The Annals of Applied Probability
2011, Vol. 21, No. 2, 464–483
DOI: 10.1214/10-AAP712

NONNORMAL APPROXIMATION BY STEIN'S METHOD OF EXCHANGEABLE PAIRS WITH APPLICATION TO THE CURIE–WEISS MODEL

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Let (W, W') be an exchangeable pair. Assume that

$$E(W - W'|W) = g(W) + r(W),$$

where $g(W)$ is a dominated term and $r(W)$ is negligible. Let $G(t) = \int_0^t g(s)\,ds$ and define $p(t) = c_1 e^{-c_0 G(t)}$, where c_0 is a properly chosen constant and $c_1 = 1/\int_{-\infty}^{\infty} e^{-c_0 G(t)}\,dt$. Let Y be a random variable with the probability density function p. It is proved that W converges to Y in distribution when the conditional second moment of $(W - W')$ given W satisfies a law of large numbers. A Berry–Esseen type bound is also given. We use this technique to obtain a Berry–Esseen error bound of order $1/\sqrt{n}$ in the noncentral limit theorem for the magnetization in the Curie–Weiss ferromagnet at the critical temperature. Exponential approximation with application to the spectrum of the Bernoulli–Laplace Markov chain is also discussed.

1. Introduction and main results. Let W be the random variable of interest. Typical examples of W include the partial sum of independent random variables and functionals of independent random variables or dependent random variables whose joint distribution is known. Since the exact distribution of W is not available for most cases, it is natural to seek the asymptotic distribution of W with a Berry–Esseen type error. Let (W, W') be an exchangeable pair. Assume that

$$E(W - W'|W) = g(W) + r(W), \tag{1.1}$$

where $g(W)$ is a dominated term while $r(W)$ is a negligible term. When $g(W) = \lambda W$, and $E((W' - W)^2|W)$ is concentrated around a constant, Stein's method for normal approximation shows that the limiting distribution of W is normal under certain regularity conditions. We refer to Stein (1986), Rinott and Rotar (1997), Chen and Shao (2005) and references therein for the general theory of Stein's method. The main aim of this paper is to find the limiting distribution of W as well

Received December 2008; revised April 2010.

[1]Supported by NSF Grant DMS-07-07054 and a Sloan Research Fellowship.

[2]Supported in part by Hong Kong RGC 602206, 602608 and RGC 603710.

MSC2010 subject classifications. Primary 60F05; secondary 60G09.

Key words and phrases. Stein's method, exchangeable pair, Berry–Esseen bound, Curie–Weiss model.

as the rate of convergence for general g. The key step is to identify the limiting density function. As soon as the limiting density function is determined, we can follow the idea of the Stein's method of exchangeable pairs for normal approximation. Let

$$G(t) = \int_0^t g(s)\,ds \quad \text{and} \quad p(t) = c_1 e^{-c_0 G(t)}, \tag{1.2}$$

where $c_0 > 0$ is a constant that will be specified later and $c_1 = 1/\int_{-\infty}^{\infty} e^{-c_0 G(t)}\,dt$ is the normalizing constant. Let Y be a random variable with the probability density function p. Set:

(H1) $g(t)$ is nondecreasing, and $g(t) \ge 0$ for $t > 0$ and $g(t) \le 0$ for $t \le 0$;
(H2) there exists $c_2 < \infty$ such that for all x,

$$\min\bigl(1/c_1, 1/|c_0 g(x)|\bigr)(|x| + 3/c_1)\max(1, c_0|g'(x)|) \le c_2;$$

(H3) there exists $c_3 < \infty$ such that for all x,

$$\min\bigl(1/c_1, 1/|c_0 g(x)|\bigr)(|x| + 3/c_1)c_0|g'(x)| \le c_3.$$

Let $\Delta = W - W'$. Our main result shows that W converges to Y in distribution as long as $c_0 E(\Delta^2|W)$ satisfies a law of large numbers.

THEOREM 1.1. *Let h be absolutely continuous with $\|h'\| = \sup_x |h'(x)| < \infty$.*

(i) *If* (H1) *and* (H2) *are satisfied*, *then*

$$\begin{aligned} &|Eh(W) - Eh(Y)| \\ &\quad \le \|h'\| \biggl\{ \frac{(1+c_2)}{c_1} E|1 - (c_0/2)E(\Delta^2|W)| \\ &\qquad + \frac{1}{2}c_0(1+c_2)E|\Delta|^3 + c_0 c_2 E|r(W)| \biggr\}. \end{aligned} \tag{1.3}$$

(ii) *If* (H1) *and* (H3) *are satisfied*, *then*

$$\begin{aligned} &|Eh(W) - Eh(Y)| \\ &\quad \le \|h'\| \biggl\{ \frac{(1+c_3)}{c_1} E|1 - (c_0/2)E(\Delta^2|W)| + \frac{1}{2}c_0(1+c_3)E|\Delta|^3 \\ &\qquad + \frac{c_0}{c_1} E\biggl(\biggl(|W| + \frac{3}{c_1}\biggr)|r(W)|\biggr) \biggr\}. \end{aligned} \tag{1.4}$$

When Δ is bounded, next theorem gives a Berry–Esseen type inequality.

THEOREM 1.2. *Assume that* $|W - W'| \le \delta$, *where* δ *is a constant. If* (H1) *and* (H3) *are satisfied, then*

$$\begin{aligned}
&|P(W \le z) - P(Y \le z)| \\
&\quad \le 3E|1 - (c_0/2)E(\Delta^2|W)| + c_1 \max(1, c_3)\delta + 2c_0 E|r(W)|/c_1 \\
&\qquad + \delta^3 c_0\{(2 + c_3/2)E|c_0 g(W)| + c_1 c_3/2\}.
\end{aligned} \tag{1.5}$$

We remark that c_0 can be chosen as follows. In order to make the error term on the right-hand side of (1.3) small, it is necessary that $E|1 - (c_0/2)E(\Delta^2|W)| \to 0$ and therefore $E(1 - (c_0/2)E(\Delta^2|W))$ must be small and we should choose c_0 so that $c_0 \sim 2/E(\Delta^2)$.

The paper is organized as follows. In Section 2, we give a concrete application of our general result to the magnetization of the Curie–Weiss model of ferromagnets at the critical temperature, and show that the rate of convergence achieves $O(n^{-1/2})$. In Section 3, we focus on approximation by the exponential distribution with an application to the spectrum of the Bernoulli–Laplace Markov chain. We present a general approach of Stein's method of exchangeable pairs in Section 4 and postpone detailed proofs of our main results to Section 5.

2. Curie–Weiss model.

Consider the Curie–Weiss model for n spins at temperature T, that is, the probability distribution on $\{-1, 1\}^n$ that puts mass

$$Z_T^{-1} \exp\left(\frac{\sum_{1\le i<j\le n} \sigma_i \sigma_j}{Tn}\right)$$

at $\sigma \in \{-1, 1\}^n$, where Z_T is the normalizing constant. Let us fix $T = 1$, which is the "critical temperature" for this model. Now let

$$W = W(\sigma) = n^{-3/4} \sum_{i=1}^{n} \sigma_i.$$

This is a simple statistical mechanical model of ferromagnetic interaction, sometimes called the Ising model on the complete graph. For a detailed mathematical treatment of this model, we refer to the book by Ellis (1985).

Following ideas in Simon and Griffiths (1973), it was proved by Ellis and Newman (1978a, 1978b) that as $n \to \infty$, the law of W converges to the distribution with density proportional to $e^{-x^4/12}$. For various interesting extensions and refinements of their results, let us refer to Ellis, Newman and Rosen (1980) and Papangelou (1989).

Below, we present a Berry–Esseen bound for this noncentral limit theorem obtained via Theorem 1.2. Incidentally, Theorem 1.2 can also be used to obtain similar error bounds for the other limit theorems in the aforementioned papers (in particular, the Curie–Weiss model at noncritical temperatures), but we prefer to stick to this example only, since it is probably the most interesting and relevant one.

Given a random element σ, construct σ' by choosing a coordinate I at random and replacing σ_I by σ'_I, where σ'_I is generated from the conditional distribution of σ_I given $(\sigma_j)_{j\neq I}$. In other words, we take one step of the Glauber dynamics. It is easy to see that (σ, σ') is an exchangeable pair. Let $W' = W(\sigma')$. We shall show that (see Section 5)

$$E\left|E(W - W'|W) - \tfrac{1}{3}n^{-3/2}W^3\right| = O(n^{-2}), \tag{2.1}$$

$$E\left|E\big((W' - W)^2|W\big) - 2n^{-3/2}\right| = O(n^{-2}), \tag{2.2}$$

$$|W' - W| = O(n^{-3/4}) \tag{2.3}$$

and

$$E|W|^3 = O(1). \tag{2.4}$$

Let us now explain roughly how we arrive at (2.1), which is the most important step. A simple computation shows that at any temperature,

$$E(W - W'|W) = n^{-3/4}\big(m - \tanh(m/T)\big) + O(n^{-2}),$$

where $m := n^{-1/4}W$ is the magnetization. Since $m \simeq 0$ with high probability when $T \geq 1$, and $\tanh x = x - x^3/3 + O(x^5)$ for $x \simeq 0$, we see that the right-hand side in the above equation is like $n^{-3/4}m(1 - 1/T)$ when $T > 1$, while it is like $n^{-3/4}m^3/3$ when $T = 1$. This is what distinguishes between the high temperature regime $T > 1$ and the critical temperature $T = 1$, and this is how we arrive at (2.1).

Let

$$g(w) = \tfrac{1}{3}n^{-3/2}w^3, \qquad c_0 = n^{3/2}, \qquad \delta = O(n^{-3/4}).$$

Then

$$G_1(w) = c_0\int_0^w g(t)\,dt = w^4/12.$$

With the above information, it can be easily checked that by Theorem 1.2, we get the following theorem.

THEOREM 2.1. *Let Y be a random variable with density function*

$$p(w) = c_1 e^{-w^4/12} \qquad \text{where } c_1 = \frac{1}{\int_{-\infty}^{\infty} e^{-w^4/12}\,dw} = \frac{2^{1/2}}{3^{1/4}\Gamma(1/4)}.$$

Then for all z,

$$|P(W \leq z) - P(Y \leq z)| \leq cn^{-1/2}, \tag{2.5}$$

where c is an absolute constant.

Incidentally, after this manuscript was submitted, it was brought to our attention that an article by Eichelsbacher and Löwe (2009) was in preparation, where the same result (Theorem 2.1) is proved, along the same lines as our proof. Eichelsbacher and Löwe (2009) has generalizations of Theorem 2.1 to some other mean-field models.

3. Exponential limit with application to spectrum of the Bernoulli–Laplace Markov chain. In this section, we focus on the exponential limit. Let (W, W') be an exchangeable pair satisfying

$$E(W - W'|W) = 1/c_0 + r(W), \tag{3.1}$$

where $c_0 > 0$ is a constant. Let $\Delta = W - W'$. As a special case of Theorems 1.1 and 1.2 with a constant function g, we have

THEOREM 3.1. *Let Y have the exponential distribution with mean* 1. *Assume* (3.1) *is satisfied.*

(i) *Let h be absolutely continuous with $\|h'\| < \infty$. Then*:

$$\begin{aligned} &|Eh(W) - Eh(Y)| \\ &\quad \le \|h'\|\{E|1 - (c_0/2)E(\Delta^2|W)| + c_0 E|\Delta|^3 + 3c_0 E|Wr(W)|\}. \end{aligned} \tag{3.2}$$

(ii) *If $|\Delta| \le \delta$ for some constant δ, then*

$$\begin{aligned} &|P(W \le z) - P(Y \le z)| \\ &\quad \le 3E|1 - (c_0/2)E(\Delta^2|W)| + \delta + 2c_0\delta^3 + 3c_0 E|Wr(W)|. \end{aligned} \tag{3.3}$$

We refer to Chatterjee, Fulman and Röllin (2008) and Peköz and Röllin (2009) for other general results for the exponential approximation.

We now apply Theorem 3.1 to the spectrum of the Bernoulli–Laplace Markov chain, a simple model of diffusion, following the work of Chatterjee, Fulman and Röllin (2008). Two urns contain n balls each. Initially the balls in each urn are all of a single color, with urn 1 containing all white balls, and urn 2 all black. At each stage, a ball is picked at random from each urn and the two are switched. Let the state of the chain be the number of white balls in the urn 1. Diaconis and Shahshahani (1987) proved that $(n/4)\log(2n) + cn$ steps suffice for this process to reach equilibrium, in the sense that the total variation distance to the stationary distribution is at most ae^{-dc} for positive universal constants a and d. In order to prove this, they used the fact that the spectrum of the Markov chain consists of the numbers

$$\lambda_i = 1 - i(2n - i + 1)/n^2 \qquad \text{for } i = 0, 1, \dots, n, \tag{3.4}$$

occurring with multiplicities

$$m_i = \binom{2n}{i} - \binom{2n}{i-1} \qquad \text{for } i = 0, 1, \dots, n.$$

Let I have distribution $P(I = i) = \pi_i$, where

$$\pi_i = \frac{\binom{2n}{i} - \binom{2n}{i-1}}{\binom{2n}{n}}$$

for $0 \le i \le n$. Then λ_I is a random eigenvalue chosen from $\{\lambda_i, 0 \le i \le n\}$ in proportion to their multiplicities. Hora (1998) proved that $W = n\lambda_I + 1$ converges in distribution to an exponential random variable with mean 1.

Noting that $n\lambda_i + 1 = (n-i)(n+1-i)/n := \mu_i$, we can rewrite $W = \mu_I$. To apply Theorem 3.1, we construct an exchangeable pair (W, W') using a reversible Markov chain on $\{0, 1, \dots, n\}$ with transition probability matrix K satisfying

$$\pi(i)K(i, j) = \pi(j)K(j, i) \qquad \text{for all } i, j \in \{0, 1, \dots, n\}.$$

Given such a K, we obtain the pair (W, W') by letting $W = u_I$ where I is chosen from the equilibrium distribution π, and $W' = \mu_J$ where J is determined by taking one step from state I according to the transition probability K. As proved in Chatterjee, Fulman and Röllin (2008), we have (with $\Delta = W - W'$)

$$E(\Delta|W) = \frac{1}{2n^2} - \frac{n+1}{2n^2} I_{\{W=0\}}, \qquad E(W) = 1,$$
$$E(\Delta^2|W) = \frac{1}{n^2} \quad \text{and} \quad E|\Delta|^3 \le 6n^{-5/2}.$$

Now applying Theorem 3.1, we have the following theorem.

THEOREM 3.2. *Let Y have the exponential distribution with mean* 1 *and h be absolutely continuous with $\|h'\| < \infty$. Then*

$$|Eh(W) - Eh(Y)| \le 12n^{-1/2}. \tag{3.5}$$

As the difference between W and W' is large when I is small, Theorem 3.1 does not provide a useful Berry–Esseen type bound. However, using a completely different approach and some heavy machinery, Chatterjee, Fulman and Röllin (2008) are able to show that

$$\sup_z |P(W \le z) - P(Y \le z)| \le Cn^{-1/2},$$

where C is a universal constant.

4. The Stein method via density approach. Let p be a strictly positive, absolutely continuous probability density function, supported on (a,b), where $-\infty \le a < b \le \infty$. Assume that a right limit $p(a+)$ at a and a left limit $p(b-)$ exist. Let p' be a version of the derivative of p and assume that

$$\int_a^b |p'(t)|\,dt < \infty.$$

Let Y be a random variable with the probability density function p. In this section, we develop the Stein method via density approach. The approach was developed in Stein et al. (2004), but the properties presented in Section 4.2 are new.

4.1. *The Stein identity and equation.* A key step is to have Stein's identity and Stein's equation. Let $\mathcal{D}$ be the set of bounded, absolutely continuous functions f with $f(b-) = f(a+) = 0$. Observe that for any $f \in \mathcal{D}$

$$\begin{aligned} E\{f'(Y) + f(Y)p'(Y)/p(Y)\} &= E\{(f(Y)p(Y))'/p(Y)\} \\ &= \int_a^b (f(y)p(y))'\,dy = 0. \end{aligned} \tag{4.1}$$

The Stein identity is

$$Ef'(Y) + Ef(Y)p'(Y)/p(Y) = 0 \qquad \text{for } f \in \mathcal{D}. \tag{4.2}$$

For any measurable function h with $E|h(Y)| < \infty$, let $f = f_h$ be the solution to Stein's equation

$$f'(w) + f(w)p'(w)/p(w) = h(w) - Eh(Y). \tag{4.3}$$

It follows from (4.3) that

$$(f(w)p(w))' = \bigl(h(w) - Eh(Y)\bigr)p(w)$$

and hence

$$\begin{aligned} f(w) &= 1/p(w)\int_a^w \bigl(h(t) - Eh(Y)\bigr)p(t)\,dt \\ &= -1/p(w)\int_w^b \bigl(h(t) - Eh(Y)\bigr)p(t)\,dt. \end{aligned} \tag{4.4}$$

Note that $f_h \in \mathcal{D}$.

Consider two classes of density functions. The first one is the family of exponential distributions. It is easy to see that if Y has the exponential distribution with parameter λ, that is, Y is a random variable with density function $p(x) = \lambda e^{-\lambda x}$ for $x > 0$ and $p(x) = 0$ for $x \le 0$. Then $p'(x)/p(x) = -\lambda$ and the Stein identity (4.2) becomes

$$Ef'(Y) - \lambda Ef(Y) = 0 \qquad \text{for } f \in \mathcal{D}. \tag{4.5}$$

The second is the family

$$p(x) = \frac{\alpha e^{-|x|^\alpha/\beta}}{2\beta^{1/\alpha}\Gamma(1/\alpha)}, \qquad -\infty < x < \infty,$$

where $\alpha > 0, \beta > 0$. Then $p'(x)/p(x) = -\frac{\alpha}{\beta}|x|^{\alpha-1}\mathrm{sign}(x)$ and hence the Stein identity reduces to

$$Ef'(Y) - \frac{\alpha}{\beta} E|Y|^{\alpha-1}\mathrm{sign}(Y)f(Y) = 0 \qquad \text{for } f \in \mathcal{D}.$$

4.2. *Properties of the Stein solution.* In order to determine error bounds for the approximation to $E(h(Y))$, we need to understand some basic properties of the Stein solution f_h. In the following, we use the notation $\|g\| := \sup_{x\in\mathbb{R}}|g(x)|$.

LEMMA 4.1. *Let h be a measurable function and f_h be the Stein solution and let $F(x) = \int_a^x p(t)\,dt$.*

(i) *Assume that h is bounded and that there exist $d_1 > 0$ and $d_2 > 0$*

$$\min\bigl(1 - F(x), F(x)\bigr) \le d_1 p(x) \tag{4.6}$$

and

$$|p'(x)| \min\bigl(F(x), 1 - F(x)\bigr) \le d_2 p^2(x). \tag{4.7}$$

Then

$$\|f_h\| \le 2d_1\|h\|, \tag{4.8}$$

$$\|f_h p'/p\| \le 2d_2\|h\| \tag{4.9}$$

and

$$\|f_h'\| \le (2 + 2d_2)\|h\|. \tag{4.10}$$

(ii) *Assume that h is absolutely continuous with bounded h'. In addition to* (4.6), (4.7), *assume that there exist d_3 and d_4 such that*

$$\begin{aligned}\min\bigl(&E|Y|I_{\{Y\le x\}} + E|Y|F(x), E|Y|I_{\{Y>x\}} + E|Y|\bigl(1 - F(x)\bigr)\bigr)|(p'/p)'| \\ &\le d_3 p(x)\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\min\bigl(&E|Y|I_{\{Y\le x\}} + E|Y|F(x), E|Y|I_{\{Y>x\}} + E|Y|\bigl(1 - F(x)\bigr)\bigr) \\ &\le d_4 p(x).\end{aligned} \tag{4.12}$$

Then if h is absolutely continuous with bounded derivative h',

$$\|f_h''\| \le (1 + d_2)(1 + d_3)\|h'\|, \tag{4.13}$$

$$\|f_h\| \le d_4\|h'\| \tag{4.14}$$

and

$$\|f_h'\| \le (1+d_3)d_1\|h'\|. \tag{4.15}$$

PROOF. (i) Let Y^* be an independent copy of Y. Then we can rewrite f_h in (4.4) as

$$\begin{aligned} f(w) &= \bigl(1/p(w)\bigr)E\bigl(h(Y)-h(Y^*)\bigr)I_{\{Y\le w\}},\\ &= -\bigl(1/p(w)\bigr)E\bigl(h(Y)-h(Y^*)\bigr)I_{\{Y>w\}}, \end{aligned} \tag{4.16}$$

which yields

$$|f(w)| \le 2\|h\| \min\bigl(F(w), 1-F(w)\bigr)/p(w). \tag{4.17}$$

Inequality (4.8) now follows from (4.6) and (4.17). Inequalities (4.17) and (4.7) imply $|f_h p'/p| \le 2d_2\|h\|$, that is (4.9), and now (4.10) follows from (4.3).

(ii) Let $g_1(x) = p'(x)/p(x)$. Recall by (4.3)

$$f'' = h' - f'g_1 - fg_1'. \tag{4.18}$$

To prove (4.13), it suffices to show that

$$\|fg_1'\| \le d_3\|h'\| \tag{4.19}$$

and

$$\|f'g_1\| \le (1+d_3)d_2\|h'\|. \tag{4.20}$$

By (4.16) again, we have

$$\begin{aligned} |f(w)p(w)| &\le \|h'\| \min\bigl(E(|Y|+|Y^*|)I_{\{Y\le w\}}, E(|Y|+|Y^*|)I_{\{Y>w\}}\bigr)\\ &= \|h'\| \min\bigl(E|Y|I_{\{Y\le w\}} + E|Y|F(w), E|Y|I_{\{Y>w\}}\\ &\qquad + E|Y|\bigl(1-F(w)\bigr)\bigr). \end{aligned} \tag{4.21}$$

This proves (4.19) by assumption (4.7). This also proves (4.14) by (4.12).

It follows from (4.18) that

$$(h' - fg_1')p = p(f'' + f'g_1) = f''p + f'p' = (f'p)'.$$

Thus

$$f'(w)p(w) = \int_a^w (h' - fg_1')p\,dx = -\int_w^b (h' - fg_1')p\,dx$$

and hence

$$|f'(w)p(w)| \le \|h'\|(1+d_3)\min\bigl(F(w), 1-F(w)\bigr),$$

which gives (4.20) as well as (4.15) by (4.12) and (4.6), respectively. $\square$

The next lemma shows that (4.6)–(4.12) are satisfied for p defined in (1.2).

LEMMA 4.2. *Let p be defined as in* (1.2). *Assume that* (H1) *and* (H2) *are satisfied. Then* (4.6)–(4.12) *hold with $d_1 = 1/c_1$, $d_2 = 1$, $d_3 = c_2$ and $d_4 = c_2$.*

PROOF. Let $g_2(t) = c_0 g(t)$, $G_1(t) = c_0 G(t)$ and $F(t) = P(Y \le t)$ be the distribution function of Y. We first show that (4.6) is satisfied with $d_1 = 1/c_1$. It suffices to show that

$$F(t) \le F(0)p(t)/c_1 \qquad \text{for } t \le 0 \tag{4.22}$$

and

$$1 - F(t) \le \bigl((1 - F(0))/c_1\bigr)p(t) \qquad \text{for } t \ge 0. \tag{4.23}$$

Let $H(t) = F(t) - (F(0)/c_1)p(t)$ for $t \le 0$. Noting that

$$\begin{aligned} H'(t) &= p(t) - \bigl(F(0)/c_1\bigr)p'(t) \\ &= p(t) + \bigl(F(0)/c_1\bigr)g_2(t)p(t) \\ &= p(t)\bigl(1 + g_2(t)F(0)/c_1\bigr). \end{aligned}$$

Since $g_2(t)$ is nondecreasing, if $H'(0) > 0$, then there is at most one t_0 such that $H'(t_0) = 0$; if $H'(0) \le 0$, then $H'(t) \le 0$ for $t < 0$. Hence, H achieves maximum either at $t = 0$ or $t = -\infty$. Notice that $H(0) = H(-\infty) = 0$, $H(t) \le 0$ for all $t < 0$. This proves (4.22). Similarly, (4.23) holds.

Next, we prove (4.7). Noting that $p' = -pg_2$, we have for $t < 0$

$$\begin{aligned} F(t) &= \int_{-\infty}^{t} p(s)\,ds \\ &\le \int_{-\infty}^{t} \frac{g_2(s)p(s)}{g_2(t)}\,ds \\ &= \int_{-\infty}^{t} \frac{-p'(s)}{g_2(t)}\,ds \\ &= \frac{p(t)}{-g_2(t)} = p(t)/|g_2(t)|. \end{aligned} \tag{4.24}$$

Similarly, we have

$$1 - F(t) \le p(t)/g_2(t) \qquad \text{for } t \ge 0. \tag{4.25}$$

Hence, (4.7) is satisfied with $d_2 = 1$.

Note that (4.6) and (4.7) imply that

$$1 - F(x) \le p(x)\min\bigl(1/c_1, 1/|g_2(x)|\bigr) \qquad \text{for } x \ge 0 \tag{4.26}$$

and

$$F(x) \le p(x)\min\bigl(1/c_1, 1/|g_2(x)|\bigr) \qquad \text{for } x \le 0. \tag{4.27}$$

To verify (4.11), with $x \ge 0$ write

$$
\begin{aligned}
E|Y|I_{\{Y>x\}} &= xP(Y>x) + \int_x^\infty P(Y \ge t)\,dt \\
&\le xp(x)\min\bigl(1/c_1, 1/|g_2(x)|\bigr) \\
&\quad + \int_x^\infty p(t)\min\bigl(1/c_1, 1/|g_2(t)|\bigr)\,dt \\
&\le xp(x)\min\bigl(1/c_1, 1/|g_2(x)|\bigr) \\
&\quad + \min\bigl(1/c_1, 1/|g_2(x)|\bigr)\int_x^\infty p(t)\,dt \\
&\le \min\bigl(1/c_1, 1/|g_2(x)|\bigr)\bigl\{xp(x) + \bigl(1-F(x)\bigr)\bigr\} \\
&\le \min\bigl(1/c_1, 1/|g_2(x)|\bigr)\{xp(x) + p(x)/c_1\} \\
&\le p(x)\min\bigl(1/c_1, 1/|g_2(x)|\bigr)\{x + 1/c_1\}.
\end{aligned} \tag{4.28}
$$

Similarly, for $x < 0$,

$$
E|Y|I_{\{Y<x\}} \le p(x)\min\bigl(1/c_1, 1/|g_2(x)|\bigr)\{|x| + 1/c_1\}. \tag{4.29}
$$

Equations (4.28) and (4.29) with $x = 0$ also give $E|Y| \le 2/c_1$. Hence, recalling (4.26)

$$
\begin{aligned}
&E|Y|I_{\{Y>x\}} + E|Y|\bigl(1-F(x)\bigr) \\
&\quad \le p(x)\min\bigl(1/c_1, 1/|g_2(x)|\bigr)\{x + 3/c_1\} \qquad \text{for } x > 0
\end{aligned} \tag{4.30}
$$

and

$$
\begin{aligned}
&E|Y|I_{\{Y<x\}} + E|Y|F(x) \\
&\quad \le p(x)\min\bigl(1/c_1, 1/|g_2(x)|\bigr)\{|x| + 3/c_1\} \qquad \text{for } x \le 0.
\end{aligned} \tag{4.31}
$$

Thus, (4.11) holds with $d_3 = c_2$ by (H2).

Equations (4.30) and (4.31) also show that (4.12) is satisfied with $d_4 = c_2$.

This completes the proof of Lemma 4.2. □

From the proof of Lemma 4.2, one can see the following remark is true.

REMARK 4.1. Assume that (H1) and (H3) are satisfied. Then (4.6)–(4.11) hold with $d_1 = 1/c_1$, $d_2 = 1$ and $d_3 = c_3$, and hence (4.13) and (4.15).

5. Proof of main results. In this section, we prove the general error bounds (Theorems 1.1 and 1.2), the result for the Curie–Weiss model (Theorem 2.1), and Theorem 3.1.

5.1. *Proof of Theorem* 1.1. Let $f = f_h$ be the solution to Stein's equation (4.3). Then

$$\begin{aligned} Eh(W) - Eh(Y) &= Ef'(W) + Ef(W)p'(W)/p(W) \\ &= Ef'(W) - c_0 Ef(W)g(W). \end{aligned} \tag{5.1}$$

Recall $\Delta = W - W'$ and observe that for any absolutely continuous function f

$$\begin{aligned} 0 &= E(W - W')\bigl(f(W') + f(W)\bigr) \\ &= 2Ef(W)(W - W') + E(W - W')\bigl(f(W') - f(W)\bigr) \\ &= 2E\bigl\{f(W)E\bigl((W - W')|W\bigr)\bigr\} - E(W - W')\int_{-\Delta}^{0} f'(W + t)\,dt \\ &= 2Ef(W)g(W) + 2Ef(W)r(W) - E\int_{-\infty}^{\infty} f'(W + t)\hat{K}(t)\,dt, \end{aligned} \tag{5.2}$$

where

$$\hat{K}(t) = E\bigl\{\Delta(I\{-\Delta \le t \le 0\} - I\{0 < t \le -\Delta\})|W\bigr\}.$$

Substituting (5.2) into (5.1) gives

$$\begin{aligned} &Ef'(W) - c_0 Ef(W)g(W) \\ &\quad = Ef'(W) - (c_0/2)\left\{E\int_{-\infty}^{\infty} f'(W + t)\hat{K}(t)\,dt - 2Ef(W)r(W)\right\} \\ &\quad = E\bigl\{f'(W)\bigl(1 - (c_0/2)E(\Delta^2|W)\bigr)\bigr\} \\ &\qquad + (c_0/2)E\int_{-\infty}^{\infty}\bigl(f'(W) - f'(W + t)\bigr)\hat{K}(t)\,dt \\ &\qquad + c_0 Ef(W)r(W). \end{aligned} \tag{5.3}$$

When (H1) and (H2) are satisfied, by Lemmas 4.1 and 4.2

$$\|f_h\| \le c_2\|h'\|, \qquad \|f_h'\| \le (1 + c_2)\|h'\|/c_1, \qquad \|f_h''\| \le 2(1 + c_2)\|h'\| \tag{5.4}$$

and hence

$$\begin{aligned} &|Ef_h'(W) - c_0 Ef_h(W)g(W)| \\ &\quad \le \frac{(1 + c_2)\|h'\|}{c_1} E\bigl|\bigl(1 - (c_0/2)E(\Delta^2|W)\bigr)\bigr| \\ &\qquad + (1 + c_2)\|h'\|c_0 E|\Delta|^3/2 + c_0 c_2\|h'\|E|r(W)|. \end{aligned}$$

This proves (1.3).

Under (H1) and (H3), by Remark 4.1

$$\|f_h'\| \le (1 + c_3)\|h'\|/c_1, \qquad \|f_h''\| \le 2(1 + c_3)\|h'\|. \tag{5.5}$$

From (4.16), (4.30) and (4.31) it follows that

$$\begin{aligned} |f(w)| &\le \bigl(1/p(w)\bigr)\|h'\| \min\bigl(E|Y - Y*|I_{\{Y\le w\}}, E|Y - Y*|I_{\{Y\ge w\}}\bigr) \\ &\le \|h'\| \min\bigl(1/c_1, 1/|g_2(w)|\bigr)(|w| + 3/c_1) \qquad (5.6) \\ &\le \|h'\|(|w| + 3/c_1)/c_1. \end{aligned}$$

This proves (1.4) by (5.3), (5.5) and (5.6).

5.2. *Proof of Theorem* 1.2. Since (1.5) is trivial when $c_1c_3\delta > 1$, we assume

$$c_1c_3\delta \le 1. \tag{5.7}$$

Let F be the distribution function of Y and let $f = f_z$ be the solution to the equation

$$f'(w) - c_0 f(w)g(w) = I(w \le z) - F(z). \tag{5.8}$$

By (5.2),

$$\begin{aligned} &2Ef(W)g(W) + 2Ef(W)r(W) \\ &\quad = E\int_{-\infty}^{\infty} f'(W+t)\hat{K}(t)\,dt \\ &\quad = E\int_{-\delta}^{\delta} \{c_0 f(W+t)g(W+t) + I(W+t \le z) - F(z)\}\hat{K}(t)\,dt \\ &\quad \ge E\int_{-\delta}^{\delta} c_0 f(W+t)g(W+t)\hat{K}(t)\,dt + EI(W \le z-\delta)\Delta^2 - F(z)E\Delta^2 \end{aligned}$$

and hence

$$\begin{aligned} &EI(W \le z - \delta)\Delta^2 - F(z)E\Delta^2 \\ &\qquad \le 2Ef(W)g(W) + 2Ef(W)r(W) \\ &\qquad\qquad - c_0E\int_{-\delta}^{\delta} f(W+t)g(W+t)\hat{K}(t)\,dt \\ &\qquad = 2Ef(W)g(W)\bigl(1 - (c_0/2)E(\Delta^2|W)\bigr) + 2Ef(W)r(W) \qquad (5.9) \\ &\qquad\qquad + c_0E\int_{-\delta}^{\delta} \{f(W)g(W) - f(W+t)g(W+t)\}\hat{K}(t)\,dt \\ &\qquad := J_1 + J_2 + J_3. \end{aligned}$$

From Lemmas 4.1 and 4.2 again, we obtain

$$\|f_z\| \le 2/c_1, \qquad \|f_z g\| \le 2/c_0 \quad \text{and} \quad \|f_z'\| \le 4. \tag{5.10}$$

Therefore,

$$|J_1| \le (4/c_0)E|1 - (c_0/2)E(\Delta^2|W)|. \tag{5.11}$$

and

$$|J_2| \le (4/c_1)E|r(W)|. \tag{5.12}$$

To bound J_3, we first show that

$$\sup_{|t|\le\delta} |g(w+t) - g(w)| \le \frac{c_1 c_3 \delta}{2c_0}\bigl(c_1 + c_0|g(w)|\bigr). \tag{5.13}$$

From (H2), it follows that

$$\begin{aligned}
|g'(x)| &\le \frac{c_1 c_3}{3c_0 \min(1/c_1, 1/|c_0 g(x)|)} \\
&= \frac{c_1 c_3}{3c_0} \max(c_1, |c_0 g(x)|) \\
&\le \frac{c_1 c_3}{3c_0}\bigl(c_1 + |c_0 g(x)|\bigr).
\end{aligned} \tag{5.14}$$

Thus, by the mean value theorem,

$$\begin{aligned}
&\sup_{|t|\le\delta} |g(w+t) - g(w)| \\
&\quad \le \delta \sup_{|t|\le\delta} |g'(w+t)| \\
&\quad \le \frac{c_1 c_3 \delta}{3c_0}\Bigl(c_1 + c_0 \sup_{|t|\le\delta} |g(w+t)|\Bigr) \\
&\quad \le \frac{c_1 c_3 \delta}{3c_0}\Bigl(c_1 + c_0|g(w)| + c_0 \sup_{|t|\le\delta} |g(w+t) - g(w)|\Bigr) \\
&\quad = \frac{c_1 c_3 \delta}{3c_0}\bigl(c_1 + c_0|g(w)|\bigr) + \frac{c_1 c_3 \delta}{3} \sup_{|t|\le\delta} |g(w+t) - g(w)| \\
&\quad \le \frac{c_1 c_3 \delta}{3c_0}\bigl(c_1 + c_0|g(w)|\bigr) + \frac{1}{3} \sup_{|t|\le\delta} |g(w+t) - g(w)|
\end{aligned}$$

by (5.7). This proves (5.13).

Now by (5.10) and (5.13), when $|t| \le \delta$

$$\begin{aligned}
&|f(w)g(w) - f(w+t)g(w+t)| \\
&\quad \le |g(w)||f(w+t) - f(w)| + |f(w+t)||g(w+t) - g(w)| \\
&\quad \le 4|g(w)||t| + \frac{2}{c_1}\frac{c_1 c_3 \delta}{2c_0}\bigl(c_1 + c_0|g(w)|\bigr) \\
&\quad \le (4 + c_3)\delta|g(w)| + \delta c_1 c_3/c_0.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
|J_3| &\le c_0(4+c_3)\delta E|g(W)|\Delta^2 + \delta c_1 c_3 E\Delta^2 \\
&\le (4+c_3)\delta^3 E|c_0 g(W)| + c_1 c_3 \delta^3.
\end{aligned}
\tag{5.15}
$$

Combining (5.9), (5.12), (5.11) and (5.15) shows that

$$
\begin{aligned}
&EI(W \le z-\delta)\Delta^2 - F(z)E\Delta^2 \\
&\quad \le (4/c_0)E|1-(c_0/2)E(\Delta^2|W)| + (4/c_1)E|r(W)| \\
&\qquad + (4+c_3)\delta^3 E|c_0 g(W)| + c_1 c_3 \delta^3.
\end{aligned}
\tag{5.16}
$$

On the other hand, using $F'(z) = p(z) \le c_1$, we have

$$
\begin{aligned}
&EI(W \le z-\delta)\Delta^2 - F(z)E\Delta^2 \\
&\quad = \frac{2}{c_0}\bigl(EI(W \le z-\delta) - F(z-\delta)\bigr) \\
&\qquad - \frac{2}{c_0}E\left\{\bigl(I(W \le z-\delta) - F(z)\bigr)\left(1 - \frac{c_0}{2}E(\Delta^2|W)\right)\right\} \\
&\qquad + \frac{2}{c_0}\bigl(F(z-\delta) - F(z)\bigr) \\
&\quad \ge \frac{2}{c_0}\bigl(P(W \le z-\delta) - F(z-\delta)\bigr) \\
&\qquad - \frac{2}{c_0}E\left|1 - \frac{c_0}{2}E(\Delta^2|W)\right| - \frac{2c_1\delta}{c_0},
\end{aligned}
\tag{5.17}
$$

which together with (5.16) yields

$$
\begin{aligned}
&P(W \le z-\delta) - F(z-\delta) \\
&\quad \le E|1-(c_0/2)E(\Delta^2|W)| + c_1\delta \\
&\qquad + \frac{c_0}{2}\bigl((4/c_0)E|1-(c_0/2)E(\Delta^2|W)| + (4/c_1)E|r(W)| \\
&\qquad\qquad + (4+c_3)\delta^3 E|c_0 g(W)| + c_1 c_3 \delta^3\bigr)
\end{aligned}
\tag{5.18}
$$

$$
\begin{aligned}
&\quad = 3E|1-(c_0/2)E(\Delta^2|W)| + c_1\delta + 2c_0 E|r(W)|/c_1 \\
&\qquad + \delta^3 c_0\{(2+c_3/2)E|c_0 g(W)| + c_1 c_3/2\}.
\end{aligned}
\tag{5.19}
$$

Similarly, we have

$$
\begin{aligned}
&F(z+\delta) - P(W \le z+\delta) \\
&\quad \le 3E|1-(c_0/2)E(\Delta^2|W)| + c_1\delta + 2c_0 E|r(W)|/c_1
\end{aligned}
\tag{5.20}
$$

$$
\qquad + \delta^3 c_0\{(2+c_3/2)E|c_0 g(W)| + c_1 c_3/2\}.
\tag{5.21}
$$

This completes the proof of (1.5).

5.3. *Proof of Theorem 2.1.* By (2.1)–(2.4)

$$E|r(W)| = O(n^{-2}),$$
$$E\big|1 - (c_0/2)E\big((W - W')^2|W\big)\big| = O(n^{-1/2}),$$
$$E|W|^3 = O(1).$$

Applying Theorem 1.2 gives Theorem 2.1.

We now show that (2.1)–(2.4) hold.

LEMMA 5.1. *With W, W' as in Section 2, we have*

$$E\left|E(W - W'|W) - \frac{n^{-3/2}}{3}W^3\right| \le 15n^{-2}, \tag{5.22}$$

$$E\big|E\big((W - W')^2|W\big) - 2n^{-3/2}\big| \le 15n^{-2} \tag{5.23}$$

and

$$E|W|^3 \le 15. \tag{5.24}$$

Also, obviously, $|W - W'| \le 2n^{-3/4}$.

PROOF. Let $m = n^{-1}\sum_{i=1}^n \sigma_i = n^{-1/4}W$, and for each i, let

$$m_i = n^{-1}\sum_{j\ne i}\sigma_j.$$

It is easy to see that for $\tau \in \{-1, 1\}$

$$P(\sigma_i' = \tau|\sigma) = \frac{e^{m_i\tau}}{e^{m_i} + e^{-m_i}}, \tag{5.25}$$

and so

$$E(\sigma_i'|\sigma) = \frac{e^{m_i}}{e^{m_i} + e^{-m_i}} - \frac{e^{-m_i}}{e^{m_i} + e^{-m_i}} = \tanh m_i.$$

Hence,

$$\begin{aligned} E(W - W'|\sigma) &= \frac{1}{n}\sum_{i=1}^n n^{-3/4}\big(\sigma_i - E(\sigma_i'|\sigma)\big) \\ &= n^{-3/4}m - n^{-7/4}\sum_{i=1}^n \tanh m_i. \end{aligned} \tag{5.26}$$

Now it is easy to verify that the function

$$\frac{d^2}{dx^2}\tanh x = \frac{-2\sinh x}{\cosh^3 x} = -2(\tanh x)(1 - \tanh^2 x)$$

has exactly two extrema $\pm x^*$ on the real line, where x^* solves the equation $\tanh^2 x^* = \frac{1}{3}$. It follows that the maximum magnitude of this function is $4/3^{3/2}$. Thus, for all $x, y \in \mathbb{R}$,

$$|\tanh x - \tanh y - (x-y)(\cosh y)^{-2}| \le \frac{2(x-y)^2}{3^{3/2}}.$$

It follows that

$$\left|\sum_{i=1}^n \tanh m_i - n \tanh m + n^{-1}(\cosh m)^{-2} \sum_{i=1}^n \sigma_i\right| \le \frac{2n^{-1}}{3^{3/2}},$$

and therefore

$$\left|\sum_{i=1}^n \tanh m_i - n \tanh m\right| \le |m| + \frac{2n^{-1}}{3^{3/2}}.$$

Using this in (5.26) and the relation $m = n^{-1/4} W$, we get

$$|E(W - W'|\sigma) + n^{-3/4} \tanh m - n^{-3/4} m| \le n^{-2}|W| + \frac{2n^{-11/4}}{3^{3/2}}. \tag{5.27}$$

Now consider the function $f(x) = \tanh x - x + \frac{x^3}{3}$. Note that $f'(x) = (\cosh x)^{-2} - 1 + x^2 \ge 0$ for all x, and hence f is an increasing function. Also $f(0) = 0$. Therefore, $f(x) \ge 0$ for all $x \ge 0$. Now, it can be easily verified that the first four derivatives of f vanish at zero, and for all $x \ge 0$,

$$\frac{d^5 f}{dx^5} = \frac{16}{\cosh^2 x} - 120 \frac{\sinh^2 x}{\cosh^4 x} + 120 \frac{\sinh^4 x}{\cosh^6 x} \le \frac{16}{\cosh^2 x} \le 16.$$

Thus, for all $x \ge 0$,

$$0 \le f(x) \le \frac{16}{5!} x^5 = \frac{2x^5}{15}.$$

Since f is an odd function, we get that for all x,

$$\left|\tanh x - x + \frac{1}{3}x^3\right| \le \frac{2|x|^5}{15}.$$

Using this information in (5.27), we get

$$\left|E(W - W'|\sigma) - \frac{n^{-3/4}}{3} m^3\right| \le \frac{2n^{-3/4}|m|^5}{15} + n^{-2}|W| + \frac{2n^{-11/4}}{3^{3/2}}.$$

Using the relation $m = n^{-1/4} W$, we get

$$\left|E(W - W'|\sigma) - \frac{n^{-3/2}}{3} W^3\right| \le \frac{2n^{-2}|W|^5}{15} + n^{-2}|W| + \frac{2n^{-11/4}}{3^{3/2}}. \tag{5.28}$$

This implies, in particular, that

$$\begin{aligned}\left|E\big((W-W')W^3\big)-\frac{n^{-3/2}}{3}E(W^6)\right|\\ \le\frac{2n^{-2}E(W^8)}{15}+n^{-2}E(W^4)+\frac{2n^{-11/4}E|W|^3}{3^{3/2}}.\end{aligned}\tag{5.29}$$

Thus,

$$\begin{aligned}E(W^6)\le 3n^{3/2}\big|E\big((W'-W)W^3\big)\big|+\frac{2n^{-1/2}E(W^8)}{5}\\ +3n^{-1/2}E(W^4)+\frac{2n^{-5/4}E|W|^3}{3^{1/2}}.\end{aligned}\tag{5.30}$$

Using the crude bound $|W|\le n^{1/4}$, we get

$$\begin{aligned}\frac{2n^{-1/2}E(W^8)}{5}+3n^{-1/2}E(W^4)+\frac{2n^{-5/4}E|W|^3}{3^{1/2}}\\ \le\frac{2E(W^6)}{5}+3E(W^2)+\frac{2n^{-1}E(W^2)}{3^{1/2}}.\end{aligned}\tag{5.31}$$

Next, note that by the exchangeability of (W, W'),

$$\begin{aligned}E\big((W'-W)W^3\big)&=\tfrac{1}{2}E\big((W'-W)(W^3-W'^3)\big)\\ &=-\tfrac{1}{2}E\big((W'-W)^2(W^2+WW'+W'^2)\big).\end{aligned}$$

Since $|W-W'|\le 2n^{-3/4}$, this gives

$$\big|E\big((W'-W)W^3\big)\big|\le 6n^{-3/2}E(W^2).\tag{5.32}$$

Combining (5.30), (5.31) and (5.32), we get

$$E(W^6)\le\left(21+\frac{2n^{-1}}{3^{1/2}}\right)E(W^2)+\frac{2E(W^6)}{5},$$

and therefore,

$$E(W^6)\le\frac{5}{3}\left(21+\frac{2n^{-1}}{3^{1/2}}\right)E(W^2)\le 36.9245E(W^2).$$

Since $E(W^2)\le(E(W^6))^{1/3}$, this gives

$$E(W^6)\le(36.9245)^{3/2}\le 224.4\tag{5.33}$$

and hence (5.24) holds.

Combined with (5.28), this gives

$$\begin{aligned} &E\left|E(W-W'|W)-\frac{n^{-3/2}}{3}W^3\right| \\ &\quad\le n^{-2}\left(\frac{2(224.4)^{5/6}}{15}+(224.4)^{1/6}\right)+\frac{2n^{-11/4}}{3^{3/2}}\le 15n^{-2}. \end{aligned}\tag{5.34}$$

By (5.25), we have

$$\begin{aligned} E\big((W-W')^2|\sigma\big) &= \frac{1}{n}\sum_{i=1}^{n}4n^{-3/2}\frac{e^{-m_i\sigma_i}}{e^{m_i\sigma_i}+e^{-m_i\sigma_i}} \\ &= 2n^{-5/2}\sum_{i=1}^{n}\big(1-\tanh(m_i\sigma_i)\big) \\ &= 2n^{-3/2}-2n^{-5/2}\sum_{i=1}^{n}\sigma_i\tanh m_i. \end{aligned}$$

Using $|\tanh m_i-\tanh m|\le|m_i-m|\le n^{-1}$, we get

$$\begin{aligned} \big|E\big((W-W')^2|\sigma\big)-2n^{-3/2}\big| &\le 2n^{-5/2}+2n^{-3/2}m\tanh m \\ &\le 2n^{-5/2}+2n^{-3/2}m^2 \\ &= 2n^{-5/2}+2n^{-2}W^2. \end{aligned}$$

Using (5.33), we get

$$E\big|E\big((W-W')^2|W\big)-2n^{-3/2}\big|\le 2n^{-5/2}+2n^{-2}(224.4)^{1/3}\le 15n^{-2}.$$

This completes the proof of the lemma. $\square$

5.4. *Proof of Theorem* 3.1. With $p(w)=e^{-w}I_{\{w>0\}}$, for given h, let f_h be the Stein solution given in (4.4)

$$f_h(w)=e^w\int_0^w\big(h(t)-Eh(Y)\big)e^{-t}\,dt=-e^w\int_w^\infty\big(h(t)-Eh(Y)\big)e^{-t}\,dt$$

for $w\ge 0$. Following the proof of Theorems 1.1 and 1.2, it suffices to show that

$$|f_h(w)|\le 3\min(\|h\|,\|h'\|)w \qquad \text{for } w\ge 0.\tag{5.35}$$

By (4.17),

$$|f_h(w)|\le 2\|h\|\min(1-e^{-w},e^{-w})e^w=2\|h\|\min(1,e^w-1)\le 3w\|h\|$$

and by (4.21)

$$\begin{aligned} |f_h(w)| &\le \|h'\|e^w\min\big(-we^{-w}+2(1-e^{-w}),(w+1)e^{-w}\big) \\ &\le \|h'\|\min\big(w+1,2(e^w-1)\big)\le 3w\|h'\|. \end{aligned}$$

This proves (5.35) and hence Theorem 3.1.

Acknowledgments. The authors thank Larry Goldstein for helping on the exponential approximation and thank an anonymous referee and an Associate Editor for their helpful comments.

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