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HAMILTON CYCLES IN RANDOM GEOMETRIC GRAPHS

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We prove that, in the Gilbert model for a random geometric graph, almost every graph becomes Hamiltonian exactly when it first becomes 2-connected. This answers a question of Penrose.

We also show that in the k-nearest neighbor model, there is a constant κ such that almost every κ-connected graph has a Hamilton cycle.

1. Introduction. In this paper we mainly consider one of the frequently studied models for random geometric graphs, namely the Gilbert model. Suppose that S_n is a $\sqrt{n} \times \sqrt{n}$ box and that $\mathcal{P}$ is a Poisson process in it with density 1. The points of the process form the vertex set of our graph. There is a parameter r governing the edges: two points are joined if their (Euclidean) distance is at most r.

Having formed this graph we can ask whether it has any of the standard graph properties, such as connectedness. As usual, we shall only consider these for large values of n. More formally, we say that $G = G_{n,r}$ has a property *with high probability* (abbreviated to whp) if the probability that G has this property tends to one as n tends to infinity.

Penrose [10] proved that the threshold for connectivity is $\pi r^2 = \log n$. In fact he proved the following very sharp result: suppose $\pi r^2 = \log n + \alpha$ for some constant α. Then the probability that $G_{n,r}$ is connected tends to $e^{-e^{-\alpha}}$.

He also generalized this result to find the threshold for κ-connectivity for $\kappa \geq 2$: namely $\pi r^2 = \log n + (2\kappa - 3) \log \log n$. [Since the reader may be surprised that

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this formula does not work for $\kappa = 1$ we remark that this is due to boundary effects: the threshold for κ-connectivity is the maximum of two quantities: $\log n + (\kappa - 1)\log\log n$ to κ-connect the central points and $\log n + (2\kappa - 3)\log\log n$ to κ-connect the points near the boundary. If one worked on the torus instead of the square, then these boundary effects would disappear.]

Moreover, he found the "obstruction" to κ-connectivity. Suppose we fix the vertex set (i.e., the point set in S_n) and "grow" r. This gradually adds edges to the graph. For a monotone graph property P let $\mathcal{H}(P)$ denote the smallest r for which the graph on this point set has the property P. Penrose showed that

$$\mathcal{H}\bigl(\delta(G) \geq \kappa\bigr) = \mathcal{H}\bigl(\text{connectivity}(G) \geq \kappa\bigr)$$

whp: that is, as soon as the graph has minimum degree κ it is κ-connected whp.

He also considered the threshold for G to have a Hamilton cycle. Obviously a necessary condition is that the graph is 2-connected. In the normal (Erdős–Rényi) random graph this is also a sufficient condition in the following strong sense. If we add edges to the graph one at a time, then the graph becomes Hamiltonian exactly when it becomes 2-connected (see [5, 8, 9] and [14]).

Penrose asked whether the same is true for a random geometric graph. In this paper we prove the following theorem answering this question.

THEOREM 1. *Suppose that $G = G_{n,r}$ is the two-dimensional Gilbert model. Then*

$$\mathcal{H}(G \textit{ is } 2\textit{-connected}) = \mathcal{H}(G \textit{ has a Hamilton cycle})$$

whp.

Combining this with Penrose's results mentioned above we see that, if $\pi r^2 = \log n + \log\log n + \alpha$, then the probability that G has a Hamilton cycle tends to $e^{-e^{-\alpha} - \sqrt{\pi} e^{-\alpha/2}}$ (the second term in the exponent is the contribution from points near the boundary of the square).

Some partial progress has been made on this question previously. Petit [13] showed that if $\pi r^2 / \log n$ tends to infinity, then G is, whp, Hamiltonian, and Díaz, Mitsche and Pérez [7] proved that if $\pi r^2 > (1 + \varepsilon)\log n$ for some $\varepsilon > 0$ then G is Hamiltonian whp. (Obviously, G is not Hamiltonian if $\pi r^2 < \log n$ since whp G is not connected!) Finally using a similar method to [7] together with significant case analysis, Balogh, Kaul and Martin [4] proved for the special case of the ℓ_∞ norm in two dimensions that the graph does become Hamiltonian exactly when it becomes 2-connected.

Our proof generalizes to higher dimensions and to other norms. The Gilbert model makes sense with any norm and in any number of dimensions: we let S_n^d be the d-dimensional hypercube with volume n. We prove the analog of Theorem 1 in this setting.

THEOREM 2. *Suppose that the dimension $d \geq 2$ and $\|\cdot\|$, a p-norm for some $1 \leq p \leq \infty$, are fixed. Let $G = G_{n,r}$ be the resulting Gilbert model. Then*

$$\mathcal{H}(G \text{ is 2-connected}) = \mathcal{H}(G \text{ has a Hamilton cycle})$$

whp.

The proof is very similar to that of Theorem 1. However, there are some significant extra technicalities.

To give an idea why these occur consider connectivity in the Gilbert model in the cube S_n^3 (with the Euclidean norm). Let A be the volume of a sphere of radius r. We count the expected number of isolated points in the process which are away from the boundary of the cube. The probability a point is isolated is e^{-A}, so the expected number of such points is ne^{-A}, so the threshold for the existence of a central isolated point is about $A = \log n$.

However, consider the probability that a point near a face of the cube is isolated: there are approximately $n^{2/3}$ such points, and the probability that they are isolated is about $e^{-A/2}$ (since about half of the sphere about the point is outside the cube S_n^3). Hence, the expected number of such points is $n^{2/3}e^{-A/2}$, so the threshold for the existence of an isolated point near a face is about $A = \frac{4}{3}\log n$. In other words isolated points are much more likely near the boundary. These boundary effects are the reason for many of the extra technicalities.

We remark that Theorem 2 is trivially true for $d = 1$: indeed, if G is 2-connected then there are two vertex disjoint paths from the left-most vertex to the right-most vertex. By adding any remaining vertices to one of these paths these two paths form a Hamilton cycle.

The k-nearest neighbor model. We also consider a second model for random geometric graphs: namely the k-nearest neighbor graph. In this model the initial setup is the same as in the Gilbert model: the vertices are given by a Poisson process of density one in the square S_n, but this time each vertex is joined to its k nearest neighbors (in the Euclidean metric) in the box. This naturally gives rise to a k-regular directed graph, but we form a simple graph $G = G_{n,k}$ by ignoring the direction of all the edges. It is easily checked that this gives us a graph with degrees between k and $6k$.

Xue and Kumar [15] showed that there are constants c_1, c_2 such that if $k < c_1 \log n$, then the graph $G_{n,k}$ is, whp, not connected, and that if $k > c_2 \log n$ then $G_{n,k}$ is, whp, connected. Balister et al. [1] proved reasonably good bounds on the constants: namely $c_1 = 0.3043$ and $c_2 = 0.5139$, and later [3] proved that there is some critical constant c such that if $k = c' \log n$ for $c' < c$, then the graph is disconnected whp, and if $k = c' \log n$ for $c' > c$, then it is connected whp. Moreover, in [2], they showed that in the latter case the graph is s-connected whp for any fixed $s \in \mathbb{N}$.

We would like to prove a sharp result like the above; that is, that as soon as the graph is 2-connected it has a Hamilton cycle. However, we prove only the weaker statement that some (finite) amount of connectivity is sufficient. Explicitly, we show the following.

THEOREM 3. *Suppose that $k = k(n)$, that $G = G_{n,k}$ is the two-dimensional k-nearest neighbor graph* (*with the Euclidean norm*) *and that G is κ-connected for $\kappa = 5 \cdot 10^7$ whp. Then G has a Hamilton cycle whp.*

Analogous results could be proved in higher dimensions and for other norms but we do not do so here.

Binomial point process. To conclude this section we briefly mention a closely related model: instead of choosing the points in S_n according to a Poisson process of density one we choose n points uniformly at random, and then form the corresponding graph. This new model is very closely related to our first model (the Gilbert model). Indeed, Penrose originally proved his results for the Binomial Point Process but it is easy to check that this implies them for the Poisson Process.

It is very easy to modify our proof to this new model. Indeed, in very broad terms each of our arguments consists of two steps: first we have an essentially trivial lemma that says the random points are "reasonably" distributed, and then we have an argument saying that if the points are reasonably distributed and the resulting graph is two-connected then the resulting graph necessarily has a Hamilton cycle. The second of these steps is entirely deterministic, so only the essentially trivial lemma needs modifying.

2. Proof of Theorem 1. We divide the proof into five parts: first we tile the square S_n with small squares in a standard tessellation argument. Second we identify "difficult" subsquares. Roughly, these will be squares containing only a few points, or squares surrounded by squares containing only a few points. Third we prove some lemmas about the structure of the difficult subsquares. In stage 4 we deal with the difficult subsquares. Finally we use the remaining easy subsquares to join everything together.

Stage 1: *Tessellation.* Let $r_0 = \sqrt{(\log n)/\pi}$ (so $\pi r_0^2 = \log n$), and let r be the random variable $\mathcal{H}(G \text{ is 2-connected})$. Let $s = r_0/c = c'\sqrt{\log n}$ where c is a large constant to be chosen later (1000 will do). We tessellate the box S_n with small squares of side length s. Whenever we talk about distances between squares we will always be referring to the distance between their centers. Moreover, we will divide all distances between squares by s, so, for example, a square's four nearest neighbors all have distance one.

By Penrose's result [11] mentioned in the Introduction we may assume that $(1 - 1/2c)r_0 < r < (1 + 1/2c)r_0$: formally the collection of point sets which do not satisfy this has measure tending to zero as n tends to infinity, and we ignore this set.

Hence points in squares at distance $\frac{r-\sqrt{2}s}{s} \geq \frac{r_0-2s}{s} = c-2$ are always joined, and points in squares at distance $\frac{r+\sqrt{2}s}{s} \leq \frac{r_0+2s}{s} = c+2$ are never joined.

Stage 2: *The "difficult" subsquares*. We call a square *full* if it contains at least M points for some M to be determined later (10^7 will do), and *nonfull* otherwise. Let N_0 be the set of nonfull squares. We say two nonfull squares are joined if their ℓ_∞ distance is at most $4c-1$ and define $\mathcal{N}$ to be the collection of nonfull components.

First we bound the size of the largest component of nonfull squares (here, and throughout this paper, we use size to refer to the number of vertices in the component).

LEMMA 4. *For any M, the largest component of nonfull squares in the above tesselation has size at most*

$$U = \lceil \pi(c+2)^2 \rceil$$

whp.

Also, the largest component of nonfull squares including a square within c of the boundary of S_n has size at most $U/2$ whp. Finally, there is no nonfull square within distance Uc of a corner whp.

PROOF. We shall make use of the following simple result: suppose that G is any graph with maximal degree Δ, and v is a vertex in G. Then the number of connected subsets of size n of G containing v is at most $(e\Delta)^n$ (see, e.g., Problem 45 of [6]).

Hence, the number of potential components of size U containing a particular square is at most $(e(8c)^2)^U$ so, since there are less than n squares, the total number of such potential components is at most $n(e(8c)^2)^U$. The probability that a square is nonfull is at most $2s^{2M}e^{-s^2}/M!$. Hence, the expected number of components of size at least U is at most

$$n\left(2s^{2M}e^{-s^2}(e(8c)^2)/M!\right)^U \leq n\left(2(\log n)^M \frac{e(8c)^2}{M!}\right)^U \exp\left(-\frac{(c+2)^2 \log n}{c^2}\right),$$

which tends to zero as n tends to infinity; that is, whp, no such component exists.

For the second part there are at most $4c\sqrt{n}$ squares within distance c of the boundary of S_n, and the result follows as above.

Finally, there are only $4U^2c^2$ squares within distance Uc of a corner. Since the probability that a square is nonfull tends to zero we see that there is no such square whp. $\square$

Note that this is true independently of M which is important since we will want to choose M depending on U.

In the rest of the argument we shall assume that there is no nonfull component of size greater than U, no nonfull component of size $U/2$ within c of an edge and no nonfull square within Uc of a corner.

Between these components of nonfull squares there are numerous full squares. To define this more precisely let $\widehat{G}$ be the graph with vertex set the small squares, and where each square is joined to all others within $(c-2)$ of this square (in the Euclidean norm). Since the probability a square is in N_0 (i.e., is nonfull) is $1-o(1)$, the graph $\widehat{G} \setminus N_0$ has one giant component consisting of almost all the squares. We call this component *sea*. (We give an equivalent formal definition just before Corollary 8.)

The idea is that it is trivial to find a cycle visiting every point of the process in a square in the sea, and that we can extend this cycle to a Hamilton cycle by adding each nonfull component (and any full squares cut off by it) one at a time. However, it is easier to phrase the argument by starting with the difficult parts and then using the sea of full squares.

Stage 3: *The structure of the difficult subsquares*. Consider one component $N \in \mathcal{N}$ of the nonfull squares, and suppose that it has size u. By Lemma 4 we know $u < U$. We will also consider N_{2c}: the $2c$-blow-up of N: that is the set of all squares with ℓ_∞ distance at most $2c$ from a square in N.

Now some full squares may be cut off from the rest of the full squares by nonfull squares in N. More precisely the graph $\widehat{G} \setminus N$ has one component $A = A(N)$ consisting of all but at most a bounded number of squares (since we have removed at most U squares from $\widehat{G}$). We call A^c the *cutoff* squares.

We split the cutoff squares into two classes: those with a neighbor in A (in $\widehat{G}$) which we think of as being "close" to A, and the rest, which we shall call *far* squares. All the close squares must be in N (since otherwise they would be part of A). However, we do not know anything about the far squares: they may be full or nonfull. See Figure 1 for a picture.

LEMMA 5. *No two far squares are more than ℓ_∞ distance $c/10$ apart.*

REMARK. This does not say whp since we are assuming this nonfull component has size at most U.

PROOF. Suppose not.

Suppose, first, that no point of N is within c of the edge of S_n, and that the two far squares are at horizontal distance at least $c/10$. Then consider the left-most far square. All squares which are to the left of this and with distance to this square less than $(c-2)$ must be close and thus in N. Similarly with the right-most far square. Also at least $(c-2)$ squares [in fact nearly $2(c-2)$] in each of at least

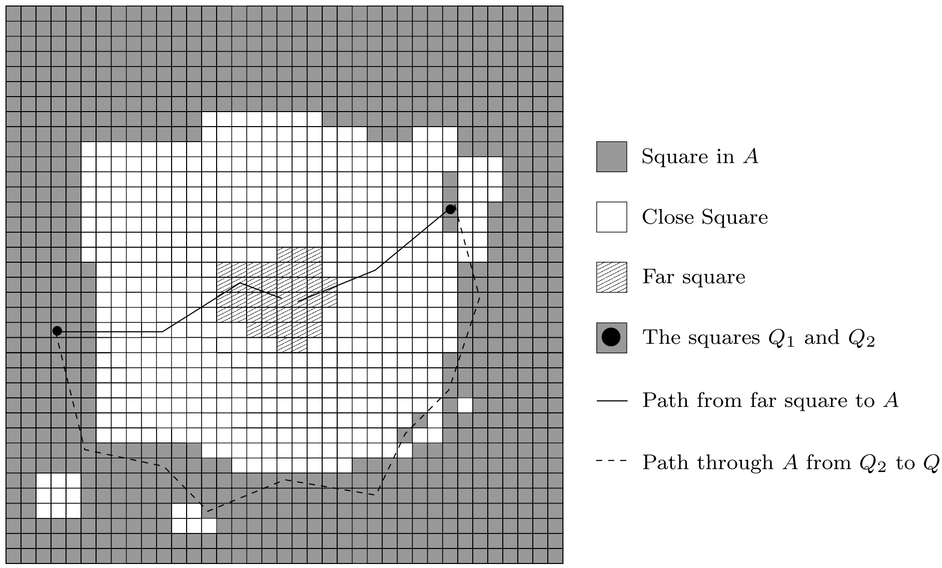

FIG. 1. *A small part of S_n containing the nonfull component N and the corresponding set A, far squares and close squares. It also shows the two vertex disjoint paths from the far squares to A and the path joining Q_2 to Q_1* (*see stage* 4).

$c/10$ columns between the original two far squares must be in N. This is a total of about $\pi(c-2)^2 + (c-2)c/10 > U$ which is a contradiction (provided we chose c reasonably large).

If there is a point of N within c of the boundary, then the above argument gives more than $U/2$ nonfull squares. Indeed, either it gives half of each part of the above construction, or it gives all of one end and all the side parts. This contradicts the second part of our assumption about the size of nonfull components.

We do not need to consider a component near two sides: it cannot be large enough to be near two sides. It also cannot go across a corner, since no square within distance Uc of a corner is nonfull. □

This result can also be deduced from a result of Penrose, as we do in the next section. We have the following instant corollary.

COROLLARY 6. *The graph $\widehat{G}$ restricted to the far squares is complete.*

COROLLARY 7. *The set of cutoff squares A^c is contained in N_c* (*the c-blow-up of N*). *In particular, the set $\Gamma(A^c)$ of neighbors in $\widehat{G}$ of A^c is contained in N_{2c}.*

PROOF. Suppose $A^c \not\subseteq N_c$. Let x be a square in $A^c \setminus N_c$. First, x cannot be a neighbor of any square in A or x would also be in A; that is, x is a far square.

Now, let y be any square with ℓ_∞ distance $c/5$ from x. The square y cannot be in N since then x would be in N_c. Therefore, y cannot be a neighbor of any square in A since then it would be in A and, since x and y are joined in $\widehat{G}$, x would be in A; that is, y is also a far square. Hence, x and y are both far squares with ℓ_∞ distance $c/5$ which contradicts Lemma 5. $\square$

In particular, Corollary 7 tells us that the sets of squares cutoff by different nonfull components and all their neighbors are disjoint (obviously the $2c$-blow-ups are disjoint).

We now formally define the *sea* $\widetilde{A} = \bigcap_{N \in \mathcal{N}} A(N)$. We show later (Corollary 11) that $\widetilde{A}$ is connected and, thus, that this is the same as our earlier informal definition. The following corollary is immediate from Corollary 7.

COROLLARY 8. *For any $N \in \mathcal{N}$ we have $\widetilde{A} \cap N_{2c} = A(N) \cap N_{2c}$.*

The final preparation we need is the following lemma.

LEMMA 9. *The set $N_{2c} \cap A$ is connected in $\widehat{G}$.*

Since the proof will be using a standard graph theoretic result, it is convenient to define one more graph $\widehat{G}_1$: again the vertex set is the set of small squares, but this time each square is joined only to its four nearest neighbors; that is, $\widehat{G}_1$ is the ordinary square lattice. We need two quick definitions. First, for a set $E \in \widehat{G}_1$ we define the *boundary* $\partial_1 E$ of E to be set of vertices in E^c that are neighbors (in $\widehat{G}_1$) of a vertex in E. Second, we say a set E in $\widehat{G}_1$ is *diagonally connected* if it is connected when we add the edges between squares which are diagonally adjacent (i.e., at distance $\sqrt{2}$) to $\widehat{G}$. The lemma we need is the following; since its proof is short we include it here for completeness. (It is also an easy consequence of the unicoherence of the square (see, e.g., page 177 of [12]).)

LEMMA 10. *Suppose that E is any subset of $\widehat{G}_1$ with E and E^c connected. Then $\partial_1 E$ is diagonally connected: in particular, it is connected in $\widehat{G}$.*

PROOF. Let F be the set of edges of $\widehat{G}_1$ from E to E^c, and let F' be the corresponding set of edges in the dual lattice. Consider the set F' as a subgraph of the dual lattice. It is easy to check that every vertex has even degree except vertices on the boundary of $\widehat{G}_1$. Thus we can decompose F' into pieces, each of which is either a cycle or a path starting and finishing at the edge of $\widehat{G}_1$. Any such cycle splits $\widehat{G}_1$ into two components, and we see that one of these must be exactly E and the other E^c. Thus F' is a single component in the dual lattice, and it is easy to check that implies that $\partial_1 E$ is diagonally connected. $\square$

PROOF OF LEMMA 9. Consider $\widehat{G}_1 \setminus N_{2c}$. This splits into components $B_1, B_2, \ldots, B_m$. By definition each B_i is connected. Moreover, each B_i^c is also connected. Indeed, suppose $x, y \in B_i^c$. Then there is an xy path in $\widehat{G}_1$. If this is contained in B_i^c we are done. If not then it must meet N_{2c}, but N_{2c} is connected. Hence we can take this path until it first meets N_{2c}, go through N_{2c} to the point where the path last leaves N_{2c} and follow the path on to y. This gives a path in B_i^c.

Hence, by Lemma 10, we see that each $\partial_1 B_i$ is connected in $\widehat{G}$ for each i (where ∂_1 denotes the boundary in $\widehat{G}_1$). Obviously $\partial_1 B_i \subset N_{2c}$.

As usual, for a set of vertices V let $\widehat{G}[V]$ denote the graph $\widehat{G}$ restricted to the vertices in V.

CLAIM. *Any two vertices in $\bigcup_{i=1}^m \partial_1 B_i$ are connected in $\widehat{G}[A \cap N_{2c}]$.*

PROOF. Suppose not. Without loss of generality assume that, for some $k < m$, $\widehat{G}[\bigcup_{i=1}^k \partial_1 B_i]$ is connected and that no other $\partial_1 B_i$ is connected via a path to it. Pick $x \in B_1$ and $y \in B_m$. Both x and y are in A (since they are not in N_{2c} and $A^c \subset N_{2c}$ by Corollary 7).

Hence there is a path from x to y in A. Consider the last time it leaves $\bigcup_{i=1}^k B_i$. The path then moves around in N_{2c} before entering some B_j with $j > k$. This gives rise to a path in $A \cap N_{2c}$ from a point in $\bigcup_{i=1}^k \partial_1 B_i$ to a point in $\partial_1 B_j$, contradicting the choice of k. □

We now complete the proof of Lemma 9. To avoid clutter we shall say that two points are *joined* if they are connected by a path. Suppose that $x, y \in A \cap N_{2c}$. Since A is connected there is a path in A from x to y. If the path is contained in N_{2c} we are done. If not, consider the first time the path leaves N_{2c}. It must enter one of the B_i, crossing the boundary $\partial_1 B_i$. Hence x is joined to some $w \in \partial_1 B_i$ in $A \cap N_{2c}$. Similarly, by considering the last time the path is not in N_{2c} we see that y is joined to some $z \in \partial_1 B_j$ for some j. However, since the claim showed that w and z are joined in $A \cap N_{2c}$, we see that x and y are joined in $A \cap N_{2c}$. □

COROLLARY 11. *The set of sea squares $\widetilde{A}$ is connected in $\widehat{G}$.*

PROOF. Given two squares x, y in $\widetilde{A}$, pick a path in $\widehat{G}$ from x to y. Now for each nonfull component N in turn do the following. If the path misses N_{2c} do nothing. Otherwise let w be the first point on the path in N_{2c} and z be the last point in N_{2c}. Replace the xy path by the path xw, any path wz in $A(N) \cap N_{2c}$ and then the path zy.

At each stage the modification ensured that the path now lies in $A(N)$. Also, the only vertices added to the path are in N_{2c} which is disjoint from all the previous N'_{2c}, and thus from all previous sets $A(N')$. Hence, when we have done this for all nonfull components the path lies in every $A(N')$, that is, in $\widetilde{A}$. Hence, $\widetilde{A}$ is connected. □

Stage 4: *Dealing with the difficult subsquares*. We deal with each nonfull component $N \in \mathcal{N}$ in turn. Fix one such component N.

Let us deal with the far squares first. There are three possibilities: the far squares contain no points at all, they contain one point in total or they contain more than one point. In the first case, do nothing and proceed to the next part of the argument.

In the second case, by the 2-connectivity of G, we can find two vertex disjoint paths from this single vertex v_1 to points in squares in A. In the third case pick two points v_1 and v_2 in the far squares. Again by 2-connectivity we can find vertex disjoint paths from these two vertices to points in squares in A.

Suppose that the path from v_1 meets A in square Q_1 at point q_1 and the other path (either from v_2 or the other path from v_1 again) meets A in square Q_2 at point q_2. Let P_1, P_2 be the squares containing the previous points on these paths. Since no two points in squares at (Euclidean) distance $(c+2)$ are joined we see that P_1 is within $(c+2)$ of Q_1. Since $P_1 \notin A$ we have that some square on a shortest P_1Q_1 path in $\widehat{G}_1$ is in N and thus that $Q_1 \in N_{2c}$. Similarly $Q_2 \in N_{2c}$. Combining we see that both Q_1 and Q_2 are in $N_{2c} \cap A$. By Lemma 9, we know that $N_{2c} \cap A$ is connected in $\widehat{G}$ so we can find a path from Q_1 to Q_2 in $N_{2c} \cap A$ in $\widehat{G}$. This "lifts" to a path in G going from q_2 to a point other than q_1 in Q_1 using at most one vertex in each subsquare on the way and never leaving N_{2c}.

Construct a path starting and finishing in Q_1 by joining together the following paths:

1. the path from q_1 to v_1;
2. a path starting at v_1 going round all points in the far region (except any such points on the q_1v_1 or q_2v_2 paths) finishing back at v_2. (Corollary 6 guarantees the existence of such a path.) We omit this piece if there is just one far vertex;
3. the path v_2 to q_2;
4. the path from q_2 through the sea back to Q_1 constructed above.

Since $Q_1 \in A \cap N_{2c}$, by Corollary 8 we have that $Q_1 \in \widetilde{A}$. Combining, we have a path starting and finishing in the same subsquare of the sea $\widetilde{A}$ (i.e., Q_1) containing all the vertices in the far region.

Next we deal with the close squares: we deal with each close square P in turn. Since P is a close square we can pick $Q \in A$ with PQ joined in $\widehat{G}$. In the following we ignore all points that we have used in the path constructed above and any points already used when dealing with other close squares.

If the square P has no point in it we ignore it. If it has one point in it, then join that point to two points in Q.

If it has two or more points in it then pick two of them x, y: and pick two points uv in Q (we choose M large enough to ensure that we can find these two unused points in Q, see below). Place the path formed by the edge ux round all the remaining unused vertices in the cutoff square finishing at y and back to the square Q with the edge yv in the cycle we are constructing.

The square Q is a neighbor of $P \in A^c$ so, by Corollary 7 is in N_{2c}. Since Q is also in A we see, by Corollary 8 as above, that $Q \in \widetilde{A}$.

When we have completed this construction we have placed every vertex in a cutoff square on one of a collection of paths, each of which starts and finishes at the same square in the sea (although different paths may start and finish in different squares in the sea).

We use at most $2U + 2$ vertices from any square in $A = A(N)$ when doing this, so, provided that $M > 2U + 2 + (2c + 1)^2$, there are at least $(2c + 1)^2$ unused vertices in each square of A when we finish this. Moreover, obviously the only squares touched by this construction are in N_{2c}, and for distinct nonfull components these are all disjoint. Hence, when we have done this for every nonfull component $N \in \mathcal{N}$ there are at least $(2c + 1)^2$ unused vertices in each square of the sea $\widetilde{A}$.

Stage 5: *Using the subsquares in the sea to join everything together.* It just remains to string everything together. This is easy. Since, by Corollary 11, the sea of squares $\widetilde{A}$ is connected, there is a spanning tree for $\widetilde{A}$. By doubling each edge we can think of this as a cycle, as in Figure 2. This cycle visits each square at most $(2c + 1)^2$ times. (In fact, by choosing a spanning tree such that the sum of the edge lengths is minimal we could assume that it visits each vertex at most six times but we do not need this.) Convert this into a Hamilton cycle as follows. Start at an unused vertex in a square of the sea. Move to any (unused) vertex in the next square in the tree cycle. Then, if this is the last time the tree cycle visits this square, visit all remaining vertices and join in all the paths constructed in the first part of the argument, then leave to the next square in the tree cycle. If it is not the last time the tree cycle visits this square, then move to any unused vertex in the next square

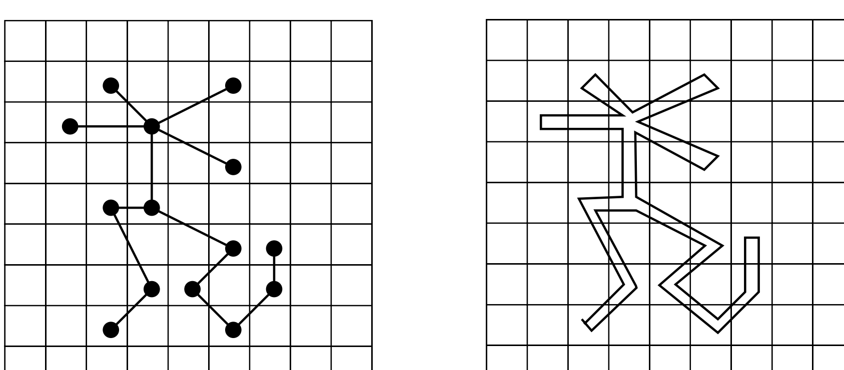

FIG. 2. *A tree of subsquares and its corresponding tree cycle.*

in the tree cycle. Repeat until we complete the tree cycle. Then join in any unused vertices and paths to this square constructed earlier before closing the cycle.

3. Higher dimensions. We generalise the proof in the previous section to higher dimensions and any p-norm. Much of the argument is the same, in particular, essentially all of stages four and five. We include details of all differences but refer the reader to the previous section where the proof is identical.

Stage 1: *Tessellation*. We work in the d-dimensional hypercube S_n^d of volume n (for simplicity we will abbreviate hypercube to cube in the following). As mentioned in the Introduction, we no longer have a nice formula for the critical radius: the boundary effects dominate.

Instead, we consider the expected number of isolated vertices $E = E(r)$. We need a little notation: let A_r denote the set $\{x \in S_n^d : d(x, A) \leq r\}$ and $|\cdot|$ denote Lebesgue measure.

We have $E = \int_{S_n^d} \exp(-|\{x\}_r|)\, dx$. Let $r_0 = r_0(n)$ be such that $E(r_0) = 1$. As before fix c a large constant to be determined later, and let $s = r_0/c$. It is easy to see that $r_0^d = \Theta(\log n)$ and $s^d = \Theta(\log n)$. We tile the cube S_n^d with small cubes of side length s.

As before, let $r = \mathcal{H}(G$ is 2-connected). By Penrose (Theorems 1.1 and 1.2 of [11] or Theorems 8.4 and 13.17 of [12]) the probability that $r \notin [r_0(1 - 1/2c), r_0(1 + 1/2c)]$ tends to zero and we ignore all these point sets. (Note that these two of Penrose's results are not claimed for $p = 1$. However, since for any $\varepsilon > 0$ we can pick $p > 1$ such that $B_1(r) \subset B_p(r) \subset B_1((1 + \varepsilon)r)$ [where $B_1(r)$ and B_p denote the l_1 and l_p balls of radius r, resp.], the above bound on r for $p = 1$ follows from Penrose's results for $p > 1$.)

This time any two points in cubes at distance $\frac{r - s\sqrt{d}}{s} \geq \frac{r_0 - ds}{s} = c - d$ are joined, and no points in cubes at distance $\frac{r + s\sqrt{d}}{s} \leq \frac{r_0 + ds}{s} = c + d$ are joined.

Stage 2: *The "difficult" subcubes*. Exactly as before we define nonfull cubes to be those containing at most M points, and we say two are joined if they have ℓ_∞ distance at most $4c - 1$.

We wish to prove a version of Lemma 4. However, we have several possible boundaries: for example, in three dimensions we have the center, the faces, the edges and the corners. We call a nonfull component containing a cube Q *bad* if it consists of at least $(1 + 1/c)|Q_{r_0}|/s^d$ cubes. (Note a component can be bad for some cubes and not others.)

LEMMA 12. *The expected number of bad components tends to zero as n tends to infinity. In particular there are no bad components whp.*

PROOF. The number of connected sets of size U containing a particular cube is at most $(e(8c)^d)^U$. The probability that a cube is nonfull is at most $2s^{dM}e^{-s^d}/M!$. Since $\min\{|Q_{r_0}|:\text{cubes } Q\} = \Theta(\log n)$ and $s^d = \Theta(\log n)$, the expected number of bad components is at most

$$\begin{aligned}
&\sum_{\text{cubes } Q} \big(2s^{dM}e^{-s^d}(e(8c)^d)/M!\big)^{(1+1/c)|Q_{r_0}|/s^d} \\
&\quad= \sum_{\text{cubes } Q} \big(2s^{dM}(e(8c)^d)/M!\big)^{(1+1/c)|Q_{r_0}|/s^d} \exp\big(-(1+1/c)|Q_{r_0}|\big) \\
&\quad= o(1) \sum_{\text{cubes } Q} \exp(-|Q_{r_0}|) \\
&\quad\le o(1) \int_{S_n^d} \exp(-|\{x\}_{r_0}|)\,dx \\
&\quad= o(1)E(r_0) \\
&\quad= o(1).
\end{aligned}$$ □

(Again, note that this is true independently of M.)

From now on we assume that there is no bad component.

Stage 3: *The structure of the difficult subcubes*. In this stage we will need one extra geometric result of Penrose, a case of Proposition 5.15 of [12] (see also Proposition 2.1 of [11]).

PROPOSITION 13. *Suppose d is fixed and that $\|\cdot\|$ is a p-norm for some $1 \le p \le \infty$. Then there exists $\eta > 0$ such that if $F \subset O^d$ (the positive orthant in $\mathbb{R}^d$) is compact with ℓ_∞ diameter at least $r/10$, and x is a point of F with minimal l_1 norm; then $|F_r| \ge |F| + |\{x\}_r| + \eta r^d$.*

We begin this stage by proving Lemma 5 for this model.

LEMMA 14. *No two far cubes are more than ℓ_∞ distance $c/10$ apart.*

PROOF. Suppose not. Then let F be the set of far cubes, let x be a point of F closest to a corner in the l_1 norm and let Q be the cube containing x (or any of the possibilities if it is on the boundary between cubes). We know that all the cubes within $(c-d)$ of a far cube are not in A. Hence all such cubes which are not far must be close, and thus nonfull.

The number of close cubes is at least

$$\begin{aligned}\frac{|F_{(c-2d)s} \setminus F|}{s^d} &\geq \frac{|\{x\}_{(c-2d)s}| + \eta((c-2d)s)^d}{s^d} && \text{by Proposition 13} \\ &\geq \frac{|Q_{(c-3d)s}| + \eta r_0^d/2}{s^d} && \text{provided } c \text{ is large enough} \\ &= \frac{|Q_{(1-3d/c)r_0}| + \eta r_0^d/2}{s^d} \\ &\geq \frac{(1-3d/c)^d |Q_{r_0}| + \eta r_0^d/2}{s^d} \\ &> \frac{(1+1/c)|Q_{r_0}|}{s^d} && \text{provided } c \text{ is large enough.}\end{aligned}$$

This shows that the component is bad which is a contradiction. □

Corollaries 6, 7 and 8 hold exactly as before. Lemma 9 also holds, we just need to replace Lemma 10 by the following higher-dimensional analogue. Note that, even in higher dimensions we say two squares are diagonally connected if their centers have distance $\sqrt{2}$.

LEMMA 15. *Suppose that E is any subset of $\widehat{G}_1$ with E and E^c connected. Then $\partial_1 E$ is diagonally connected: in particular, it is connected in $\widehat{G}$.*

REMARK. Again the final conclusion of connectivity in $\widehat{G}$ is an easy consequence of unicoherence, this time of the hypercube.

PROOF. Let I be a (diagonally connected) component of $\partial_1 E$. We aim to show the $I = \partial_1 E$ and, thus, that $\partial_1 E$ is diagonally connected.

CLAIM. *Suppose that C is any circuit in $\widehat{G}_1$. Then the number of edges of C with one end in E and the other end in I is even.*

PROOF. We say that a circuit is contractible to a single point using the following operations. First, we can remove an out and back edge. Second, we can do the following two-dimensional move. Suppose that two consecutive edges of the circuit form two sides of a square; then we can replace them by the other two sides of the square keeping the rest of the circuit the same. For example, we can replace $(x, y+1, \vec{z}) \to (x+1, y+1, \vec{z}) \to (x+1, y, \vec{z})$ in the circuit by $(x, y+1, \vec{z}) \to (x, y, \vec{z}) \to (x+1, y, \vec{z})$.

Next we show that C is contractible. Let $w(C)$ denote the weight of the circuit: that is, the sum of all the coordinates of all the vertices in C. We show that, if C is nontrivial, we can apply one of the above operations and reduce w. Indeed, let

v be a vertex on C with maximal coordinate sum, and suppose that v_- and v_+ are the vertices before and after v on the circuit. If $v_- = v_+$ then we can apply the first operation removing v and v_+ from the circuit which obviously reduces w. If not, then both v_- and v_+ have strictly smaller coordinate sums than v, and we can apply the second operation reducing w by two. We repeat the above until we reach the trivial circuit.

Now, let J be the number of edges of C with an end in each of E and I. The first operation obviously does not change the parity of J. A simple finite check yields the same for the second operation. Indeed, assume that we are changing the path from $(x, y+1), (x+1, y+1), (x+1, y)$ to $(x, y+1), (x, y), (x+1, y)$. Let F be the set of these four vertices. If no vertex of I is in F, then obviously J does not change. If there is a vertex of I in F, then, by the definition of diagonally connected, $F \cap I = F \cap \partial_1 E$. Hence the parity of J does not change. [It is even if $(x, y+1)$ and $(x+1, y)$ are both in E or both in E^c and odd otherwise.] □

Suppose that there is some vertex $v \in \partial_1 E \setminus I$ and that $u \in E$ is a neighbor of v. Let $y \in I$ and $x \in E$ be neighbors. Since E and E^c are connected we can find paths P_{xu} and P_{vy} in E and E^c, respectively. The circuit P_{xu}, uv, P_{vy}, yx contains a single edge from E to I which contradicts the claim. □

To complete this stage observe that Corollary 11 holds as before.

Stage 4: *Dealing with the difficult subcubes*, *and Stage* 5: *Using the subcubes in the sea to join everything together.* These two stages go through exactly as before [with one trivial change: replace $(2c+1)^2$ by $(2c+1)^d$]. This completes the proof of Theorem 2.

4. Proof of Theorem 3. In this section we prove Theorem 3. Once again, the proof is very similar to that in Section 2. We shall outline the key differences, and emphasise why we are only able to prove the weaker version of the result.

Stage 1: *Tessellation.* The tessellation is similar to before, but this time some edges may be much longer than some nonedges.

Let $k = \mathcal{H}(G \text{ is } \kappa\text{-connected})$ be the smallest k that $G_{n,k}$ is κ-connected. Since G is connected we may assume that $0.3 \log n < k < 0.52 \log n$ (see [1] and [2]). Let r_- be such that any two points at distance r_- are joined whp; for example, Lemma 8 of [1] implies that this is true provided $\pi r_-^2 \le 0.3 e^{-1-1/0.3} \log n$, so we can take $r_- = 0.035\sqrt{\log n}$.

Let r_+ be such that no edge in the graph has length more than r_+. Then, again by Lemma 8 of [1], we have

$$\pi r_+^2 \le 4e(1+0.52) \log n$$

whp, so we can take $r_+ = 2.3\sqrt{\log n} \le 66 r_-$.

From here on, we ignore all point sets with an edge longer than r_+ or a nonedge shorter than r_-.

Let $s = r_-/\sqrt{8}$. We tessellate the box S_n with small squares of side length s. (Since we are proving only this weaker result our tesselation does not need to be very fine.) By the choice of s and the bound on r_- any two points in neighboring or diagonally neighbouring squares are joined in G. Also, by the bound on r_+ no two points in squares with centers at distance more than $(66\sqrt{5}+2)s < 150s$ are joined. Let $D = 10^4$; we have that no two points in squares with centers distance Ds apart are joined.

Stage 2: *The "difficult" subsquares*. We call a square *full* if it contains at least $M = 10^9$ points and *nonfull* otherwise. We say two nonfull squares are joined if they are at ℓ_∞ distance at most $2D-1$.

First we bound the size of the largest component of nonfull squares.

LEMMA 16. *The largest component of nonfull squares has size less than* 7000 *whp*.

PROOF. The number of connected subgraphs of $\widehat{G}$ of size 7000 containing a particular square is at most $(e(4D)^2)^{7000}$, so, since there are less than n squares, the total number of such connected subgraphs is at most $n(e(4D)^2)^{7000}$. The probability that a square is nonfull is at most $2s^{2M}e^{-s^2}/M!$. Hence, the expected number of components of nonfull squares of size at least 7000 is at most

$$n\big(2s^{2M}e^{-s^2}(e(4D)^2)/M!\big)^{7000}$$
$$\leq n\bigg(2\bigg(\frac{(0.035)^2\log n}{8}\bigg)^M\frac{e(4D)^2}{M!}\bigg)^{7000}\exp\bigg(\frac{-7000(0.035)^2\log n}{8}\bigg),$$

which tends to zero as n tends to infinity [since $7000(-0.035)^2/8 > 1.07 > 1$]; that is, whp, no such component exists. $\square$

In the rest of the argument we shall assume that there is no nonfull component of size greater than 7000.

Stage 3: *The structure of the difficult subsquares*. As usual we fix one component N of the nonfull squares, and suppose that it has size u (so we know $u < 7000$). This time we define $\widehat{G}$ to be the graph on the small squares where each square is joined to its eight nearest neighbors (i.e., adjacent and diagonal). Let $A = A(N)$ be the giant component of $G \setminus N$, and again split the cutoff squares into close and far depending whether they have a neighbor (in $\widehat{G}$) in A.

By the vertex isoperimetric inequality in the square there are at most $u^2/2$ squares in $A^c \setminus N$ so $|A^c| \leq u^2/2 + u < 2.5 \cdot 10^7$.

Next we prove a result similar to Corollary 7.

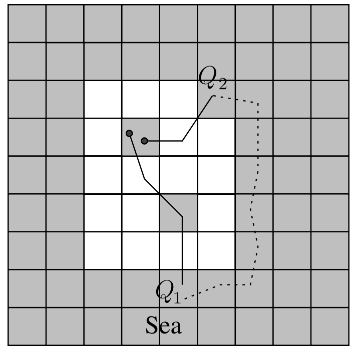

FIG. 3. *Two paths from one cutoff square to the sea together with the path from the meeting point in Q_2 to the square Q_1.*

LEMMA 17. *The set of cutoff squares A^c is in N_D (where $D = 10^4$ as above).*

PROOF. Suppose not, and that Q is a square in A^c not in N_D. Then all squares within ℓ_∞ distance of Q at most D are not in N. Hence they must be in A^c (since otherwise there would be a path from Q to a square in A not going through any square in N). Hence $|A^c| > D^2 = 10^8$ which contradicts Lemma 16. □

Finally, we need the analogue of Lemma 9 whose proof is exactly the same.

LEMMA 18. *The set $N_D \cap A$ is connected in $\widehat{G}$.*

Stage 4: *Dealing with the difficult subsquares*. Let us deal with these cutoff squares now. From each cutoff square that contains at least two vertices, pick any 2 vertices, and from each cutoff square that contains a single vertex pick that vertex with multiplicity two. We have picked at most $5 \cdot 10^7$ vertices, so since G is $\kappa = 5 \cdot 10^7$ connected we can simultaneously find vertex disjoint paths from each of our picked vertices to vertices in squares in A (two paths from those vertices that are repeated).

We remark that these are not just single edges; these paths may go through other cutoff squares.

Call the first point of such a path which is in A a *meeting point*, and the square containing this point a *meeting* square.

Fix a cutoff square and let v_1, v_2 be the two vertices picked above from this square (let $v_1 = v_2$ if the square only contains one vertex). This cutoff square has two meeting points, say q_1 and q_2 in subsquares Q_1 and Q_2, respectively. Since the longest edge is at most r_+, both Q_1 and Q_2 are in N_D. Since $A \cap N_D$ is connected in $\widehat{G}$ we construct a path in the squares in $A \cap N_D$ from the meeting point in Q_2 to a vertex in Q_1 using at most one vertex in each subsquare on the way, and missing all the other meeting points. This is possible since each full square contains at least $M = 10^9$ vertices.

Construct a path starting and finishing in Q_1 containing all the (unused) vertices in this cutoff square by joining together the following paths:

1. the path from q_1 to v_1;
2. a path starting at v_1 going round all points in the cutoff square finishing back at v_2 (omit this piece if there is just one far vertex);
3. the path v_2 to q_2;
4. the path from q_2 through $A \cap N_D$ back to Q_1 constructed above.

Do this for every cutoff square. For each cutoff square this construction uses at most two vertices from any square in A. Moreover, it obviously only touches squares in N_D. Since nonfull squares in distinct components are at distance at least $2D$ the squares touched by different nonfull components are distinct. Thus in total we have used at most $4 \cdot 10^7$ vertices in any square in the sea, and since $M = 10^9$ there are many (we shall only need 8) unused vertices left in each full square in the sea.

Stage 5: *Using the subsquares in the sea to join everything together.* This is exactly the same as before.

5. Comments on the k-nearest neighbor proof. We start by giving some reasons why the proof in the k-nearest neighbor model only yields the weaker Theorem 3. The first superficial problem is that we use squares in the tesselation which are of "large" size rather than relatively small as in the proof of Theorem 1, (in other words we did not introduce the constant c when setting s depending on r).

Obviously we could have introduced this constant. The difficulty when trying to mimic the proof of Theorem 1 is the large difference between r_- and r_+, which corresponds to having a very large number of squares (many times πc^2) in our nonfull component N. This means that we cannot easily prove anything similar to Lemma 5. Indeed, a priori, we could have two far squares with πc^2 nonfull squares around each of them.

A different way of viewing this difficulty is that, in the k-nearest neighbor model, the graph $\widehat{G}$ on the small squares does not approximate the real graph G very well, whereas in the Gilbert model it is a good approximation. Thus, it is not surprising that we only prove a weaker result.

This is typical of results about the k-nearest neighbor model; the results tend to be weaker than for the Gilbert model. This is primarily because the obstructions tend to be more complex; for example, the obstruction for connectivity in the Gilbert model is the existence of an isolated vertex. Obviously in the k-nearest neighbor model we never have an isolated vertex; the obstruction must have at least $k + 1$ vertices.

Extensions of Theorem 3. When proving Theorem 3 we only used two facts about the random geometric graph. First, that any two points at distance $r_- = 0.035\sqrt{\log n}$ are joined whp. Secondly, that the ratio of r_+ (the longest edge) to r_- (the shortest nonedge) was at most 60 whp. Obviously, we could prove the theorem (with different constants) in any graph with $r_- = \Theta(\sqrt{\log n})$ and r_+/r_- bounded. This includes higher dimensions and different norms and to different shaped regions instead of S_n (e.g., to disks or toruses). Indeed, the only place we used the norm was in obtaining the bounds on r_+ and r_- in stage 1 of the proof.

Indeed, it also generalizes to irregular distributions of vertices provided that the above bounds on r_- and r_+ hold. For example, it holds in the square S_n where the density of points in the Poisson Process decrease linearly from 10 to 1 across the square.

6. Closing remarks and open questions.

A related model where the result does not seem to follow easily from our methods is the directed version of the k-nearest neighbor graph. As mentioned above, the k-nearest neighbor model naturally gives rise to a directed graph, and we can ask whether this has a directed Hamilton cycle. Note that this directed model is significantly different from the undirected. For example, it is likely (see [1]) that the obstruction to directed connectivity (i.e., the existence of a directed path between any two vertices) is a single vertex with in-degree zero; obviously this cannot occur in the undirected case where every vertex has degree at least k. In some other random graph models a sufficient condition for the existence of a Hamilton cycle (whp) is that there are no vertices of in-degree or out-degree zero. Of course, in the directed k-nearest neighbor model every vertex has out-degree k so we ask the following question.

QUESTION. *Let $\vec{G} = \vec{G}_{n,k}$ be the directed k-nearest neighbor model. Is*

$$\mathcal{H}(\vec{G} \text{ has a Hamilton cycle}) = \mathcal{H}(\vec{G} \text{ has no vertex of in-degree zero})$$

whp?

It is obvious that the bound on connectivity in the k-nearest neighbor model can be improved, but the key question is "should it be two?" We make the following natural conjecture:

CONJECTURE. *Suppose that $k = k(n)$ such that the k-nearest neighbor graph $G = G(k, n)$ is a 2-connected whp. Then, whp, G has a Hamilton cycle.*

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