathcal{H}$ contains the class of measurable functions ϕ on $\mathbb{R}_+\times\mathbb{R}^d$ such that

$$\int_{\mathbb{R}_+^2\times\mathbb{R}^{2d}} |\phi(s,x)\phi(t,y)||s-t|^{2H_0-2}\prod_{i=1}^{d}|x_i-y_i|^{2H_i-2}\,ds\,dt\,dx\,dy < \infty. \tag{2.2}$$

We will denote by D the derivative operator in the sense of Malliavin calculus. That is, if F is a smooth and cylindrical random variable of the form

$$F = f(W(\phi_1),\dots,W(\phi_n)),$$

$\phi_i \in \mathcal{H}$, $f \in C_p^\infty(\mathbb{R}^n)$ (f and all its partial derivatives have polynomial growth), then DF is the $\mathcal{H}$-valued random variable defined by

$$DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W(\phi_1), \ldots, W(\phi_n))\phi_j.$$

The operator D is closable from $L^2(\Omega)$ into $L^2(\Omega; \mathcal{H})$ and we define the Sobolev space $\mathbb{D}^{1,2}$ as the closure of the space of smooth and cylindrical random variables under the norm

$$\|DF\|_{1,2} = \sqrt{E(F^2) + E(\|DF\|_{\mathcal{H}}^2)}.$$

We denote by δ the adjoint of the derivative operator given by duality formula

$$E(\delta(u)F) = E(\langle DF, u\rangle_{\mathcal{H}}) \tag{2.3}$$

for any $F \in \mathbb{D}^{1,2}$ and any element $u \in L^2(\Omega; \mathcal{H})$ in the domain of δ. The operator δ is also called the *Skorokhod integral* because in the case of the Brownian motion, it coincides with an extension of the Itô integral introduced by Skorokhod. We refer to Nualart [10] for a detailed account of the Malliavin calculus with respect to a Gaussian process. If DF and u are almost surely measurable functions on $\mathbb{R}_+ \times \mathbb{R}^d$ verifying condition (2.2), then the duality formula (2.3) can be written using the expression of the inner product in $\mathcal{H}$ given in (2.1):

$$E(\delta(u)F) = \alpha_H E\biggl(\int_{\mathbb{R}_+^2 \times \mathbb{R}^{2d}} D_{s,x}Fu(t,y)|s-t|^{2H_0-2} \times \prod_{i=1}^d |x_i - y_i|^{2H_i-2}\, ds\, dt\, dx\, dy\biggr).$$

We recall the following formula, which we will use in the paper:

$$FW(\phi) = \delta(F\phi) + \langle DF, \phi\rangle_{\mathcal{H}} \tag{2.4}$$

for any $\phi \in \mathcal{H}$ and any random variable F in the Sobolev space $\mathbb{D}^{1,2}$.

Throughout the paper, C will denote a positive constant which may vary from one formula to another.

3. Definition and exponential integrability of the stochastic Feynman–Kac functional.

For any $\varepsilon > 0$, we denote by $p_\varepsilon(x)$ the d-dimensional heat kernel:

$$p_\epsilon(x) = (2\pi\varepsilon)^{-d/2}e^{-|x|^2/2\epsilon}, \qquad x \in \mathbb{R}^d.$$

On the other hand, for any $\delta > 0$, we define the function

$$\varphi_\delta(x) = \frac{1}{\delta}I_{[0,\delta]}(x).$$

$\varphi_\delta(t)p_\varepsilon(x)$ then provides an approximation of the Dirac delta function $\delta(t,x)$ as ε and δ tend to zero. We denote by $W^{\epsilon,\delta}$ the approximation of the fractional Brownian sheet $W(t,x)$ defined by

$$W^{\epsilon,\delta}(t,x)=\int_0^t\int_{\mathbb{R}^d}\varphi_\delta(t-s)p_\epsilon(x-y)W(s,y)\,ds\,dy. \tag{3.1}$$

Fix $x\in\mathbb{R}^d$ and $t>0$. Suppose that $B=\{B_t,t\geq 0\}$ is a d-dimensional standard Brownian motion independent of W. We denote by $B_t^x=B_t+x$ the Brownian motion starting at the point x. We are going to define the random variable $\int_0^t\int_{\mathbb{R}^d}\delta(B_{t-r}^x-y)W(dr,dy)$ by approximating the Dirac delta function $\delta(B_{t-r}^x-y)$ by

$$A_{t,x}^{\epsilon,\delta}(r,y)=\int_0^t\varphi_\delta(t-s-r)p_\varepsilon(B_s^x-y)\,ds. \tag{3.2}$$

We will show that for any $\varepsilon>0$ and $\delta>0$, the function $A_{t,x}^{\epsilon,\delta}$ belongs to the space $\mathcal{H}$ almost surely and the family of random variables

$$V_{t,x}^{\varepsilon,\delta}=\int_0^t\int_{\mathbb{R}^d}A_{t,x}^{\epsilon,\delta}(r,y)W(dr,dy) \tag{3.3}$$

converges in L^2 as ε and δ tend to zero.

The specific approximation chosen here will allow us, in Section 4, to construct an approximate Feynman–Kac formula with the random potential $\dot{W}^{\epsilon,\delta}(t,x)$ given in (4.1). Moreover, this approximation has the useful properties proved in Lemmas A.2 and A.3. We could have used other types of approximation schemes with similar results. Also, we can restrict ourselves to the special case $\delta=\varepsilon$, but the slightly more general case considered here does not need any additional effort.

Throughout the paper, we denote by $E^B(\Phi(B,W))$ [resp., by $E^W(\Phi(B,W))$] the expectation of a functional $\Phi(B,W)$ with respect to B (resp., with respect to W). We will use E for the composition E^BE^W and also in the case of a random variable depending only on B or W.

THEOREM 3.1. *Suppose that $2H_0+\sum_{i=1}^d H_i>d+1$. Then, for any $\varepsilon>0$ and $\delta>0$, $A_{t,x}^{\varepsilon,\delta}$ defined in (3.2) belongs to $\mathcal{H}$ and the family of random variables $V_{t,x}^{\epsilon,\delta}$ defined in (3.3) converges in L^2 to a limit denoted by (this being the stochastic Feynman–Kac functional)*

$$V_{t,x}=\int_0^t\int_{\mathbb{R}^d}\delta(B_{t-r}^x-y)W(dr,dy). \tag{3.4}$$

Conditional on B, $V_{t,x}$ is a Gaussian random variable with mean 0 and variance

$$\operatorname{Var}^W(V_{t,x})=\alpha_H\int_0^t\int_0^t|r-s|^{2H_0-2}\prod_{i=1}^d|B_r^i-B_s^i|^{2H_i-2}\,dr\,ds. \tag{3.5}$$

PROOF. Fix ε, ε', δ and $\delta' > 0$. Let us compute the inner product

$$\begin{aligned}\langle A_{t,x}^{\varepsilon,\delta}, A_{t,x}^{\varepsilon',\delta'}\rangle_{\mathcal{H}} \\ &= \alpha_H \int_{[0,t]^4}\int_{\mathbb{R}^{2d}} p_\epsilon(B_s^x - y)p_{\epsilon'}(B_r^x - z) \\ &\qquad \times \varphi_\delta(t-s-u)\varphi_{\delta'}(t-r-v)|u-v|^{2H_0-2} \\ &\qquad \times \prod_{i=1}^d |y_i - z_i|^{2H_i-2}\, dy\, dz\, du\, dv\, ds\, dr.\end{aligned} \tag{3.6}$$

By Lemmas A.2 and A.3, we have the estimate

$$\begin{aligned}&\int_{[0,t]^2}\int_{\mathbb{R}^{2d}} p_\epsilon(B_s^x - y)p_{\epsilon'}(B_r^x - z)\varphi_\delta(t-s-u)\varphi_{\delta'}(t-r-v) \\ &\qquad \times |u-v|^{2H_0-2}\prod_{i=1}^d |y_i - z_i|^{2H_i-2}\, dy\, dz\, du\, dv \\ &\qquad \le C|s-r|^{2H_0-2}\prod_{i=1}^d |B_s^i - B_r^i|^{2H_i-2}\end{aligned} \tag{3.7}$$

for some constant $C > 0$. The expectation of this random variable is integrable in $[0, t]^2$ because

$$\begin{aligned}&E^B \int_0^t\int_0^t |s-r|^{2H_0-2}\prod_{i=1}^d |B_s^i - B_r^i|^{2H_i-2}\, ds\, dr \\ &\qquad = \prod_{i=1}^d E|\xi|^{2H_i-2}\int_0^t\int_0^t |s-r|^{2H_0+\sum_{i=1}^d H_i - d-2}\, ds\, dr \\ &\qquad = \frac{2\prod_{i=1}^d E|\xi|^{2H_i-2}t^{\kappa+1}}{\kappa(\kappa+1)} < \infty,\end{aligned} \tag{3.8}$$

where

$$\kappa = 2H_0 + \sum_{i=1}^d H_i - d - 1 > 0 \tag{3.9}$$

and ξ is a $N(0, 1)$ random variable.

As a consequence, taking the mathematical expectation with respect to B in (3.6), letting $\varepsilon = \varepsilon'$ and $\delta = \delta'$ and using the estimates (3.7) and (3.8) yields

$$E^B\|A_{t,x}^{\varepsilon,\delta}\|_{\mathcal{H}}^2 \le C.$$

This implies that almost surely $A_{t,x}^{\varepsilon,\delta}$ belongs to the space $\mathcal{H}$ for all ε and $\delta > 0$. Therefore, the random variables $V_{t,x}^{\varepsilon,\delta} = W(A_{t,x}^{\varepsilon,\delta})$ are well defined and we have

$$E^B E^W(V_{t,x}^{\epsilon,\delta}V_{t,x}^{\epsilon',\delta'}) = E^B\langle A_{t,x}^{\varepsilon,\delta}, A_{t,x}^{\varepsilon',\delta'}\rangle_{\mathcal{H}}.$$

For any $s \neq r$ and $B_s \neq B_r$, as ε, ε', δ and δ' tend to zero, the left-hand side of the inequality (3.7) converges to $|s-r|^{2H_0-2}\prod_{i=1}^d |B_s^i - B_r^i|^{2H_i-2}$. Therefore, by the dominated convergence theorem, we obtain that $E^B E^W(V_{t,x}^{\epsilon,\delta} V_{t,x}^{\epsilon',\delta'})$ converges to Σ_t as ε, ε', δ and δ' tend to zero, where

$$\Sigma_t = \frac{2\alpha_H \prod_{i=1}^d E|\xi|^{2H_i-2} t^{\kappa+1}}{\kappa(\kappa+1)}.$$

Thus, we obtain

$$E(V_{t,x}^{\varepsilon,\delta} - V_{t,x}^{\varepsilon',\delta'})^2 = E(V_{t,x}^{\varepsilon,\delta})^2 - 2E(V_{t,x}^{\varepsilon,\delta} V_{t,x}^{\varepsilon',\delta'}) + E(V_{t,x}^{\varepsilon',\delta'})^2 \to 0.$$

This implies that $V_{t,x}^{\epsilon_n,\delta_n}$ is a Cauchy sequence in L^2 for all sequences ε_n and δ_n converging to zero. As a consequence, $V_{t,x}^{\epsilon_n,\delta_n}$ converges in L^2 to a limit denoted by $V_{t,x}$ which does not depend on the choice of the sequences ε_n and δ_n. Finally, by a similar argument, we show (3.5). □

Condition $2H_0 + \sum_{i=1}^d H_i > d+1$ is sharp and cannot be improved. In fact, if this condition does not hold, then almost surely $(r, y) \mapsto \delta(B_{t-r}^x - y)$ is not an element of the space $\mathcal{H}$, as follows from the next proposition.

PROPOSITION 3.2. *Suppose that $H_i > 1/2$, $i = 0, 1, \dots, d$, and $2H_0 + \sum_{i=1}^d H_i \le d+1$. Then, conditionally on B, the family $V_{t,x}^{\varepsilon,\delta}$ does not converge in probability as ε and δ tend to zero for almost all trajectories of B.*

PROOF. Given B, $V_{t,x}^{\varepsilon,\delta}$ is a Gaussian family of random variables and it suffices to show that they do not converge in L^2. This follows from the fact that the variance limit is infinite almost surely. In fact, from the Lévy modulus of continuity of the Brownian motion, it is easy to show that if $2H_0 + \sum_{i=1}^d H_i \le d+1$, then

$$\int_0^t \int_0^t |s-r|^{2H_0-2} \prod_{i=1}^d |B_s^i - B_r^i|^{2H_i-2}\, ds\, dr = \infty$$

almost surely. □

The next result provides the exponential integrability of the random variable $V_{t,x}$ defined in (3.4).

THEOREM 3.3. *Suppose that $2H_0 + \sum_{i=1}^d H_i > d+1$. Then, for any $\lambda \in \mathbb{R}$, we have*

$$E \exp\left(\lambda \int_0^t \int_{\mathbb{R}^d} \delta(B_{t-r}^x - y) W(dr, dy)\right) < \infty. \tag{3.10}$$

PROOF. The proof involves several steps.

Step 1. From (3.5), we obtain

$$Ee^{\lambda V_{t,x}} = E^B \exp\left(\frac{\lambda^2}{2}\alpha_H \int_0^t \int_0^t |s-r|^{2H_0-2} \prod_{i=1}^d |B_s^i - B_r^i|^{2H_i-2}\, ds\, dr\right)$$

and the scaling property of the Brownian motion yields

$$Ee^{\lambda V_{t,x}} = Ee^{\mu Y}, \tag{3.11}$$

where $\mu = \frac{\lambda^2}{2}\alpha_H t^{\kappa+1}$, κ is as defined in (3.9) and

$$Y = \int_0^1 \int_0^1 |s-r|^{2H_0-2} \prod_{i=1}^d |B_s^i - B_r^i|^{2H_i-2}\, ds\, dr. \tag{3.12}$$

It then suffices to show that the random variable Y has exponential moments of all orders.

Step 2. Our approach to proving that $E\exp(\lambda Y) < \infty$ for any $\lambda \in \mathbb{R}$ is motivated by the method of Le Gall [8]. For $k = 1, \dots, 2^{n-1}$, we define $A_{n,k} = [\frac{2k-2}{2^n}, \frac{2k-1}{2^n}] \times [\frac{2k-1}{2^n}, \frac{2k}{2^n}]$ and

$$\alpha_{n,k} = \int_{A_{n,k}} |s-r|^{2H_0-2} \prod_{i=1}^d |B_s^i - B_r^i|^{2H_i-2}\, ds\, dr.$$

The random variables $\alpha_{n,k}$ have the following two properties:

(i) for every $n \geq 1$, the variables $\alpha_{n,1}, \dots, \alpha_{n,2^{n-1}}$ are independent;

(ii) $\alpha_{n,k} \overset{d}{=} 2^{-n(\kappa+1)}\alpha_0$, where

$$\alpha_0 = \int_0^1 \int_0^1 (s+r)^{2H_0-2} \prod_{i=1}^d |B_s^i - \tilde{B}_r^i|^{2H_i-2}\, ds\, dr,$$

and $\tilde{B}$ is a standard Brownian motion independent of B.

The condition $2H_0 + \sum_{i=1}^d H_i > d+1$ implies that $E\alpha_0 < \infty$ and we deduce that

$$Y = 2\sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} \alpha_{n,k},$$

where the series converges in the L^1 sense.

Step 3. For any integer $n \geq 1$, we claim that

$$E\alpha_0^n \leq E\left(C \int_0^1 \prod_{i=1}^d |B_s^i|^{2H_i-2}\, ds\right)^n \tag{3.13}$$

for some constant $C > 0$. In fact, we have

$$E\alpha_0^n = E\int_{[0,1]^{2n}} \prod_{j=1}^n \prod_{i=1}^d (|s_j + t_j|^{2H_0-2} |B_{s_j}^i - \tilde{B}_{t_j}^i|^{2H_i-2})\, ds\, dt. \tag{3.14}$$

Using the formula

$$c^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty e^{-c\tau} \tau^{z-1}\, d\tau,$$

we obtain, for each $i = 1, \dots, d$,

$$\begin{aligned} E \prod_{j=1}^n |B_{s_j}^i - \tilde{B}_{t_j}^i|^{2H_i - 2} &= \Gamma(1 - H_i)^{-n} \\ &\quad \times \int_{[0,\infty)^n} E \exp\Biggl(-\sum_{j=1}^n |B_{s_j}^i - \tilde{B}_{t_j}^i|^2 \tau_j\Biggr) \\ &\quad \times \prod_{j=1}^n \tau_j^{-H_i}\, d\tau. \end{aligned} \tag{3.15}$$

For any $\tau_1, \dots, \tau_n > 0$ and $s_1, t_1, \dots, s_n, t_n \in (0, 1)$, we define

$$Q_1 = \bigl(E(B_{s_j}^i B_{s_k}^i)\sqrt{\tau_j \tau_k}\bigr)_{n\times n}, \qquad Q_2 = \bigl(E(\tilde{B}_{t_j}^i \tilde{B}_{t_k}^i)\sqrt{\tau_j \tau_k}\bigr)_{n\times n}.$$

We know that

$$E \exp\Biggl(-\sum_{j=1}^n |B_{s_j}^i - \tilde{B}_{t_j}^i|^2 \tau_j\Biggr) = \det(I + 2Q_1 + 2Q_2)^{-1/2}. \tag{3.16}$$

Substituting (3.16) into (3.15) yields

$$\begin{aligned} E \prod_{j=1}^n &|B_{s_j}^i - \tilde{B}_{t_j}^i|^{2H_i - 2} \\ &= \Gamma(1 - H_i)^{-n} \int_{[0,\infty)^n} \det(I + 2Q_1 + 2Q_2)^{-1/2} \prod_{j=1}^n \tau_j^{-H_i}\, d\tau \\ &\le \Gamma(1 - H_i)^{-n} \\ &\qquad \times \int_{[0,\infty)^n} \det(I + 2Q_1)^{-1/4} \det(I + 2Q_2)^{-1/4} \prod_{j=1}^n \tau_j^{-H_i}\, d\tau \qquad (3.17) \\ &\le \Gamma(1 - H_i)^{-n} \Biggl[\int_{[0,\infty)^n} \det(I + 2Q_1)^{-1/2} \prod_{j=1}^n \tau_j^{-H_i} d\tau\Biggr]^{1/2} \\ &\qquad \times \Biggl[\int_{[0,\infty)^n} \det(I + 2Q_2)^{-1/2} \prod_{j=1}^n \tau_j^{-H_i}\, d\tau\Biggr]^{1/2} \\ &= \Biggl[E \prod_{j=1}^n |B_{s_j}^i|^{2H_i - 2} E \prod_{j=1}^n |\tilde{B}_{t_j}^i|^{2H_i - 2}\Biggr]^{1/2}, \end{aligned}$$

where, in the above first inequality, we have used the estimates

$$(I+2Q_1+2Q_2) \ge \tfrac{1}{2}[(I+2Q_1)+(I+2Q_2)] \\ \ge (I+2Q_1)^{1/2}(I+2Q_2)^{1/2}.$$

Substituting (3.17) into (3.14) and using the inequality $(s_j+t_j)^{2H_0-2} \le s_j^{H_0-1} \times t_j^{H_0-1}$, we obtain

$$E\alpha_0^n \le \int_{[0,1]^{2n}} \prod_{j=1}^n (s_j+t_j)^{2H_0-2} \prod_{i=1}^d \left[E\prod_{j=1}^n |B_{s_j}^i|^{2H_i-2} E\prod_{j=1}^n |\tilde{B}_{t_j}^i|^{2H_i-2}\right]^{1/2} ds\,dt$$
$$\le \left(\int_{[0,1]^n} \prod_{j=1}^n s_j^{H_0-1}\left(E\prod_{j=1}^n\prod_{i=1}^d |B_{s_j}^i|^{2H_i-2}\right)^{1/2} ds\right)^2.$$

Finally, using Hölder's inequality with $\frac{1}{H_0} < p < 2$, we get

$$E\alpha_0^n \le C^n\left(\int_{[0,1]^n}\left(E\prod_{i=1}^d\prod_{j=1}^n |B_{s_j}^i|^{2H_i-2}\right)^{p/2} ds\right)^{2/p}$$
$$\le C^n \int_{[0,1]^n} E\prod_{i=1}^d\prod_{j=1}^n |B_{s_j}^i|^{2H_i-2}\,ds$$
$$= E\left(C\int_0^1 \prod_{i=1}^d |B_s^i|^{2H_i-2}\,ds\right)^n.$$

This completes the proof of (3.13).

Step 4. For any $\lambda > 0$, using (3.13) and Lemma A.5 in the Appendix, we obtain

$$Ee^{\lambda\alpha_0} \le E\exp\left(C\lambda\int_0^1\prod_{i=1}^d |B_s^i|^{2H_i-2}\,ds\right) < \infty, \tag{3.18}$$

because $\rho < 1$.

Step 5. Define $\varphi(\lambda) = E(e^{\lambda(\alpha_0 - E\alpha_0)})$. By (3.18), $\varphi(\lambda) < \infty$ for all $\lambda \in \mathbb{R}$. Since $\varphi'(0) = 0$, for every $K > 0$, we can find a positive constant C_K such that for all $\lambda \in [0, K]$,

$$\varphi(\lambda) \le 1 + C_K\lambda^2.$$

Define $\overline{\alpha}_{n,k} = \alpha_{n,k} - E(\alpha_{n,k})$. Fix $K > 0$ and $a \in (0, \kappa+1)$, where κ is as defined in (3.9). Recall that by property (ii) in step 3, $\overline{\alpha}_{n,k} \overset{d}{=} 2^{-n(\kappa+1)}\overline{\alpha}_0$. For every $N \ge 2$, set $b_N = 2K\prod_{j=2}^{j=N}(1 - 2^{-a(j-1)})$ and $b_1 = 2K$. Then, by Hölder's inequality and properties (i) and (ii) of $\alpha_{n,k}$, we have, for $N \geq 2$,

$$
\begin{aligned}
E \exp\Biggl(b_N \sum_{n=1}^{N} \sum_{k=1}^{2^{n-1}} \overline{\alpha}_{n,k}\Biggr)
&\leq \Biggl[E \exp\Biggl(\frac{b_N}{1-2^{-a(N-1)}} \sum_{n=1}^{N-1} \sum_{k=1}^{2^{n-1}} \overline{\alpha}_{n,k}\Biggr)\Biggr]^{1-2^{-a(N-1)}} \\
&\quad \times \Biggl[E \exp\Biggl(2^{a(N-1)} b_N \sum_{k=1}^{2^{N-1}} \overline{\alpha}_{N,k}\Biggr)\Biggr]^{2^{-a(N-1)}} \\
&\leq E \exp\Biggl(b_{N-1} \sum_{n=1}^{N-1} \sum_{k=1}^{2^{n-1}} \overline{\alpha}_{n,k}\Biggr) \varphi\bigl(b_N 2^{a(N-1)-(\kappa+1)N}\bigr)^{2^{(1-a)(N-1)}}.
\end{aligned}
$$

Notice that $b_N 2^{a(N-1)-(\kappa+1)N} \leq 2K$. It follows that

$$
\begin{aligned}
\varphi\bigl(b_N 2^{a(N-1)-(\kappa+1)N}\bigr)^{2^{(1-a)(N-1)}} &\leq \bigl(1 + C_K b_N^2 2^{2((a-\kappa-1)N-a)}\bigr)^{2^{(1-a)(N-1)}} \\
&\leq \exp\bigl(C 2^{(a+1-2(\kappa+1))N}\bigr)
\end{aligned}
$$

for a constant C independent of N. By induction, we get

$$
\begin{aligned}
E \exp\Biggl(b_N \sum_{n=1}^{N} \sum_{k=1}^{2^{n-1}} \overline{\alpha}_{n,k}\Biggr) &\leq \exp\Biggl(C \sum_{n=2}^{N} 2^{(a+1-2(\kappa+1))n}\Biggr) E \exp(b_1 \overline{\alpha}_{1,1}) \\
&\leq \exp\bigl(C\bigl(1 - 2^{a+1-2(\kappa+1)}\bigr)^{-1}\bigr)\varphi(K).
\end{aligned}
$$

Letting N tend to infinity and using Fatou's lemma, we obtain

$$
E \exp\bigl(b_\infty (Y - EY)/2\bigr) < \infty,
$$

where $b_\infty = 2K \prod_{j=1}^{\infty}(1 - 2^{-aj}) > 0$. Since $K > 0$ is arbitrary, we conclude that $E \exp(\lambda Y) < \infty$ for all $\lambda \in \mathbb{R}$. This completes the proof, in view of (3.11). □

4. Feynman–Kac formula. We recall that W is a fractional Brownian sheet on $\mathbb{R}_+ \times \mathbb{R}^d$ with Hurst parameters $(H_0, H_1, \ldots, H_d)$, where $H_i \in (\frac{1}{2}, 1)$ for $i = 0, \ldots, d$. For any $\varepsilon, \delta > 0$, we define

$$
\dot{W}^{\epsilon,\delta}(t,x) := \int_0^t \int_{\mathbb{R}^d} \varphi_\delta(t-s) p_\epsilon(x-y) W(ds, dy). \tag{4.1}
$$

In order to provide a notion of solution for the heat equation with fractional noise (1.2), we need the following definition of the Stratonovich integral, which is equivalent to that of Russo and Vallois in [11].

DEFINITION 4.1. Given a random field $v = \{v(t,x), t \geq 0, x \in \mathbb{R}^d\}$ such that

$$\int_0^T \int_{\mathbb{R}^d} |v(t,x)|\, dx\, dt < \infty$$

almost surely for all $T > 0$, the Stratonovich integral $\int_0^T \int_{\mathbb{R}^d} v(t,x) W(dt,dx)$ is defined as the following limit in probability, if it exists:

$$\lim_{\epsilon,\delta \downarrow 0} \int_0^T \int_{\mathbb{R}^d} v(t,x) \dot{W}^{\epsilon,\delta}(t,x)\, dx\, dt.$$

We are going to consider the following notion of solution for (1.2).

DEFINITION 4.2. A random field $u = \{u(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a weak solution of (1.2) if, for any C^∞ function φ with compact support on $\mathbb{R}^d$, we have

$$\begin{aligned}\int_{\mathbb{R}^d} u(t,x)\varphi(x)\, dx &= \int_{\mathbb{R}^d} f(x)\varphi(x)\, dx + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} u(s,x)\Delta\varphi(x)\, dx\, ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} u(s,x)\varphi(x) W(ds,dx)\end{aligned}$$

almost surely for all $t \geq 0$, where the last term is a Stratonovich stochastic integral in the sense of Definition 4.1.

The following is the main result of this section.

THEOREM 4.3. *Suppose that* $2H_0 + \sum_{i=1}^d H_i > d+1$ *and that* f *is a bounded measurable function. Then, the process*

$$u(t,x) = E^B\left(f(B_t^x) \exp\left(\int_0^t \int_{\mathbb{R}^d} \delta(B_{t-r}^x - y) W(dr,dy) \right) \right) \tag{4.2}$$

is a weak solution of (1.2).

PROOF. Consider the approximation of (1.2) given by the following heat equation with a random potential:

$$\begin{cases} \dfrac{\partial u^{\varepsilon,\delta}}{\partial t} = \dfrac{1}{2}\Delta u^{\varepsilon,\delta} + u^{\varepsilon,\delta} \dot{W}_{t,x}^{\epsilon,\delta}, \\ u^{\varepsilon,\delta}(0,x) = f(x). \end{cases} \tag{4.3}$$

From the classical Feynman–Kac formula, we know that

$$u^{\varepsilon,\delta}(t,x) = E^B\left(f(B_t^x) \exp\left(\int_0^t \dot{W}^{\epsilon,\delta}(t-s, B_s^x)\, ds \right) \right),$$

where B_t^x is a d-dimensional Brownian motion independent of W starting at x. By Fubini's theorem, we can write

$$\begin{aligned}\int_0^t \dot{W}^{\varepsilon,\delta}(t-s, B_s^x)\,ds &= \int_0^t \left(\int_0^t \int_{\mathbb{R}^d} \varphi_\delta(t-s-r) p_\varepsilon(B_s^x - y) W(dr, dy)\right) ds \\ &= \int_0^t \int_{\mathbb{R}^d} \left(\int_0^t \varphi_\delta(t-s-r) p_\varepsilon(B_s^x - y)\,ds\right) W(dr, dy) \\ &= V_{t,x}^{\varepsilon,\delta},\end{aligned}$$

where $V_{t,x}^{\varepsilon,\delta}$ is defined in (3.3). Therefore,

$$u^{\varepsilon,\delta}(t,x) = E^B(f(B_t^x)\exp(V_{t,x}^{\varepsilon,\delta})).$$

Step 1. We will prove that for any $x \in \mathbb{R}^d$ and any $t > 0$, we have

$$\lim_{\varepsilon,\delta\downarrow 0} E^W|u^{\varepsilon,\delta}(t,x) - u(t,x)|^p = 0 \tag{4.4}$$

for all $p \geq 2$, where $u(t,x)$ is defined in (4.2). Notice that

$$\begin{aligned}E^W|u^{\varepsilon,\delta}(t,x) - u(t,x)|^p &= E^W\left|E^B\left(f(B_t^x)[\exp(V_{t,x}^{\varepsilon,\delta}) - \exp(V_{t,x})]\right)\right|^p \\ &\leq \|f\|_\infty^p E|\exp(V_{t,x}^{\varepsilon,\delta}) - \exp(V_{t,x})|^p,\end{aligned}$$

where $V_{t,x}$ is defined in (3.4). Since $\exp(V_{t,x}^{\varepsilon,\delta})$ converges to $\exp(V_{t,x})$ in probability by Theorem 3.1, to show (4.4), it suffices to prove that for any $\lambda \in \mathbb{R}$,

$$\sup_{\epsilon,\delta} E\exp(\lambda V_{t,x}^{\varepsilon,\delta}) < \infty. \tag{4.5}$$

The estimate (4.5) follows from (3.3), (3.7) and (3.10):

$$\begin{aligned}E\exp(\lambda V_{t,x}^{\varepsilon,\delta}) &= E\exp\left(\frac{\lambda^2}{2}\|A_{t,x}^{\varepsilon,\delta}\|_{\mathcal{H}}^2\right) \\ &\leq E\exp\left(\frac{\lambda^2}{2} C \int_0^t \int_0^t |r-s|^{2H_0-2} \prod_{i=1}^d |B_r^i - B_s^i|^{2H_i-2}\,dr\,ds\right) \qquad (4.6)\\ &< \infty.\end{aligned}$$

Step 2. We now prove that $u(t,x)$ is a weak solution of (1.2) in the sense of Definition 4.2. Suppose that φ is a smooth function with compact support. We know that

$$\begin{aligned}&\int_{\mathbb{R}^d} u^{\varepsilon,\delta}(t,x)\varphi(x)\,dx \\ &\quad = \int_{\mathbb{R}^d} f(x)\varphi(x)\,dx + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} u^{\varepsilon,\delta}(t,x)\Delta\varphi(x)\,dx\,ds \qquad (4.7)\\ &\qquad + \int_0^t \int_{\mathbb{R}^d} u^{\varepsilon,\delta}(t,x)\varphi(x)\dot{W}^{\varepsilon,\delta}(s,x)\,ds\,dx.\end{aligned}$$

Therefore, it suffices to prove that

$$\lim_{\varepsilon,\delta\downarrow 0}\int_0^t\int_{\mathbb{R}^d} u^{\varepsilon,\delta}(s,x)\varphi(x)\dot W^{\varepsilon,\delta}(s,x)\,ds\,dx = \int_0^t\int_{\mathbb{R}^d} u(s,x)\varphi(x)W(ds,dx)$$

in probability. From (4.7) and (4.4), it follows that $\int_0^t\int_{\mathbb{R}^d} u^{\epsilon,\delta}(s,x)\varphi(x)\dot W^{\epsilon,\delta}(s,x)\,ds\,dx$ converges in L^2 to the random variable

$$G = \int_{\mathbb{R}^d} u(t,x)\varphi(x)\,dx - \int_{\mathbb{R}^d} f(x)\varphi(x)\,dx - \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} u(t,x)\Delta\varphi(x)\,dx\,ds$$

as ε and δ tend to zero. Hence, if

$$B_{\varepsilon,\delta} = \int_0^t\int_{\mathbb{R}^d}\big(u^{\epsilon,\delta}(s,x)-u(s,x)\big)\varphi(x)\dot W^{\varepsilon,\delta}(s,x)\,ds\,dx$$

converges in L^2 to zero, then

$$\int_0^t\int_{\mathbb{R}^d} u(s,x)\varphi(x)\dot W^{\varepsilon,\delta}\,ds\,dx = \int_0^t\int_{\mathbb{R}^d} u^{\varepsilon,\delta}(s,x)\varphi(x)\dot W^{\varepsilon,\delta}\,ds\,dx - B_{\varepsilon,\delta}$$

converges to G in L^2. Thus, $u(s,x)\varphi(x)$ will be Stratonovich integrable and we will have

$$\int_0^t\int_{\mathbb{R}^d} u(s,x)\varphi(x)W(ds,dx) = G,$$

which will complete the proof. In order to show the convergence to zero of $B_{\varepsilon,\delta}$, we will express the product $(u^{\varepsilon,\delta}(s,x)-u(s,x))\dot W^{\varepsilon,\delta}(s,x)$ as the sum of a divergence integral plus a trace term [see (2.4)]:

$$\begin{aligned}
&\big(u^{\varepsilon,\delta}(s,x)-u(s,x)\big)\dot W^{\varepsilon,\delta}(s,x)\\
&\quad=\int_0^t\int_{\mathbb{R}^d}\big(u^{\varepsilon,\delta}(s,x)-u(s,x)\big)\varphi_\delta(s-r)p_\varepsilon(x-z)\delta W_{r,z}\\
&\qquad+\big\langle D\big(u^{\varepsilon,\delta}(s,x)-u(s,x)\big),\varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\big\rangle_{\mathcal{H}}.
\end{aligned}$$

We then have

$$\begin{aligned}
B_{\varepsilon,\delta} &= \int_0^t\int_{\mathbb{R}^d}\phi_{r,z}^{\epsilon,\delta}\delta W_{r,z}\\
&\quad+\int_0^t\int_{\mathbb{R}^d}\varphi(x)\big\langle D\big(u^{\varepsilon,\delta}(s,x)-u(s,x)\big),\varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\big\rangle_{\mathcal{H}}\,ds\,dx \qquad (4.8)\\
&= B_{\varepsilon,\delta}^1 + B_{\varepsilon,\delta}^2,
\end{aligned}$$

where

$$\phi_{r,z}^{\varepsilon,\delta} = \int_0^t\int_{\mathbb{R}^d}\big(u^{\varepsilon,\delta}(s,x)-u(s,x)\big)\varphi(x)\varphi_\delta(s-r)p_\varepsilon(x-z)\,ds\,dx$$

and $\delta(\phi^{\varepsilon,\delta}) = \int_0^t \int_{\mathbb{R}^d} \phi^{\varepsilon,\delta}_{r,z} \delta W_{r,z}$ denotes the divergence or the Skorokhod integral of $\phi^{\varepsilon,\delta}$.

Step 3. For the term $B^1_{\varepsilon,\delta}$, we use the following L^2 estimate for the Skorokhod integral:

$$E[(B^1_{\varepsilon,\delta})^2] \le E(\|\phi^{\varepsilon,\delta}\|^2_{\mathcal{H}}) + E(\|D\phi^{\varepsilon,\delta}\|^2_{\mathcal{H}\otimes\mathcal{H}}). \tag{4.9}$$

The first term in (4.9) is estimated as follows:

$$\begin{aligned} E(\|\phi^{\varepsilon,\delta}\|^2_{\mathcal{H}}) &= \int_0^t \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} E\big[\big(u^{\varepsilon,\delta}(s,x) - u(s,x)\big) \\ &\qquad \times \big(u^{\varepsilon,\delta}(r,y) - u(r,y)\big)\big]\varphi(x)\varphi(y) \\ &\qquad \times \langle \varphi_\delta(s-\cdot) p_\varepsilon(x-\cdot), \\ &\qquad \varphi_\delta(r-\cdot) p_\varepsilon(y-\cdot)\rangle_{\mathcal{H}} \, ds\, dx\, dr\, dy. \end{aligned} \tag{4.10}$$

Using Lemmas A.2 and A.3, we can write

$$\begin{aligned} &\langle \varphi_\delta(s-\cdot) p_\epsilon(x-\cdot), \varphi_\delta(r-\cdot) p_\varepsilon(y-\cdot)\rangle_{\mathcal{H}} \\ &\quad = \alpha_H \left(\int_{[0,t]^2} \varphi_\delta(s-\sigma)\varphi_\delta(r-\tau)|\sigma-\tau|^{2H_0-2} \, d\sigma\, d\tau \right) \\ &\qquad \times \left(\int_{\mathbb{R}^{2d}} p_\varepsilon(x-z) p_\varepsilon(y-w) \prod_{i=1}^d |z_i - w_i|^{2H_i-2} \, dz\, dw \right) \\ &\quad \le C|s-r|^{2H_0-2} \prod_{i=1}^d |x-y|^{2H_i-2} \end{aligned} \tag{4.11}$$

for some constant $C > 0$. As a consequence, the integrand on the right-hand side of (4.10) converges to zero as ε and δ tend to zero for any s, r, x, y due to (4.4). From (4.6), we get

$$\begin{aligned} &\sup_{\varepsilon,\delta} \sup_{x\in\mathbb{R}^d} \sup_{0\le s\le t} E(u^{\epsilon,\delta}(s,x))^2 \\ &\qquad \le \|f\|^2_\infty \sup_{\varepsilon,\delta} \sup_{x\in\mathbb{R}^d} \sup_{0\le s\le t} E\exp(2V^{\varepsilon,\delta}_{s,x}) < \infty. \end{aligned} \tag{4.12}$$

Hence, from (4.11) and (4.12), we get that the integrand on the right-hand side of (4.10) is bounded by $C|s-r|^{2H_0-2}\prod_{i=1}^d |x_i - y_i|^{2H_i-2}$ for some constant $C > 0$. Therefore, by dominated convergence, we get that $E(\|\phi^{\varepsilon,\delta}\|^2_{\mathcal{H}})$ converges to zero as ε and δ tend to zero.

Step 4. On the other hand, we have

$$D(u^{\varepsilon,\delta}(t,x)) = E^B[f(B_t+x)\exp(V_{t,x}^{\varepsilon,\delta})A_{t,x}^{\varepsilon,\delta}],$$

where $A_{t,x}^{\varepsilon,\delta}$ is defined as in (3.2). Therefore,

$$\begin{aligned}
&E\langle D(u^{\varepsilon,\delta}(t,x)), D(u^{\varepsilon',\delta'}(t,x))\rangle_{\mathcal{H}} \\
&\quad = E^W E^B\big(f(B_t^1+x)f(B_t^2+x)\exp\big(V_{t,x}^{\varepsilon,\delta}(B^1)+V_{t,x}^{\varepsilon,\delta}(B^2)\big) \\
&\qquad\qquad \times \langle A_{t,x}^{\varepsilon,\delta}(B^1), A_{t,x}^{\varepsilon',\delta'}(B^2)\rangle_{\mathcal{H}}\big),
\end{aligned}\tag{4.13}$$

where B^1 and B^2 are two independent d-dimensional Brownian motions and where E^B denotes the expectation with respect to (B^1, B^2). Then, from the previous results it is easy to show that

$$\begin{aligned}
&\lim_{\varepsilon,\delta\downarrow 0} E\langle D(u^{\varepsilon,\delta}(t,x)), D(u^{\varepsilon',\delta'}(t,x))\rangle_{\mathcal{H}} \\
&\quad = E\Bigg[f(B_t^1+x)f(B_t^2+x) \\
&\qquad \times \exp\Bigg(\frac{\alpha_H}{2}\sum_{j,k=1}^{2}\int_0^t\int_0^t |s-r|^{2H_0-2}\prod_{i=1}^{d}|B_s^{j,i}-B_r^{k,i}|^{2H_i-2}\,ds\,dr\Bigg) \\
&\qquad \times \alpha_H\int_0^t\int_0^t |s-r|^{2H_0-2}\prod_{i=1}^{d}|B_s^{1,i}-B_r^{2,i}|^{2H_i-2}\,ds\,dr\Bigg].
\end{aligned}\tag{4.14}$$

This implies that $u^{\varepsilon,\delta}(t,x)$ converges in the space $\mathbb{D}^{1,2}$ to $u(t,x)$ as $\delta\downarrow 0$ and $\varepsilon\downarrow 0$. Letting $\varepsilon'=\varepsilon$ and $\delta'=\delta$ in (4.13) and using the same argument as for (4.12), we obtain

$$\sup_{\varepsilon,\delta}\sup_{x\in\mathbb{R}^d}\sup_{0\le s\le t} E\|D(u^{\varepsilon,\delta}(s,x))\|_{\mathcal{H}}^2 < \infty.$$

Then,

$$\begin{aligned}
&E\|D\phi^{\varepsilon,\delta}\|_{\mathcal{H}\otimes\mathcal{H}}^2 \\
&\quad = \int_0^t\int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}} E\big\langle D\big(u^{\varepsilon,\delta}(s,x)-u(s,x)\big), D\big(u^{\varepsilon,\delta}(r,y)-u(r,y)\big)\big\rangle_{\mathcal{H}} \\
&\qquad \times \varphi(x)\varphi(y)\langle\varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot), \\
&\qquad\qquad \varphi_\delta(r-\cdot)p_\varepsilon(y-\cdot)\rangle_{\mathcal{H}}\,ds\,dx\,dr\,dy
\end{aligned}$$

converges to zero as ε and δ tend to zero. Hence, by (4.9), $B_{\varepsilon,\delta}^1$ converges to zero in L^2 as ε and δ tend to zero.

Step 5. The second summand in the right-hand side of (4.8) can be written as

$$
\begin{aligned}
B^2_{\varepsilon,\delta} &= \int_0^t \int_{\mathbb{R}^d} \varphi(x) \langle D\big(u^{\varepsilon,\delta}(s,x) - u(s,x)\big), \varphi_\delta(s-\cdot) p_\varepsilon(x-\cdot)\rangle_{\mathcal{H}} \, ds\, dx \\
&= \int_0^t \int_{\mathbb{R}^d} \varphi(x) E^B\big(f(B^x_s)\exp(V^{\varepsilon,\delta}_{s,x}) \langle A^{\varepsilon,\delta}_{s,x}, \varphi_\delta(s-\cdot) p_\epsilon(x-\cdot)\rangle_{\mathcal{H}}\big)\, ds\, dx \\
&\quad - \int_0^t \int_{\mathbb{R}} \varphi(x) E^B\big(f(B^x_s)\exp(V_{s,x}) \\
&\qquad\qquad \times \langle \delta(B^x_{s-\cdot} - \cdot), \varphi_\delta(s-\cdot) p_\varepsilon(x-\cdot)\rangle_{\mathcal{H}}\big)\, ds\, dx \\
&= B^3_{\varepsilon,\delta} - B^4_{\varepsilon,\delta},
\end{aligned}
$$

where

$$
\begin{aligned}
&\langle A^{\varepsilon,\delta}_{s,x}, \varphi_\delta(s-\cdot) p_\epsilon(x-\cdot)\rangle_{\mathcal{H}} \\
&\quad = \alpha_H \int_{[0,s]^3} \int_{\mathbb{R}^{2d}} |r-v|^{2H_0-2} \\
&\qquad\qquad \times \prod_{i=1}^d |y_i - z_i|^{2H_i-2} \varphi_\delta(s-r) p_\varepsilon(B^x_r - y) \\
&\qquad\qquad \times \varphi_\delta(s-v) p_\varepsilon(x-z)\, dy\, dz\, dr\, dr\, dv
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle \delta(B^x_{s-\cdot} - \cdot), \varphi_\delta(s-\cdot) p_\varepsilon(x-\cdot)\rangle_{\mathcal{H}} \\
&\quad = \alpha_H \int_{[0,s]^2} \int_{\mathbb{R}^d} v^{2H_0-2} \prod_{i=1}^d |B^{x_i}_r - y_i|^{2H_i-2} \varphi_\delta(r-v) p_\varepsilon(x-y)\, dy\, dv\, dr.
\end{aligned}
$$

Lemma A.2 and Lemma A.3 imply that

$$
\langle A^{\varepsilon,\delta}_{s,x}, \varphi_\delta(s-\cdot) p_\varepsilon(x-\cdot)\rangle_{\mathcal{H}} \le C \int_0^s r^{2H_0-2} \prod_{i=1}^d |B^i_r|^{2H_i-2}\, dr \tag{4.15}
$$

and

$$
\langle \delta(B^x_{s-\cdot} - \cdot), \varphi_\delta(s-\cdot) p_\varepsilon(x-\cdot)\rangle_{\mathcal{H}} \le C \int_0^s r^{2H_0-2} \prod_{i=1}^d |B^i_r|^{2H_i-2}\, dr \tag{4.16}
$$

for some constant $C > 0$. Then, from (4.15), (4.16) and the fact that the random variable $\int_0^s r^{2H_0-2} \prod_{i=1}^d |B^i_r|^{2H_i-2}\, dr$ is square integrable because of Lemma A.4, we can apply the dominated convergence theorem and get that $B^3_{\varepsilon,\delta}$ and $B^4_{\varepsilon,\delta}$ both converge in L^2 to

$$
\alpha_H \int_0^t \int_{\mathbb{R}^d} \varphi(x) E^B\left(f(B^x_s)\exp(V_{s,x}) \int_0^s r^{2H_0-2} \prod_{i=1}^d |B^i_r|^{2H_i-2}\, dr \right) ds\, dx
$$

as ε and δ tend to zero. Therefore, $B^2_{\varepsilon,\delta}$ converges to zero in L^2 as ε and δ tend to zero. This completes the proof. □

We can also show that the process $u(t,x)$ given in (4.2) is a mild solution to (1.2), in the sense that the following equation holds:

$$u(t,x) = p_t f(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) u(s,y)\, dW_{s,y},$$

where p_t denotes the heat kernel and $p_t f(x) = \int_{\mathbb{R}^d} p_t(x-y) f(y)\, dy$. In fact, as in the proof of Theorem 4.3, we need to show that

$$\int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\big(u(s,y) - u^{\varepsilon,\delta}(s,y)\big)\, dW^{\varepsilon,\delta}_{s,y}$$

converges to zero in L^2. This can be proven with the same arguments as in the proof of Theorem 4.3, replacing φ by the heat kernel. For instance, instead of the estimate (4.11), we should have

$$\begin{aligned}\int_0^t \int_0^t \int_{\mathbb{R}^{2d}} & p_{t-r}(x-y) p_{t-s}(x-z) |s-r|^{2H_0-2} \prod_{i=1}^d |y-z|^{2H_i-2}\, dy\, dz\, dr\, ds \\ &= \int_0^t \int_0^t |s-r|^{2H_0-2} E\left(\prod_{i=1}^d |B^{1,i}_{t-r} - B^{2,i}_{t-s}|^{2H_i-2}\right) dr\, ds < \infty.\end{aligned}$$

We omit the details of this proof.

REMARK 4.4. The uniqueness of the solution remains to be investigated in a future work. The definition of the Stratonovich integral as a limit in probability makes the uniqueness problem nontrivial and it is not clear how to proceed.

As a corollary of Theorem 4.3, we obtain the following result.

COROLLARY 4.5. *Suppose that $2H_0 + \sum_{i=1}^d H_i > d+1$. Then, the solution $u(t,x)$ given by (4.2) has finite moments of all orders. Moreover, for any positive integer p, we have*

$$\begin{aligned} E(u(t,x)^p) & \\ &= E\Bigg(\prod_{j=1}^p f(B^j_t + x) \\ &\qquad \times \exp\Bigg[\frac{\alpha_H}{2} \sum_{j,k=1}^p \int_0^t \int_0^t |s-r|^{2H_0-2} \\ &\qquad\qquad \times \prod_{i=1}^d |B^{j,i}_s - B^{k,i}_r|^{2H_i-2}\, ds\, dr\Bigg]\Bigg), \end{aligned} \tag{4.17}$$

where $B_1, \dots, B_p$ are independent d-dimensional standard Brownian motions.

REMARK 4.6. In the previous work [5], a formula similar to (4.17) was obtained in the special case $H_1 = \cdots = H_d = \frac{1}{2}$, without the condition $2H_0 + \sum_{i=1}^d H_i > d + 1$. This type of formula was proven assuming $d = 1$ and $H_0 > \frac{3}{4}$. In the case of the Skorokhod-type equation, a formula for the moments of the solution similar to (4.17) was established in [5] if $d = 1$ or 2, $H_0 > \frac{1}{2}$ and t is small enough.

5. Behavior of the Feynman–Kac formula. In this section, we present two applications of the Feynman–Kac formula.

5.1. *Hölder continuity of the solution.* In this subsection, we study the Hölder continuity of the solution of (1.2). The main result of this section is the following theorem.

THEOREM 5.1. *Suppose that $2H_0 + \sum_{i=1}^d H_i > d + 1$ and let $u(t, x)$ be the solution of (1.2). Then, $u(t, x)$ has a continuous modification such that for any $\rho \in (0, \frac{\kappa}{2})$ [where κ is defined as in (3.9)] and any compact rectangle $I \subset \mathbb{R}_+ \times \mathbb{R}^d$, there exists a positive random variable K_I such that almost surely, for any $(s, x), (t, y) \in I$, we have*

$$|u(t, y) - u(s, x)| \le K_I(|t - s|^\rho + |y - x|^{2\rho}).$$

PROOF. The proof involves several steps.

Step 1. Recall that $V_{t,x} = \int_0^t \int_{\mathbb{R}^d} \delta(B_{t-r}^x - y) W(dr, dy)$ denotes the random variable introduced in (3.4) and

$$u(t, x) = E^B(f(B_t^x) \exp(V_{t,x}).$$

Set $V = V_{s,x}$ and $\tilde{V} = V_{t,y}$. We can then write

$$\begin{aligned}
&E^W |u(s, x) - u(t, y)|^p \\
&\quad = E^W |E^B(e^V - e^{\tilde{V}})|^p \\
&\quad \le E^W \big(E^B[|\tilde{V} - V| e^{\max(V, \tilde{V})}]\big)^p \\
&\quad \le E^W \big[\big(E^B e^{2\max(V, \tilde{V})}\big)^{p/2} \big(E^B (\tilde{V} - V)^2\big)^{p/2}\big] \\
&\quad \le \big[E^W E^B e^{2p\max(V, \tilde{V})}\big]^{1/2} \big[E^W \big(E^B (\tilde{V} - V)^2\big)^p\big]^{1/2}.
\end{aligned}$$

Applying Minkowski's inequality, the equivalence between the L^2-norm and the L^p-norm for a Gaussian random variable and using the exponential integrability property (3.10), we obtain

$$\begin{aligned}
E^W |u(s, x) - u(t, y)|^p &\le C\big[E^W \big(E^B (\tilde{V} - V)^2\big)^p\big]^{1/2} \\
&\le C_p [E^B E^W |\tilde{V} - V|^2]^{p/2}.
\end{aligned} \tag{5.1}$$

In a similar way to (3.5), we can deduce the following formula for the conditional variance of $\tilde{V} - V$:

$$
\begin{aligned}
E^W|\tilde{V} - V|^2 \\
&= \alpha_H E^B \Biggl(\int_0^s \int_0^s |r-v|^{2H_0-2} \prod_{i=1}^d |B^i_{s-r} - B^i_{s-v}|^{2H_i-2}\, dr\, dv \\
&\qquad + \int_0^t \int_0^t |r-v|^{2H_0-2} \prod_{i=1}^d |B^i_{t-r} - B^i_{t-v}|^{2H_i-2}\, dr\, dv \\
&\qquad - 2 \int_0^s \int_0^t |r-v|^{2H_0-2} \\
&\qquad\qquad \times \prod_{i=1}^d |B^i_{s-r} - B^i_{t-v} + x_i - y_i|^{2H_i-2}\, dr\, dv \Biggr) \qquad (5.2) \\
&:= \alpha_H C(s,t,x,y).
\end{aligned}
$$

Step 2. Fix $1 \le j \le d$. Let us estimate $C(s,t,x,y)$ when $s = t$ and $x_i = y_i$ for all $i \ne j$. We can write

$$
\begin{aligned}
C(t,t,x,y) \\
&= 2 \int_0^t \int_0^t |r-v|^{\kappa-1} \qquad (5.3) \\
&\qquad \times \prod_{i \ne j}^d E(|\xi|^{2H_i-2}) E(|\xi|^{2H_j-2} - |z+\xi|^{2H_j-2})\, dr\, dv,
\end{aligned}
$$

where $z = \frac{x_j - y_j}{\sqrt{|r-v|}}$ and ξ is a standard normal variable. Set $\beta_j = 2H_j + 1 > 2$. By Lemma A.6, the factor $E(|\xi|^{2H_j-2} - |z+\xi|^{2H_j-2})$ can be bounded by a constant if $|r-v| \le (x_j - y_j)^2$ and it can be bounded by $C|x_j - y_j|^{\beta_j} |r-v|^{-\beta_j/2}$ if $|r-s| > (x_j - y_j)^2$. In this way, we obtain

$$
\begin{aligned}
C(t,t,x,y) &\le C \int_{\{0<r,v<t, |r-v| \le (x_j-y_j)^2\}} |r-v|^{\kappa-1}\, dr\, dv \\
&\qquad + C|x_j - y_j|^{\beta_j} \int_{\{0<r,v<t, |r-v| > (x_j-y_j)^2\}} |r-v|^{\kappa-1-\beta_j/2}\, dr\, dv \\
&\le C|x_j - y_j|^{2\kappa}.
\end{aligned}
$$

So, from (5.1), we have

$$
E^W |u(t,x) - u(t,y)|^p \le C|x_j - y_j|^{\kappa p}. \qquad (5.4)
$$

Step 3. Now, suppose that $s < t$ and $x = y$. Set $\delta = \sum_{i=1}^d H_i - d$. We have

$$\begin{aligned}
&C(s,t,x,x)\\
&\quad = C\biggl[\int_s^t \int_s^t |r-v|^{\kappa-1}\,dr\,dv\\
&\qquad + \int_0^s \int_0^t |r-v|^{2H_0-2}(|r-v|^{\delta} - |r-v+t-s|^{\delta})\,dr\,dv\biggr].
\end{aligned}$$

The first integral is $O((t-s)^{\kappa+1})$ when $t-s$ is small. For the second integral, we use the change of variable $\sigma = r - v$, $v = \tau$ and we have

$$\begin{aligned}
&\int_0^s \int_0^t |r-v|^{2H_0-2}(|r-v|^{\delta} - |r-v+t-s|^{\delta})\,dr\,dv\\
&\quad \le \int_0^t d\tau \int_{-t}^s |\sigma|^{2H_0-2}\bigl||\sigma|^{\delta} - |\sigma+t-s|^{\delta}\bigr|\,d\sigma\\
&\quad = t\biggl[\int_0^s \sigma^{2H_0-2}\bigl(\sigma^{\delta} - (\sigma+t-s)^{\delta}\bigr)\,d\sigma\\
&\qquad + \int_{-t}^{s-t} (-\sigma)^{2H_0-2}\bigl((-\sigma-t+s)^{\delta} - (-\sigma)^{\delta}\bigr)\,d\sigma\\
&\qquad + \int_{s-t}^{0} (-\sigma)^{2H_0-2}|(-\sigma)^{\delta} - (\sigma+t-s)^{\delta}|\,d\sigma\biggr]\\
&\quad = t[A' + B' + C'].
\end{aligned}$$

For the first term in the above decomposition, we can write

$$\begin{aligned}
A' &= (t-s)^{\kappa-1}\int_0^{t_1/(t-s)} \sigma^{2H_0-2}\bigl(\sigma^{\delta} - (\sigma+1)^{\delta}\bigr)\,d\sigma\\
&\le (t-s)^{\kappa-1}\int_0^{\infty} \sigma^{2H_0-2}\bigl(\sigma^{\delta} - (\sigma+1)^{\delta}\bigr)\,d\sigma\\
&\le C(t-s)^{\kappa},
\end{aligned}$$

because $2H_0 + \sum_{i=1}^d -d - 3 < -1$. Similarly, we can get that

$$B' \le (t-s)^{\kappa}\int_1^{\infty} \sigma^{2H_0-2}\bigl(\sigma^{\delta} - (\sigma+1)^{\delta}\bigr)\,d\sigma.$$

Finally,

$$C' \le \int_0^{t-s} \sigma^{2H_0-2}\bigl(\sigma^{\delta} + (t-s-\sigma)^{\delta}\bigr)\,d\sigma = C(t-s)^{\kappa}.$$

So, we have

$$E^W|u(s,x) - u(t,y)|^p \le C(t-s)^{\kappa/2p}. \tag{5.5}$$

Step 4. Combining equations (5.4) and (5.5) with the estimates (5.1) and (5.2), the result of this theorem can now be concluded from Theorem 1.4.1 of Kunita [7] if we choose p large enough. □

5.2. *Regularity of the density.* In this subsection, we shall use the Feynman–Kac formula established in the previous section to show that for any t and x, the probability law of the solution $u(t,x)$ of (1.2) has a smooth density with respect to the Lebesgue measure. To this end, we shall show that $\|Du(t,x)\|_{\mathcal{H}}$ has negative moments of all orders.

THEOREM 5.2. *Suppose that* $2H_0+\sum_{i=1}^d H_i > d+1$. *Fix* $t>0$ *and* $x\in\mathbb{R}^d$. *Assume that for any positive number* p, $E|f(B_t+x)|^{-p}<\infty$. *Then, the law of* $u(t,x)$ *has a smooth density.*

PROOF. From Theorem 4.3, we can write

$$u(t,x)=E^B[f(B_t^x)\exp(V_{t,x})].$$

The Malliavin derivative of the solution is given by

$$D_{r,y}u(t,x)=E^B[f(B_t^x)\exp(V_{t,x})\delta(B_{t-r}^x-y)].$$

It is not difficult to show that $u(t,x)\in\mathbb{D}^\infty$. Thus, by the general criterion for the smoothness of densities (see [10]), it suffices to show that $E(\|Du(t,x)\|_{\mathcal{H}}^{-2p})<\infty$ for any $t>0$ and $x\in\mathbb{R}^d$. We have

$$\begin{aligned}\|Du(t,x)\|_{\mathcal{H}}^2 &= E^B\big[f(B_t^1+x)f(B_t^2+x)\exp\big(V_{t,x}(B^1)+V_{t,x}(B^2)\big)\\ &\qquad\times\langle\delta(B_{t-r}^{1,x}-y),\delta(B_{t-r}^{2,x}-y)\rangle_{\mathcal{H}}\big]\\ &=\alpha_H E^B\Bigg[f(B_t^1+x)f(B_t^2+x)\exp\big(V_{t,x}(B^1)+V_{t,x}(B^2)\big)\\ &\qquad\times\int_0^t\int_0^t|r-s|^{2H_0-2}\prod_{i=1}^d|B_{t-r}^{1,i}-B_{t-s}^{2,i}|^{2H_i-2}\,dr\,ds\Bigg],\end{aligned}$$

where B^1 and B^2 are independent d-dimensional Brownian motions. By Jensen's inequality, we have, for any $p>0$, that

$$\begin{aligned}&\|Du(t,x)\|_{\mathcal{H}}^{-2p}\\ &\quad\le(\alpha_H)^{-p}E^B\Bigg[|f(B_t^1+x)f(B_t^2+x)|^{-p}\exp\big(-p[V_{t,x}(B^1)+V_{t,x}(B^2)]\big)\\ &\qquad\times\bigg(\int_0^t\int_0^t|r-s|^{2H_0-2}\prod_{i=1}^d|B_{t-r}^{i,1}-B_{t-s}^{2,i}|^{2H_i-2}\,dr\,ds\bigg)^{-p}\Bigg].\end{aligned}$$

Hence, by Hölder's inequality, we obtain

$$
\begin{aligned}
&E\|Du(t,x)\|_{\mathcal{H}}^{-2p}\\
&\quad\le (\alpha_H)^{-p}\bigl(E|f(B_t^1+x)f(B_t^2+x)|^{-pp_1}\bigr)^{1/p_1}\\
&\qquad\times\bigl(E\exp(-pp_2[V_{t,x}(B^1)+V_{t,x}(B^2)])\bigr)^{1/p_2}\\
&\qquad\times\left(E\left(\int_0^t\int_0^t |r-s|^{2H_0-2}\prod_{i=1}^d |B_{t-r}^{1,i}-B_{t-s}^{2,i}|^{2H_i-2}\,dr\,ds\right)^{-pp_3}\right)^{1/p_3}\\
&\quad= I_1 I_2 I_3,
\end{aligned}
$$

where $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$. The first factor, I_1, is finite by the assumption on f and Hölder's inequality. The second factor is finite by Theorem 3.3. Finally, from Jensen's inequality, we have

$$
\begin{aligned}
I_3^{p_3} &= E\left[t^{-2pp_3}\left\{\frac{1}{t^2}\int_0^t\int_0^t |r-s|^{2H_0-2}\prod_{i=1}^d |B_{t-r}^{1,i}-B_{t-s}^{2,i}|^{2H_i-2}\,dr\,ds\right\}^{-pp_3}\right]\\
&\le E\Biggl[t^{-2pp_3-2}\Biggl\{\int_0^t\int_0^t |r-s|^{-(2H_0-2)pp_3}\\
&\qquad\qquad\times\prod_{i=1}^d |B_{t-r}^{1,i}-B_{t-s}^{2,i}|^{-(2H_i-2)pp_3}\,dr\,ds\Biggr\}\Biggr]\\
&\le C\int_0^t\int_0^t |r-s|^{-(2H_0-2)pp_3} E\left\{\prod_{i=1}^d |B_{t-r}^{1,i}-B_{t-s}^{2,i}|^{-(2H_i-2)pp_3}\right\}dr\,ds\\
&< \infty.
\end{aligned}
$$

This completes the proof. □

6. The case $H_0 > \frac{3}{4}$, $H_1 = \frac{1}{2}$ and $d = 1$.

6.1. *Preliminaries.* In this case, all the setup is the same as before, except that if ϕ and ψ are functions in $\mathcal{E}$, then

$$
\begin{aligned}
E(W(\phi)W(\psi)) &= \langle\phi,\psi\rangle_{\mathcal{H}}\\
&= \alpha_{H_0}\int_0^\infty\int_0^\infty\int_{\mathbb{R}}\phi(s,x)\psi(t,x)|s-t|^{2H_0-2}\,ds\,dt\,dx,
\end{aligned}
$$

where $\alpha_{H_0}=H_0(2H_0-1)$.

6.2. *Definition and exponential integrability of the stochastic Feynman–Kac functional.* Similarly, we also have the following theorem.

THEOREM 6.1. *Suppose that* $H_1 = 1/2$ *and* $H_0 > 3/4$. *Then, for any* $\varepsilon > 0$ *and* $\delta > 0$, $A_{t,x}^{\varepsilon,\delta}$ *defined in* (3.2) *belongs to* $\mathcal{H}$ *and the family of random variables* $V_{t,x}^{\epsilon,\delta}$ *defined in* (3.3) *converges in* L^2 *to a limit denoted by*

$$V_{t,x} = \int_0^t \int_{\mathbb{R}} \delta(B_{t-r}^x - y) W(dr, dy). \tag{6.1}$$

Conditional on B, $V_{t,x}$ *is a Gaussian random variable with mean* 0 *and variance*

$$\mathrm{Var}^W(V_{t,x}) = \alpha_{H_0} \int_0^t \int_0^t |r-s|^{2H_0-2} \delta(B_r - B_s)\, dr\, ds. \tag{6.2}$$

PROOF. Fix ε, ε', δ and $\delta' > 0$.

$$\begin{aligned}
&E^B E^W(V_{t,x}^{\epsilon,\delta}, V_{t,x}^{\epsilon',\delta'}) \\
&\quad = E^B \langle A_{t,x}^{\varepsilon,\delta}, A_{t,x}^{\varepsilon',\delta'} \rangle_{\mathcal{H}} \\
&\quad = \alpha_{H_0} E^B \biggl(\int_{[0,t]^4} \int_{\mathbb{R}} p_\epsilon(B_s^x - y) p_{\epsilon'}(B_r^x - y) \varphi_\delta(t-s-u) \\
&\qquad\qquad \times \varphi_{\delta'}(t-r-v) |u-v|^{2H_0-2}\, dy\, du\, dv\, ds\, dr \biggr) \\
&\quad = \alpha_{H_0} \biggl(\int_{[0,t]^4} E^B p_{\epsilon+\epsilon'}(B_s - B_r) \varphi_\delta(t-s-u) \\
&\qquad\qquad \times \varphi_{\delta'}(t-r-v) |u-v|^{2H_0-2}\, du\, dv\, ds\, dr \biggr) \\
&\quad = \alpha_{H_0} \biggl(\int_{[0,t]^4} \frac{1}{\sqrt{2\pi}} (\epsilon + \epsilon' + |s-r|)^{-1/2} \varphi_\delta(t-s-u) \\
&\qquad\qquad \times \varphi_{\delta'}(t-r-v) |u-v|^{2H_0-2}\, du\, dv\, ds\, dr \biggr).
\end{aligned}$$

By Lemma A.3,

$$\begin{aligned}
&\int_{[0,t]^2} (\epsilon + \epsilon' + |s-r|)^{-1/2} \times \varphi_\delta(t-s-u) \varphi_{\delta'}(t-r-v) |u-v|^{2H_0-2}\, du\, dv \\
&\qquad \le C|s-r|^{2H_0-5/2}.
\end{aligned}$$

Then, by the dominated convergence theorem, $E^B E^W(V_{t,x}^{\epsilon,\delta}, V_{t,x}^{\epsilon',\delta'})$ converges to

$$\frac{\alpha_{H_0}}{\sqrt{2\pi}} \int_{[0,t]^2} |s-r|^{2H_0-5/2}\, ds\, dr$$

as ε, ε', δ and δ' tend to zero. This implies that $V_{t,x}^{\epsilon,\delta}$ converges in L^2, as ε and δ tend to zero, to a limit denoted by $V_{t,x}$. On the other hand, from the above computations, we have

$$E^W[(V_{t,x}^{\epsilon,\delta})^2] = \alpha_{H_0} \int_{[0,t]^4} p_{2\epsilon}(B_s - B_r)\varphi_\delta(t-s-u) \times \varphi_\delta(t-r-v)|u-v|^{2H_0-2}\, du\, dv\, ds\, dr$$

and this expression converges to right-hand side of (6.2) almost surely. Moreover, because of the above arguments, the convergence is also in L^1 and this implies (6.2). □

6.3. *Feynman–Kac formula*. By Proposition 3.3 and Theorem 6.2 in [5], we have the following theorem.

THEOREM 6.2. *Suppose that $H_1 = 1/2$ and $H_0 > 3/4$. Then, for any $\lambda \in \mathbb{R}$, we have*

$$E \exp\left(\lambda \int_0^t \int_{\mathbb{R}} \delta(B_{t-r}^x - y) W(dr, dy)\right) < \infty$$

and, for any measurable and bounded function f, the process

$$u(t,x) = E^B\left(f(B_t^x) \exp\left(\int_0^t \int_{\mathbb{R}} \delta(B_{t-r}^x - y) W(dr, dy)\right)\right) \tag{6.3}$$

is a weak solution of (1.2).

6.4. *Hölder continuity.* We also have the following theorem, whose proof is similar to that of Theorem 6.1.

THEOREM 6.3. *Suppose that $H_1 = 1/2$, $H_0 > 3/4$ and let $u(t,x)$ be the solution of* (1.2). *Then, $u(t,x)$ has a continuous modification such that for any $\rho \in (0, H_0 - 3/4)$ and any compact rectangle $I \subset \mathbb{R}_+ \times \mathbb{R}$, there exists a positive random variable K_I such that almost surely, for any $(t_1, x_1), (t_2, x_2) \in I$, we have*

$$|u(t_2, x_2) - u(t_1, x_1)| \le K_I(|t_2 - t_1|^\rho + |x_2 - x_1|^{2\rho}).$$

PROOF. As in the proof of Theorem 6.1, we have

$$E^W|u(s,x) - u(t,y)|^p \le C_p[E^B E^W|\tilde{V} - V|^2]^{p/2},$$

where $V = \int_0^t \int_{\mathbb{R}} \delta(B_{t-r}^x - z) W(dr, dz)$ and $\tilde{V} = \int_0^s \int_{\mathbb{R}} \delta(B_{s-r}^y - z) W(dr, dz)$. If $s = t$, then we can write

$$\begin{aligned} E^B E^W|\tilde{V} - V|^2 &= 2 \int_0^t \int_0^t |r-v|^{2H_0-2} \\ &\qquad \times E[\delta(B_r - B_v) - \delta(B_r - B_v + x - y)]\, dr\, dv \\ &= \frac{2}{\sqrt{2\pi}} \int_0^t \int_0^t |r-v|^{2H_0-5/2}\left(1 - e^{-(x-y)^2/(2|r-v|)}\right) dr\, dv. \end{aligned}$$

For any $2\rho < \gamma < 2H_0 - 3/2$, we have $1 - e^{-(x-y)^2/(2|r-v|)} \le (\frac{(x-y)^2}{2|r-v|})^\gamma$. Thus, $E^B E^W |\tilde{V} - V|^2 \le C_\gamma |x-y|^{2\gamma}$. Consequently, we have

$$E^W |u(t,x) - u(t,y)|^p \le C|x-y|^{\gamma p}. \tag{6.4}$$

On the other hand, if $x = y$, then

$$\begin{aligned} &E^B E^W |\tilde{V} - V|^2 \\ &\quad = C\biggl[\int_s^t \int_s^t |r-v|^{2H_0-5/2}\,dr\,dv \\ &\qquad + \int_0^s \int_0^t |r-v|^{2H_0-2}(|r-v|^{-1/2} - |r-v+t-s|^{-1/2})\,dr\,ds\biggr] \end{aligned}$$

and, by a similar computation to step 3 before, we can obtain

$$E^W |u(s,x) - u(t,x)|^p \le C(t-s)^{(H_0-3/4)p}. \tag{6.5}$$

Combining (6.4) and (6.5), we prove the theorem. □

6.5. *Regularity of the density.* We can also show the following result.

THEOREM 6.4. *Suppose that $d = 1$, $H_1 = 1/2$ and $H_0 > 3/4$. Fix $t > 0$ and $x \in \mathbb{R}$. Assume that for any positive number p, $E|f(B_t + x)|^{-p} < \infty$. The law of $u(t,x)$ then has a smooth density.*

PROOF. The proof is similar to that of Theorem 5.2, using the existence of finite moments of all orders for the self-intersection local time of the Brownian motion proved in the Appendix (see Proposition A.7). □

7. Skorokhod-type equations and chaos expansion. In this section, we consider the following heat equation on $\mathbb{R}^d$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\Delta u + u \diamond \dfrac{\partial^{d+1}}{\partial t\,\partial x_1 \cdots \partial x_d} W, \\ u(0,x) = f(x). \end{cases} \tag{7.1}$$

The difference between the above equation and (1.2) is that here we use the Wick product $\diamond$ (see, e.g., [6]). This equation is studied in [5] in the case $H_1 = \cdots = H_d = 1/2$. As in that paper, we can define the following notion of mild solution.

DEFINITION 7.1. An adapted random field $u = \{u(t,x), t \ge 0, x \in \mathbb{R}^d\}$ such that $E(u^2(t,x)) < \infty$ for all (t,x) is a mild solution to equation (7.1) if, for any $(t,x) \in [0,\infty) \times \mathbb{R}^d$, the process $\{p_{t-s}(x-y)u(s,y)\mathbf{1}_{[0,t]}(s), s \ge 0, y \in \mathbb{R}^d\}$ is Skorokhod integrable and the following equation holds:

$$u(t,x) = p_t f(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)u(s,y)\,\delta W_{s,y}, \tag{7.2}$$

where $p_t(x)$ denotes the heat kernel and $p_t f(x) = \int_{\mathbb{R}^d} p_t(x-y)f(y)\,dy$.

As in the paper [5], the mild solution $u(t,x)$ of (7.1) admits the following Wiener chaos expansion:

$$u(t,x) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t,x)), \tag{7.3}$$

where I_n denotes the multiple stochastic integral with respect to W and $f_n(\cdot,t,x)$ is a symmetric element in $\mathcal{H}^{\otimes n}$, defined explicitly as

$$\begin{aligned} &f_n(s_1, y_1, \dots, s_n, y_n, t, x) \\ &\quad = \frac{1}{n!} p_{t-s_{\sigma(n)}}\big(x - y_{\sigma(n)}\big) \cdots p_{s_{\sigma(2)}-s_{\sigma(1)}}\big(y_{\sigma(2)} - y_{\sigma(1)}\big) p_{s_{\sigma(1)}} f\big(y_{\sigma(1)}\big). \end{aligned} \tag{7.4}$$

In the above equation, σ denotes a permutation of $\{1, 2, \dots, n\}$ such that $0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t$. Moreover, the solution, if it exists, will be unique because the kernels in the Wiener chaos expansion are uniquely determined.

The following theorem is the main result of this section.

THEOREM 7.2. *Suppose that $2H_0 + \sum_{i=1}^d H_i > d + 1$ and that f is a bounded measurable function. Then, the process*

$$\begin{aligned} u(t,x) = E^B\bigg[f(B_t^x) \exp\bigg(&\int_0^t \int_{\mathbb{R}^d} \delta(B_{t-r}^x - y) W(dr, dy) \\ &- \frac{1}{2}\alpha_H \int_0^t \int_0^t |r-s|^{2H_0-2} \\ &\times \prod_{i=1}^d |B_r^i - B_s^i|^{2H_i-2}\, dr\, ds \bigg) \bigg] \end{aligned} \tag{7.5}$$

is the unique mild solution to equation (1.2).

PROOF. From Theorem 3.3, we obtain that the expectation E^B in (7.5) is well defined. It then suffices to show that the random variable $u(t,x)$ has the Wiener chaos expansion (7.3). This can be easily proven by expanding the exponential and then taking the expectation with respect to B.

Theorem 3.1 implies that almost surely $\delta(B_{t-\cdot}^x - \cdot)$ is an element of $\mathcal{H}$ with a norm given by (3.4). As a consequence, almost surely with respect to the Brownian motion B, we have the following chaos expansion for the exponential factor in equation (7.5):

$$\begin{aligned} \exp\bigg(&\int_0^t \int_{\mathbb{R}^d} \delta(B_{t-r}^x - y) W(dr, dy) \\ &- \frac{1}{2}\alpha_H \int_0^t \int_0^t |r-s|^{2H_0-2} \prod_{i=1}^d |B_r^i - B_s^i|^{2H_i-2}\, dr\, ds \bigg) = \sum_{n=0}^{\infty} I_n(g_n), \end{aligned}$$

where g_n is the symmetric element in $\mathcal{H}^{\otimes n}$ given by

$$g_n(s_1, y_1, \dots, s_n, y_n, t, x) = \frac{1}{n!}\delta(B^x_{t-s_1} - y_1)\cdots\delta(B^x_{t-s_n} - y_n). \tag{7.6}$$

Thus, the right-hand side of (7.5) admits the chaos expansion

$$u(t, x) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(h_n(\cdot, t, x)) \tag{7.7}$$

with

$$h_n(t, x) = E^B[f(B^x_t)\delta(B^x_{t-s_1} - y_1)\cdots\delta(B^x_{t-s_n} - y_n)]. \tag{7.8}$$

This can be regarded as a Feynman–Kac formula for the coefficients of the chaos expansion of the solution of (7.1). To compute the above expectation, we shall use the following identity:

$$\begin{aligned} E^B[f(B^x_t)\delta(B^x_t - y)|\mathcal{F}_s] &= \int_{\mathbb{R}^d} p_{t-s}(B^x_s - z) f(z)\delta(z - y)\, dz \\ &= p_{t-s}(B^x_s - y) f(y). \end{aligned} \tag{7.9}$$

Assume that $0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t$ for some permutation σ of $\{1, 2, \dots, n\}$. Then, conditioning with respect to $\mathcal{F}_{t-s_{\sigma(1)}}$ and using the Markov property of the Brownian motion, we have

$$\begin{aligned} h_n(t, x) &= E^B\{E^B[\delta(B^x_{t-s_{\sigma(n)}} - y_{\sigma(n)})\cdots \\ &\qquad \times \delta(B^x_{t-s_{\sigma(1)}} - y_{\sigma(1)}) f(B^x_t)|\mathcal{F}_{t-s_{\sigma(1)}}]\} \\ &= E^B[\delta(B^x_{t-s_{\sigma(n)}} - y_{\sigma(n)})\cdots\delta(B^x_{t-s_{\sigma(1)}} - y_{\sigma(1)}) p_{s_{\sigma(1)}} f(B^x_{t-s_{\sigma(1)}})]. \end{aligned}$$

Conditioning with respect to $\mathcal{F}_{t-s_{\sigma(2)}}$ and using (7.9), we have

$$\begin{aligned} h_n(t, x) &= E^B\{E^B[\delta(B_{t-s^x_{\sigma(n)}} - y_{\sigma(n)}) \\ &\qquad \times \delta(B^x_{t-s_{\sigma(1)}} - y_{\sigma(1)}) p_{s_{\sigma(1)}} f(B^x_{t-s_{\sigma(1)}})]|\mathcal{F}_{t-s_{\sigma(2)}}\} \\ &= E^B\{\delta(B_{t-s^x_{\sigma(n)}} - y_{\sigma(n)})\cdots\delta(B^x_{t-s_{\sigma(2)}} - y_{\sigma(2)}) \\ &\qquad \times E^B[\delta(B^x_{t-s_{\sigma(1)}} - y_{\sigma(1)}) p_{s_{\sigma(1)}} f(B^x_{t-s_{\sigma(1)}})|\mathcal{F}_{t-s_{\sigma(2)}}]\} \\ &= E^B[\delta(B_{t-s^x_{\sigma(n)}} - y_{\sigma(n)})\cdots\delta(B^x_{t-s_{\sigma(2)}} - y_{\sigma(2)}) \\ &\qquad \times p_{s_{\sigma(2)}-s_{\sigma(1)}}(B^x_{t-s_{\sigma(2)}} - y_{\sigma(1)}) p_{s_{\sigma(1)}} f(y_{\sigma(1)})]. \end{aligned}$$

Continuing in this way, we find that

$$h_n(t, x) = p_{t-s_{\sigma(n)}}(x - y_{\sigma(n)})\cdots p_{s_{\sigma(2)}-s_{\sigma(1)}}(y_{\sigma(2)} - y_{\sigma(1)}) p_{s_{\sigma(1)}} f(y_{\sigma(1)}),$$

which is the same as (7.4). □

REMARK 7.3. The method of this section can be applied to obtain a Feynman–Kac formula for the coefficients of the chaos expansion of the solution of equation (1.2):

$$u(t,x)=\sum_{n=0}^{\infty}\frac{1}{n!}I_n(h_n(\cdot,t,x))$$

with

$$\begin{aligned}h_n(t,x)=E^B\Bigg[&f(B_t^x)\delta(B_{t-s_1}^x-y_1)\cdots\delta(B_{t-s_n}^x-y_n)\\&\times\exp\Bigg(\frac{1}{2}\alpha_H\int_0^t\int_0^t|r-s|^{2H_0-2}\\&\qquad\times\prod_{i=1}^d|B_r^i-B_s^i|^{2H_i-2}\,dr\,ds\Bigg)\Bigg].\end{aligned}\tag{7.10}$$

REMARK 7.4. We can also consider equation (1.2) when $d=1$, $H_1=1/2$ and $H_0>3/4$. In this case, we easily see that the solution $u(t,x)$ admits the following chaos expansion:

$$u(t,x)=\sum_{n=0}^{\infty}\frac{1}{n!}I_n(h_n(\cdot,t,x))$$

with

$$\begin{aligned}h_n(t,x)=E^B\Bigg[&f(B_t^x)\delta(B_{t-s_1}^x-y_1)\cdots\delta(B_{t-s_n}^x-y_n)\\&\times\exp\Bigg(\frac{1}{2}\alpha_{H_0}\int_0^t\int_0^t|r-s|^{2H_0-2}\delta(B_r-B_s)\,dr\,ds\Bigg)\Bigg].\end{aligned}\tag{7.11}$$

From the Feynman–Kac formula, we can derive the following formula for the moments of the solution analogous to (4.17), which can be compared with the formulas obtained in [5] in the case $H_1=\cdots=H_d=\frac{1}{2}$:

$$\begin{aligned}&E(u(t,x)^p)\\&\quad=E\Bigg(\prod_{j=1}^p f(B_t^j+x)\\&\qquad\times\exp\Bigg[\alpha_H\sum_{j,k=1,j<k}^p\int_0^t\int_0^t|s-r|^{2H_0-2}\\&\qquad\qquad\times\prod_{i=1}^d|B_s^{j,i}-B_r^{k,i}|^{2H_i-2}\,ds\,dr\Bigg]\Bigg),\end{aligned}$$

where $p \geq 1$ is an integer and $B^j, 1 \leq j \leq d$, are independent d-dimensional Brownian motions.

APPENDIX

LEMMA A.1. *Suppose that* $0 < \alpha < 1$, $\epsilon > 0$, $x > 0$ *and that* X *is a standard normal random variable. Then, there is a constant* C, *independent of* x *and* ϵ (*it may depend on* α), *such that*

$$E|x + \epsilon X|^{-\alpha} \leq C \min(\epsilon^{-\alpha}, x^{-\alpha}).$$

PROOF. It is straightforward to check that $K = \sup_{z \geq 0} E|z + X|^{-\alpha} < \infty$. Thus,

$$E|x + \epsilon X|^{-\alpha} = \epsilon^{-\alpha} E\left|\frac{x}{\epsilon} + X\right|^{-\alpha} \leq K\epsilon^{-\alpha}. \tag{A.1}$$

On the other hand,

$$\begin{aligned} E|x + \epsilon X|^{-\alpha} &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |x + \epsilon y|^{-\alpha} e^{-y^2/2}\, dy \\ &= \frac{1}{\sqrt{2\pi}} \biggl(\int_{\{|x+\epsilon y| > x/2\}} |x + \epsilon y|^{-\alpha} e^{-y^2/2}\, dy \\ &\qquad + \int_{\{|x+\epsilon y| \leq x/2\}} |x + \epsilon y|^{-\alpha} e^{-y^2/2}\, dy \biggr). \end{aligned}$$

It is easy to see that the first integral is bounded by $Cx^{-\alpha}$ for some constant C. The second integral, denoted by B, is bounded as follows:

$$\begin{aligned} B &= C\frac{1}{\epsilon} \int_{|z| < x/2} |z|^{-\alpha} e^{-(z-x)^2/(2\epsilon^2)}\, dz \leq C\frac{1}{\epsilon} \int_{|z| < x/2} |z|^{-\alpha} e^{-x^2/(8\epsilon^2)}\, dz \\ &= C\frac{x}{\epsilon} e^{-x^2/(8\epsilon^2)} x^{-\alpha} \leq Cx^{-\alpha}. \end{aligned}$$

Thus, we have $E|x + \epsilon X|^{-\alpha} \leq C|x|^{-\alpha}$. Combining this with (A.1), we obtain the lemma. □

LEMMA A.2. *Suppose that* $\alpha \in (0, 1)$. *There exists a constant* $C > 0$ *such that*

$$\sup_{\epsilon, \epsilon'} \int_{\mathbb{R}^2} p_\epsilon(x_1 + y_1) p_{\epsilon'}(x_2 + y_2) |y_1 - y_2|^{-\alpha}\, dy_1\, dy_2 \leq C|x_1 - x_2|^{-\alpha}.$$

PROOF. We can write

$$\int_{\mathbb{R}^2} p_\epsilon(x_1 + y_1) p_{\epsilon'}(x_2 + y_2) |y_1 - y_2|^{-\alpha}\, dy_1\, dy_2 = E(|\varepsilon X_1 - x_1 - \varepsilon' X_2 + x_2|^{-\alpha}).$$

Thus, Lemma A.2 follows directly from Lemma A.1. □

LEMMA A.3. *Suppose that* $\alpha \in (0,1)$. *There exists a constant* $C > 0$ *such that*

$$\sup_{\delta,\delta'} \int_0^t \int_0^t \varphi_\delta(t-s_1-r_1)\varphi_{\delta'}(t-s_2-r_2)|r_1-r_2|^{-\alpha}\, dr_1\, dr_2 \le C|s_1-s_2|^{-\alpha}.$$

PROOF. Since

$$p_\delta(x) \ge p_\delta(x) I_{[0,\sqrt{\delta}]}(x) = \frac{1}{\sqrt{2\pi\delta}} e^{-x^2/(2\delta)} I_{[0,\sqrt{\delta}]}(x) \ge \frac{1}{\sqrt{2\pi e}} \varphi_{\sqrt{\delta}}(x),$$

the lemma follows from Lemma A.2. □

LEMMA A.4. *Suppose that* $2H_0 + \sum_{i=1}^d H_i > d+1$. *Let* $B^1, \dots, B^d$ *be independent one-dimensional Brownian motions. We then have*

$$E\left(\int_0^t s^{2H_0-2} \prod_{i=1}^d |B_s^i|^{2H_i-2}\, ds\right)^2 < \infty.$$

PROOF. We can write

$$\begin{aligned} E&\left(\int_0^t s^{2H_0-2} \prod_{i=1}^d |B_s^i|^{2H_i-2}\, ds\right)^2 \\ &= 2\int_0^t \int_0^s (sr)^{2H_0-2} \prod_{i=1}^d E(|B_s^i|^{2H_i-2}|B_r^i|^{2H_i-2})\, dr\, ds. \end{aligned}$$

Let X be a standard normal random variable. From Lemma A.1, taking into account that $2-2H_i < 1$, we have, when $r < s$, that

$$\begin{aligned} E(|B_r^i|^{2H_i-2}|B_s^i|^{2H_i-2}) &= E\big[|B_r^i|^{2H_i-2} E\big[|\sqrt{s-r}X + x|^{2H_i-2}|_{x=B_r^i}\big]\big] \\ &\le C E\big[|B_r^i|^{2H_i-2}(s-r)^{H_i-1}\big) \qquad \text{(A.2)} \\ &\le C r^{H_i-1}(s-r)^{H_i-1}. \end{aligned}$$

As a consequence, the conclusion of the lemma follows from the fact that

$$\int_0^t \int_0^s r^{2H_0+\sum_{i=1}^d H_i-d-2} s^{2H_0-2}(s-r)^{\sum_{i=1}^d H_i-d}\, dr\, ds < \infty,$$

because $2H_0 + \sum_{i=1}^d H_i - d - 2 > -1$ and $\sum_{i=1}^d H_i - d > -1$. □

LEMMA A.5. *Let* $B^1, \dots, B^d$ *be independent one-dimensional Brownian motions. If* $\alpha_i \in (-1,0)$, $i = 1, \dots, d$, *and* $\sum_{i=1}^d \alpha_i > -2$, *then*

$$E \exp\left(\lambda \int_0^1 \prod_{i=1}^d |B_s^i|^{\alpha_i}\, ds\right) < \infty$$

for all $\lambda > 0$.

PROOF. The proof is based on the method of moments. We can write

$$E\exp\left(\lambda\int_0^1\prod_{i=1}^d |B_s^i|^{\alpha_i}\,ds\right)=\sum_{n=1}^\infty \frac{\lambda^n}{n!}E\int_{[0,1]^n}\prod_{k=1}^n\prod_{i=1}^d |B_{s_k}^i|^{\alpha_i}\,ds$$
$$=\sum_{n=1}^\infty \lambda^n\int_{[0<s_1<\cdots<s_n<1]}\prod_{i=1}^d E\left(\prod_{k=1}^n |B_{s_k}^i|^{\alpha_i}\right)ds.$$

From Lemma A.1, since $\alpha_i\in(-1,0)$, we obtain

$$E[|B_{s_k}^i|^{\alpha_i}|\mathcal{F}_{s_{k-1}}^i]=E[|B_{s_k}^i-B_{s_{k-1}}^i+B_{s_{k-1}}^i|^{\alpha_i}|\mathcal{F}_{s_{k-1}}^i]\le C(s_k-s_{k-1})^{\alpha_i/2},$$

where $\mathcal{F}_t$ is the filtration generated by the Brownian motion B^i. As a consequence, taking the conditional expectation of $\prod_{k=1}^n |B_{s_k}^i|^{\alpha_i}$ with respect to the σ-fields $\mathcal{F}_{s_{n-1}}^i,\mathcal{F}_{s_{n-2}}^i,\dots,\mathcal{F}_{s_1}^i$ and $\mathcal{F}_0^i$, we get

$$E\left(\prod_{k=1}^n |B_{s_k}^i|^{\alpha_i}\right)\le C^n(s_n-s_{n-1})^{\alpha_i/2}\cdots(s_2-s_1)^{\alpha_i/2}s_1^{\alpha_i/2}.$$

Letting $\alpha=\sum_{i=1}^d\alpha_i$, we have

$$E\exp\left(\lambda\int_0^1\prod_{i=1}^d |B_s^i|^{\alpha_i}\,ds\right)$$
$$\le\sum_{n=1}^\infty (C\lambda)^n\int_{[0<s_1<\cdots<s_n<1]}(s_n-s_{n-1})^{\alpha/2}\cdots$$
$$\times(s_2-s_1)^{\alpha/2}s_1^{\alpha/2}\,ds.$$

Since $\alpha>-2$, the integrals on the right-hand side are equal to $\frac{(\Gamma(\alpha/2+1))^n}{(n+n\alpha/2)\Gamma(n+n\alpha/2)}$ and the series converges for any $\lambda>0$. □

LEMMA A.6. *For any* $0<\alpha<1$, *define*

$$C_\alpha(y)=E(|\xi|^{-\alpha}-|y+\xi|^{-\alpha}),$$

where $y>0$ *and* ξ *is a standard normal random variable. Then*,

$$C_\alpha(y)\le C\min(1,(y^2+y^{3-\alpha}))$$

for some constant $C>0$.

PROOF. First, note that $C_\alpha(y)<C$, where $C>0$ is a constant, since $\lim_{y\to\infty}E|y+\xi|^{-\alpha}=0$. On the other hand, we can decompose the function $C_\alpha(y)$

as follows:

$$
\begin{aligned}
C_\alpha(y) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (|x|^{-\alpha} - |y+x|^{-\alpha}) e^{-x^2/2}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \biggl(\int_{\{x\ge 0\}\cup\{x\le -y\}} (|x|^{-\alpha} - |y+x|^{-\alpha}) e^{-x^2/2}\, dx \\
&\qquad + \int_{\{-y<x<0\}} (|x|^{-\alpha} - |y+x|^{-\alpha}) e^{-x^2/2}\, dx \biggr) \\
&= \frac{1}{\sqrt{2\pi}} (A+B),
\end{aligned}
$$

where A and B denote the first and second integrals, respectively, in the second-to-last line. For integral A, we can write

$$
\begin{aligned}
A &= \int_0^\infty \bigl(x^{-\alpha} - (x+y)^{-\alpha}\bigr)\bigl(e^{-x^2/2} - e^{-(x+y)^2/2}\bigr)\, dx \\
&\le \int_0^\infty x^{-\alpha}(x+y)^{1-\alpha}[(x+y)^\alpha - x^\alpha] y e^{-x^2/2}\, dx.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
A &\le \int_0^\infty x^{1-2\alpha}[(x+y)^\alpha - x^\alpha] y e^{-x^2/2}\, dx \\
&\quad + \int_0^\infty x^{-\alpha}[(x+y)^\alpha - x^\alpha] y^{2-\alpha} e^{-x^2/2}\, dx.
\end{aligned}
$$

For the first integral in the above expression, we use the estimate $(x+y)^\alpha - x^\alpha \le \alpha y x^{\alpha-1}$ and for the second, we use $(x+y)^\alpha - x^\alpha \le y^\alpha$. In this way, we obtain

$$
A \le C y^2
$$

for some constant $C > 0$. On the other hand,

$$
B = \int_0^y x^{-\alpha}\bigl(e^{-x^2/2} - e^{-(x+y)^2/2}\bigr)\, dx \le \int_0^y x^{-\alpha}(x+y) y\, dx \le C y^{3-\alpha}
$$

for some constant $C > 0$, which completes the proof of the lemma. □

PROPOSITION A.7. *Let B be a one-dimensional standard Brownian motion. Then, for any $p > 0$,*

$$
E\left| \int_0^1 \int_0^1 \delta(B_t - B_s)\, ds\, dt \right|^{-p} < \infty.
$$

PROOF. For $k = 1, \dots, 2^{n-1}$, we define $A_{n,k} = [\frac{2k-2}{2^n}, \frac{2k-1}{2^n}] \times [\frac{2k-1}{2^n}, \frac{2k}{2^n}]$ and

$$
\alpha_{n,k} = \int_{A_{n,k}} \delta(B_t - B_s)\, ds\, dt.
$$

The random variables $\alpha_{n,k}$ have the following two properties:

(i) for every $n \geq 1$, the variables $\alpha_{n,1}, \dots, \alpha_{n,2^{n-1}}$ are independent;

(ii) $\alpha_{n,k} \stackrel{d}{=} 2^{-n/2} \int_0^1 \int_0^1 \delta(B_t - \tilde{B}_s)\, ds\, dt$ and $\tilde{B}$ is a standard Brownian motion independent of B.

For any $p > 0$, we may choose a integer $n > 0$ such that $p2^{1-n} < 1/3$. Then, we can write

$$E\left|\int_0^1 \int_0^1 \delta(B_t - B_s)\, ds\, dt\right|^{-p} \leq E\left|\sum_{k=1}^{2^{n-1}} \alpha_{n,k}\right|^{-p} \leq E\left|\prod_{k=1}^{2^{n-1}} \alpha_{n,k}\right|^{-p2^{1-n}}$$

and it suffices to show that $E|\int_0^1 \int_0^1 \delta(B_t - \tilde{B}_s)\, ds\, dt|^{-p} < \infty$ for some $p > 0$. Notice that

$$L := \int_0^1 \int_0^1 \delta(B_t - \tilde{B}_s)\, ds\, dt = \int_{\mathbb{R}} L_1^x \tilde{L}_1^x\, dx,$$

where L_t^x (resp., $L_t^{\tilde{x}}$) denotes the local time of the Brownian motion B (resp., $\tilde{B}$). As a consequence, for any $0 < \alpha < 1$,

$$\begin{aligned} P(L < \epsilon) &\leq P\left(\int_0^{\epsilon^{4/5}} L_1^x \tilde{L}_1^x\, dx\right) \\ &\leq P\left(L_1^0 \tilde{L}_1^0 < \frac{1}{2}\epsilon^{1/5}\right) + P\left(\int_0^{\epsilon^{4/5}} |L_1^0 \tilde{L}_1^0 - L_1^x \tilde{L}_1^x|\, dx \geq \frac{\epsilon}{2}\right) \\ &\leq \frac{1}{\sqrt{2}} \epsilon^{1/10} (E(L_1^0)^{-1/2})^2 + \frac{2}{\epsilon} \int_0^{\epsilon^{4/5}} E|L_1^0 \tilde{L}_1^0 - L_1^x \tilde{L}_1^x|\, dx \\ &\leq C\epsilon^{1/10}, \end{aligned}$$

which implies that $E(L^{1/10}) < \infty$. □

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KANSAS
LAWRENCE, KANSAS, 66045
USA
E-MAIL: hu@math.ku.edu
nualart@math.ku.edu
jsong@math.ku.edu