NETWORK ROUTING IN A DYNAMIC ENVIRONMENT[1,2]

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Recently, there has been an explosion of work on network routing in hostile environments. Hostile environments tend to be dynamic, and the motivation for this work stems from the scenario of IED placements by insurgents in a logistical network. For discussion, we consider here a sub-network abstracted from a real network, and propose a framework for route selection. What distinguishes our work from related work is its decision theoretic foundation, and statistical considerations pertaining to probability assessments. The latter entails the fusion of data from diverse sources, modeling the socio-psychological behavior of adversaries, and likelihood functions that are induced by simulation. This paper demonstrates the role of statistical inference and data analysis on problems that have traditionally belonged in the domain of computer science, communications, transportation science, and operations research.

1. Introduction: Background and overview. Network routing problems involve the selection of a pathway from a source to a sink in a network. Network routing is encountered in logistics, communications, the internet, mission planning for unmanned aerial vehicles, telecommunications, and transportation, wherein the cost effective and safe movement of goods, personnel, or information is the driving consideration. In transportation science and operations research, network routing goes under the label *vehicle routing problem* (VRP); see Bertsimas and Simchi-Levi (1996) for a survey. The flow of any commodity within a network is hampered by the failure of one or more pathways that connect any two nodes. Pathway failures could be due to natural and physical causes, or due to the capricious actions of an adversary. For example, a cyber-attack on the internet, or the placement of an improvised explosive device (IED) on a pathway by an insurgent. Generally, the occurrence of all types of failures is taken to be probabilistic. See, for example, Gilbert (1959), or Savla, Temple and Frazzoli (2008) who assume that the placement of mines in a region can be described by a spatio-temporal Poisson process.

The traditional approach in network routing assumes that the failure probabilities are fixed for all time, and known; see, for example, Colburn (1987). Mod-

ern approaches recognize that networks operate in dynamic environments which cause the failure probabilities to be dynamic. Dynamic probabilities are the manifestations of new information, updated knowledge, or new developments (circumstances); de Vries, Roefs and Theunissen (2007) articulate this matter for unmanned aerial vehicles.

The work described here is motivated by the placement of IED's on the pathways of a logistical network; see Figure 1. Our aim is to prescribe an optimal course of action that a decision maker $\mathcal{D}$ is to take vis-à-vis choosing a route from the source to the sink. By optimal action we mean selecting that route which is both cost effective and safe. $\mathcal{D}$'s efforts are hampered by the actions of an adversary $\mathcal{A}$, who unknown to $\mathcal{D}$, may place IED's in the pathways of the network. In military logistics, $\mathcal{A}$ is an insurgent; in cyber security, $\mathcal{A}$ is a hacker. $\mathcal{D}$'s uncertainty about IED presence on a particular route is encapsulated by $\mathcal{D}$'s personal probability, and $\mathcal{D}$'s actions determined by a judicious combination of probabilities and $\mathcal{D}$'s utilities. For an interesting discussion on a military planner's attitude to risk, see de Vries, Roefs and Theunissen (2007) who claim that individuals tend to be risk prone when the information presented is in terms of losses, and risk averse when it is in terms of gains. Methods for a meaningful assessment of $\mathcal{D}$'s utilities are not on the agenda of this paper; our focus is on an assessment of $\mathcal{D}$'s probabilities, and the unconventional statistical issues that such assessments spawn.

To cast this paper in the context of recent work in route selection under dynamic probabilities, we cite Ye et al. (2010) who consider minefield detection and clearing. For these authors, dynamic probabilities are a consequence of improved estimation as detection sensors get close to their targets. The focus of their work is otherwise different from the decision theoretic focus of ours.

We suppose that $\mathcal{D}$ is a coherent Bayesian and thus an expected utility maximizer; see Lindley (1985). This point of view has been questioned by de Vries, Roefs and Theunissen (2007) who claim that humans use heuristics to make decisions. The procedures we endeavor to prescribe are on behalf of $\mathcal{D}$. We do not simultaneously model $\mathcal{A}$'s actions, which is what would be done by game theorists. Rather, our appreciation of $\mathcal{A}$'s actions are encapsulated via likelihood functions, and modeling socio-psychological behavior via subjectively specified likelihoods is a novel feature of this paper. Fienberg and Thomas (2010) give a nice survey of the diverse aspects of network routing dating from the 1950s, covering the spectrum of probabilistic, statistical, operations research, and computer science literatures. In Thomas and Fienberg (2010) an approach more comprehensive than that of this paper is proposed; their approach casts the problem in the framework of social network analysis, generalized linear models, and expert testimonies.

1.1. *Overview of the paper.* We start Section 2 by presenting a subnetwork, which is part of a real logistical network in Iraq, and some IED data experienced by this subnetwork. For security reasons, we are unable to present the entire network and do not have access to all its IED experience. Section 3 pertains to the decision-
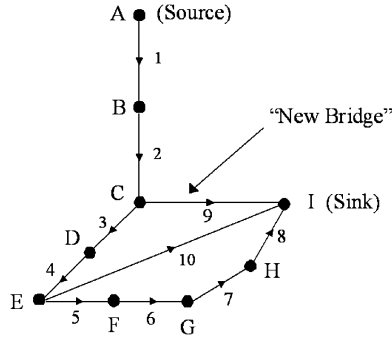
FIG. 1. *Subnetwork for transportation from A to I.*

theoretic aspects of optimal route selection. We discuss both the nonsequential and the sequential protocols. The latter raises probabilistic issues, pertaining to the "Principle of Conditionalization," that appear to have been overlooked by the network analyses communities. The material of Section 3 constitutes the general architecture upon which the material of Section 4 rests. Section 4 is about the inferential and statistical matters that the architecture of Section 3 raises. It pertains to the dynamic assessment of failure probabilities, and describes an approach for the integration of data from multiple sources. Such data help encapsulate the actions of $\mathcal{A}$, and $\mathcal{D}$'s efforts to defeat them. The approach of Section 4 is Bayesian; it entails the use of logistic regression and an unusual way of constructing the necessary likelihood functions. Section 5 summarizes the paper, and portrays the manner in which the various pieces of Sections 3 and 4 fit together. Section 5 also closes the paper by showing the workings of our approach on the network of Section 2.

2. A network for transportation logistics. Figure 1 is a subnetwork abstracted from a real logistics network used in Iraq. The subnetwork has nine nodes, labeled A (not to be confused with adversary $\mathcal{A}$) to I, and ten links, labeled 1 to 10. The source node is A and the sink node is I.

There are thirteen bridges dispersed over the ten links of Figure 1, with link 9 having one bridge, the "*new bridge.*" This bridge is a mile away from a park, the old city, the bus station, and the mosque. The precise locations of the remaining 12 bridges in the subnetwork are classified. There have been four crossings on the "new bridge," and none of these have experienced an IED attack. To plan an optimal route from source to sink, $\mathcal{D}$ needs to know the probability of experiencing an IED attack on the next crossing on each of the ten links. However, we focus discussion on link 9, because it is for this link that we have information on the number of previous crossings.

To assess the required probabilities, we need to have all possible kinds of information, including that given in Table 1, which gives the history of IED placements on the remaining twelve bridges of the subnetwork. The data of Table 1, though

TABLE 1
Historical data on IED placements on 12 *bridges in Iraq*

Bridge	Attack	Park	Old city	Bus station	Mosque
Aimma	0	0	0	1	0.1
Adhimiya	0	0.25	0.75	1.5	1
Sarafiya	1	0	1	1	0.5
Sabataash	0	1	0	0.75	0.2
Shuhada	0	2	0	0.75	0.1
Ahrar	0	1	0	1	0.75
Sinak	0	0.5	0	1	0.3
Jumhuriya	0	0.1	0	0.75	0.3
Arbataash	1	0	3	3.5	2
Jadriya	1	0	5.5	5	2
SJadriya	0	0	6	5.5	3
Dora	1	2	5	4	4

public, were painstakingly generated via information from multiple sources—such as Google Maps—by the so-called process of "connecting the dots." Generally, such data are hard to come by via the public domain. The recently released WikiLeaks (2010) data has some covariate information on IED experiences in Afghanistan. However, there are very few well-defined logistical routes in Afghanistan, and those that may be there are not identified in the WikiLeaks database. Furthermore, the covariate information that is available is not of the kind relevant to route selection. Thus, for this paper, the WikiLeaks–Afghanistan data are of marginal value.

In Table 1, the column labeled "Attack" is 1 whenever the bridge has experienced an attack; otherwise it is 0. The other columns give the distance of the bridge, in miles, from population centers like a park, old city, bus station, and mosque. An entry of zero denotes that the bridge is next to the landmark. Whereas data on IED attacks tends to be public (because of press reports), data on the number of crossings by convoys, the number of IEDs cleared, the composition of the convoys, etc., remains classified.

The three routes suggested by Figure 1 are as follows: $(1, 2, 3, 4, 5, 6, 7, 8)$, $(1, 2, 3, 4, 10)$, and $(1, 2, 9)$. Since IEDs are placed by adversaries, $\mathcal{D}$ is generally uncertain of their presence when planning begins. Additionally, there are pros and cons with each route in terms of distance traversed, route conditions (such as the number of curves and bends, terrain topology), proximity to hostile territory, receptiveness of the local population to harbor insurgents, and so on. In actuality, $\mathcal{D}$ will have access to historical data of the type shown in Table 1, and also information about the nature of the cargo, the convoy speed, intelligence about the cunningness and sophistication of the insurgents, the number of previous unencumbered crossings on a link, etc.

$\mathcal{D}$'s problem is to select an optimal route between the three routes given above. A variant is to specify the optimal route *sequentially*. That is, start by going from A to C via links 1 and 2, and then, upon arrival at C, make a decision to proceed along link 9 to the sink, or to take the circuitous routes via the links 3 to 8, and 10, to get to the sink. Similarly, upon arrival at node E, $\mathcal{D}$ could proceed along link 10, or via the links 5, 6, 7, and 8 to arrive at the sink. $\mathcal{D}$'s decision as to which choice to make will be based on $\mathcal{D}$'s uncertainty of IED presence on the links 3 to 10, assessed when $\mathcal{D}$ is at node C and at node E.

Thus, optimal route selection is a problem of decision under uncertainty. Because of the dynamic environment in which convoys operate, $\mathcal{D}$'s uncertainties change over time. In Section 3 we prescribe a decision-theoretic architecture for route selection. This requires that $\mathcal{D}$ assess his (her) uncertainties about IED placements, as well as utilities for a successful or failed traversal. Since $\mathcal{D}$'s uncertainties are dynamic, the prescription of Section 3 is also dynamic; that is, the selected route is optimal only for an upcoming trip. The main challenge therefore is an assessment of the dynamic probabilities; see Section 4.

3. $\mathcal{D}$'s decision-theoretic architecture. Under the nonsequential protocol, $\mathcal{D}$ needs to choose, at decision time, from the following: $D_1 \equiv$ take route $(1, 2, 9)$; $D_2 \equiv$ take route $(1, 2, 3, 4, 5, 6, 7, 8)$; or $D_3 \equiv$ take route $(1, 2, 3, 4, 10)$. Figure 2 shows $\mathcal{D}$'s decision tree for these choices, with each D_i leading to a random node R_i, with each R_i leading to an outcome S (for success) and F (for failure), $i = 1, 2$. Here S is the event that an IED is not encountered on any link of the route, and F the event that an IED is encountered. If $\mathcal{D}$ is aware of any route clearing activity, then this becomes a part of $\mathcal{D}$'s covariates used to assess probabilities. The presence of an IED does not necessarily imply an explosion. Unexploded IEDs cause disruptions, and $\mathcal{D}$'s aim is to choose that route which minimizes the risk of damage and disruption.

In Figure 2, $p_1(S)$ and $p_1(F) = 1 - p_1(S)$ denote $\mathcal{D}$'s probabilities for success and failure, and $U(D_1, S)$ and $U(D_1, F)$, $\mathcal{D}$'s utilities under D_1. The quantities $p_2(S)$, $p_2(F)$, $U(D_2, S)$, and $U(D_2, F)$ pertain to D_2; similarly, for D_3.

Assessing utilities is a substantive task [cf. Singpurwalla (2010)] entailing rewards, penalties, and attitudes to risk. This task is not pursued here. However, one often assumes binary loss functions, so that $U(D_i, S) = 1$ and $U(D_i, F) = 0$.
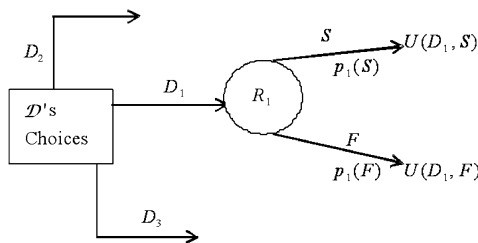

FIG. 2. *$\mathcal{D}$'s decision tree for nonsequential actions.*

Per the principle of *maximization of expected utility*, $\mathcal{D}$ chooses that D_i for which the expected utility is a maximum. Thus, at each R_i, $\mathcal{D}$ computes, for $i = 1, 2, 3$,

$$\mathbf{E}[U(D_i)] = p_i(S)U(D_i, S) + p_i(F)U(D_i, F),$$

and chooses that D_i which maximizes $\mathbf{E}[U(D_i)]$.

3.1. *$\mathcal{D}$'s assessment of $p_i(S)$.* The building blocks of $p_i(S)$ are the $p(j)$'s, $\mathcal{D}$'s probabilities of an IED placement on link j, $j = 1, \ldots, 10$. Under action D_1, the event S will occur at the terminus of the tree if there is no IED placement on the links 1, 2, and 9. If $E(j)$ denotes the event that an IED is placed on link j, then $p(j)$ is an abbreviation for $P(E(j))$. If $\mathcal{D}$ assumes that the $E(j)$'s, $j = 1, 2, 9$, are independent, then

$$p_1(S) = (1 - p(1))(1 - p(2))(1 - p(9)) \quad \text{and} \quad p_1(F) = 1 - p_1(S);$$

otherwise,

$$p_1(F) = p(1) + p(2) + p(9) - p(1, 2) - p(1, 9) - p(2, 9) + p(1, 2, 9),$$

where $p(j, k)$ is $\mathcal{D}$'s joint probability that both $E(j)$ and $E(k)$ occur, $j \neq k$; similarly with $p(j, k, l)$, $j \neq k \neq l$. If $p(j|k)$ denotes $\mathcal{D}$'s conditional probability of $E(j)$ given $E(k)$, and if $\mathcal{D}$ judges $E(j)$ independent of $E(l)$, given $E(k)$, then

$$p(j, k, l) = p(j|k)p(k|l)p(l).$$

Conditional independence in networks is often invoked when dependence between $E(j)$ and $E(k)$ matters only when links j and k are neighbors. Since links 1 and 9 are not neighbors, $\mathcal{D}$ may judge $E(1)$ and $E(9)$ independent given $E(2)$.

$\mathcal{D}$'s main task is to assess the probabilities of the type $p(j)$ and $p(j|k)$. The material of Section 4 pertains to this exercise.

3.2. *Decision making under a sequential protocol.* Here, $\mathcal{D}$ starts with a single choice, namely, getting to node C via links 1 and 2, and then, upon arriving at C, making one of two choices: get to the sink via link 9, or via the links 3 through 8, and 10. With three choices, the decision tree for the sequential protocol will be analogous to that of Figure 2, save for the fact that the decision nodes will be at nodes C and E, instead of being at node A. The rest of the analysis parallels that described in the material following Figure 2 [cf. Singpurwalla (2009)], save for one matter, namely, the caveat of conditionalization.

3.2.1. *The caveat of conditionalization.* The *principle of conditionalization* (POC) pertains to probability assessments of two (or more) events, and the disposition of one of them becomes known [cf. Singpurwalla (2006), page 21, and (2007)]. It arises because conditional probabilities are in the *subjunctive mood*.

When the disposition of the conditioning event becomes known, and the POC is upheld, the probability of the unconditioned event is its previously assessed conditional probability. When the POC is not upheld, one assesses the probability of the unconditioned event via a likelihood and Bayes' Law, using the revealed value of the conditioned event as data. When sequential routing is done for strategic reasons, socio-psychological issues come into play, and then it is realistic to assess the probability of the unconditioned event via a likelihood.

To illustrate the above, consider the scenario of $\mathcal{D}$ choosing a sequential protocol, and having arrived at node C needs to assess the quantities $p_{2,9}(S)$ and $p_{2,3}(S)$, where $p_{2,9}(S)$ is the probability of successfully arriving at the sink via links 2 and 9. If the POC is upheld, then $p_{2,9}(S)$ is obtained as $P(E^c(9)|E^c(2))$; $E^c(2)$ is the probability of no IED presence on link 2. If the POC is not upheld, then

$$p_{2,9}(S) = P(E^c(9); E^c(2)) \propto \mathcal{L}(E^c(9); E^c(2))(1 - p(9)),$$

where the middle term is $\mathcal{D}$'s likelihood of an IED absence on link 9, under the sure knowledge of an IED absence on link 2. Similarly with $p_{2,3}(S)$.

The likelihood is specified by $\mathcal{D}$ and is the price to be paid for rejecting the POC. Such likelihoods may encapsulate the socio-psychological considerations that $\mathcal{D}$ chooses to exercise. Since the likelihood is a weight that $\mathcal{D}$ assigns to a prior probability, $\mathcal{D}$ may upgrade (downgrade) the prior via the likelihood depending on whether the absence of an IED on link 2 would make the presence of an IED on link 9 more (or less) likely. Here much depends on what $\mathcal{D}$ thinks of the abilities and resources of insurgents.

4. Dynamic assessment of link probabilities.
By link probabilities, we mean unconditional probabilities of the type $p(j)$, $j = 1, \ldots, 10$. By a dynamic assessment, we mean an updating of each $p(j)$ due to additional information that can come in the form of hard data, expert testimonies, socio-psychological considerations, or new covariate information. The updating of a $p(j)$ can come into play at any time, most often at the commencement of each route scheduling session, or in the case of sequential routing, at any time during the cycle at an intermediate node. In what follows, we focus on link j, and discuss the assessment of $p(j)$. A dynamic assessment of the conditional probabilities $p(j|k)$ is discussed in Section 4.4.

Factors that influence any $p(j)$ would be covariates such as route topography (the number of bends, curves, bridges, and surface conditions), convoy size and composition (materials or humans), convoy speed, time of transport (day or night), weather conditions, political climate, etc. A second factor would be historical data on IED placements on link j, and on all the other links in the region. Finally, also relevant would be $\mathcal{D}$'s subjective view about $p(j)$, encapsulated via a prior.

4.1. *Notation and terminology.*   Let $(X = 1)$ denote the event that one or more IEDs are placed on link j; $(X = 1)$ is a proxy for $E(j)$, and $P(X = 1)$ a proxy for $p(j)$. To avoid cumbersome notation, we will not endow X with the index j. Let $Z_1, \ldots, Z_k$ be k covariates that influence $p(j)$, and denote these by the vector $\mathbf{Z} = (Z_1, \ldots, Z_k)$; $\mathbf{Z}$ is assumed known to $\mathcal{D}$. Suppose that there have been n crossings on link j, with $X_m = 1(0)$ if the mth crossing experienced (did not experience) an IED, $m = 1, \ldots, n$. Let $\mathbf{X} = (X_1, \ldots, X_n)$ denote the historical IED experience on link j. Assume that $X_1 = X_2 = \cdots = X_n = 0$, or that $X_1 = X_2 = \cdots = X_{n-1} = 0$, and that $X_n = 1$. That is, $\mathcal{D}$ has observed a series of n successes on link j, or has just experienced a failure. Motivation for these extreme cases is given later.

The IED experience for the entire region is in matrix $\mathbf{D}$, where

$$\mathbf{D} = \begin{bmatrix} Y_1 & Z_{11} & \cdots & Z_{k1} \\ Y_2 & Z_{12} & \cdots & Z_{k2} \\ \vdots & & & \\ Y_l & Z_{1l} & \cdots & Z_{kl} \\ \vdots & & & \\ Y_s & Z_{1s} & \cdots & Z_{ks} \end{bmatrix}.$$

In the lth row of $\mathbf{D}$, $Y_l = 1(0)$ if an IED presence has been encountered (not encountered) under condition $Z_{1l}, \ldots, Z_{kl}$, for $l = 1, \ldots, s$. Thus, at disposal to $\mathcal{D}$ are the s IED related experiences in the region, and associated with each experience are the values of the k covariates that influence each experience. To avoid any duplicate weighting of data, $\mathbf{X}$ will *not* be a part of $\mathbf{D}$. The motivation for excluding $\mathbf{X}$ from $\mathbf{D}$ is to give link j a special emphasis by incorporating the effect of $\mathbf{X}$, which is specific to link j, in a vein that is different from $\mathbf{D}$.

Let x_i be the realization of X_i, and y_l of Y_l, $i = 1, \ldots, n$ and $l = 1, \ldots, s$. Each $x_i = 1$ or 0; similarly, y_l. $\mathbf{D}$ is assumed known to $\mathcal{D}$; its elements may not be controlled by $\mathcal{D}$.

$\mathcal{D}$'s task is to assess $P(X = 1; \mathbf{x}, \mathbf{Z}, \mathbf{D}^*)$, where $\mathbf{x} = (x_1, \ldots, x_n)$, and $\mathbf{D}^*$ is $\mathbf{D}$ with the Y_l's replaced by y_l, $l = 1, \ldots, s$. The above expression is $\mathcal{D}$'s probability of an IED presence on link j, knowing $\mathbf{x}$, $\mathbf{Z}$, and $\mathbf{D}^*$. Assessing this probability is tantamount to fusing data from two sources: IED experience on link j, and historical IED experience in the region wherein j resides. It is a form of weighting wherein one borrows strength based on individual and population characteristics.

4.1.1. *The proposed model.*   Start by assuming $\mathbf{x}$ unknown, so that $P(X = 1; \mathbf{x}, \mathbf{Z}, \mathbf{D}^*)$ is $P(X = 1 | \mathbf{X}; \mathbf{Z}, \mathbf{D}^*)$, and invoke the law of total probability to write

$$P(X = 1 | \mathbf{X}; \mathbf{Z}, \mathbf{D}^*) = \int_0^1 P(X = 1 | p, \mathbf{X}; \mathbf{Z}, \mathbf{D}^*) \pi(p | \mathbf{X}; \mathbf{Z}, \mathbf{D}^*) \, dp,$$

where p is a *propensity* [see Singpurwalla (2006), page 50], and $\pi(p | \mathbf{X}; \mathbf{Z}, \mathbf{D}^*)$ is $\mathcal{D}$'s uncertainty about p, given $\mathbf{X}$, with $\mathbf{Z}$ and $\mathbf{D}^*$ known. The propensity of event $\mathcal{E}$ is the proportion of times $\mathcal{E}$ occurs in an infinite number of trials.

If we assume that, given p, the event $(X = 1)$ is independent of $\mathbf{X}$, $\mathbf{Z}$, and $\mathbf{D}^*$, then

$$(4.1) \qquad P(X = 1|\mathbf{X}; \mathbf{Z}, \mathbf{D}^*) = \int_0^1 p \cdot \pi(p|\mathbf{X}; \mathbf{Z}, \mathbf{D}^*)\, dp,$$

and by Bayes' Law,

$$\pi(p|\mathbf{X}; \mathbf{Z}, \mathbf{D}^*) \propto \pi(\mathbf{X}|p; \mathbf{Z}, \mathbf{D}^*) \cdot \pi(p; \mathbf{Z}, \mathbf{D}^*)$$
$$= \pi(\mathbf{X}|p) \cdot \pi(p; \mathbf{Z}, \mathbf{D}^*),$$

if given p, $\mathbf{X}$ is independent of $\mathbf{Z}$ and $\mathbf{D}^*$. Here $\pi(p; \mathbf{Z}, \mathbf{D}^*)$ is $\mathcal{D}$'s uncertainty about p in light of $\mathbf{Z}$ and $\mathbf{D}^*$, and $\pi(\mathbf{X}|p)$ is $\mathcal{D}$'s probability model for $\mathbf{X}$. Equation (4.1) now becomes

$$(4.2) \qquad P(X = 1|\mathbf{X}; \mathbf{Z}, \mathbf{D}^*) \propto \int_0^1 p \cdot \pi(\mathbf{X}|p) \cdot \pi(p; \mathbf{Z}, \mathbf{D}^*)\, dp.$$

However, $\mathbf{X}$ is observed as $\mathbf{x}$, and, thus, a probability model for $\mathbf{X}$ does not make sense. We therefore write $P(X = 1|\mathbf{X}; \mathbf{Z}, \mathbf{D}^*)$ as $P(X = 1; \mathbf{x}, \mathbf{Z}, \mathbf{D}^*)$, and $\pi(\mathbf{X}|p)$ as $\mathcal{L}(p; \mathbf{x})$, the likelihood of p under $\mathbf{x}$. Now equation (4.2) becomes

$$(4.3) \qquad P(X = 1; \mathbf{x}, \mathbf{Z}, \mathbf{D}^*) \propto \int_0^1 p \cdot \mathcal{L}(p; \mathbf{x}) \cdot \pi(p; \mathbf{Z}, \mathbf{D}^*)\, dp.$$

Equation (4.3) is our proposed model for assessing $p(j)$. To proceed, $\mathcal{D}$ needs to specify the likelihood $\mathcal{L}(p; \mathbf{x})$ and $\pi(p; \mathbf{Z}, \mathbf{D}^*)$, the posterior of p.

4.2. *Psychological considerations in specifying likelihoods.* The IED scenario entails special considerations for specifying $\mathcal{L}(p; \mathbf{x})$. These arise because $\mathcal{D}$ needs to incorporate an insurgent's *socio-psychological behavior* in the IED placement process, and also $\mathcal{D}$'s strategy for outfoxing the insurgent.

Recall that with $\mathbf{x} = (0, \ldots, 0)$ or $\mathbf{x} = (0, \ldots, 0, 1)$, the conventional likelihood of p would be $\mathcal{L}(p; \mathbf{x}) = p^{\sum x_i}(1 - p)^{n - \sum x_i}$, which for the aforementioned $\mathbf{x}$ would be $(1 - p)^n$ or $(1 - p)^{n-1} \cdot p$. The motivation for the conventional specification is that a preponderance of failures (i.e., non-IED placements) should decrease the propensity of an IED placement, and vice versa. However, the conventional approach, though appropriate for scenarios which are nonadversarial, is inappropriate for IED placement which embodies an adversary with a *socio-psychological agenda*. It seems that here a preponderance of failures should eventually increase the propensity of success. Insurgents are opportunistic adversaries who may allow a series of successful link crossings only to impart to $\mathcal{D}$ a sense of false security, while all the time preparing to do damage on the next crossing. Similarly, an astute $\mathcal{D}$ would view the occurrence of a success that is preceded by a sequence of failures (i.e., non-IED placements) with much pessimism, as a dramatic change in the operating environment. Essentially, $\mathcal{D}$ would downgrade the impact of the

observed sequence of $(n-1)$ failures and strongly weigh the impact of the last success. With the above behavioristic considerations, our proposed likelihood for p, for $\mathbf{x} = (x_1, \ldots, x_n)$ fixed, is of the form

$$\mathcal{L}(p; \mathbf{x}) = (1-p)^{\sqrt[n]{n-\sum x_i}} \cdot p^{\sum x_i}.$$

When $\mathbf{x} = (0, \ldots, 0)$, the above likelihood becomes

$$(4.4) \qquad \mathcal{L}(p; \mathbf{x}) = (1-p)^{\sqrt[n]{n}},$$

and when $\mathbf{x} = (0, \ldots, 0, 1)$, it is

$$(4.5) \qquad \mathcal{L}(p; \mathbf{x}) = (1-p)^{\sqrt[n]{n-1}} \cdot p.$$

As $n \to \infty$, equation (4.4) tends to $(1-p)$, the conventional likelihood for a single Bernoulli trial that results in a failure. With $n \to \infty$, equation (4.5) tends to $(1-p) \cdot p$, the conventional likelihood for the case of two Bernoulli trials resulting in one failure and one success. In an adversarial context, this is tantamount to $\mathcal{D}$ regarding a long series of failures as only a single failure (i.e., $\mathcal{D}$ does not become complacent), and a long series of failures followed by a success as only one failure and one success. In this latter case, $\mathcal{D}$ gives equal weight to the $(n-1)$ failures and the one success; that is, $\mathcal{D}$ becomes deeply concerned when the first success is observed. Figure 3 illustrates the likelihood.

The proposed likelihood of p is in the envelope bounded by $(1-p)$ and $(1-p)^{\sqrt[3]{3}}$. Thus, after three successive failures $\mathcal{D}$ gives more and more weight to larger values of p, suggesting an absence of $\mathcal{D}$'s complacence with a long series of failures. The specification of the likelihoods as embodied in equations (4.4) and (4.5) is a novel feature of this paper; it is a possible approach to *adversarial modeling*.

4.3. $\mathcal{D}$'s assessment of the posterior $\pi(p; \mathbf{Z}, \mathbf{D}^*)$.

An assessment of the posterior of p in the light of known covariates $\mathbf{Z}$ and the historical data $\mathbf{D}^*$ is developed in two stages. The challenge here is with the specification of the likelihood.
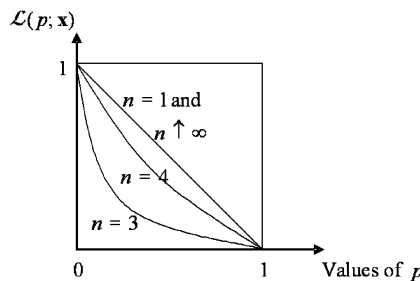

FIG. 3. *The likelihood of p as a function of n.*

Stage I: Logistic regression for extracting the information in $\mathbf{D}^*$. Information provided by $\mathbf{D}^*$ lies in an assessment of the posterior of $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_l, \ldots, \beta_k)$, where $\boldsymbol{\beta}$ appears in a logistic regression model

$$P(Y_l = 1; \boldsymbol{\beta}, \mathbf{Z}_l) = \frac{1}{1 + \exp(-\sum_{u=1}^{k} Z_{lu}\beta_u)}$$

for $l = 1, \ldots, s$, with $\mathbf{Z}_l = (Z_{1l}, \ldots, Z_{kl})$. Recall, Y_l and $\mathbf{Z}_l$ are the lth row of $\mathbf{D}^*$.

Using standard but computationally intensive simulation procedures, we can obtain the posterior of $\boldsymbol{\beta}$ in light of $\mathbf{D}^*$. Denote this posterior as $\pi(\boldsymbol{\beta}; \mathbf{D}^*)$.

Stage II: The likelihood of p *under* $\mathbf{Z}$ *and* $\mathbf{D}^*$. To assess the posterior $\pi(p; \mathbf{Z}, \mathbf{D}^*)$, invoke Bayes' Law to write

(4.6) $\pi(p; \mathbf{Z}, \mathbf{D}^*) \propto \mathcal{L}(p; \mathbf{Z}, \mathbf{D}^*)\pi(p),$

where $\mathcal{L}(p; \mathbf{Z}, \mathbf{D}^*)$ is the likelihood of p in light of the known $\mathbf{Z}$ and $\mathbf{D}^*$, and $\pi(p)$ is $\mathcal{D}$'s prior for p. Note that p and $\mathbf{Z}$ are specific to link i, whereas $\mathbf{D}^*$ is common to all the links of the network. The prior on p could be any suitable distribution, such as a beta distribution over $(0, 1)$. The main theme of Stage II, however, is a development of the likelihood $\mathcal{L}(p; \mathbf{Z}, \mathbf{D}^*)$.

Whereas likelihoods may be subjectively specified, the conventional method is to invert a probability model by juxtaposing the parameter(s) and the random variables. This is the strategy we use, but to do so we need a probability model for p with $\mathbf{Z}$ and $\mathbf{D}^*$ as background information. Since p depends on $\mathbf{Z}$, we denote this dependence by replacing p with $p(\mathbf{Z})$. Thus, we seek a probability model for $p(\mathbf{Z})$ with $\mathbf{D}^*$ as a background, namely, $P[p(\mathbf{Z}); \mathbf{D}^*]$. But knowing $\mathbf{D}^*$ is equivalent to knowing $\boldsymbol{\beta}$ with its posterior probability, $\pi(\boldsymbol{\beta}; \mathbf{D}^*)$, developed in Stage I. Thus, for $\boldsymbol{\beta} = \boldsymbol{\beta}^*$, $[p(\mathbf{Z}); \boldsymbol{\beta}^*]$ has probability $\pi(\boldsymbol{\beta}^*; \mathbf{D}^*)$. However, per the logistic regression model,

$$[p(\mathbf{Z}); \boldsymbol{\beta}^*] = \frac{1}{1 + \exp(-\sum_u^k Z_u\beta_u^*)},$$

where β_u^* appears as the uth element of $\boldsymbol{\beta}^* = (\beta_1^*, \ldots, \beta_k^*)$.

To summarize, the event $[p(\mathbf{Z}); \boldsymbol{\beta}^*] = 1/[1 + \exp(-\sum Z_u\beta_u^*)]$ has probability $\pi(\boldsymbol{\beta}^*; \mathbf{D}^*)$, and this provides us with a probability model for $[p(\mathbf{Z}); \mathbf{D}^*]$. Consequently, a plot of $(p(\mathbf{Z}); \boldsymbol{\beta}^*)$ versus $\pi(\boldsymbol{\beta}^*; \mathbf{D}^*)$ provides the required likelihood function.

To implement this idea, we sample a $\boldsymbol{\beta}^*$ from $\pi(\boldsymbol{\beta}; \mathbf{D}^*)$ to obtain

$$[p(\mathbf{Z}); \boldsymbol{\beta}^*] = \frac{1}{1 + \exp(-\sum_u^k Z_u\beta_u^*)},$$

and also $\pi(\boldsymbol{\beta}^*; \mathbf{D})$. A plot of $(p(\mathbf{Z}); \boldsymbol{\beta}^*)$ versus $\pi(\boldsymbol{\beta}^*; \mathbf{D}^*)$ is then the likelihood function of p in light of $\mathbf{Z}$ and $\mathbf{D}^*$; see Figure 4.
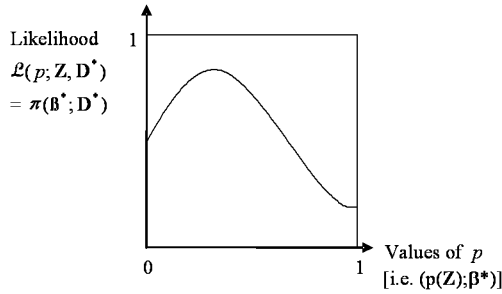
FIG. 4. *The likelihood of p with* **Z** *and* **D*** *known.*

With $\pi(p(\mathbf{Z}))$ the prior on p specified, and the likelihood $\mathcal{L}(p; \mathbf{Z}, \mathbf{D}^*)$ induced via a logistic regression model governing $p(\mathbf{Z})$ and $\mathbf{D}^*$, the desired posterior

$$\pi(p; \mathbf{Z}, \mathbf{D}^*) \propto \mathcal{L}(p; \mathbf{Z}, \mathbf{D}^*) \cdot \pi(p(\mathbf{Z}))$$

can be numerically assessed.

Once the above is done, all the necessary ingredients for obtaining equation (4.3), which can now be written as

$$(4.7) \qquad P(X = 1; \mathbf{x}, \mathbf{Z}, \mathbf{D}^*) \propto \int p \cdot \mathcal{L}(p; \mathbf{x}) \cdot \mathcal{L}(p; \mathbf{Z}, \mathbf{D}^*) \cdot \pi(p) \, dp,$$

are at hand. The above expression can be numerically evaluated.

4.4. *Dynamic assessment of conditional probabilities.* For both the nonsequential and sequential protocols wherein the POC is upheld, we need to assess conditional probabilities of the type $P(n|m)$, where links m and n are adjacent to each other, and traversing on m precedes that on n. There are two possible strategies. The first one is for $\mathcal{D}$ to subjectively change the assessed $p(n)$ by either increasing it because an insurgent might find it easy to populate neighboring links with IEDs, or to decrease it if $\mathcal{D}$ thinks that an insurgent has limited resources for placing IEDs.

The second approach is less subjective because it incorporates data on IED placements or nonplacements on neighboring links. The idea here is to treat the conditioning event $E(m)$ as a covariate, so that the vectors $\mathbf{Z}$ and $\boldsymbol{\beta}$ of Sections 4.1 and 4.3 get expanded by an additional term, as $\mathbf{Z} = (Z_1, \ldots, Z_k, 1)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_k, \beta_{k+1})$. Correspondingly, the matrix $\mathbf{D}$ of Section 4.1 also gets expanded to include an additional column whose lth term $Z_{(k+1)l}$ is 1 whenever there has been an IED experience in a preceding link; otherwise $Z_{(k+1)l}$ is 0. With the above in place, a repeat of the exercise described in Section 4.3 would enable a formal assessment of the conditional probabilities. The only other matter that remains to be addressed pertains to the likelihood of p as discussed in Section 4.2. Since the likelihood is a weight assigned to the posterior of p, $\mathcal{D}$ may either increase the $\mathcal{L}(p; \mathbf{x})$ of equations (4.4) and (4.5), or decrease it depending on what

$\mathcal{D}$ thinks of an insurgent's abilities and resources. $\mathcal{D}$ would increase $\mathcal{L}(p; \mathbf{x})$ if $\mathcal{D}$ feels that the insurgent's resources are plentiful; otherwise $\mathcal{D}$ downgrades $\mathcal{L}(p; \mathbf{x})$.

5. Summary and conclusions. Equation (4.7) shows how $\mathcal{D}$ can assess $p(j)$, the probability of one or more IED placements on link j in a unified manner by a systematic application of the Bayesian approach. It entails a fusion of information on past IED experience on link j (encapsulated by $\mathbf{X}$), historical data on IED experience in the region (encapsulated by the matrix $\mathbf{D}^*$), and $\mathcal{D}$'s subjective views about $p(j)$, encapsulated via the likelihood $\mathcal{L}(p; \mathbf{x})$ and the prior $\pi(p)$. The essence of equation (4.7) is that its right-hand side is the expected value of a weighted prior distribution of p. The weighting of the prior is by the product of two likelihoods, one reflecting historical IED experience specific to link i, and the other reflecting historical IED experience in the region as well as the relevant covariates specific to the forthcoming trip contemplated by $\mathcal{D}$. The entire development being grounded in the calculus of probability is therefore *coherent*.

Though cumbersome to plough through, there are novel features to the two likelihoods. The first likelihood—equations (4.4) and (4.5)—is an unconventional likelihood for use with Bernoulli trials. It is motivated by socio-psychological considerations attributed to both the insurgents who place the IED's, as well as to $\mathcal{D}$, who does not become complacent upon a sequence of successful crossings and who upon the occurrence of the first failure adopts the posture of extreme caution. The second likelihood—that of Figure 4—is induced in an unusual manner by leaning on the posterior distribution of the parameter vector of a logistic regression.

The approach of Section 4 displays the manner in which information from different sources can be fused by decomposing the likelihood of p. Equation (4.7) shows this. The material of Section 4 feeds into that of Section 3 which pertains to sequential and nonsequential decision making under uncertainty.

The computational and simulation work spawned by Section 4 entails logistic regression, generating k-dimensional samples from the posterior distribution of $\boldsymbol{\beta}$, numerically assessing $\pi(p; \mathbf{Z}, \mathbf{D}^*)$—equation (4.6)—and numerical integration to obtain $P(X = 1; \mathbf{x}, \mathbf{Z}, \mathbf{D}^*)$—equation (4.7). None of these pose any obstacles. Section 4.4 pertains to conditional probabilities. It expands on Sections 4.1 through 4.3, by treating the conditioning events as covariates.

5.1. *Data and information requirements.* The one major obstacle pertains to the paucity of the data for validating the approach. The required data, namely, $\mathbf{x}$, $\mathbf{Z}$, and $\mathbf{D}^*$, are available to the military logisticians, but are almost always classified. The WikiLeaks data tend to focus on IED explosions and not on success stories wherein IED's get cleared, similarly with other publicly available data. Information that is relevant to constructing the likelihood based on socio-psychological

considerations is highly individualized, and perhaps not even recorded. It is *desirable* to collect this kind of information via experiments pertaining to the psychology of logisticians and route planners, and also insurgents via what is known as "red teaming."

The text of this paper can be seen as a template for addressing network routing in a dynamic environment. The network architecture of Figure 1 brings out the necessary caveats that problems of this type pose, one such caveat being the caveat of conditionalization, discussed in Section 3.2.1. Real logistical networks are more elaborate. In actual practice the matrix $\mathbf{D}^*$ could have a very large dimension and thus be unmanageable. However, given the role that $\mathbf{D}^*$ plays, one may simply sample from a high dimensional $\mathbf{D}^*$ to work with a more manageable matrix. Besides a prior for p, $\pi(p)$, all that is required of $\mathcal{D}$ are the utilities mentioned in Section 3. However, these utilities are proxies for costs, and no form of optimization can be achieved without cost considerations. Finally, this paper shows how statistical methodologies can be constructively brought to bear in network routing problems which generically belong in the domain of computer science, network analysis, and operations research.

We close this paper by illustrating in Section 5.1 the workings of Sections 3 and 4 by using the data of Table 1 to assess the probability of encountering an IED on the next crossing on the "new bridge."

5.2. *The logistics network revisited.*   With respect to the network of Figure 1, the data of Table 1 maps to the matrix $\mathbf{D}^*$ of Section 4.1, with its column 2 corresponding to Y_l, $l = 1, \ldots, 12$, column 3 corresponding to $Z_{1,1}, \ldots, Z_{1,12}$, and so on, with column 6 corresponding to $Z_{4,1}, \ldots, Z_{4,12}$.

A logistic regression model

$$P(Y_l = 1; \boldsymbol{\beta}, \mathbf{Z}_l) = \frac{1}{1 + \exp(-\sum_{u=0}^{4} Z_{ul}\beta_u)}$$

for $l = 1, \ldots, 12$, with $Z_{0l} = 1$, was fitted to the data of Table 1 using independent Gaussian priors with means 0 and standard deviations 10. This choice of priors is arbitrary. The joint posterior distribution of $(\beta_0, \ldots, \beta_4)$ was obtained via Gibbs sampling with 10,000 simulations after a burn-in of 1,000 simulations.

The marginal posterior distributions of β_0, β_2, and β_4 were symmetric looking, but those of β_1 and β_3 were skewed to the left; plots of these distributions are not shown. Table 2 compares posterior means against their maximum likelihood estimates, showing a good agreement between the two, save for β_0.

About 60 samples from the joint posterior distribution of $(\beta_0, \ldots, \beta_4)$ were generated, and for each sample, the quantity $[1 + \exp(-\sum_{u=0}^{4} \beta_u)]^{-1}$ computed. Here $\mathbf{Z} = (1, 1, 1, 1)$, suggesting that the next crossing is to be on the new bridge which is one mile away from all the four city centers of interest. Associated with each generated sample is also the probability of the sample; this is provided by the joint

TABLE 2
Comparison of Bayes' versus maximum likelihood estimates

Approach	β_0	β_1	β_2	β_3	β_4
Bayes	0.635	1.583	3.584	4.382	1.579
Maximum likelihood	1.811	1.817	3.299	4.402	1.311

probability density. Figure 5 shows a plot of the computed quantity mentioned above [our $(p(\mathbf{Z}), \boldsymbol{\beta}^*)$ of Section 4.3] versus the joint probability. A smoothed plot, smoothed by a moving average of five consecutive points, is the Monte Carlo induced likelihood.

Since the new bridge has experienced 4 previous crossings and none of these crossings have experienced an IED attack, $\mathbf{x} = (0, 0, 0, 0)$; thus, $\mathcal{L}(p; \mathbf{x}) = (1 - p)^{\sqrt{2}}$, see equation (4.4). With the above in place, all the ingredients needed to compute $P(X = 1; \mathbf{x}; \mathbf{Z}, \mathbf{D}^*)$—equation (4.7)—are at hand, save for $\pi(p)$ the prior. Supposing $\pi(p)$ uniform on $(0, 1)$, we have

$$P(X = 1; \mathbf{x}; \mathbf{Z}, \mathbf{D}^*) \propto \int_0^1 p(1 - p)^{\sqrt{2}} \mathcal{L}(p; \mathbf{Z}, \mathbf{D}^*) \, dp,$$

with $\mathcal{L}(p; \mathbf{Z}, \mathbf{D}^*)$ given by the likes of Figure 5. This can be numerically evaluated for a range of p, say, $p = 0.05, 0.1, \ldots, 0.95, 1$, to obtain $P(X = 1; \mathbf{x}, \mathbf{Z}, \mathbf{D}^*) \propto 0.129$. Similarly, we obtain $P(X = 0; \mathbf{x}, \mathbf{Z}, \mathbf{D}^*) \propto 0.293$. The normalizing constant is 0.422, giving $P(X = 1; \mathbf{x}, \mathbf{Z}, \mathbf{D}^*) = 0.306$ and $P(X = 0; \mathbf{x}, \mathbf{Z}, \mathbf{D}^*) = 0.694$.
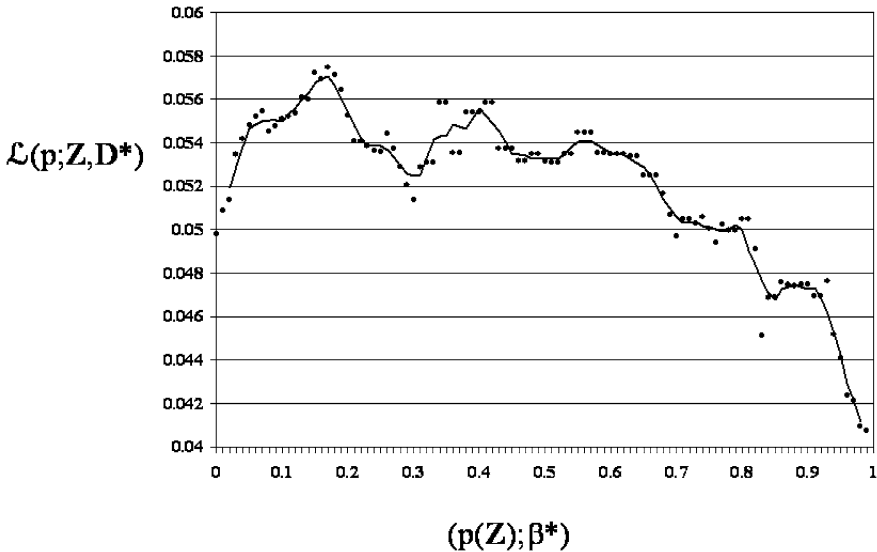

FIG. 5. *Monte Carlo induced likelihood function.*

Thus, the probability of encountering an IED on the next crossing on the "new bridge" is 0.306.

5.2.1. *Optimal route selection for logistical network.* In order to prescribe an optimal route for the network of Figure 1, we need to calculate the probability of encountering an IED on each of the remaining 9 links of the network in a manner akin to that given above for link 9, the "new bridge." This requires that we have the vectors **x** and **Z** for each of these links, where **x** is the historic IED experience for a link, and **Z** is the vector of covariates associated with the links. This we do not have and are unable to obtain for reasons of security. Consequently, and purely with the intent of illustrating how our decision theoretic framework can be put to work, we shall make some meaningful specifications about the $p(j)$'s, $j = 1, \ldots, 8, 10$. These will be based on the relative lengths of each link, relative to the length of link 9 for which $p(9)$ has been assessed as 0.306; that is, calibrate the required $p(j)$'s in terms of $p(9)$.

To do the above, we start by remarking that links 1 and 2 are of almost equal length, and are about two-thirds the length of link 9. Links 3 to 8 are of equal length and are about one-fifth the length of link 9, whereas link 10 is about half the length of link 9. Note that Figure 1 is not drawn to scale. Thus, we set $p(1) = p(2) = (0.66)(0.306) = 0.20$, $p(10) = (0.50)(0.306) = 0.15$ and $p(3) = p(4) = p(5) = p(6) = p(7) = p(8) = (0.20)(0.306) = 0.06$. These choices are purely illustrative; we could have used other methods of scaling such as the logarithmic or the square root.

In addition to specifying the $p(j)$'s, we also need to specify utilities. For this we propose a utility function of the form $1 - n/x$ for a successful route traversal. Here n is the number of links in the route, and x is a constant which ensures that a successful traversal does not result in a negative utility. Specifically, the idea here is that a successful traversal yields a utility of one, but each link in the route contributes to a disutility to which is assigned a weight $1/x$. Choice D_1 entails the route $(1, 2, 9)$ and with x chosen to be 100, the utility of a successful traversal on this route will be $1 - 3/100$. Similarly, the failure to achieve a successful traversal yields a utility of $0 - n/x$, yielding a negative utility of $-n/x$, which in the case of route $(1, 2, 9)$ with $x = 100$ is $-3/100$.

The above choices for utility do not take into consideration things such as composition of the convoys, traversal time, vicinity to hostile territory, costs of disruption, etc. With the above in place, and assuming independence of the IED placement events, it can be easily seen that the expected utilities of choices D_1, D_2, and D_3 are 0.414, 0.361, and 0.430, respectively. Thus, for the given choices of probabilities and utilities, $\mathcal{D}$'s optimal route will be D_3, which is $(1, 2, 3, 4, 10)$. Observe that neither the shortest nor the longest routes are optimal. Sensitivity of $\mathcal{D}$'s final choice to values of x other than 100 can be explored. For example, were x taken to be 10, then D_1 will turn out to be $\mathcal{D}$'s optimal choice. This is because it

turns out the probability of a successful traversal via choices D_1, D_2, and D_3 turns out to be rather close to each other, namely, 0.444, 0.441, and 0.480, respectively.

This completes our discussion on illustrating the workings of the proposed approach vis-à-vis the network of Figure 1, and closes the paper.

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