

EFFICIENT SIMULATION OF NONLINEAR PARABOLIC SPDES WITH ADDITIVE NOISE

BY ARNULF JENTZEN^{1,2}, PETER KLOEDEN² AND GEORG WINKEL

*Princeton University, Johann Wolfgang Goethe University
and Johann Wolfgang Goethe University*

Recently, in a paper by Jentzen and Kloeden [*Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **465** (2009) 649–667], a new method for simulating nearly linear stochastic partial differential equations (SPDEs) with additive noise has been introduced. The key idea was to use suitable linear functionals of the noise process in the numerical scheme which allow a higher approximation order to be obtained. Following this approach, a new simplified version of the scheme in the above named reference is proposed and analyzed in this article. The main advantage of the convergence result given here is the higher convergence order for nonlinear parabolic SPDEs with additive noise, although the used numerical scheme is very simple to simulate and implement.

1. Introduction. In this article, the numerical approximation of nonlinear parabolic stochastic partial differential equations (SPDEs) is considered. Following the idea in [10] for somewhat linear SPDEs, a new numerical method for simulating nonlinear SPDEs with additive noise is proposed and analyzed in this article. The main advantage of the convergence result in this article is the higher convergence order for nonlinear parabolic SPDEs with additive noise in comparison to convergence results of classical schemes such as the linear implicit Euler scheme. Nevertheless, the here presented scheme is very simple to simulate and implement.

More precisely, let $T \in (0, \infty)$ be a real number, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $H = L^2((0, 1), \mathbb{R})$ be the \mathbb{R} -Hilbert space of equivalence classes of square integrable functions from $(0, 1)$ to \mathbb{R} . Moreover, let $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function with bounded partial derivatives, let $\xi : [0, 1] \rightarrow \mathbb{R}$ with $\xi(0) = \xi(1) = 0$ be a smooth function and let $W^Q : [0, T] \times \Omega \rightarrow H$ be a standard Q -Wiener process with a trace class operator $Q : H \rightarrow H$ (see, e.g., Definition 2.1.9 in [13]). It is a classical result (see, e.g., Proposition 2.1.5 in [13]) that the covariance operator $Q : H \rightarrow H$ of the Wiener process $W^Q : [0, T] \times \Omega \rightarrow H$

Received November 2009; revised April 2010.

¹Supported in part by the DFG Collaborative Research Centre 701 “Spectral Structures and Topological Methods in Mathematics.”

²Supported by the DFG project “Pathwise numerics and dynamics of stochastic evolution equations.”

MSC2010 subject classifications. Primary 60H15; secondary 35R60, 65C30.

Key words and phrases. Exponential Euler scheme, linear implicit Euler scheme, computational cost, stochastic reaction diffusion equations.

has an orthonormal basis $g_j \in H, j \in \mathbb{N}$, of eigenfunctions with summable eigenvalues $\mu_j \in [0, \infty), j \in \mathbb{N}$. In order to have a more concrete example, we consider the choice $g_j(x) = \sqrt{2} \sin(j\pi x)$ and $\mu_j = cj^{-(r+1)}$ for all $x \in (0, 1)$ and all $j \in \mathbb{N}$ with some $c \in [0, \infty)$ and some arbitrarily small $r \in (0, \infty)$ in the following and refer to Section 2 for our general setting. Then we consider the SPDE

$$(1) \quad \begin{aligned} dX_t &= \left[\frac{\partial^2}{\partial x^2} X_t + f(x, X_t) \right] dt + dW_t^Q, \\ X_t(0) = X_t(1) &= 0, \quad X_0 = \xi, \end{aligned}$$

for $x \in (0, 1)$ and $t \in [0, T]$. Under the assumptions above, the SPDE (1) has a unique mild solution. Specifically, there exists an up to indistinguishability unique stochastic process $X : [0, T] \times \Omega \rightarrow H$ with continuous sample paths which satisfies

$$(2) \quad X_t = e^{At}\xi + \int_0^t e^{A(t-s)} F(X_s) ds + \int_0^t e^{A(t-s)} dW_s^Q, \quad \mathbb{P}\text{-a.s.}$$

for all $t \in [0, T]$ where $A : D(A) \subset H \rightarrow H$ is the Laplacian with Dirichlet boundary conditions on $(0, 1)$ and where $F : H \rightarrow H$ is the Nemytskii operator $(F(v))(x) := f(x, v(x))$ for all $x \in (0, 1)$ and all $v \in H$.

Then our goal is to solve the strong approximation problem of the SPDE (1). More precisely, we want to compute a $\mathcal{F}/\mathcal{B}(H)$ -measurable numerical approximation $Y : \Omega \rightarrow H$ such that

$$(3) \quad \left(\mathbb{E} \left[\int_0^1 |X_T(x) - Y(x)|^2 dx \right] \right)^{1/2} < \varepsilon$$

holds for a given precision $\varepsilon > 0$ with the least possible computational effort (number of computational operations and independent standard normal random variables needed to compute $Y : \Omega \rightarrow H$). A computational operation is here an arithmetical operation (addition, subtraction, multiplication, division), a trigonometrical operation (sine, cosine) or an evaluation of $f : (0, 1) \times \mathbb{R} \rightarrow \mathbb{R}$ or the exponential function.

In order to be able to calculate such a numerical approximation on a computer, both the time interval $[0, T]$ and the infinite-dimensional \mathbb{R} -Hilbert space $H = L^2((0, 1), \mathbb{R})$ have to be discretized. While for temporal discretizations the linear implicit Euler scheme is often used, spatial discretizations are usually achieved with finite elements, finite differences and spectral Galerkin methods. For instance, the linear implicit Euler scheme combined with spectral Galerkin methods which we denote by $\mathcal{F}/\mathcal{B}(H)$ -measurable mappings $Z_n^N : \Omega \rightarrow H, n \in \{0, 1, \dots, N^2\}, N \in \mathbb{N} := \{1, 2, \dots\}$, is given by $Z_0^N := P_N(\xi)$ and

$$(4) \quad \begin{aligned} Z_{n+1}^N &:= \left(I - \frac{T}{N^2} A \right)^{-1} \\ &\times \left(Z_n^N + \frac{T}{N^2} \cdot (P_N F)(Z_n^N) + P_N(W_{(n+1)T/N^2}^Q - W_{nT/N^2}^Q) \right) \end{aligned}$$

for every $n \in \{0, 1, \dots, N^2 - 1\}$ and every $N \in \mathbb{N}$ where the bounded linear operators $P_N : H \rightarrow H$, $N \in \mathbb{N}$, are given by

$$(5) \quad (P_N(v))(x) := \sum_{n=1}^N 2 \sin(n\pi x) \int_0^1 \sin(n\pi s) v(s) ds$$

for all $x \in (0, 1)$, $v \in H$ and all $N \in \mathbb{N}$. Note that the infinite-dimensional \mathbb{R} -Hilbert space H is projected down to the N -dimensional \mathbb{R} -Hilbert space $P_N(H)$ for the spatial discretization and the time interval $[0, T]$ is divided into N^2 subintervals, that is, N^2 time steps are used, for the temporal discretization in the scheme Z_n^N , $n \in \{0, 1, \dots, N^2\}$, above for $N \in \mathbb{N}$. The exact solution $X : [0, T] \times \Omega \rightarrow H$ of the SPDE (1) enjoys at least twice the regularity in space than in time and therefore, the quadratic number of time steps is used in the scheme (4) above (see also Walsh [15] for details).

We now review how efficiently the numerical method (4) solves the strong approximation problem (3) of the SPDE (1). Standard results in the literature (see, e.g., Theorem 2.1 in Hausenblas [7]) yield the existence of a real number $C > 0$ such that

$$(6) \quad \left(\mathbb{E} \left[\int_0^1 |X_T(x) - Z_{N^2}^N(x)|^2 dx \right] \right)^{1/2} \leq C \cdot N^{-1}$$

holds for all $N \in \mathbb{N}$. Since $P_N(H)$ is N -dimensional and since N^2 time steps are used in (4), $O(N^3 \log(N))$ computational operations and independent standard normal random variables are needed to compute $Z_{N^2}^N$ for $N \in \mathbb{N}$. The log term in $O(N^3 \log(N))$ for $N \in \mathbb{N}$ arises due to computing the nonlinearity with fast Fourier transform (aliasing errors are neglected here). Combining the computational effort $O(N^3 \log(N))$ and the estimate (6) shows that the linear implicit Euler scheme needs about $O(\varepsilon^{-3})$ computational operations and independent standard normal random variables to achieve a precision of size $\varepsilon > 0$ in the sense of (3). In fact, we have demonstrated that the linear implicit Euler scheme method (4) needs $O(\varepsilon^{-(3+\delta)})$ computational operations and random variables to solve (3) for every arbitrarily small $\delta \in (0, \infty)$ but for simplicity we write about $O(\varepsilon^{-3})$ computational operations and random variables here and below.

Recently, in [10], a new numerical method for simulating somewhat linear SPDEs with additive noise has been introduced. The key idea in [10] is to use suitable linear functionals of the noise process in the numerical scheme which allows a higher approximation order to be obtained. In this paper, we extend this idea to the case of nonlinear SPDEs of the form (1). More precisely, we introduce the following numerical scheme which is a simplified version of the scheme considered in [10]. Let $Y_n^N : \Omega \rightarrow H$, $n \in \{0, 1, \dots, N\}$, $N \in \mathbb{N}$, be $\mathcal{F}/\mathcal{B}(H)$ -measurable

mappings given by $Y_0^N := P_N(\xi)$ and

$$(7) \quad \begin{aligned} Y_{n+1}^N &:= e^{AT/N} \left(Y_n^N + \frac{T}{N} \cdot (P_N F)(Y_n^N) \right) \\ &+ P_N \left(\int_{nT/N}^{(n+1)T/N} e^{A((n+1)T/N-s)} dW_s^Q \right) \end{aligned}$$

\mathbb{P} -a.s. for every $n \in \{0, 1, \dots, N - 1\}$ and every $N \in \mathbb{N}$. Note that the infinite-dimensional \mathbb{R} -Hilbert space H is projected down to the N -dimensional \mathbb{R} -Hilbert space $P_N(H)$ for the spatial discretization and the time interval $[0, T]$ is divided into N subintervals, that is, N time steps are used, for the temporal discretization in the scheme $Y_n^N, n \in \{0, 1, \dots, N\}$, above for $N \in \mathbb{N}$.

We now illustrate the main result of this article (Theorem 1) and show how efficiently the method (7) solves the strong approximation problem (3) of the SPDE (1). Theorem 1 shows the existence of real numbers $C_\delta > 0, \delta \in (0, 1)$, such that

$$(8) \quad \left(\mathbb{E} \left[\int_0^1 |X_T(x) - Y_N^N(x)|^2 dx \right] \right)^{1/2} \leq C_\delta \cdot N^{(\delta-1)}$$

holds for all $N \in \mathbb{N}$ and all arbitrarily small $\delta \in (0, 1)$. The stochastic integrals

$$(9) \quad P_N \left(\int_{nT/N}^{(n+1)T/N} e^{A((n+1)T/N-s)} dW_s^Q \right)$$

for $n \in \{0, 1, \dots, N\}$ and $N \in \mathbb{N}$ in (7) provide more information about the exact solution and this allows us to obtain the estimate (8) although only N time steps (instead of N^2 time steps in the case of the linear implicit Euler scheme) are used in (7). Nevertheless, since the stochastic integrals (9) in (7) depend linearly on the Wiener process $W^Q : [0, T] \times \Omega \rightarrow H$, they are again normally distributed and hence easy to simulate. More precisely, since $P_N(H)$ is N -dimensional and since N time steps are used in (7), $O(N^2 \log(N))$ computational operations and independent standard normal random variables are needed to compute Y_N^N for $N \in \mathbb{N}$. The log term in $O(N^2 \log(N))$ for $N \in \mathbb{N}$ also arises due to computing the nonlinearity with fast Fourier transform (aliasing errors are neglected here). Combining the computational effort $O(N^2 \log(N))$ and the estimate (8) shows that the numerical scheme (7) needs about $O(\varepsilon^{-2})$ computational operations and independent standard normal random variables to achieve a precision of size $\varepsilon > 0$ in the sense of (3).

The estimates (6) and (8) are both asymptotic results since there is no information about the size of the corresponding error constants. In particular, the error constants $C_\delta \in (0, \infty), \delta \in (0, 1)$, in (8) could be much bigger than in (6). Therefore, from a practical point of view, one may ask whether the numerical method (7) solves the strong approximation problem (3) more efficiently than the linear implicit Euler scheme (4) for a given example of the form (1) and a given concrete

$\varepsilon > 0$. In order to analyze this question, we compare both methods in the case of a simple reaction diffusion SPDE of the form (1) (see Section 4.1 for details) and assume that the strong approximation problem (3) should be solved with the precision $\varepsilon = \frac{1}{300}$. In that example, it turns out that the linear implicit Euler scheme precisely needs $2^{21} = 2,097,152$ independent standard normal random variables while the numerical method (7) precisely needs $2^{16} = 65,536$ independent standard normal random variables to achieve an approximation error of size $\varepsilon = \frac{1}{300}$ (see Tables 1 and 2 in Section 4.1). We also emphasize that the numerical scheme (7) is very simple to implement and refer to Figure 2 for a short MATLAB code.

Having illustrated the main result of this article, we now sketch the key idea in the proof of Theorem 1. The main difficulty was to estimate the discretization error for nonlinear F . In that case, the main problem was to establish estimates of the form

$$(10) \quad \left\| \sum_{n=0}^{N-1} \int_{nT/N}^{(n+1)T/N} e^{A(T-s)} (F(X_s) - F(X_{nT/N})) ds \right\|_{L^2(\Omega; H)} \leq C_\delta \cdot N^{(\delta-1)}$$

for all $N \in \mathbb{N}$ and all $\delta \in (0, 1)$ where $C_\delta \in (0, \infty)$, $\delta \in (0, 1)$, are appropriate constants and where we write $\|Y\|_{L^2(\Omega; H)} := (\mathbb{E}[\int_0^1 |Y(x)|^2 dx])^{1/2} \in [0, \infty]$ for every $\mathcal{F}/\mathcal{B}(H)$ -measurable mapping $Y : \Omega \rightarrow H$ for simplicity. The smoothness of the Nemytskii operator F on an appropriate subspace $V \subset H$ shows that it remains to estimate

$$(11) \quad \left\| \sum_{n=0}^{N-1} \int_{nT/N}^{(n+1)T/N} e^{A(T-s)} F'(X_{nT/N})(X_s - X_{nT/N}) ds \right\|_{L^2(\Omega; H)} \leq C_\delta \cdot N^{(\delta-1)}$$

for all $N \in \mathbb{N}$ and all $\delta \in (0, 1)$. In [10], the linear operators $F'(v)$ for $v \in H$ and $A : D(A) \subset H \rightarrow H$ are assumed to commute in some sense which is fulfilled in the case of linear F such as $F(v) = v$, $v \in H$, but excludes nonlinear Nemytskii operators such as $F(v) = \frac{(1-v)}{(1+v^2)}$, $v \in H$ (see Assumption 2.4 in [10] for details). Under this commutativity condition, (11) can easily be established by using the smoothing effect of the semigroup e^{At} , $t \in [0, T]$ (see Section 5.b.i in [10]). Instead of this condition, our key assumption on the nonlinearity is an appropriate estimate on the adjoint operators of the Fréchet derivative operators of F [see (13)]. Since in our examples F is a (nonlinear) Nemytskii operator, the derivative operators $F'(v)$, $v \in V$, are self-adjoint and hence, it can easily be seen that this assumption is fulfilled [see (17) in Section 4 for details]. Moreover, this assumption enables use to show (11) and hence (10) [see Section 6.1.1 and particularly estimate (31)]. We also mention that the difficulty to estimate (10) can be avoided by using a more complicated scheme with a second linear functional (see Section 6.4 in [11]).

Finally, we would like to point out limitations of the here presented numerical method. The following assumption is essential to apply our algorithm. The eigenfunctions of the dominating linear operator and of the covariance operator of the

driving additive noise process of the SPDE must coincide and must be known explicitly.

The rest of this article is organized as follows. The basic setting and the assumptions that we use (including our key assumption on the adjoint of the Fréchet derivative of the nonlinearity) are presented in Section 2. The new numerical scheme and its convergence theorem which is the main result of this article are given in Section 3. This result is illustrated with some examples and some numerical simulations in Section 4. Although our setting in Section 2 uses the standard global Lipschitz assumption on the nonlinearity of the SPDE, we demonstrate the efficiency of our method numerically for a SPDE with a cubic nonglobally Lipschitz nonlinearity in Section 5. Proofs are postponed to the final section.

2. Setting and assumptions. Fix $T \in (0, \infty)$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t \in [0, T]}$ which means $\mathcal{F}_{t+} = \mathcal{F}_t$ for all $t \in [0, T)$ and $\{A \in \mathcal{F} | \mathbb{P}[A] = 0\} \subset \mathcal{F}_0$ (see, e.g., Definition 2.1.11 in [13]). In addition, let $(V, \|\cdot\|_V)$ be a separable \mathbb{R} -Banach space and let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ be a separable \mathbb{R} -Hilbert space with $V \subset H$ continuously. The following assumptions will be used.

ASSUMPTION 1 (Linear operator A). Let $(\lambda_n)_{n \in \mathbb{N}} \subset (0, \infty)$ be an increasing sequence of real numbers and let $(e_n)_{n \in \mathbb{N}} \subset H$ be an orthonormal basis of H . Assume that the linear operator $A : D(A) \subset H \rightarrow H$ is given by

$$Av = \sum_{n=1}^{\infty} -\lambda_n \langle e_n, v \rangle_H e_n$$

for all $v \in D(A)$ with $D(A) = \{w \in H | \sum_{n=1}^{\infty} |\lambda_n|^2 |\langle e_n, w \rangle_H|^2 < \infty\}$.

Let $D((-A)^r)$ with $\|v\|_{D((-A)^r)} = \|(-A)^r v\|_H$ for $v \in D((-A)^r)$ and $r \in \mathbb{R}$ denote the domains of fractional powers of the linear operator $-A$ (see, e.g., Section 3.7 in [14]).

ASSUMPTION 2 (Nonlinearity F). Assume $D((-A)^{1/2}) \subset V$ continuously and let $F : V \rightarrow V$ be a twice continuously Fréchet differentiable mapping with

$$(12) \quad \|F'(v)w\|_H \leq c\|w\|_H,$$

$$\|F'(v)\|_{L(V)} \leq c, \quad \|F''(v)\|_{L^2(V)} \leq c,$$

$$(13) \quad \|(F'(u))^*\|_{L(D((-A)^{1/2}))} \leq c(1 + \|u\|_{D((-A)^{1/2})})$$

for every $v, w \in V$ and every $u \in D((-A)^{1/2})$ where $c \in [0, \infty)$ is a given real number.

By definition $F'(v) \in L(V)$ is a bounded linear mapping from V to V for every $v \in V$. Due to the first condition in (12), we also have that $F'(v) \in L(H)$ is a bounded linear mapping from H to H for every $v \in V$. In that sense, the adjoint operator $(F'(v))^* \in L(H)$ given by

$$\langle (F'(v))^* u, w \rangle_H = \langle u, F'(v)w \rangle_H$$

for all $u, w \in H$ is well defined for every $v \in V$. Due to (13), the operator $(F'(v))^* \in L(H)$ is also a bounded linear mapping from $D((-A)^{1/2})$ to $D((-A)^{1/2})$ for every $v \in D((-A)^{1/2})$.

ASSUMPTION 3 (Stochastic process O). Let $O : [0, T] \times \Omega \rightarrow D((-A)^\gamma)$ be a centered and adapted stochastic process with continuous sample paths such that $O_{t_2} - e^{A(t_2-t_1)} O_{t_1}$ is independent of \mathcal{F}_{t_1} for all $0 \leq t_1 < t_2 \leq T$ and such that

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|(-A)^\gamma O_t\|_H^4 \right] + \sup_{0 \leq t_1 < t_2 \leq T} ((t_2 - t_1)^{-4\theta} \mathbb{E}[\|O_{t_2} - O_{t_1}\|_V^4]) < \infty$$

holds where $\gamma \in [\frac{1}{2}, 1)$ and $\theta \in (0, \frac{1}{2}]$ are given real numbers.

ASSUMPTION 4 (Initial value ξ). Let $\xi : \Omega \rightarrow D(A)$ be a $\mathcal{F}_0/\mathcal{B}(D(A))$ -measurable mapping with $\mathbb{E}[\|A\xi\|_H^4] < \infty$.

These assumptions suffice to ensure the existence of a unique solution of the SPDE (14).

LEMMA 1 (Existence of the solution). *Let Assumptions 1–4 be fulfilled. Then there exists a unique adapted stochastic process $X : [0, T] \times \Omega \rightarrow D((-A)^\gamma)$ with continuous sample paths which fulfills*

$$(14) \quad X_t(\omega) = e^{At} \xi(\omega) + \int_0^t e^{A(t-s)} F(X_s(\omega)) ds + O_t(\omega)$$

for all $t \in [0, T]$ and all $\omega \in \Omega$. Moreover, $X : [0, T] \times \Omega \rightarrow D((-A)^\gamma)$ satisfies $\mathbb{E}[\sup_{0 \leq t \leq T} \|(-A)^\gamma X_t\|_H^4] < \infty$.

The proof of Lemma 1 is given in Section 6. Some examples satisfying Assumptions 1–4 are presented in Section 4.

3. Numerical scheme and main result. For numerical approximations of the SPDE (14), we have to discretize both the time interval $[0, T]$ and the \mathbb{R} -Hilbert space H . To this end, we use projections $P_N : H \rightarrow H$ given by $P_N(v) := \sum_{n=1}^N \langle e_n, v \rangle_H e_n$ for every $v \in H$, $N \in \mathbb{N}$ and finite-dimensional \mathbb{R} -Hilbert spaces

$H_N \subset H$ given by $H_N := P_N(H)$ for every $N \in \mathbb{N}$. Finally, we define $\mathcal{F}/\mathcal{B}(H_N)$ -measurable mappings $Y_m^{N,M} : \Omega \rightarrow H_N$ for $m \in \{0, 1, \dots, M\}$ and $N, M \in \mathbb{N}$ by $Y_0^{N,M}(\omega) := P_N(\xi(\omega)) + P_N(O_0(\omega))$ and by

$$(15) \quad \begin{aligned} Y_{m+1}^{N,M}(\omega) := & e^{AT/M} \left(Y_m^{N,M}(\omega) + \frac{T}{M} \cdot (P_N F)(Y_m^{N,M}(\omega)) \right) \\ & + P_N(O_{(m+1)T/M}(\omega) - e^{AT/M} O_{mT/M}(\omega)) \end{aligned}$$

for every $m \in \{0, 1, \dots, M - 1\}$, $N, M \in \mathbb{N}$ and every $\omega \in \Omega$. In many examples, this scheme is as easy to simulate as the classical linear implicit Euler scheme. We refer to Section 4 for a detailed description of the implementation of our numerical scheme including a short MATLAB code.

THEOREM 1. *Let Assumptions 1–4 be fulfilled. Then there is a real number $C > 0$ such that*

$$(16) \quad (\mathbb{E}[\|X_{mT/M} - Y_m^{N,M}\|_H^2])^{1/2} \leq C \left(\frac{1}{(\lambda_N)^\gamma} + \frac{(1 + \log(M))}{M^{2\theta}} \right)$$

holds for every $m \in \{0, 1, \dots, M\}$ and every $N, M \in \mathbb{N}$ where $(\lambda_N)_{N \in \mathbb{N}} \subset (0, \infty)$ is given in Assumption 1 where $\gamma \in [\frac{1}{2}, 1)$ and $\theta \in (0, \frac{1}{2}]$ are given in Assumption 3 where $X : [0, T] \times \Omega \rightarrow D((-A)^\gamma)$ is the solution of the SPDE (14) and where $Y_m^{N,M} : \Omega \rightarrow H_N$, $m \in \{0, 1, \dots, M\}$, $N, M \in \mathbb{N}$, is given by (15).

Here and below \log is the natural logarithm. While the expression $\frac{1}{(\lambda_N)^\gamma}$ for $N \in \mathbb{N}$ in (16) arises due to discretizing the infinite-dimensional \mathbb{R} -Hilbert space H , the expression $\frac{(1 + \log(M))}{M^{2\theta}}$ for $M \in \mathbb{N}$ arises due to discretizing the time interval $[0, T]$. We would like to remark that the logarithmic term in $\frac{(1 + \log(M))}{M^{2\theta}}$ for $M \in \mathbb{N}$ can be avoided by assuming $F(D((-A)^{1/2})) \subset D((-A)^\varepsilon)$ and an appropriate linear growth condition on F for some $\varepsilon > 0$. Although this condition is fulfilled in our examples below, we use this logarithmic term in Theorem 1 here in order to formulate Assumption 2 in our abstract setting as simple as possible.

A similar result could be obtained for SPDEs of the form (14) but with a time dependent nonlinearity F . However, we omit the time dependency of the nonlinearity here for simplicity.

4. Examples. Let $H = L^2((0, 1), \mathbb{R})$ be the \mathbb{R} -Hilbert space of equivalence classes of $\mathcal{B}((0, 1))/\mathcal{B}(\mathbb{R})$ -measurable and square integrable functions from $(0, 1)$ to \mathbb{R} with the scalar product and the norm given by

$$\langle v, w \rangle_H = \int_0^1 v(s)w(s) ds, \quad \|v\|_H = \left(\int_0^1 |v(s)|^2 ds \right)^{1/2}$$

for every $v, w \in H$. In addition, let $V = C([0, 1], \mathbb{R})$ be the \mathbb{R} -Banach space of continuous functions from $[0, 1]$ to \mathbb{R} equipped with the norm $\|v\|_V = \sup_{0 \leq x \leq 1} |v(x)|$ for every $v \in V$.

Let $\kappa \in (0, \infty)$ be a given positive real number and let $(\lambda_n)_{n \in \mathbb{N}} \subset (0, \infty)$ and $(e_n)_{n \in \mathbb{N}} \subset H$ be given by

$$\lambda_n := \kappa n^2 \pi^2, \quad e_n(x) := \sqrt{2} \sin(n\pi x)$$

for every $x \in (0, 1)$ and every $n \in \mathbb{N}$. Hence, the linear operator $A : D(A) \subset H \rightarrow H$ reduces to the Laplacian with Dirichlet boundary conditions on the interval $(0, 1)$ times the constant $\kappa \in (0, \infty)$ (see, e.g., Section 3.8.1 in Sell and You [14]). In particular, $D((-A)^{1/2})$ reduces to the \mathbb{R} -Sobolev space $H_0^1((0, 1), \mathbb{R})$ equipped with the norm

$$\begin{aligned} \|u\|_{D((-A)^{1/2})} &= \|(-A)^{1/2}u\|_H \\ &= \left(\sum_{n=1}^{\infty} \kappa n^2 \pi^2 |\langle e_n, u \rangle_H|^2 \right)^{1/2} \\ &= \sqrt{\kappa} \left(\int_0^1 |u'(x)|^2 dx \right)^{1/2} \end{aligned}$$

for all $u \in D((-A)^{1/2})$. (See Sell and You [14] for more information about this space.)

Furthermore, let $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be a twice continuously differentiable function with the bounded partial derivatives

$$\left| \left(\frac{\partial f}{\partial y} \right)(x, y) \right| \leq K, \quad \left| \left(\frac{\partial^2 f}{\partial x \partial y} \right)(x, y) \right| \leq K, \quad \left| \left(\frac{\partial^2 f}{\partial y^2} \right)(x, y) \right| \leq K$$

for all $x \in [0, 1]$ and all $y \in \mathbb{R}$ with an arbitrary constant $K \in [0, \infty)$. Then the Nemytskii operator $F : V \rightarrow V$ given by $(F(v))(x) = f(x, v(x))$ for every $x \in [0, 1]$ and every $v \in V$ satisfies Assumption 2. To see this note that

$$\begin{aligned} F'(u)(v) &= \left(\frac{\partial f}{\partial y} \right)(x, u(x)) \cdot v(x), \\ F''(u)(v, w) &= \left(\frac{\partial^2 f}{\partial y^2} \right)(x, u(x)) \cdot v(x) \cdot w(x) \end{aligned}$$

holds for all $u, v, w \in V$. Therefore, we have

$$\begin{aligned} &\|(F'(u))^* v\|_{D((-A)^{1/2})}^2 \\ &= \|(-A)^{1/2} F'(u)v\|_H^2 \\ &= \kappa \int_0^1 \left| \frac{\partial}{\partial x} \left\{ \left(\frac{\partial f}{\partial y} \right)(x, u(x)) \cdot v(x) \right\} \right|^2 dx \\ &= \kappa \int_0^1 \left| \frac{\partial}{\partial x} \left[\left(\frac{\partial f}{\partial y} \right)(x, u(x)) \right] v(x) + \left(\frac{\partial f}{\partial y} \right)(x, u(x)) \cdot v'(x) \right|^2 dx \end{aligned}$$

$$\begin{aligned} &\leq 2\kappa \int_0^1 \left| \left\{ \frac{\partial}{\partial x} \left[\left(\frac{\partial f}{\partial y} \right) (x, u(x)) \right] \right\} v(x) \right|^2 dx \\ &\quad + 2\kappa \int_0^1 \left| \left(\frac{\partial f}{\partial y} \right) (x, u(x)) \cdot v'(x) \right|^2 dx \end{aligned}$$

and

$$\begin{aligned} &\| (F'(u))^* v \|_{D((-A)^{1/2})}^2 \\ &\leq 2\kappa \|v\|_V^2 \int_0^1 \left| \frac{\partial}{\partial x} \left[\left(\frac{\partial f}{\partial y} \right) (x, u(x)) \right] \right|^2 dx \\ &\quad + 2\kappa K^2 \int_0^1 |v'(x)|^2 dx \\ &\leq 4\|v\|_{D((-A)^{1/2})}^2 \int_0^1 \left| \left(\frac{\partial^2 f}{\partial x \partial y} \right) (x, u(x)) \right|^2 dx \\ &\quad + 4\|v\|_{D((-A)^{1/2})}^2 \int_0^1 \left| \left(\frac{\partial^2 f}{\partial y^2} \right) (x, u(x)) \cdot u'(x) \right|^2 dx \\ &\quad + 2K^2 \|v\|_{D((-A)^{1/2})}^2 \end{aligned}$$

for all $u, v \in D((-A)^{1/2})$. Hence, we obtain

$$\begin{aligned} &\| (F'(u))^* v \|_{D((-A)^{1/2})}^2 \\ &\leq 4K^2 \|v\|_{D((-A)^{1/2})}^2 \\ &\quad + 4K^2 \|v\|_{D((-A)^{1/2})}^2 \int_0^1 |u'(x)|^2 dx \\ &\quad + 2K^2 \|v\|_{D((-A)^{1/2})}^2 \\ &= 6K^2 \|v\|_{D((-A)^{1/2})}^2 + 4K^2 \kappa^{-1} \|v\|_{D((-A)^{1/2})}^2 \|u\|_{D((-A)^{1/2})}^2 \end{aligned}$$

and

$$\begin{aligned} &\| (F'(u))^* v \|_{D((-A)^{1/2})} \\ &\leq \sqrt{6K^2 \|v\|_{D((-A)^{1/2})}^2 + 4K^2 \kappa^{-1} \|v\|_{D((-A)^{1/2})}^2 \|u\|_{D((-A)^{1/2})}^2} \\ (17) \quad &\leq \sqrt{6}K \|v\|_{D((-A)^{1/2})} + 2K\kappa^{-1/2} \|v\|_{D((-A)^{1/2})} \|u\|_{D((-A)^{1/2})} \\ &\leq K \|v\|_{D((-A)^{1/2})} (3 + 2\kappa^{-1/2} \|u\|_{D((-A)^{1/2})}) \\ &\leq (3 + 2\kappa^{-1/2})K \|v\|_{D((-A)^{1/2})} (1 + \|u\|_{D((-A)^{1/2})}) \end{aligned}$$

for all $u, v \in D((-A)^{1/2})$. This shows that F indeed satisfies Assumption 2 with $c = 3K$.

Let $(b_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ be a sequence of real numbers with $\sum_{n=1}^\infty n^\varepsilon |b_n|^2 < \infty$ for some arbitrarily small $\varepsilon \in (0, \infty)$. Lemma 4.3 in [1] then gives the existence of an up to indistinguishability unique stochastic process $O : [0, T] \times \Omega \rightarrow V$ which satisfies Assumption 3 for $\theta = \frac{1}{2}$ and $\gamma = \frac{1}{2}$ and which satisfies

$$\mathbb{P} \left[O_t = \sum_{n=1}^\infty b_n \left(\int_0^t e^{-\lambda_n(t-s)} d\beta_s^n \right) e_n \right] = 1$$

for all $t \in [0, T]$ where the $\beta^n : [0, T] \times \Omega \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, are independent standard Brownian motions with respect to a given normal filtration $(\mathcal{F}_t)_{t \in [0, T]}$.

Moreover, the mapping $\xi : \Omega \rightarrow V$ given by

$$(\xi(\omega))(x) = \frac{1}{\sqrt{2}} \sin(\pi x) + \frac{3\sqrt{2}}{5} \sin(3\pi x)$$

for all $\omega \in \Omega$ and all $x \in (0, 1)$ obviously satisfies Assumption 4.

In view of the above choice, the SPDE (14) reduces to

$$\begin{aligned} dX_t &= \left[\kappa \frac{\partial^2}{\partial x^2} X_t + f(x, X_t) \right] dt + B dW_t, \\ (18) \quad X_t(0) &= X_t(1) = 0, \\ X_0(x) &= \frac{\sin(\pi x)}{\sqrt{2}} + \frac{3\sqrt{2}}{5} \sin(3\pi x) \end{aligned}$$

for $x \in [0, 1]$ and $t \in [0, T]$ where the linear operator $B : H \rightarrow H$ is given by

$$Bv = \sum_{n=1}^\infty b_n \langle e_n, v \rangle e_n$$

for all $v \in H$ and where $(W_t)_{t \in [0, T]}$ is a cylindrical I -Wiener process on H .

Since Assumption 3 is fulfilled for $\theta = \frac{1}{2}$ and $\gamma = \frac{1}{2}$, Theorem 1 shows the existence of a real number $C > 0$, such that

$$(19) \quad \left(\mathbb{E} \left[\int_0^1 |X_{nT/M}(x) - Y_n^{N,M}(x)|^2 dx \right] \right)^{1/2} \leq C \left(\frac{1}{N} + \frac{(1 + \log(M))}{M} \right)$$

holds for all $n \in \{0, 1, \dots, M\}$ and all $N, M \in \mathbb{N}$. While the expression $\frac{1}{N}$ for $N \in \mathbb{N}$ in (19) corresponds to the spatial discretization error, the expression $\frac{(1 + \log(M))}{M}$ for $M \in \mathbb{N}$ in (19) corresponds to the temporal discretization error. Since these error terms are nearly of the same size, we choose $M = N$ and consider the numerical approximations $Y_n^{N,N} : \Omega \rightarrow H_N$, $n \in \{0, 1, \dots, N\}$, $N \in \mathbb{N}$, in the following. Due to (19), we obtain the existence of real numbers $C_\delta > 0$, $\delta \in (0, 1)$,

such that

$$(20) \quad \left(\mathbb{E} \left[\int_0^1 |X_T(x) - Y_N^{N,N}(x)|^2 dx \right] \right)^{1/2} \leq C_\delta \cdot N^{(\delta-1)}$$

holds for all $N \in \mathbb{N}$ and all arbitrarily small $\delta \in (0, 1)$.

In order to describe the implementation of the numerical scheme (15) in this example, we use the $\mathcal{F}/\mathcal{B}(\mathbb{R})$ -measurable mappings $Y_{n,m}^{N,M} : \Omega \rightarrow \mathbb{R}$ given by $Y_{n,m}^{N,M}(\omega) := \langle e_n, Y_m^{N,M}(\omega) \rangle_H$ for all $n \in \{1, 2, \dots, N\}$, $m \in \{0, 1, \dots, M\}$ and all $N, M \in \mathbb{N}$. The numerical scheme (15) for the SPDE (18) with $M = N$ then reduces to $Y_{1,0}^{N,N} = \frac{1}{2}$, $Y_{2,0}^{N,N} = 0$, $Y_{3,0}^{N,N} = \frac{3}{5}$, $Y_{4,0}^{N,N} = Y_{5,0}^{N,N} = \dots = 0$ and

$$(21) \quad \begin{aligned} Y_{1,n+1}^{N,N} &= e^{-\kappa\pi^2 T/N} \left(Y_{1,n}^{N,N} + \frac{T}{N} \langle e_1, F(Y_n^{N,N}) \rangle_H \right) \\ &\quad + \frac{\sqrt{1 - e^{-2\kappa\pi^2 T/N}}}{b_1 \cdot \pi \sqrt{2\kappa}} \chi_{1,n}^N, \\ Y_{2,n+1}^{N,N} &= e^{-\kappa\pi^2 2^2 T/N} \left(Y_{2,n}^{N,N} + \frac{T}{N} \langle e_2, F(Y_n^{N,N}) \rangle_H \right) \\ &\quad + \frac{\sqrt{1 - e^{-2\kappa 2^2 \pi^2 T/N}}}{b_2 \cdot 2\pi \sqrt{2\kappa}} \chi_{2,n}^N, \\ &\quad \vdots \\ Y_{N,n+1}^{N,N} &= e^{-\kappa\pi^2 N^2 T/N} \left(Y_{N,n}^{N,N} + \frac{T}{N} \langle e_N, F(Y_n^{N,N}) \rangle_H \right) \\ &\quad + \frac{\sqrt{1 - e^{-2\kappa N^2 \pi^2 T/N}}}{b_N \cdot N\pi \sqrt{2\kappa}} \chi_{N,n}^N \end{aligned}$$

for all $n \in \{0, 1, \dots, N - 1\}$ and all $N \in \mathbb{N}$ where the $\mathcal{F}/\mathcal{B}(\mathbb{R})$ -measurable mappings $\chi_{n,m}^N : \Omega \rightarrow \mathbb{R}$ for $n \in \{1, 2, \dots, N\}$, $m \in \{0, 1, \dots, N - 1\}$ and $N \in \mathbb{N}$ are independent standard normal random variables. Since $O(N^2 \log(N))$ computational operations and independent standard normal random variables (computational effort) are needed to compute the numerical solution $Y_N^{N,N}$ given by (21) for $N \in \mathbb{N}$, it follows that $Y_N^{N,N}$ converges with order $\frac{1}{2}^-$ with respect to the computational effort to the exact solution $X : [0, T] \times \Omega \rightarrow D((-A)^{1/2})$ of the SPDE (18) in the sense of (20). We remark that the log term in the computational effort $O(N^2 \log(N))$ for $N \in \mathbb{N}$ arises if one computes the nonlinearity in (21) with fast Fourier transform (see Figure 2 for details).

In order to compare the new numerical scheme (21) with classical schemes, we consider the well-known linear implicit Euler scheme combined with spectral Galerkin methods applied to the SPDE (18). The linear implicit Euler scheme is

denoted by $\mathcal{F}/\mathcal{B}(H_N)$ -measurable mappings $Z_n^N : \Omega \rightarrow H_N, n \in \{0, 1, \dots, N^2\}, N \in \mathbb{N}$, given by $Z_0^N(\omega) := P_N(\xi(\omega)) + P_N(O_0(\omega))$ and

$$(22) \quad Z_{n+1}^N(\omega) := \left(I - \frac{T}{N^2}A \right)^{-1} \left\{ Z_n^N(\omega) + \frac{T}{N^2}(P_N F)(Z_n^N(\omega)) + P_N(B(W_{(n+1)T/N^2}(\omega) - W_{nT/N^2}(\omega))) \right\}$$

for every $n \in \{0, 1, \dots, N^2 - 1\}$ and every $N \in \mathbb{N}$. It has been shown in the literature (see, e.g., Walsh [15], Gyöngy [4] and Hausenblas [7]) that the linear implicit Euler scheme (22) and other classical numerical schemes such as the linear implicit Crank–Nicolson scheme combined with finite elements, finite differences and spectral Galerkin methods converge with order $\frac{1}{3}^-$ with respect to the computational effort.

The following two numerical examples illustrate the convergence order $\frac{1}{2}^-$ of the numerical scheme (21) and the convergence order $\frac{1}{3}^-$ of the linear implicit Euler scheme (22).

4.1. *A stochastic reaction diffusion equation.* In this example, we set $\kappa = \frac{1}{100}, T = 1, b_n = \frac{n^{-0.55}}{3.5}$ for all $n \in \mathbb{N}$ and consider $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x, y) = 5 \frac{(1-y)}{(1+y^2)}$ for all $x \in [0, 1], y \in \mathbb{R}$. The SPDE (18) then reduces to

$$(23) \quad \begin{aligned} dX_t &= \left[\frac{1}{100} \frac{\partial^2}{\partial x^2} X_t + 5 \frac{(1 - X_t)}{(1 + X_t^2)} \right] dt + B dW_t, \\ X_t(0) &= X_t(1) = 0, \\ X_0(x) &= \frac{\sin(\pi x)}{\sqrt{2}} + \frac{3\sqrt{2}}{5} \sin(3\pi x) \end{aligned}$$

for $x \in [0, 1]$ and $t \in [0, 1]$. In Figure 1 (see also Tables 1 and 2), we plot the root mean square discretization error

$$(24) \quad \left(\mathbb{E} \left[\int_0^1 |X_T(x) - Y_N^{N,N}(x)|^2 dx \right] \right)^{1/2}$$

of the numerical scheme (21) versus $N^2 \log(N)$ (up to a constant the computational effort) and the root mean square discretization error

$$(25) \quad \left(\mathbb{E} \left[\int_0^1 |X_T(x) - Z_{N^2}^N(x)|^2 dx \right] \right)^{1/2}$$

of the linear implicit Euler scheme (22) versus $N^3 \log(N)$ (up to a constant the computational effort) for different $N \in \mathbb{N}$. The “expectations” are based on 40 independent random realizations and the unknown “exact” solution is approximated with a very high accuracy there.

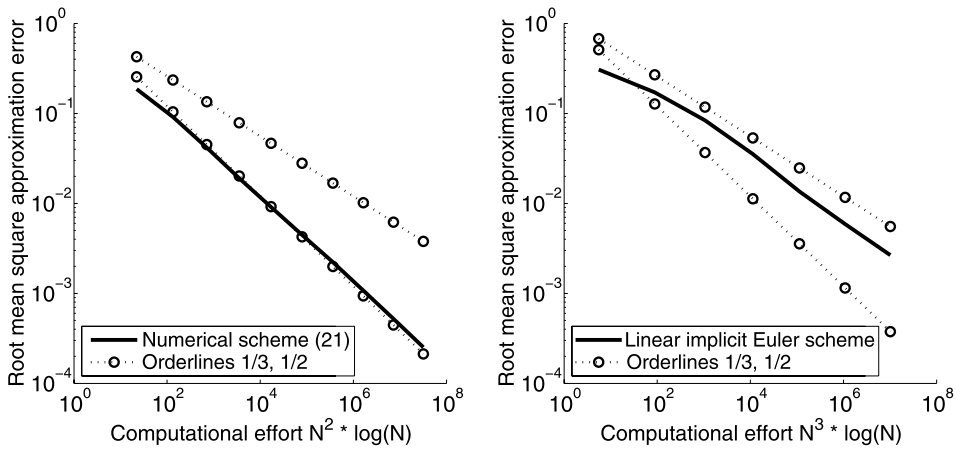


FIG. 1. Root mean square approximation error (24) of the numerical scheme (21) and root mean square approximation error (25) of the linear implicit Euler scheme (22) applied to SPDE (23) versus up to a constant the computational effort.

TABLE 1
 Root mean square approximation error (24) of $Y_N^{N,N}$ given by (21)
 applied to the SPDE (23) for $N \in \{2^2, 2^3, \dots, 2^{11}\}$

Numerical scheme (21)	Independent standard normal random variables N^2	Computational effort $N^2 \log(N)$ (up to a constant)	Root mean square approximation error (24)
$Y_{2^2}^{2^2,2^2}$	16	22	0.1864
$Y_{2^3}^{2^3,2^3}$	64	133	0.0914
$Y_{2^4}^{2^4,2^4}$	256	710	0.0417
$Y_{2^5}^{2^5,2^5}$	1024	3549	0.0191
$Y_{2^6}^{2^6,2^6}$	4096	17,035	0.0091
$Y_{2^7}^{2^7,2^7}$	16,384	79,496	0.0045
$Y_{2^8}^{2^8,2^8}$	65,536	363,408	0.0022
$Y_{2^9}^{2^9,2^9}$	262,144	1,635,339	0.0011
$Y_{2^{10}}^{2^{10},2^{10}}$	1,048,576	7,268,174	0.0005
$Y_{2^{11}}^{2^{11},2^{11}}$	4,194,304	31,979,969	0.0003

TABLE 2
 Root mean square approximation error (25) of $Z_{N^2}^N$ given by (22) applied
 to the SPDE (23) for $N \in \{2^1, 2^2, \dots, 2^7\}$

Linear implicit Euler scheme (22)	Independent standard normal random variables N^3	Computational effort $N^3 \log(N)$ (up to a constant)	Root mean square approximation error (25)
$Z_{2^2}^{2^1}$	8	6	0.3066
$Z_{2^4}^{2^2}$	64	88	0.1715
$Z_{2^6}^{2^3}$	512	1064	0.0837
$Z_{2^8}^{2^4}$	4096	11,356	0.0353
$Z_{2^{10}}^{2^5}$	32,768	113,565	0.0135
$Z_{2^{12}}^{2^6}$	262,144	1,090,226	0.0058
$Z_{2^{14}}^{2^7}$	2,097,152	10,175,444	0.0027

The short MATLAB code in Figure 2 shows that the solution of SPDE (23) can be simulated quite easily with the numerical scheme (21). Figure 3 is the result of the MATLAB code in Figure 2. It shows the solution of the stochastic reaction diffusion equation (23) at time $t = T = 1$ for one sample path $\omega \in \Omega$ approximated with the numerical method (21).

4.2. A stochastic partial differential equation with a spatially dependent f . This time let $\kappa = \frac{1}{50}$, $T = 1$, $b_n = \frac{n^{-0.6}}{5}$ for all $n \in \mathbb{N}$ and consider $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x, y) = (3.8x^2 - 2)y$ for all $x \in [0, 1]$, $y \in \mathbb{R}$ to obtain the SPDE

$$dX_t = \left[\frac{1}{50} \frac{\partial^2}{\partial x^2} X_t + (3.8x^2 - 2)X_t \right] dt + B dW_t,$$

(26) $X_t(0) = X_t(1) = 0,$

$$X_0(x) = \frac{\sin(\pi x)}{\sqrt{2}} - \frac{3\sqrt{2}}{5} \sin(3\pi x)$$

```
N = 1000; T = 1; A = - pi^2 * (1:N).^2 / 100; Y = [1/2,0,3/5,zeros(1,N-3)];
S = sqrt( ( exp(2*T/N*A) - 1 ) ./ A / 2 ) / 3.5 .* (1:N).^ -0.55;
for n=1:N
    y = dst(Y) * sqrt(2);
    FY = idst( 5 * ( 1 - y ) ./ ( 1 + y.^2 ) ) / sqrt(2);
    Y = exp( A * T/N ) .* ( Y + T/N * FY ) + S .* randn(1,N);
end
plot( (0:N+1)/(N+1), [0,dst(Y)*sqrt(2),0], 'k', 'Linewidth', 2 );
```

FIG. 2. MATLAB code for the numerical scheme (21) applied to the SPDE (23).

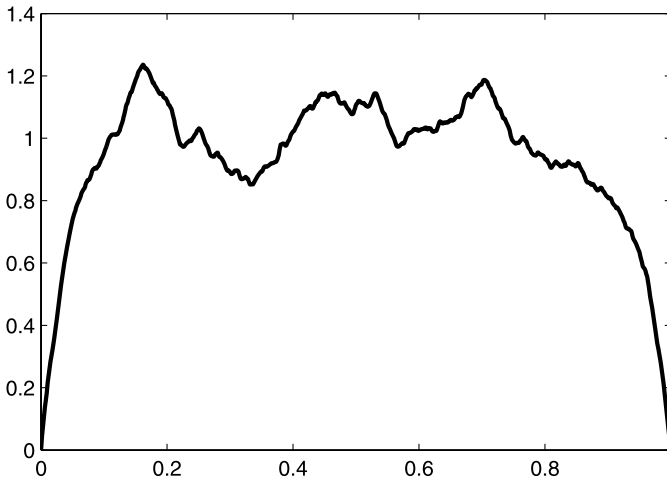


FIG. 3. Result of the MATLAB code in Figure 2: Solution of the stochastic reaction diffusion equation (23) at $t = T = 1$ for one sample path $\omega \in \Omega$ approximated with the numerical method (21).

for $x \in [0, 1]$ and $t \in [0, T]$. Here too, the numerical approximation (21) converges to the exact solution with order $\frac{1}{2}^-$ with respect to up to a constant the computational effort (see Figure 4). Finally, in Figure 5 we illustrate how the two differ-

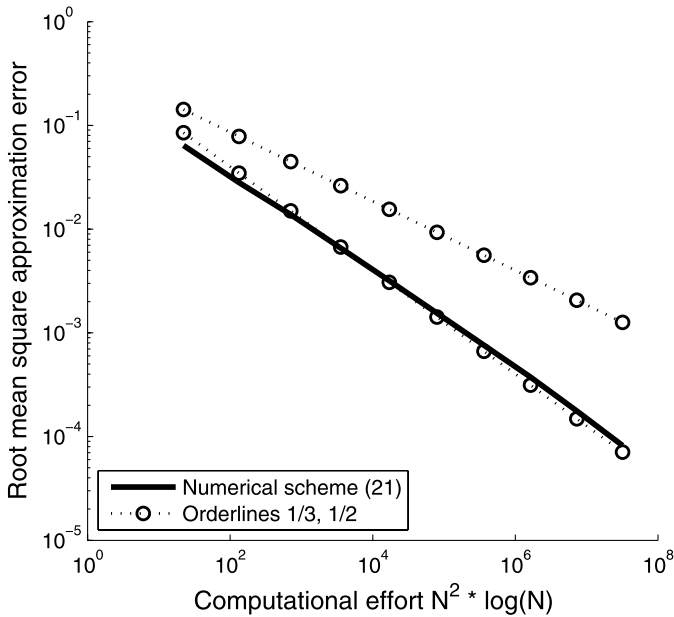


FIG. 4. Root mean square approximation error (24) of the numerical scheme (21) applied to SPDE (26) versus up to a constant the computational effort.

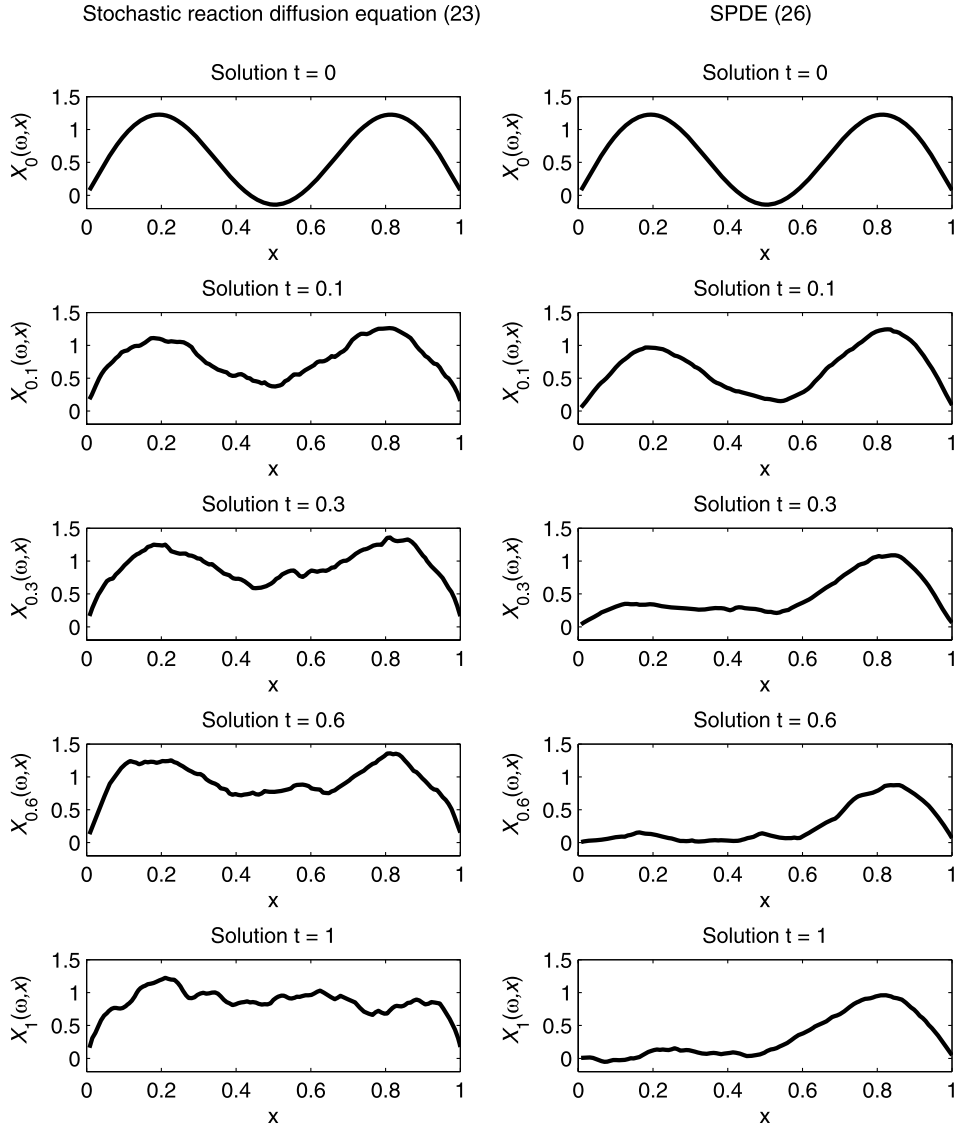


FIG. 5. Solution $X_t(\omega, x)$, $x \in [0, 1]$, of the stochastic reaction diffusion equation (23) and of the SPDE (26) for $t \in \{0, \frac{1}{10}, \frac{3}{10}, \frac{6}{10}, 1\}$ and one sample path $\omega \in \Omega$ approximated with the numerical method (21).

ent f from examples (23) and (26) affect the evolution of the respective solution $X_t(\omega, x)$, $x \in [0, 1]$, for $t \in \{0, \frac{1}{10}, \frac{3}{10}, \frac{6}{10}, 1\}$ and one sample path $\omega \in \Omega$.

5. A further numerical example. Although our setting in Section 2 uses the standard global Lipschitz assumption on the nonlinearity of the SPDE, we demon-

strate the efficiency of our method numerically for a SPDE with a cubic nonglobally Lipschitz nonlinearity in this section. More formally, we consider the SPDE

$$(27) \quad dX_t = \left[\frac{1}{10} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) X_t + X_t - X_t^3 \right] dt + dW_t^Q,$$

with

$$X_t|_{\partial(0,1)^2} \equiv 0$$

and

$$X_0(x_1, x_2) = \sin(\pi x_1) \sin(\pi x_2)$$

for $t, x_1, x_2 \in [0, 1]$ on the \mathbb{R} -Hilbert space $H = L^2((0, 1)^2, \mathbb{R})$ of equivalence classes of $\mathcal{B}((0, 1)^2)/\mathcal{B}(\mathbb{R})$ -measurable and square integrable functions from $(0, 1)^2$ to \mathbb{R} here where $(W_t^Q)_{t \in [0,1]}$ is a cylindrical Q -Wiener process on H with the covariance operator $Q: H \rightarrow H$ given by

$$\begin{aligned} (Qv)(x_1, x_2) &= \sum_{n,m=1}^{\infty} \frac{4 \sin(n\pi x_1) \sin(m\pi x_2)}{(n+m)^2} \\ &\quad \times \int_0^1 \int_0^1 \sin(n\pi y_1) \sin(m\pi y_2) v(y_1, y_2) dy_1 dy_2 \end{aligned}$$

for all $x_1, x_2 \in (0, 1)$ and all $v \in H$. Of course, (27) is not included in our setting in Section 2. Even worse, it has recently been shown in [9] that many numerical methods fail to converge to the solution of a stochastic differential equation with super linearly growing coefficients in the strong root mean square sense. However, convergence in the pathwise sense often holds due to Gyöngy’s result [3]. Therefore, we plot in Figure 6 the pathwise difference

$$\left(\int_0^1 \int_0^1 |X_T(\omega, x_1, x_2) - Y_N^{N,N}(\omega, x_1, x_2)|^2 dx_1 dx_2 \right)^{1/2}$$

of the exact solution $X_T(\omega)$ and of the numerical approximation $Y_N^{N,N}(\omega)$ [see (15)] applied to the SPDE (27) versus up to a constant the computational effort $N^3 \log(N)$ for $N \in \{2^2, 2^3, \dots, 2^7\}$ and one random $\omega \in \Omega$. It turns out that the method (15) converges with order $\frac{1}{3}^-$ with respect to the computational effort. The linear implicit Euler scheme is known to converge in the pathwise sense with order $\frac{1}{4}^-$ with respect to the computational effort to the solution of the SPDE (27). Further pathwise approximation results for the SPDE (27) and other SPDEs with nonglobally Lipschitz coefficients can be found in [4–6] and [12], for instance. Finally, we plot the solution of SPDE (27) for $t \in \{0, \frac{6}{10}\}$ and one random $\omega \in \Omega$ in Figure 7.

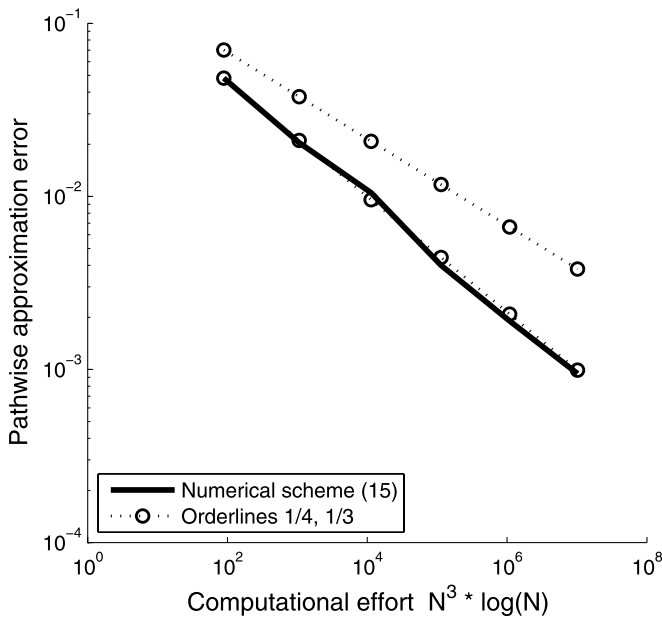


FIG. 6. Pathwise approximation error of the numerical scheme (15) applied to SPDE (27) versus up to a constant the computational effort for one random $\omega \in \Omega$.

6. Proofs. The notation

$$\|Z\|_{L^p(\Omega;W)} := (\mathbb{E}[\|Z\|_W^p])^{1/p} \in [0, \infty]$$

is used throughout this section for an \mathbb{R} -Banach space $(W, \|\cdot\|_W)$, a $\mathcal{F}/\mathcal{B}(W)$ -measurable mapping $Z : \Omega \rightarrow W$ and a real number $p \in [1, \infty)$.

6.1. Proof of Theorem 1. The $\mathcal{F}/\mathcal{B}(H)$ -measurable mappings $Y_m^M : \Omega \rightarrow H$ for $m \in \{0, 1, \dots, M\}$ and $M \in \mathbb{N}$ given by

$$(28) \quad \begin{aligned} Y_m^M(\omega) &:= e^{Amh} \xi(\omega) + h \left(\sum_{k=0}^{m-1} e^{A(mh-kh)} F(X_{kh}(\omega)) \right) \\ &\quad + O_{mh}(\omega) \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$, $\omega \in \Omega$ and $M \in \mathbb{N}$ are used throughout this proof. Here and below h is the time stepsize $h = h_M = \frac{T}{M}$ with $M \in \mathbb{N}$. This proof is divided into three parts. In the first part (see Section 6.1.1), we estimate

$$\|X_{mh} - Y_m^M\|_{L^2(\Omega;H)}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$ which corresponds to the temporal discretization error. In the second part (see Section 6.1.2), we estimate

$$\|Y_m^M - P_N(Y_m^M)\|_{L^2(\Omega;H)}$$

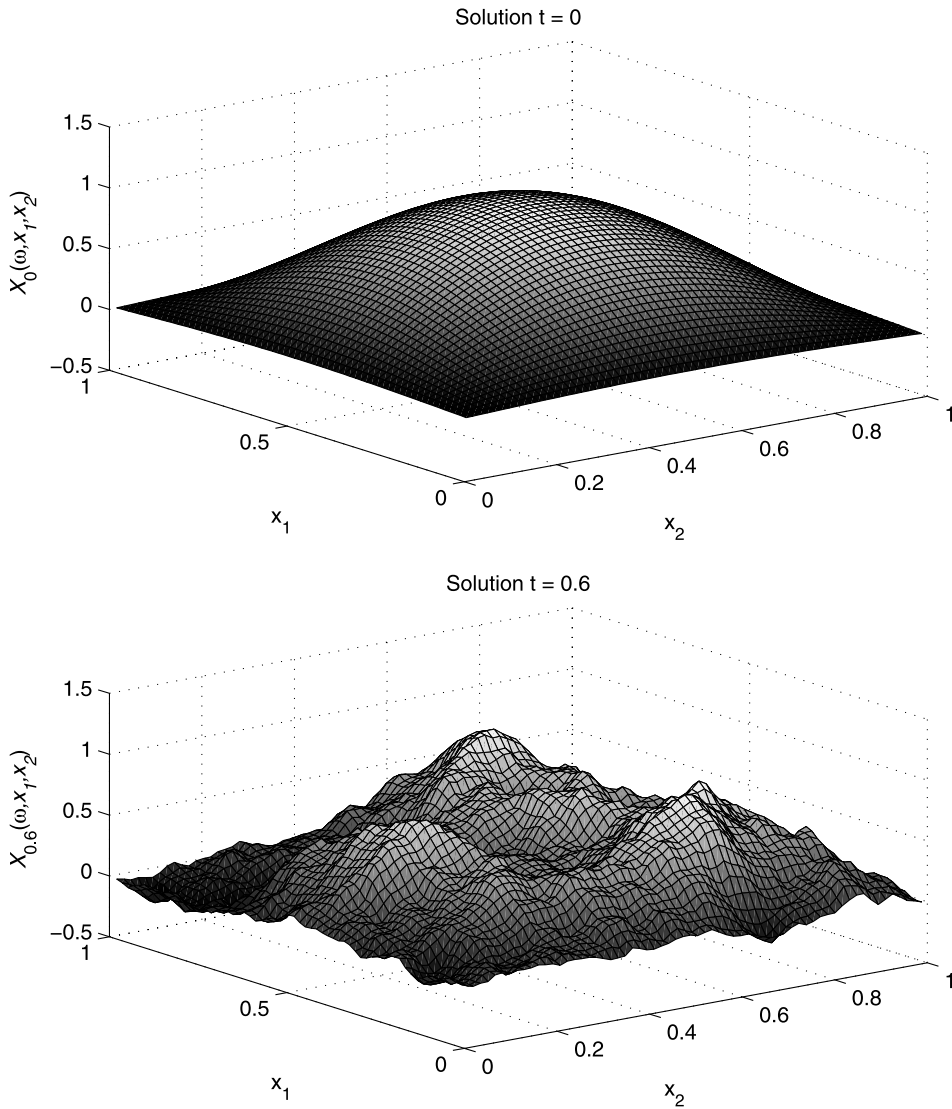


FIG. 7. Solution $X_t(\omega, x_1, x_2)$, $x_1, x_2 \in [0, 1]$, of the SPDE (27) for $t \in \{0, \frac{6}{10}\}$ and one random $\omega \in \Omega$ approximated with the numerical method (15).

for every $m \in \{0, 1, \dots, M\}$ and every $N, M \in \mathbb{N}$ which corresponds to the spatial discretization error. Finally, we estimate

$$\|P_N(Y_m^M) - Y_m^{N,M}\|_{L^2(\Omega; H)}$$

for every $m \in \{0, 1, \dots, M\}$ and every $N, M \in \mathbb{N}$ in the third part (see Section 6.1.3). Combining these three parts will then yield the desired assertion via Gronwall’s lemma as we will see below.

Before we begin with the first part, we introduce a universal constant $R > 0$ which is needed throughout this proof. More precisely, let $R \in (0, \infty)$ be a real number which satisfies

$$\begin{aligned} \|F(X_t)\|_{L^2(\Omega; H)} &\leq R, \\ \|(X_{t_2} - O_{t_2}) - (X_{t_1} - O_{t_1})\|_{L^2(\Omega; H)} &\leq R|t_2 - t_1|, \\ \frac{1}{\lambda_1} + \frac{1}{(1 - \gamma)} + T + c &\leq R, \\ \|v\|_H &\leq R\|v\|_V, \\ \|O_{t_2} - O_{t_1}\|_{L^4(\Omega; V)} &\leq R|t_2 - t_1|^\theta, \\ \|\xi\|_{L^2(\Omega; D((-A)^\gamma))} &\leq R, \\ \|O_t\|_{L^4(\Omega; D((-A)^\gamma))} &\leq R, \\ \|X_t\|_{L^4(\Omega; D((-A)^{1/2}))} &\leq R \end{aligned}$$

for every $t, t_1, t_2 \in [0, T]$ and every $v \in V$ where $\lambda_1 \in (0, \infty)$ is given in Assumption 1 where $c \in [0, \infty)$ is given in Assumption 2 and where $\gamma \in [\frac{1}{2}, 1)$ and $\theta \in (0, \frac{1}{2}]$ are given in Assumption 3. Indeed, such a real number exists due to Assumptions 1–4 and Lemma 4 in Section 6.2.

6.1.1. *Temporal discretization error.* Due to (14), we have

$$\begin{aligned} X_{mh} &= e^{Amh} \xi + \int_0^{mh} e^{A(mh-s)} F(X_s) ds + O_{mh} \\ &= e^{Amh} \xi + \sum_{k=0}^{m-1} \int_{kh}^{(k+1)h} e^{A(mh-s)} F(X_s) ds + O_{mh} \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$. From (28), we have

$$\begin{aligned} &\|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \\ &= \left\| \sum_{k=0}^{m-1} \int_{kh}^{(k+1)h} e^{A(mh-s)} F(X_s) ds - h \left(\sum_{k=0}^{m-1} e^{A(mh-kh)} F(X_{kh}) \right) \right\|_{L^2(\Omega; H)} \\ &\leq \left\| \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} e^{A(mh-s)} F(X_s) ds - h \left(\sum_{k=0}^{m-2} e^{A(mh-kh)} F(X_{kh}) \right) \right\|_{L^2(\Omega; H)} \\ &\quad + \int_{\max(m-1, 0)h}^{mh} \|e^{A(mh-s)} F(X_s)\|_{L^2(\Omega; H)} ds \\ &\quad + h \|e^{Ah} F(X_{\max(m-1, 0)h})\|_{L^2(\Omega; H)} \end{aligned}$$

and

$$\begin{aligned} & \|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \\ & \leq \left\| \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} e^{A(mh-s)} F(X_s) ds - h \left(\sum_{k=0}^{m-2} e^{A(mh-kh)} F(X_{kh}) \right) \right\|_{L^2(\Omega; H)} \\ & \quad + \int_{\max(m-1, 0)h}^{mh} \|e^{A(mh-s)}\|_{L(H)} \|F(X_s)\|_{L^2(\Omega; H)} ds \\ & \quad + h \|e^{Ah}\|_{L(H)} \|F(X_{\max(m-1, 0)h})\|_{L^2(\Omega; H)} \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$. Therefore, we obtain

$$\begin{aligned} & \|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \\ & \leq \left\| \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} e^{A(mh-s)} F(X_s) ds \right. \\ & \quad \left. - h \left(\sum_{k=0}^{m-2} e^{A(mh-kh)} F(X_{kh}) \right) \right\|_{L^2(\Omega; H)} + 2Rh \\ & \leq \left\| \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} e^{A(mh-s)} (F(X_s) - F(X_{kh})) ds \right\|_{L^2(\Omega; H)} \\ & \quad + 2Rh \\ & \quad + \left\| \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} e^{A(mh-s)} F(X_{kh}) ds \right. \\ & \quad \left. - h \left(\sum_{k=0}^{m-2} e^{A(mh-kh)} F(X_{kh}) \right) \right\|_{L^2(\Omega; H)} \end{aligned}$$

and

$$\begin{aligned} & \|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \\ & \leq \left\| \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} e^{A(mh-s)} (F(X_s) - F(X_{kh} + O_s - O_{kh})) ds \right\|_{L^2(\Omega; H)} \\ & \quad + \left\| \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} e^{A(mh-s)} (F(X_{kh} + O_s - O_{kh}) - F(X_{kh})) ds \right\|_{L^2(\Omega; H)} \\ & \quad + \sum_{k=0}^{m-2} \left\| \int_{kh}^{(k+1)h} (e^{A(mh-s)} - e^{A(mh-kh)}) F(X_{kh}) ds \right\|_{L^2(\Omega; H)} + 2Rh \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$. Hence, we obtain

$$\begin{aligned} & \|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \\ & \leq \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} \|e^{A(mh-s)}\|_{L(H)} \|F(X_s) - F(X_{kh} + O_s - O_{kh})\|_{L^2(\Omega; H)} ds \\ & \quad + \left\| \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} e^{A(mh-s)} F'(X_{kh})(O_s - O_{kh}) ds \right\|_{L^2(\Omega; H)} \\ & \quad + \left\| \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} e^{A(mh-s)} \int_0^1 F''(X_{kh} + r(O_s - O_{kh})) \right. \\ & \quad \quad \quad \times (O_s - O_{kh}, O_s - O_{kh}) \\ & \quad \quad \quad \left. \times (1-r) dr ds \right\|_{L^2(\Omega; H)} \\ & \quad + \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} \|(e^{A(mh-s)} - e^{A(mh-kh)})F(X_{kh})\|_{L^2(\Omega; H)} ds + 2R^2M^{-1} \end{aligned}$$

and

$$\begin{aligned} & \|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \\ & \leq c \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} \|X_s - (X_{kh} + O_s - O_{kh})\|_{L^2(\Omega; H)} ds \\ & \quad + \left\| \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} e^{A(mh-s)} F'(X_{kh})(O_s - e^{A(s-kh)} O_{kh}) ds \right\|_{L^2(\Omega; H)} \\ & \quad + \left\| \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} e^{A(mh-s)} F'(X_{kh})(e^{A(s-kh)} - I) O_{kh} ds \right\|_{L^2(\Omega; H)} \\ & \quad + \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} \|e^{A(mh-s)}\|_{L(H)} \\ & \quad \quad \quad \times \int_0^1 \|F''(X_{kh} + r(O_s - O_{kh}))\|_{L^2(\Omega; H)} \\ & \quad \quad \quad \times (O_s - O_{kh}, O_s - O_{kh}) dr ds \\ & \quad + \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} \|e^{A(mh-s)} - e^{A(mh-kh)}\|_{L(H)} \|F(X_{kh})\|_{L^2(\Omega; H)} ds \\ & \quad + 2R^2M^{-1} \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$. Therefore, we have

$$\begin{aligned}
 & \|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \\
 & \leq c \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} \|(X_s - O_s) - (X_{kh} - O_{kh})\|_{L^2(\Omega; H)} ds \\
 & \quad + \left\| \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} e^{A(mh-s)} F'(X_{kh})(O_s - e^{A(s-kh)} O_{kh}) ds \right\|_{L^2(\Omega; H)} \\
 (29) \quad & + \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} \|e^{A(mh-s)} F'(X_{kh})((e^{A(s-kh)} - I)O_{kh})\|_{L^2(\Omega; H)} ds \\
 & + cR \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} \int_0^1 \|O_s - O_{kh}\|_V^2 \|L^2(\Omega; \mathbb{R})\| dr ds \\
 & + \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} \frac{(mh - kh - mh + s)}{(mh - s)} \|F(X_{kh})\|_{L^2(\Omega; H)} ds \\
 & + 2R^2 M^{-1}
 \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$ due to Lemma 2 below (see Section 6.2). Furthermore, we have

$$\begin{aligned}
 & \mathbb{E} \left[\left\| \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} e^{A(mh-s)} F'(X_{kh})(O_s - e^{A(s-kh)} O_{kh}) ds \right\|_H^2 \right] \\
 & = \sum_{k, \tilde{k}=0}^{m-2} \mathbb{E} \left[\left\langle \int_{kh}^{(k+1)h} e^{A(mh-s)} F'(X_{kh})(O_s - e^{A(s-kh)} O_{kh}) ds, \right. \right. \\
 & \quad \left. \left. \int_{\tilde{k}h}^{(\tilde{k}+1)h} e^{A(mh-s)} F'(X_{\tilde{k}h})(O_s - e^{A(s-\tilde{k}h)} O_{\tilde{k}h}) ds \right\rangle_H \right]
 \end{aligned}$$

and hence

$$\begin{aligned}
 & \mathbb{E} \left[\left\| \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} e^{A(mh-s)} F'(X_{kh})(O_s - e^{A(s-kh)} O_{kh}) ds \right\|_H^2 \right] \\
 & = \sum_{k=0}^{m-2} \mathbb{E} \left[\left\| \int_{kh}^{(k+1)h} e^{A(mh-s)} F'(X_{kh}) \right. \right. \\
 & \quad \left. \left. \times (O_s - e^{A(s-kh)} O_{kh}) ds \right\|_H^2 \right]
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{\substack{k, \tilde{k}=0 \\ k \neq \tilde{k}}}^{m-2} \mathbb{E} \left[\left\langle \int_{kh}^{(k+1)h} e^{A(mh-s)} F'(X_{kh})(O_s - e^{A(s-kh)} O_{kh}) ds, \right. \right. \\
 & \qquad \qquad \qquad \left. \left. \int_{\tilde{k}h}^{(\tilde{k}+1)h} e^{A(mh-s)} F'(X_{\tilde{k}h})(O_s - e^{A(s-\tilde{k}h)} O_{\tilde{k}h}) ds \right\rangle_H \right]
 \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$. This yields

$$\begin{aligned}
 & \mathbb{E} \left[\left\| \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} e^{A(mh-s)} F'(X_{kh})(O_s - e^{A(s-kh)} O_{kh}) ds \right\|_H^2 \right] \\
 & = \sum_{k=0}^{m-2} \mathbb{E} \left[\left\| \int_{kh}^{(k+1)h} e^{A(mh-s)} F'(X_{kh})(O_s - e^{A(s-kh)} O_{kh}) ds \right\|_H^2 \right] \\
 & \quad + 2 \sum_{\substack{k, \tilde{k}=0 \\ k < \tilde{k}}}^{m-2} \mathbb{E} \left[\left\langle \int_{kh}^{(k+1)h} e^{A(mh-s)} F'(X_{kh}) \right. \right. \\
 & \qquad \qquad \qquad \times (O_s - e^{A(s-kh)} O_{kh}) ds, \\
 & \qquad \qquad \qquad \left. \int_{\tilde{k}h}^{(\tilde{k}+1)h} e^{A(mh-s)} F'(X_{\tilde{k}h}) \right. \\
 & \qquad \qquad \qquad \times (O_s - e^{A(s-\tilde{k}h)} O_{\tilde{k}h}) ds \Big\rangle_H \Big]
 \end{aligned}$$

and

$$\begin{aligned}
 & \mathbb{E} \left[\left\| \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} e^{A(mh-s)} F'(X_{kh})(O_s - e^{A(s-kh)} O_{kh}) ds \right\|_H^2 \right] \\
 & = \sum_{k=0}^{m-2} \mathbb{E} \left[\left\| \int_{kh}^{(k+1)h} e^{A(mh-s)} F'(X_{kh})(O_s - e^{A(s-kh)} O_{kh}) ds \right\|_H^2 \right] \\
 & \quad + 2 \sum_{\substack{k, \tilde{k}=0 \\ k < \tilde{k}}}^{m-2} \mathbb{E} \left[\mathbb{E} \left[\left\langle \int_{kh}^{(k+1)h} e^{A(mh-s)} F'(X_{kh}) \right. \right. \right. \\
 & \qquad \qquad \qquad \times (O_s - e^{A(s-kh)} O_{kh}) ds, \\
 & \qquad \qquad \qquad \left. \int_{\tilde{k}h}^{(\tilde{k}+1)h} e^{A(mh-s)} F'(X_{\tilde{k}h}) \right. \\
 & \qquad \qquad \qquad \times (O_s - e^{A(s-\tilde{k}h)} O_{\tilde{k}h}) ds \Big\rangle_H \Big| \mathcal{F}_{\tilde{k}h} \Big]
 \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$. Hence, we obtain

$$\begin{aligned} & \mathbb{E} \left[\left\| \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} e^{A(mh-s)} F'(X_{kh})(O_s - e^{A(s-kh)} O_{kh}) ds \right\|_H^2 \right] \\ &= \sum_{k=0}^{m-2} \mathbb{E} \left[\left\| \int_{kh}^{(k+1)h} e^{A(mh-s)} F'(X_{kh})(O_s - e^{A(s-kh)} O_{kh}) ds \right\|_H^2 \right] \\ &+ 2 \sum_{\substack{k, \tilde{k}=0 \\ k < \tilde{k}}}^{m-2} \mathbb{E} \left[\left\langle \int_{kh}^{(k+1)h} e^{A(mh-s)} F'(X_{kh}) \right. \right. \\ &\quad \times (O_s - e^{A(s-kh)} O_{kh}) ds, \\ &\quad \left. \left. \int_{\tilde{k}h}^{(\tilde{k}+1)h} e^{A(mh-s)} F'(X_{\tilde{k}h}) \right. \right. \\ &\quad \left. \left. \times (\mathbb{E}[O_s - e^{A(s-\tilde{k}h)} O_{\tilde{k}h} | \mathcal{F}_{\tilde{k}h}^-]) ds \right\rangle_H \right] \end{aligned}$$

and

$$\begin{aligned} (30) \quad & \mathbb{E} \left[\left\| \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} e^{A(mh-s)} F'(X_{kh})(O_s - e^{A(s-kh)} O_{kh}) ds \right\|_H^2 \right] \\ &= \sum_{k=0}^{m-2} \mathbb{E} \left[\left\| \int_{kh}^{(k+1)h} e^{A(mh-s)} F'(X_{kh})(O_s - e^{A(s-kh)} O_{kh}) ds \right\|_H^2 \right] \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$ due to Assumption 3. Combining (29) and (30) then shows

$$\begin{aligned} & \|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \\ & \leq c \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} \|(X_s - O_s) - (X_{kh} - O_{kh})\|_{L^2(\Omega; H)} ds \\ & \quad + \left(\sum_{k=0}^{m-2} \left\| \int_{kh}^{(k+1)h} e^{A(mh-s)} F'(X_{kh})(O_s - e^{A(s-kh)} O_{kh}) ds \right\|_{L^2(\Omega; H)}^2 \right)^{1/2} \\ & \quad + \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} \|e^{A(mh-s)} F'(X_{kh})((e^{A(s-kh)} - I)O_{kh})\|_{L^2(\Omega; H)} ds \\ & \quad + cR \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} \|O_s - O_{kh}\|_{L^4(\Omega; V)}^2 ds + \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} R \frac{(s - kh)}{(mh - s)} ds \\ & \quad + 2R^2 M^{-1} \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$. Hence, we obtain

$$\begin{aligned} & \|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \\ & \leq cR \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} (s - kh) ds \\ & \quad + \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} R \frac{(s - kh)}{(mh - (k + 1)h)} ds \\ & \quad + \left(\sum_{k=0}^{m-2} \left\| \int_{kh}^{(k+1)h} e^{A(mh-s)} F'(X_{kh})(O_s - e^{A(s-kh)} O_{kh}) ds \right\|_{L^2(\Omega; H)}^2 \right)^{1/2} \\ & \quad + \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} \|e^{A(mh-s)} (-A)^{1/2}\|_{L(H)} \\ & \quad \quad \times \|(-A)^{-1/2} F'(X_{kh})((e^{A(s-kh)} - I) O_{kh})\|_{L^2(\Omega; H)} ds \\ & \quad + cR \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} \|O_s - O_{kh}\|_{L^4(\Omega; V)}^2 ds + 2R^2 M^{-1} \end{aligned}$$

and

$$\begin{aligned} & \|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \\ & \leq \frac{1}{2} cRMh^2 \\ & \quad + R \sum_{k=0}^{m-2} \frac{h}{2(m - k - 1)} + 2R^2 M^{-1} \\ & \quad + \left\{ \sum_{k=0}^{m-2} \left(\int_{kh}^{(k+1)h} \|e^{A(mh-s)} F'(X_{kh}) \right. \right. \\ & \quad \quad \left. \left. \times (O_s - e^{A(s-kh)} O_{kh})\|_{L^2(\Omega; H)} ds \right)^2 \right\}^{1/2} \\ & \quad + \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} (mh - s)^{-1/2} \\ & \quad \quad \times \|(-A)^{-1/2} F'(X_{kh})((e^{A(s-kh)} - I) O_{kh})\|_{L^2(\Omega; H)} ds \\ & \quad + cR \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} (R(s - kh)^\theta)^2 ds \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$. This yields

$$\begin{aligned} & \|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \\ & \leq \frac{1}{2}cRTh + \frac{1}{2}Rh \left(\sum_{k=1}^{m-1} \frac{1}{k} \right) + 2R^2M^{-1} \\ & \quad + cR^3 \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} (s - kh)^{2\theta} ds \\ & \quad + \left\{ \sum_{k=0}^{m-2} \left(\int_{kh}^{(k+1)h} \|F'(X_{kh})(O_s - e^{A(s-kh)}O_{kh})\|_{L^2(\Omega; H)} ds \right)^2 \right\}^{1/2} \\ & \quad + \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} (mh - (k + 1)h)^{-1/2} \\ & \quad \quad \quad \times \|(-A)^{-1/2}F'(X_{kh})(e^{A(s-kh)} - I)O_{kh}\|_{L^2(\Omega; H)} ds \end{aligned}$$

and

$$\begin{aligned} & \|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \\ & \leq \frac{1}{2}cRT^2M^{-1} + \frac{1}{2}R^2M^{-1} \left(1 + \sum_{k=2}^{m-1} \frac{1}{k} \right) \\ & \quad + 2R^2M^{-1} + cR^3Mh^{(1+2\theta)} \\ & \quad + \sqrt{h} \left\{ \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} \|F'(X_{kh})(O_s - e^{A(s-kh)}O_{kh})\|_{L^2(\Omega; H)}^2 ds \right\}^{1/2} \\ & \quad + \sqrt{T} \sum_{k=0}^{m-2} \frac{1}{(m - k - 1)h} \int_{kh}^{(k+1)h} \|(-A)^{-1/2}F'(X_{kh}) \\ & \quad \quad \quad \times ((e^{A(s-kh)} - I)O_{kh})\|_{L^2(\Omega; H)} ds \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$. Hence, we have

$$\begin{aligned} & \|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \\ & \leq \frac{1}{2}R^4M^{-1} + \frac{1}{2}R^2M^{-1} \left(1 + \int_1^M \frac{1}{s} ds \right) + 2R^2M^{-1} + cR^3Th^{2\theta} \\ & \quad + \sqrt{h} \left\{ \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} c^2 \|O_s - e^{A(s-kh)}O_{kh}\|_{L^2(\Omega; H)}^2 ds \right\}^{1/2} \end{aligned}$$

$$\begin{aligned}
 &+ R \sum_{k=0}^{m-2} \frac{1}{(m-k-1)h} \\
 &\quad \times \int_{kh}^{(k+1)h} \|(-A)^{-1/2} F'(X_{kh})(e^{A(s-kh)} - I) O_{kh}\|_{L^2(\Omega; H)} ds
 \end{aligned}$$

and

$$\begin{aligned}
 &\|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \\
 &\leq \frac{1}{2} R^4 M^{-1} + \frac{1}{2} R^2 M^{-1} (1 + \log(M)) + 2R^2 M^{-1} + R^6 M^{-2\theta} \\
 &\quad + \sqrt{T} c M^{-1/2} \left\{ \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} \|O_s - e^{A(s-kh)} O_{kh}\|_{L^2(\Omega; H)}^2 ds \right\}^{1/2} \\
 &\quad + \sum_{k=0}^{m-2} \frac{R}{(m-k-1)h} \\
 &\quad \times \int_{kh}^{(k+1)h} \left\| \sup_{\|w\|_H \leq 1} \langle (w, (-A)^{-1/2} F'(X_{kh}) \right. \\
 &\quad \quad \left. \times (e^{A(s-kh)} - I) O_{kh} \rangle_H \right\|_{L^2(\Omega; \mathbb{R})} ds
 \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$. This yields

$$\begin{aligned}
 &\|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \\
 &\leq \left(\frac{1}{2} R^4 + \frac{1}{2} R^2 + 2R^2 + R^6 \right) \frac{(1 + \log(M))}{M^{2\theta}} \\
 &\quad + \sum_{k=0}^{m-2} \frac{R}{(m-k-1)h} \\
 &\quad \times \int_{kh}^{(k+1)h} \left\| \sup_{\|w\|_H \leq 1} \langle (F'(X_{kh}))^* (-A)^{-1/2} w, \right. \\
 &\quad \quad \left. (e^{A(s-kh)} - I) O_{kh} \rangle_H \right\|_{L^2(\Omega; \mathbb{R})} ds \\
 &\quad + R^2 M^{-1/2} \left\{ \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} (\|O_s - O_{kh}\|_{L^2(\Omega; H)} \right. \\
 &\quad \quad \left. + \|e^{A(s-kh)} O_{kh} - O_{kh}\|_{L^2(\Omega; H)})^2 ds \right\}^{1/2}
 \end{aligned}$$

and

$$\begin{aligned}
 & \|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \\
 & \leq 4R^6 \frac{(1 + \log(M))}{M^{2\theta}} \\
 & \quad + \sum_{k=0}^{m-2} \frac{R}{(m - k - 1)h} \\
 & \quad \quad \times \int_{kh}^{(k+1)h} \left\| \sup_{\|w\|_H \leq 1} \|(F'(X_{kh}))^* (-A)^{-1/2} w\|_{D((-A)^{1/2})} \right. \\
 & \quad \quad \quad \times \left. \|(e^{A(s-kh)} - I) O_{kh}\|_{D((-A)^{-1/2})} \right\|_{L^2(\Omega; \mathbb{R})} ds \\
 & \quad + R^3 M^{-1/2} \\
 & \quad \times \left\{ \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} (\|O_s - O_{kh}\|_{L^2(\Omega; V)} \right. \\
 & \quad \quad \left. + \|(e^{A(s-kh)} - I) O_{kh}\|_{L^2(\Omega; H)})^2 ds \right\}^{1/2}
 \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$. Using now condition (13) in Assumption 2 shows

$$\begin{aligned}
 & \|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \\
 & \leq 4R^6 \frac{(1 + \log(M))}{M^{2\theta}} \\
 & \quad + \sum_{k=0}^{m-2} \frac{R}{(m - k - 1)h} \\
 & \quad \quad \times \int_{kh}^{(k+1)h} \|c(1 + \|X_{kh}\|_{D((-A)^{1/2})})\| \\
 (31) \quad & \quad \quad \times \|(e^{A(s-kh)} - I) O_{kh}\|_{D((-A)^{-1/2})} \|_{L^2(\Omega; \mathbb{R})} ds \\
 & \quad + R^3 M^{-1/2} \\
 & \quad \times \left\{ \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} (R(s - kh)^\theta \right. \\
 & \quad \quad \left. + (s - kh)^\nu \|O_{kh}\|_{L^2(\Omega; D((-A)^\nu))})^2 ds \right\}^{1/2}
 \end{aligned}$$

and therefore

$$\begin{aligned} & \|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \\ & \leq 4R^6 \frac{(1 + \log(M))}{M^{2\theta}} \\ & \quad + \sum_{k=0}^{m-2} \frac{cR}{(m - k - 1)h} \int_{kh}^{(k+1)h} \|1 + \|X_{kh}\|_{D((-A)^{1/2})}\|_{L^4(\Omega; \mathbb{R})} \\ & \quad \quad \quad \times \|(e^{A(s-kh)} - I)O_{kh}\|_{L^4(\Omega; D((-A)^{-1/2})} ds \\ & \quad + R^3 M^{-1/2} \left\{ \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} (Rh^\theta + Rh^\theta T^{(\gamma-\theta)})^2 ds \right\}^{1/2} \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$. Hence, we obtain

$$\begin{aligned} & \|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \\ & \leq 4R^6 \frac{(1 + \log(M))}{M^{2\theta}} \\ & \quad + \sum_{k=0}^{m-2} \frac{cR}{(m - k - 1)h} (1 + \|X_{kh}\|_{L^4(\Omega; D((-A)^{1/2})}) \\ & \quad \quad \quad \times \int_{kh}^{(k+1)h} \|(e^{A(s-kh)} - I)O_{kh}\|_{L^4(\Omega; D((-A)^{-1/2})} ds \\ & \quad + R^3 M^{-1/2} \left\{ \sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} (2R^2 h^\theta)^2 ds \right\}^{1/2} \end{aligned}$$

and

$$\begin{aligned} & \|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \\ & \leq 4R^6 \frac{(1 + \log(M))}{M^{2\theta}} \\ & \quad + 2R^5 M^{-1/2} \left(\sum_{k=0}^{m-2} \int_{kh}^{(k+1)h} h^{2\theta} ds \right)^{1/2} \\ & \quad + \sum_{k=0}^{m-2} \frac{cR(1 + R)}{(m - k - 1)h} \\ & \quad \quad \quad \times \int_{kh}^{(k+1)h} \|(-A)^{-(\gamma+1/2)}(e^{A(s-kh)} - I)\|_{L(H)} \\ & \quad \quad \quad \times \|O_{kh}\|_{L^4(\Omega; D((-A)^\gamma)} ds \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$. This yields

$$\begin{aligned} & \|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \\ & \leq 4R^6 \frac{(1 + \log(M))}{M^{2\theta}} + 2R^5 M^{-1/2} (Mh^{(1+2\theta)})^{1/2} \\ & \quad + \sum_{k=0}^{m-2} \frac{2cR^3}{(m-k-1)h} \int_{kh}^{(k+1)h} \|(-A)^{-(\gamma+1/2)} (e^{A(s-kh)} - I)\|_{L(H)} ds \end{aligned}$$

and hence

$$\begin{aligned} & \|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \\ & \leq 4R^6 \frac{(1 + \log(M))}{M^{2\theta}} + 2R^5 \sqrt{T} M^{-1/2} h^\theta \\ & \quad + \sum_{k=0}^{m-2} \frac{2R^4}{(m-k-1)h} \int_{kh}^{(k+1)h} \|(-A)^{(1/2-\gamma)}\|_{L(H)} \\ & \quad \quad \quad \times \|A^{-1}(e^{A(s-kh)} - I)\|_{L(H)} ds \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} & \|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \\ & \leq 4R^6 \frac{(1 + \log(M))}{M^{2\theta}} + 2R^6 M^{-(1/2+\theta)} \\ & \quad + \sum_{k=0}^{m-2} \frac{2R^4}{(m-k-1)h} \int_{kh}^{(k+1)h} \left(\frac{1}{\lambda_1}\right)^{(\gamma-1/2)} (s-kh) ds \end{aligned}$$

and, finally,

$$\begin{aligned} (32) \quad & \|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \leq 6R^6 \frac{(1 + \log(M))}{M^{2\theta}} + \sum_{k=0}^{m-2} \frac{R^5 h}{(m-k-1)} \\ & \leq 6R^6 \frac{(1 + \log(M))}{M^{2\theta}} + R^6 M^{-1} \left(\sum_{k=1}^M \frac{1}{k}\right) \\ & \leq 6R^6 \frac{(1 + \log(M))}{M^{2\theta}} + R^6 M^{-2\theta} \left(1 + \int_1^M \frac{1}{s} ds\right) \\ & = 7R^6 \frac{(1 + \log(M))}{M^{2\theta}} \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$.

6.1.2. *Spatial discretization error.* Due to (28), we obtain

$$\begin{aligned} & \|Y_m^M - P_N(Y_m^M)\|_{L^2(\Omega; H)} \\ &= \left\| e^{Amh}(\xi - P_N(\xi)) \right. \\ &\quad \left. + h \left(\sum_{k=0}^{m-1} (e^{A(mh-kh)} - P_N e^{A(mh-kh)}) F(X_{kh}) \right) \right. \\ &\quad \left. + O_{mh} - P_N(O_{mh}) \right\|_{L^2(\Omega; H)} \\ &\leq \|e^{Amh}(\xi - P_N(\xi))\|_{L^2(\Omega; H)} + \|O_{mh} - P_N(O_{mh})\|_{L^2(\Omega; H)} \\ &\quad + \left\| h \left(\sum_{k=0}^{m-1} (e^{A(mh-kh)} - P_N e^{A(mh-kh)}) F(X_{kh}) \right) \right\|_{L^2(\Omega; H)} \end{aligned}$$

and

$$\begin{aligned} & \|Y_m^M - P_N(Y_m^M)\|_{L^2(\Omega; H)} \\ &\leq \|\xi - P_N(\xi)\|_{L^2(\Omega; H)} + \|O_{mh} - P_N(O_{mh})\|_{L^2(\Omega; H)} \\ &\quad + h \left(\sum_{k=0}^{m-1} \|e^{A(mh-kh)} - P_N e^{A(mh-kh)}\|_{L(H)} \|F(X_{kh})\|_{L^2(\Omega; H)} \right) \\ &\leq \|(I - P_N)\xi\|_{L^2(\Omega; H)} + \|(I - P_N)O_{mh}\|_{L^2(\Omega; H)} \\ &\quad + Rh \left(\sum_{k=0}^{m-1} \|(I - P_N)e^{A(mh-kh)}\|_{L(H)} \right) \end{aligned}$$

and hence

$$\begin{aligned} & \|Y_m^M - P_N(Y_m^M)\|_{L^2(\Omega; H)} \\ &\leq \|(-A)^{-\gamma}(I - P_N)\|_{L(H)} \|(-A)^\gamma \xi\|_{L^2(\Omega; H)} \\ &\quad + \|(-A)^{-\gamma}(I - P_N)\|_{L(H)} \|(-A)^\gamma O_{mh}\|_{L^2(\Omega; H)} \\ &\quad + Rh \left(\sum_{k=0}^{m-1} \|(-A)^{-\gamma}(I - P_N)\|_{L(H)} \|(-A)^\gamma e^{A(mh-kh)}\|_{L(H)} \right) \\ &\leq (\lambda_N)^{-\gamma} (\|(-A)^\gamma \xi\|_{L^2(\Omega; H)} + \|(-A)^\gamma O_{mh}\|_{L^2(\Omega; H)}) \\ &\quad + Rh \left(\sum_{k=0}^{m-1} (\lambda_N)^{-\gamma} \|(-A)^\gamma e^{A(mh-kh)}\|_{L(H)} \right) \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} & \|Y_m^M - P_N(Y_m^M)\|_{L^2(\Omega; H)} \\ & \leq (\lambda_N)^{-\gamma} (\|\xi\|_{L^2(\Omega; D((-A)^\gamma))} + \|O_{mh}\|_{L^2(\Omega; D((-A)^\gamma))}) \\ & \quad + Rh(\lambda_N)^{-\gamma} \left(\sum_{k=0}^{m-1} \frac{1}{(mh - kh)^\gamma} \|(-A(mh - kh))^\gamma e^{A(mh - kh)}\|_{L(H)} \right) \\ & \leq 2R(\lambda_N)^{-\gamma} + Rh^{(1-\gamma)}(\lambda_N)^{-\gamma} \left(\sum_{k=0}^{m-1} \frac{1}{(m - k)^\gamma} \left(\sup_{x>0} x^\gamma e^{-x} \right) \right) \\ & \leq 2R(\lambda_N)^{-\gamma} + Rh^{(1-\gamma)}(\lambda_N)^{-\gamma} \left(\sum_{k=1}^m \frac{1}{k^\gamma} \right) \\ & \leq 2R(\lambda_N)^{-\gamma} + Rh^{(1-\gamma)}(\lambda_N)^{-\gamma} \left(1 + \sum_{k=2}^m \frac{1}{k^\gamma} \right) \end{aligned}$$

and

$$\begin{aligned} & \|Y_m^M - P_N(Y_m^M)\|_{L^2(\Omega; H)} \\ & \leq R(\lambda_N)^{-\gamma} \left(2 + h^{(1-\gamma)} \left(1 + \int_1^M \frac{1}{s^\gamma} ds \right) \right) \\ (33) \quad & = R(\lambda_N)^{-\gamma} \left(2 + h^{(1-\gamma)} \left(1 + \left[\frac{s^{(1-\gamma)}}{(1-\gamma)} \right]_{s=1}^{s=M} \right) \right) \\ & = R(\lambda_N)^{-\gamma} \left(2 + h^{(1-\gamma)} \left(1 + \frac{M^{(1-\gamma)}}{(1-\gamma)} - \frac{1}{(1-\gamma)} \right) \right) \\ & \leq R(\lambda_N)^{-\gamma} \left(2 + \frac{T^{(1-\gamma)}}{(1-\gamma)} \right) \leq 3R^3(\lambda_N)^{-\gamma} \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $M \in \mathbb{N}$.

6.1.3. *Lipschitz estimates.* Note that $Y_m^{N,M} : \Omega \rightarrow V$ satisfies

$$(34) \quad Y_m^{N,M} = e^{Amh}(P_N(\xi)) + h \left(\sum_{k=0}^{m-1} P_N e^{A(mh - kh)} F(Y_k^{N,M}) \right) + P_N(O_{mh})$$

for every $m \in \{0, 1, \dots, M\}$ and every $N, M \in \mathbb{N}$. Indeed, in the case $m = 0$ we have

$$\begin{aligned} Y_0^{N,M} & = P_N(\xi) + P_N(O_0) \\ & = e^{A0}(P_N(\xi)) + h \left(\sum_{k=0}^{-1} P_N e^{A(0 - kh)} F(Y_k^{N,M}) \right) + P_N(O_0) \end{aligned}$$

for every $N, M \in \mathbb{N}$. Moreover, if (34) holds for one $m \in \{0, 1, \dots, M - 1\}$, then we obtain

$$\begin{aligned} Y_{m+1}^{N,M} &= e^{Ah}(Y_m^{N,M} + h \cdot (P_N F)(Y_m^{N,M})) + P_N(O_{(m+1)h} - e^{Ah} O_{mh}) \\ &= e^{Ah} Y_m^{N,M} + h \cdot P_N e^{Ah} F(Y_m^{N,M}) + P_N(O_{(m+1)h}) - e^{Ah} P_N(O_{mh}) \\ &= e^{Ah}(Y_m^{N,M} - P_N(O_{mh})) + h \cdot P_N e^{Ah} F(Y_m^{N,M}) + P_N(O_{(m+1)h}) \end{aligned}$$

and

$$\begin{aligned} Y_{m+1}^{N,M} &= e^{Ah} \left(e^{Amh}(P_N(\xi)) + h \left(\sum_{k=0}^{m-1} P_N e^{A(mh-kh)} F(Y_k^{N,M}) \right) \right) \\ &\quad + h \cdot P_N e^{Ah} F(Y_m^{N,M}) + P_N(O_{(m+1)h}) \\ &= e^{A(m+1)h}(P_N(\xi)) + h \left(\sum_{k=0}^{m-1} P_N e^{A((m+1)h-kh)} F(Y_k^{N,M}) \right) \\ &\quad + h \cdot P_N e^{Ah} F(Y_m^{N,M}) + P_N(O_{(m+1)h}) \\ &= e^{A(m+1)h}(P_N(\xi)) + h \left(\sum_{k=0}^m P_N e^{A((m+1)h-kh)} F(Y_k^{N,M}) \right) \\ &\quad + P_N(O_{(m+1)h}) \end{aligned}$$

for every $N, M \in \mathbb{N}$, which shows (34) by induction. In the next step, (34) yields

$$\begin{aligned} P_N(Y_m^N) - Y_m^{N,M} &= h \left(\sum_{k=0}^{m-1} P_N e^{A(mh-kh)} F(X_{kh}) \right) \\ &\quad - h \left(\sum_{k=0}^{m-1} P_N e^{A(mh-kh)} F(Y_k^{N,M}) \right) \\ &= h \left(\sum_{k=0}^{m-1} P_N e^{A(mh-kh)} (F(X_{kh}) - F(Y_k^{N,M})) \right) \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $N, M \in \mathbb{N}$. Therefore, we obtain

$$\begin{aligned} &\|P_N(Y_m^N) - Y_m^{N,M}\|_{L^2(\Omega; H)} \\ &\leq h \sum_{k=0}^{m-1} \|P_N e^{A(mh-kh)} (F(X_{kh}) - F(Y_k^{N,M}))\|_{L^2(\Omega; H)} \\ (35) \quad &\leq h \sum_{k=0}^{m-1} (\|P_N e^{A(mh-kh)}\|_{L(H)} \|F(X_{kh}) - F(Y_k^{N,M})\|_{L^2(\Omega; H)}) \end{aligned}$$

$$\begin{aligned} &\leq h \sum_{k=0}^{m-1} \|F(X_{kh}) - F(Y_k^{N,M})\|_{L^2(\Omega;H)} \\ &\leq ch \sum_{k=0}^{m-1} \|X_{kh} - Y_k^{N,M}\|_{L^2(\Omega;H)} \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $N, M \in \mathbb{N}$. Combining (32), (33) and (35) finally yields

$$\begin{aligned} &\|X_{mh} - Y_m^{N,M}\|_{L^2(\Omega;H)} \\ &\leq \|X_{mh} - Y_m^M\|_{L^2(\Omega;H)} \\ &\quad + \|Y_m^M - P_N(Y_m^M)\|_{L^2(\Omega;H)} + \|P_N(Y_m^M) - Y_m^{N,M}\|_{L^2(\Omega;H)} \\ &\leq 7R^6 \frac{(1 + \log(M))}{M^{2\theta}} + 3R^3 \frac{1}{(\lambda_N)^\gamma} + ch \sum_{k=0}^{m-1} \|X_{kh} - Y_k^{N,M}\|_{L^2(\Omega;H)} \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $N, M \in \mathbb{N}$. Hence, Gronwall’s lemma yields

$$\begin{aligned} (36) \quad &\|X_{mh} - Y_m^{N,M}\|_{L^2(\Omega;H)} \leq \left(7R^6 \frac{(1 + \log(M))}{M^{2\theta}} + 3R^3 \frac{1}{(\lambda_N)^\gamma}\right) e^{cT} \\ &\leq \left(7R^6 \frac{(1 + \log(M))}{M^{2\theta}} + 7R^6 \frac{1}{(\lambda_N)^\gamma}\right) e^{cT} \\ &= (e^{cT} 7R^6) \left(\frac{(1 + \log(M))}{M^{2\theta}} + \frac{1}{(\lambda_N)^\gamma}\right) \end{aligned}$$

for every $m \in \{0, 1, \dots, M\}$ and every $N, M \in \mathbb{N}$, which shows the assertion.

6.2. Properties of the SPDE (1).

PROOF OF LEMMA 1. A standard application of Banach’s fix point theorem (see, e.g., Section 7.1 in [2]) yields the existence of a unique adapted stochastic process $X : [0, T] \times \Omega \rightarrow V$ with continuous sample paths which fulfills (14). Moreover, we have

$$(37) \quad \int_0^t e^{A(t-s)} F(X_s(\omega)) ds \in D((-A)^\gamma)$$

for every $t \in [0, T]$ and every $\omega \in \Omega$, since

$$\begin{aligned} &\int_0^t \|(-A)^\gamma e^{A(t-s)} F(X_s(\omega))\|_H ds \\ &\leq \int_0^t \|(-A)^\gamma e^{A(t-s)}\|_{L(H)} \|F(X_s(\omega))\|_H ds \end{aligned}$$

$$\begin{aligned} &\leq \int_0^t (t-s)^{-\gamma} \|F(X_s(\omega))\|_H ds \\ &\leq \int_0^t (t-s)^{-\gamma} (c\|X_s(\omega)\|_H + \|F(0)\|_H) ds \\ &\leq \left(\int_0^t s^{-\gamma} ds\right) \left(c\left(\sup_{0 \leq s \leq T} \|X_s(\omega)\|_H\right) + \|F(0)\|_H\right) \end{aligned}$$

and

$$\begin{aligned} &\int_0^t \|(-A)^\gamma e^{A(t-s)} F(X_s(\omega))\|_H ds \\ &\leq \left[\frac{s^{(1-\gamma)}}{(1-\gamma)}\right]_{s=0}^{s=T} \left(c\left(\sup_{0 \leq s \leq T} \|X_s(\omega)\|_H\right) + \|F(0)\|_H\right) \\ &\leq \frac{T^{(1-\gamma)}}{(1-\gamma)} \left(c\left(\sup_{0 \leq s \leq T} \|X_s(\omega)\|_H\right) + \|F(0)\|_H\right) < \infty \end{aligned}$$

holds for every $t \in [0, T]$ and every $\omega \in \Omega$. Assumptions 3, 4 and (37) hence imply $X_t(\omega) \in D((-A)^\gamma)$ for every $t \in [0, T]$ and every $\omega \in \Omega$. Furthermore, we have

$$\begin{aligned} &\|(-A)^\gamma X_t\|_H \\ &\leq \|(-A)^\gamma e^{At} \xi\|_H + \int_0^t \|(-A)^\gamma e^{A(t-s)} F(X_s)\|_H ds + \|(-A)^\gamma O_t\|_H \\ &\leq \|(-A)^\gamma \xi\|_H + \int_0^t \|(-A)^\gamma e^{A(t-s)}\|_{L(H)} \|F(X_s)\|_H ds \\ &\quad + \sup_{0 \leq s \leq T} \|(-A)^\gamma O_s\|_H \\ &\leq \|(-A)^\gamma \xi\|_H + \int_0^t (t-s)^{-\gamma} \|F(X_s)\|_H ds + \sup_{0 \leq s \leq T} \|(-A)^\gamma O_s\|_H \\ &\leq \left(\|(-A)^\gamma \xi\|_H + \sup_{0 \leq s \leq T} \|(-A)^\gamma O_s\|_H\right) \\ &\quad + \int_0^t (t-s)^{-\gamma} (c\|X_s\|_H + \|F(0)\|_H) ds \end{aligned}$$

for every $t \in [0, T]$. This yields

$$\begin{aligned} &\|(-A)^\gamma X_t\|_H \\ &\leq c \int_0^t (t-s)^{-\gamma} \|X_s\|_H ds \\ &\quad + \left(\|(-A)^\gamma \xi\|_H + \sup_{0 \leq s \leq T} \|(-A)^\gamma O_s\|_H + \|F(0)\|_H \left(\int_0^t s^{-\gamma} ds\right)\right) \end{aligned}$$

$$\begin{aligned} &\leq \left(\|(-A)^\gamma \xi\|_H + \sup_{0 \leq s \leq T} \|(-A)^\gamma O_s\|_H + \frac{T^{(1-\gamma)} \|F(0)\|_H}{(1-\gamma)} \right) \\ &\quad + c \|(-A)^{-\gamma}\|_{L(H)} \int_0^t (t-s)^{-\gamma} \|(-A)^\gamma X_s\|_H ds \end{aligned}$$

for every $t \in [0, T]$. Hence, Lemma 7.1.1 in [8] shows

$$\begin{aligned} &\sup_{0 \leq t \leq T} \|(-A)^\gamma X_t\|_H \\ &\leq E_{(1-\gamma)}(T(c\|(-A)^{-\gamma}\|_{L(H)}\Gamma(1-\gamma))^{1/(1-\gamma)}) \\ &\quad \times \left(\|(-A)^\gamma \xi\|_H + \sup_{0 \leq s \leq T} \|(-A)^\gamma O_s\|_H + \frac{T^{(1-\gamma)} \|F(0)\|_H}{(1-\gamma)} \right) \end{aligned}$$

and therefore

$$\begin{aligned} &\left\| \sup_{0 \leq t \leq T} \|(-A)^\gamma X_t\|_H \right\|_{L^4(\Omega; \mathbb{R})} \\ &\leq E_{(1-\gamma)}(T(c\|(-A)^{-\gamma}\|_{L(H)}\Gamma(1-\gamma))^{1/(1-\gamma)}) \\ &\quad \times \left(\|(-A)^\gamma \xi\|_{L^4(\Omega; H)} + \left\| \sup_{0 \leq s \leq T} \|(-A)^\gamma O_s\|_H \right\|_{L^4(\Omega; \mathbb{R})} \right. \\ &\quad \left. + \frac{T^{(1-\gamma)} \|F(0)\|_H}{(1-\gamma)} \right) < \infty, \end{aligned}$$

which shows the assertion. Here $E_{(1-\gamma)} : [0, \infty) \rightarrow [0, \infty)$ is given by

$$E_{(1-\gamma)}(x) := \sum_{n=0}^{\infty} \frac{x^{n(1-\gamma)}}{\Gamma(n(1-\gamma) + 1)}$$

for every $x \in [0, \infty)$ where $\Gamma : (0, \infty) \rightarrow (0, \infty)$ is the Gamma function. \square

LEMMA 2. *Let Assumptions 1–4 be fulfilled. Then we have*

$$\|e^{At_2} - e^{At_1}\|_{L(H)} \leq \frac{(t_2 - t_1)}{t_1}$$

for every $t_1, t_2 \in (0, T]$ with $t_1 \leq t_2$.

PROOF. By definition, we have

$$\begin{aligned} &\|e^{At_2} - e^{At_1}\|_{L(H)} \\ &= \|(e^{A(t_2-t_1)} - I)e^{At_1}\|_{L(H)} \\ &\leq \|A^{-1}(e^{A(t_2-t_1)} - I)\|_{L(H)} \|Ae^{At_1}\|_{L(H)} \end{aligned}$$

$$\begin{aligned} &= \|(A(t_2 - t_1))^{-1}(e^{A(t_2-t_1)} - I)\|_{L(H)} \\ &\quad \times \|At_1 e^{At_1}\|_{L(H)} \frac{(t_2 - t_1)}{t_1} \\ &\leq \left(\sup_{x \in (0, \infty)} \frac{(1 - e^{-x})}{x} \right) \left(\sup_{x \in (0, \infty)} x e^{-x} \right) \frac{(t_2 - t_1)}{t_1} \\ &\leq \frac{(t_2 - t_1)}{t_1} \end{aligned}$$

for every $t_1, t_2 \in (0, T]$ with $t_1 < t_2$. \square

LEMMA 3. *Let Assumptions 1–4 be fulfilled. Then we obtain*

$$\sup_{0 \leq t_1 < t_2 \leq T} \frac{\|X_{t_2} - X_{t_1}\|_{L^2(\Omega; H)}}{(t_2 - t_1)^\theta} < \infty,$$

where $\theta \in (0, \frac{1}{2}]$ is given in Assumption 3 and where $X : \Omega \times [0, T] \rightarrow D((-A)^\nu)$ is the solution of the SPDE (14).

PROOF. First, let $R \in [0, \infty)$ be the real number given by

$$\begin{aligned} R := & \|\xi\|_{L^2(\Omega; D(A))} + \sup_{t \in [0, T]} \|F(X_t)\|_{L^2(\Omega; H)} \\ & + \sup_{0 \leq t_1 < t_2 \leq T} \left(\frac{\|O_{t_2} - O_{t_1}\|_{L^2(\Omega; H)}}{(t_2 - t_1)^\theta} \right), \end{aligned}$$

which is finite due to Assumptions 1–4. Then we have

$$\begin{aligned} &\|e^{At_2}\xi - e^{At_1}\xi\|_{L^2(\Omega; H)} \\ (38) \quad &= \|e^{At_1}(e^{A(t_2-t_1)}\xi - \xi)\|_{L^2(\Omega; H)} \leq \|e^{A(t_2-t_1)}\xi - \xi\|_{L^2(\Omega; H)} \\ &\leq \|A^{-1}(e^{A(t_2-t_1)} - I)\|_{L(H)} \|\xi\|_{L^2(\Omega; D(A))} \leq R(t_2 - t_1) \end{aligned}$$

for every $0 \leq t_1 < t_2 \leq T$. Moreover, we obtain

$$\begin{aligned} &\left\| \int_0^{t_2} e^{A(t_2-s)} F(X_s) ds - \int_0^{t_1} e^{A(t_1-s)} F(X_s) ds \right\|_{L^2(\Omega; H)} \\ &= \left\| \int_{t_1}^{t_2} e^{A(t_2-s)} F(X_s) ds + \int_0^{t_1} (e^{A(t_2-s)} - e^{A(t_1-s)}) F(X_s) ds \right\|_{L^2(\Omega; H)} \\ &\leq \int_{t_1}^{t_2} \|F(X_s)\|_{L^2(\Omega; H)} ds + \left\| \int_0^{t_1} (e^{A(t_2-s)} - e^{A(t_1-s)}) F(X_s) ds \right\|_{L^2(\Omega; H)} \end{aligned}$$

and hence

$$\begin{aligned} & \left\| \int_0^{t_2} e^{A(t_2-s)} F(X_s) ds - \int_0^{t_1} e^{A(t_1-s)} F(X_s) ds \right\|_{L^2(\Omega; H)} \\ & \leq R(t_2 - t_1) + \int_0^{t_1} \|e^{A(t_2-s)} - e^{A(t_1-s)}\|_{L(H)} \|F(X_s)\|_{L^2(\Omega; H)} ds \\ & \leq R(t_2 - t_1) + R \int_0^{t_1} \|e^{A(t_2-s)} - e^{A(t_1-s)}\|_{L(H)}^{(1-\theta)} \|e^{A(t_2-s)} - e^{A(t_1-s)}\|_{L(H)}^\theta ds \end{aligned}$$

for every $0 \leq t_1 < t_2 \leq T$. This yields

$$\begin{aligned} & \left\| \int_0^{t_2} e^{A(t_2-s)} F(X_s) ds - \int_0^{t_1} e^{A(t_1-s)} F(X_s) ds \right\|_{L^2(\Omega; H)} \\ & \leq R(t_2 - t_1) + 2^{(1-\theta)} R \int_0^{t_1} \left(\frac{t_2 - t_1}{t_1 - s} \right)^\theta ds \\ & \leq R(t_2 - t_1) + 2R(t_2 - t_1)^\theta \int_0^{t_1} (t_1 - s)^{-\theta} ds \end{aligned}$$

due to Lemma 2 and therefore, we obtain

$$\begin{aligned} & \left\| \int_0^{t_2} e^{A(t_2-s)} F(X_s) ds - \int_0^{t_1} e^{A(t_1-s)} F(X_s) ds \right\|_{L^2(\Omega; H)} \\ (39) \quad & \leq R(t_2 - t_1) + 2R(t_2 - t_1)^\theta \int_0^{t_1} s^{-\theta} ds \\ & \leq R(t_2 - t_1) + \left(\frac{2}{(1-\theta)} RT^{(1-\theta)} \right) (t_2 - t_1)^\theta \end{aligned}$$

for every $0 \leq t_1 < t_2 \leq T$. Combining (38), (39) and Assumption 3 yields the assertion. \square

LEMMA 4. *Let Assumptions 1–4 be fulfilled. Then we obtain*

$$\sup_{0 \leq t_1 < t_2 \leq T} \frac{\|(X_{t_2} - O_{t_2}) - (X_{t_1} - O_{t_1})\|_{L^2(\Omega; H)}}{(t_2 - t_1)} < \infty,$$

where $O : [0, T] \times \Omega \rightarrow D((-A)^\gamma)$ is given in Assumption 3 and where $X : [0, T] \times \Omega \rightarrow D((-A)^\gamma)$ is the solution of the SPDE (14).

PROOF. First, let $R \in [0, \infty)$ be the real number given by

$$R := \|\xi\|_{L^2(\Omega; D(A))} + \sup_{t \in [0, T]} \|F(X_t)\|_{L^2(\Omega; H)} + \sup_{0 \leq t_1 < t_2 \leq T} \frac{\|X_{t_2} - X_{t_1}\|_{L^2(\Omega; H)}}{(t_2 - t_1)^\theta},$$

which exists due to Lemma 3. Then we have

$$\begin{aligned}
 (40) \quad & \|e^{At_2}\xi - e^{At_1}\xi\|_{L^2(\Omega;H)} = \|e^{At_1}(e^{A(t_2-t_1)}\xi - \xi)\|_{L^2(\Omega;H)} \\
 & \leq \|e^{A(t_2-t_1)}\xi - \xi\|_{L^2(\Omega;H)} \\
 & \leq R(t_2 - t_1)
 \end{aligned}$$

for every $0 \leq t_1 < t_2 \leq T$. Moreover, we have

$$\begin{aligned}
 & \left\| \int_0^{t_2} e^{A(t_2-s)} F(X_s) ds - \int_0^{t_1} e^{A(t_1-s)} F(X_s) ds \right\|_{L^2(\Omega;H)} \\
 &= \left\| \int_{t_1}^{t_2} e^{A(t_2-s)} F(X_s) ds \right. \\
 &\quad \left. + \int_0^{t_1} (e^{A(t_2-s)} - e^{A(t_1-s)}) F(X_s) ds \right\|_{L^2(\Omega;H)} \\
 &\leq \int_{t_1}^{t_2} \|e^{A(t_2-s)}\|_{L(H)} \|F(X_s)\|_{L^2(\Omega;H)} ds \\
 &\quad + \left\| \int_0^{t_1} (e^{A(t_2-s)} - e^{A(t_1-s)})(F(X_s) - F(X_{t_1})) ds \right\|_{L^2(\Omega;H)} \\
 &\quad + \left\| \int_0^{t_1} (e^{A(t_2-s)} - e^{A(t_1-s)}) F(X_{t_1}) ds \right\|_{L^2(\Omega;H)}
 \end{aligned}$$

and therefore

$$\begin{aligned}
 & \left\| \int_0^{t_2} e^{A(t_2-s)} F(X_s) ds - \int_0^{t_1} e^{A(t_1-s)} F(X_s) ds \right\|_{L^2(\Omega;H)} \\
 &\leq R(t_2 - t_1) \\
 &\quad + \int_0^{t_1} \|e^{A(t_2-s)} - e^{A(t_1-s)}\|_{L(H)} \|F(X_s) - F(X_{t_1})\|_{L^2(\Omega;H)} ds \\
 &\quad + R \left\| \int_0^{t_1} (e^{A(t_2-s)} - e^{A(t_1-s)}) ds \right\|_{L(H)}
 \end{aligned}$$

for every $0 \leq t_1 < t_2 \leq T$. Hence, we obtain

$$\begin{aligned}
 & \left\| \int_0^{t_2} e^{A(t_2-s)} F(X_s) ds - \int_0^{t_1} e^{A(t_1-s)} F(X_s) ds \right\|_{L^2(\Omega;H)} \\
 &\leq R(t_2 - t_1) \\
 &\quad + c \int_0^{t_1} \|e^{A(t_2-s)} - e^{A(t_1-s)}\|_{L(H)} \|X_s - X_{t_1}\|_{L^2(\Omega;H)} ds \\
 &\quad + R \left\| \int_0^{t_1} e^{A((t_2-t_1)+s)} ds - \int_0^{t_1} e^{As} ds \right\|_{L(H)}
 \end{aligned}$$

and

$$\begin{aligned} & \left\| \int_0^{t_2} e^{A(t_2-s)} F(X_s) ds - \int_0^{t_1} e^{A(t_1-s)} F(X_s) ds \right\|_{L^2(\Omega; H)} \\ & \leq R(t_2 - t_1) + cR \int_0^{t_1} \|e^{A(t_2-s)} - e^{A(t_1-s)}\|_{L(H)} |s - t_1|^\theta ds \\ & \quad + R \left\| \int_{(t_2-t_1)}^{t_2} e^{As} ds - \int_0^{t_1} e^{As} ds \right\|_{L(H)} \end{aligned}$$

for every $0 \leq t_1 < t_2 \leq T$. This shows

$$\begin{aligned} & \left\| \int_0^{t_2} e^{A(t_2-s)} F(X_s) ds - \int_0^{t_1} e^{A(t_1-s)} F(X_s) ds \right\|_{L^2(\Omega; H)} \\ & \leq R(t_2 - t_1) \\ & \quad + cR \int_0^{t_1} \frac{(t_2 - t_1)}{(t_1 - s)} |s - t_1|^\theta ds + R \left\| \int_{t_1}^{t_2} e^{As} ds - \int_0^{(t_2-t_1)} e^{As} ds \right\|_{L(H)} \end{aligned}$$

due to Lemma 2 and

$$\begin{aligned} & \left\| \int_0^{t_2} e^{A(t_2-s)} F(X_s) ds - \int_0^{t_1} e^{A(t_1-s)} F(X_s) ds \right\|_{L^2(\Omega; H)} \\ & \leq R(t_2 - t_1) + cR(t_2 - t_1) \int_0^{t_1} (t_1 - s)^{(\theta-1)} ds + 2R(t_2 - t_1) \\ & = R(t_2 - t_1) + cR(t_2 - t_1) \int_0^{t_1} s^{(\theta-1)} ds + 2R(t_2 - t_1) \\ & \leq (R + cR(T + 1)\theta^{-1} + 2R)(t_2 - t_1) \end{aligned}$$

for every $0 \leq t_1 < t_2 \leq T$. Combining this and (40) shows the assertion. \square

Acknowledgments. We strongly thank the anonymous referee for his careful reading and his very valuable advice.

REFERENCES

- [1] BLÖMKER, D. and JENTZEN, A. (2009). Galerkin approximations for the stochastic burgers equation. Preprint, Institute for Mathematics, Univ. Augsburg. Available at <http://opus.bibliothek.uni-augsburg.de/volltexte/2009/1444/>.
- [2] DA PRATO, G. and ZABCZYK, J. (1992). *Stochastic Equations in Infinite Dimensions*. *Encyclopedia of Mathematics and Its Applications* **44**. Cambridge Univ. Press, Cambridge. MR1207136
- [3] GYÖNGY, I. (1998). A note on Euler's approximations. *Potential Anal.* **8** 205–216. MR1625576
- [4] GYÖNGY, I. (1999). Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise. II. *Potential Anal.* **11** 1–37. MR1699161

- [5] GYÖNGY, I. and MILLET, A. (2005). On discretization schemes for stochastic evolution equations. *Potential Anal.* **23** 99–134. [MR2139212](#)
- [6] GYÖNGY, I. and MILLET, A. (2009). Rate of convergence of space–time approximations for stochastic evolution equations. *Potential Anal.* **30** 29–64. [MR2465711](#)
- [7] HAUSENBLAS, E. (2003). Approximation for semilinear stochastic evolution equations. *Potential Anal.* **18** 141–186. [MR1953619](#)
- [8] HENRY, D. (1981). *Geometric Theory of Semilinear Parabolic Equations. Lecture Notes in Math.* **840**. Springer, Berlin. [MR610244](#)
- [9] HUTZENTHALER, M., JENTZEN, A. and KLOEDEN, P. E. (2011). Strong and weak divergence in finite time of Euler’s method for SDEs with non-globally Lipschitz continuous coefficients. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **467** 1563–1576.
- [10] JENTZEN, A. and KLOEDEN, P. E. (2009). Overcoming the order barrier in the numerical approximation of stochastic partial differential equations with additive space–time noise. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **465** 649–667. [MR2471778](#)
- [11] JENTZEN, A. and KLOEDEN, P. E. (2010). Taylor expansions of solutions of stochastic partial differential equations with additive noise. *Ann. Probab.* **38** 532–569.
- [12] PETERSSON, R. and SIGNAHL, M. (2005). Numerical approximation for a white noise driven SPDE with locally bounded drift. *Potential Anal.* **22** 375–393. [MR2135265](#)
- [13] PRÉVÔT, C. and RÖCKNER, M. (2007). *A Concise Course on Stochastic Partial Differential Equations. Lecture Notes in Math.* **1905**. Springer, Berlin. [MR2329435](#)
- [14] SELL, G. R. and YOU, Y. (2002). *Dynamics of Evolutionary Equations. Applied Mathematical Sciences* **143**. Springer, New York. [MR1873467](#)
- [15] WALSH, J. B. (2005). Finite element methods for parabolic stochastic PDE’s. *Potential Anal.* **23** 1–43. [MR2136207](#)

A. JENTZEN
 PROGRAM IN APPLIED
 AND COMPUTATIONAL MATHEMATICS
 PRINCETON UNIVERSITY
 FINE HALL, WASHINGTON ROAD
 PRINCETON, NEW JERSEY 08544-1000
 USA
 E-MAIL: ajentzen@math.princeton.edu

P. KLOEDEN
 G. WINKEL
 INSTITUTE OF MATHEMATICS
 JOHANN WOLFGANG GOETHE UNIVERSITY
 D-60054 FRANKFURT AM MAIN
 GERMANY
 E-MAIL: kloeden@math.uni-frankfurt.de
georg.winkel@gmx.net