The Annals of Applied Probability
2011, Vol. 21, No. 2, 609–644
DOI: 10.1214/10-AAP706

HYBRID ATLAS MODELS

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Dedicated to Professor J. Michael Harrison on the occasion of his 65th Birthday

We study Atlas-type models of equity markets with local characteristics that depend on both name and rank, and in ways that induce a stable capital distribution. Ergodic properties and rankings of processes are examined with reference to the theory of reflected Brownian motions in polyhedral domains. In the context of such models we discuss properties of various investment strategies, including the so-called *growth-optimal* and *universal* portfolios.

1. Introduction. In modeling equity market behavior, the goal is to construct models that are simple enough to be amenable to mathematical analysis, yet complicated enough to capture the salient characteristics of real equity markets. A particularly salient characteristic of an equity market is its *capital distribution curve*,

$$\log k \mapsto \log \mu_{(k)}(t), \qquad k = 1, \dots, n, \tag{1.1}$$

that is, the logarithms of the individual companies' relative capitalizations (market weights) $\mu_{(\cdot)}(t)$ at time t, arranged in descending order $\mu_{(1)}(t) \ge \mu_{(2)}(t) \ge \cdots \ge \mu_{(n)}(t)$, versus the logarithms of their respective ranks from the largest company $k = 1$ down to the smallest $k = n$.

The capital distribution curve for the US equity market has shown remarkable stability over the last century (see, for instance, Figure 5.1 of Fernholz [13]), and this stability has been captured in the Atlas and first-order models introduced in [13] and studied by Banner, Fernholz and Karatzas [3] and others. These models assign growth rates and volatilities to the different stocks based purely on the stocks' rank in terms of relative capitalization at any given time, and roughly speaking, if the smallest stocks are assigned big growth rates and big variances, then a stable capital distribution does indeed emerge.

While Atlas and first-order models are able to reproduce the shape and stability of the capital distribution curve, they still fail to provide an accurate representation

Received September 2009; revised March 2010.

[1]Supported by NSF Grants DMS-06-01774 and DMS-09-05754. This author is on partial leave from the Department of Mathematics, Columbia University.

MSC2010 subject classifications. Primary 60G44, 91B28; secondary 70F10.

Key words and phrases. Diffusion processes interacting through their ranks, reflected Brownian motions in polyhedral domains, invariant measure of diffusion, growth-optimal and universal portfolios, local times of Bessel processes.

of market behavior. It was shown in [3] that in these models each stock spends about the same proportion of time in each rank over the long term. While this kind of ergodicity may be a nice mathematical property, it does not seem to hold for real markets: in real markets the largest stocks seem to retain their status for long periods of time, while most stocks never reach the upper echelons of capitalization. Hence, a more elaborate model is needed.

In this paper we generalize the first-order models by introducing name-based effects of companies, in addition to the purely rank-based effects of the simpler models studied in [3]. The resulting *hybrid model* (2.1) has more flexibility to describe faithfully the complexity of the entire market; in particular, the model has both stability properties and occupation time properties that are realistic.

Relation to extant literature. From a different point of view, the Atlas model can be seen as a physical particle system with each company represented by a particle diffusing on the positive real line. These individual diffusive motions have drift and volatility coefficients that depend on the entire configuration of particles at any given moment, but not on the individual particles' "identities" (tags). Recently, Pal and Pitman [22] and Chaterjee and Pal [7, 8] studied such systems, specifically when the drift coefficient is a function of the particle's rank and all volatility coefficients are equal to a given constant. Under appropriate conditions on the drift coefficients, the system has a unique invariant probability measure in a lower-dimensional space; to wit, the system of the n particles is itself not ergodic, but the projected system in a lower-dimensional hyperplane turns out to be ergodic, and with invariant probability measure that has an explicit exponential-product-form probability density function. Moreover, when the number of particles increases to infinity, the system converges weakly to one described by a Poisson–Dirichlet distribution on the real line. These analyses are useful in studying the Atlas model for an equity market, when the volatility coefficients are all equal.

The model is still tractable when its volatility coefficients depend on the rankings. Questions of existence and uniqueness for such systems in this generality are settled through the theory of martingale problems studied by Stroock and Varadhan [26] and Bass and Pardoux [5]. An important new feature of such models is that three or more particles may now collide with each other at the same time with positive probability, or even with probability one, under a suitably "uneven" volatility structure. This is a very significant departure from the constant-volatility case. Some sufficient conditions on the volatility coefficients for the occurrence and for the avoidance of triple (or higher-order) collisions, are derived in [18], by comparison with Bessel processes and with help from properties of reflected Brownian motion.

The ranked particle system has a deep relation with the theory of multidimensional reflected Brownian motion studied intensively in the context of stochastic network systems by Harrison, Reiman and Williams [15–17] and their collaborators. Recently, Dieker and Moriarty [11] provided necessary and sufficient conditions for the invariant density of semimartingale reflected Brownian motions in a two-dimensional wedge to be expressed as a finite sum of terms of product-of-exponential form, by extending the geometric considerations on the so-called "skew-symmetry" condition. In the present paper we use this skew-symmetry condition [see (5.8) in Lemma 3] for the n-dimensional reflected Brownian motions to solve the basic adjoint relation introduced in the context of a piece-wise constant drift coefficient structure, and thus compute an invariant density for the ranked process of the hybrid Atlas model as a sum of products of exponentials. With this explicit formula we compute the invariant distribution of the capital distribution curve as well as the long-term average occupation times.

Another interesting system of ranked particles is Dyson's process of noncolliding Brownian motions, which are the ordered eigenvalues of a Brownian motion on the space of Hermitian matrices. Recent work by Warren [27] constructs Dyson's process using Doob's h-transform and Brownian motion in the Gelfand–Tsetlin cone, as an extension of Dubédat's work [12] on the relation between reflected Brownian motions on the wedge and a Bessel process of dimension three. The (infinite) ranked particle systems also appear in mean-field spin glass theory of mathematical physics. In another recent development, Arguin and Aizenman [1] analyze robust quasi-stationary competing particle systems with overlapping hierarchical structures where the Poisson–Dirichlet distribution emerges as in [22]. Instead of taking Dyson's process or the spin glass theory as our model for rankings in equity markets, we obtain the ranked particle system through a general formula of Banner and Ghomrasni [4] for continuous semimartingales in the hybrid Atlas model.

Preview. This paper follows the following structure. We describe our model in Section 2, its lower-dimensional ergodic properties in Section 3, the dynamics of its rankings in Section 4, its invariant measure and occupation times in Section 5 and some portfolio analysis in its context in Section 6. In the Appendix we prove some auxiliary results stated in the main sections.

Notation. The following notions and notation are useful to describe rankings as in [3]. We consider a collection $\{Q_k^{(i)}\}_{1\le i,k\le n}$ of polyhedral domains in $\mathbb{R}^n$, where $y = (y_1, \dots, y_n) \in Q_k^{(i)}$ means that the coordinate y_i is ranked kth among $y_1, \dots, y_n$, with ties resolved in favor of the lowest index (or "name"). Note that for every index $i = 1, \dots, n$ and rank $k = 1, \dots, n$, we have the partition properties $\bigcup_{\ell=1}^n Q_\ell^{(i)} = \mathbb{R}^n = \bigcup_{j=1}^n Q_k^{(j)}$.

We shall denote by Σ_n the symmetric group of permutations of $\{1, \dots, n\}$. For each permutation $\mathbf{p} \in \Sigma_n$ we consider $\mathcal{R}_{\mathbf{p}} := \bigcap_{k=1}^n Q_k^{(\mathbf{p}(k))}$, the polyhedral chamber consisting of all points $y \in \mathbb{R}^n$ such that $y_{\mathbf{p}(k)}$ is ranked kth among $y_1, \dots, y_n$, for every $k = 1, \dots, n$. The collection of polyhedral chambers $\{\mathcal{R}_{\mathbf{p}}\}_{\mathbf{p}\in\Sigma_n}$ is a partition of all of $\mathbb{R}^n$.

Since for each $y \in \mathbb{R}^n$ there exists a unique $\mathbf{p} \in \Sigma_n$ such that $y \in \mathcal{R}_{\mathbf{p}}$ (because of the way ties are resolved), we shall find it useful to define an *indicator map* $\mathbb{R}^n \ni (x_1, \dots, x_n)' = x \mapsto \mathfrak{p}^x \in \Sigma_n$ such that $x_{\mathfrak{p}^x(1)} \geq \cdots \geq x_{\mathfrak{p}^x(n)}$. In other words, $\mathfrak{p}^x(k)$ is the index of the coordinate in the vector x that occupies the kth rank among $x_1, \dots, x_n$.

When matrices and vectors are used, the vector norm $\|x\| := (\sum_{i=1}^n x_i^2)^{1/2}$ and the inner product $\langle x, y\rangle := \sum_{i=1} x_i y_i = x'y$ for $x, y \in \mathbb{R}^n$, where $'$ stands for transposition, are defined in the usual manner. The gradient ∇ and the Laplacian Δ operators on the space C^2 of twice continuously differentiable functions are used in Section 5, as well as the notation $C_c^2(\cdot)$ [resp., $C_b^2(\cdot)$] for the spaces of twice continuously differentiable functions which have compact support (resp., are bounded functions).

2. Model. We shall study an equity market that consists of n assets (stocks) with capitalizations $\mathfrak{X}(t) = (X_1(t), \dots, X_n(t))'$ which are positive for all times $0 \leq t < \infty$. The random variable $X_i(t)$ represents the capitalization at time t of the asset with index (name) i.

We shall assume that the log-capitalizations $Y_i(t) := \log X_i(t)$, $i = 1, \dots, n$, satisfy the system of stochastic differential equations

$$\begin{aligned} dY_i(t) &= \left(\sum_{k=1}^n g_k \mathbf{1}_{Q_k^{(i)}}(Y(t)) + \gamma_i + \gamma\right) dt + \sum_{j=1}^n \rho_{i,j}\, dW_j(t) \\ &\quad + \sum_{k=1}^n \sigma_k \mathbf{1}_{Q_k^{(i)}}(Y(t))\, dW_i(t), \qquad Y_i(0) = y_i, \qquad 0 \leq t < \infty \end{aligned} \tag{2.1}$$

with given initial condition $y = (y_1, \dots, y_n)'$. The process $W(\cdot) := (W_1(\cdot), \dots, W_n(\cdot))'$ is an n-dimensional Brownian motion. As long as the n-dimensional process $Y(\cdot) := (Y_1(\cdot), \dots, Y_n(\cdot))'$ of log-capitalizations is in the polyhedron $Q_k^{(i)}$, the ith-coordinate $Y_i(\cdot)$ is ranked kth among $Y_1(\cdot), \dots, Y_n(\cdot)$ and behaves like a Brownian motion with drift $g_k + \gamma_i + \gamma$ and variance $(\sigma_k + \rho_{i,i})^2 + \sum_{j \neq i} \rho_{i,j}^2$. The constants γ, γ_i and g_k represent respectively a common, a name-based and a rank-based drift (growth rate) whereas the constants σ_k and $\rho_{i,j}$ represent rank-based volatilities and name-based correlations, respectively.

ASSUMPTION. Throughout this paper we assume (without loss of generality) that the drift constants satisfy the stability condition

$$\sum_{k=1}^n g_k + \sum_{i=1}^n \gamma_i = 0. \tag{2.2}$$

We shall assume that the $(n \times n)$ matrices

$$\mathfrak{s}_{\mathbf{p}} := \operatorname{diag}(\sigma_{\mathbf{p}^{-1}(1)}, \dots, \sigma_{\mathbf{p}^{-1}(n)}) + (\rho_{i,j})_{1 \leq i,j \leq n} \text{ are positive definite} \tag{2.3}$$

for every $\mathbf{p} \in \Sigma_n$, with $\sigma_k > 0$ for every $k = 1, \dots, n$.

Equation (2.1) can be cast in vector form as

$$dY(t) = G(Y(t))\,dt + S(Y(t))\,dW(t), \qquad Y(0) = y \in \mathbb{R}^n \tag{2.4}$$

for $0 \le t < \infty$, where the functions $G : \mathbb{R}^n \to \mathbb{R}^n$ and $S : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$ are

$$G(y) := \sum_{\mathbf{p}\in\Sigma_n} \mathbf{1}_{\mathcal{R}_\mathbf{p}}(y)\cdot \big(g_{\mathbf{p}^{-1}(1)} + \gamma_1 + \gamma, \dots, g_{\mathbf{p}^{-1}(n)} + \gamma_n + \gamma\big)',$$

$$S(y) := \sum_{\mathbf{p}\in\Sigma_n} \mathbf{1}_{\mathcal{R}_\mathbf{p}}(y)\cdot \mathfrak{s}_\mathbf{p}, \qquad y \in \mathbb{R}^n.$$

Thus (2.1) is a system of stochastic differential equations with coefficients that are piecewise constant, the same in each polyhedral chamber $\mathcal{R}_\mathbf{p}$, $\mathbf{p} \in \Sigma_n$. Under the assumption of positive definiteness in (2.3), the system (2.1) admits a weak solution (Y, W) on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ satisfying the usual conditions. By the martingale-problem theory of Stroock and Varadhan [26] and the results in Bass and Pardoux [5], this weak solution is unique in the sense of the probability distribution.

3. Ergodicity. Thanks to assumption (2.2) on the drifts, and taking the average of both sides of (2.1), we obtain the average log-capitalization process $\overline{Y}(\cdot) := \sum_{i=1}^n Y_i(\cdot)/n$ in the form

$$\begin{gathered}\overline{Y}(t) = \frac{1}{n}\sum_{i=1}^n y_i + \gamma t + \frac{1}{n}\sum_{k=1}^n \sigma_k B_k(t) + \frac{1}{n}\sum_{i,j=1}^n \rho_{i,j} W_j(t),\\ \text{where } B_k(t) := \sum_{i=1}^n \int_0^t \mathbf{1}_{Q_k^{(i)}}(Y(s))\,dW_i(s),\ k = 1,\dots,n,\end{gathered} \tag{3.1}$$

for $0 \le t < \infty$ because of $\bigcup_{i=1}^n Q_k^{(i)} = \mathbb{R}^n$. Here $B_1(\cdot), \dots, B_n(\cdot)$ are continuous local martingales with quadratic (cross-)variations $\langle B_k, B_\ell\rangle\ (t) = t\delta_{k,\ell}$, and hence are independent standard Brownian motions by the Knight theorem. It follows that the average $\overline{Y}(\cdot)$ of the log-capitalizations $Y_1(\cdot), \dots, Y_n(\cdot)$ grows at a rate equal to the common drift γ, that is,

$$\lim_{T\to\infty} \frac{\overline{Y}(T)}{T} = \gamma \qquad \text{holds a.s.,} \tag{3.2}$$

by the strong law of large numbers for Brownian motion.

In order to study the long-term behavior of the whole log-capitalizations, let us quote Theorems 4.1 and 5.1 on pages 119–121 of Khas'minskii [20], since our argument relies on them rather decisively.

PROPOSITION 1 (Khas'minskii). *Consider a diffusion $\xi(\cdot)$ with values in a subset E of Euclidean space. Assume that there exists a bounded domain $U \subset E$ with regular boundary, having the following properties*:

(B.1) *In the domain U the smallest eigenvalue of the diffusion matrix of the process $\xi(\cdot)$ is bounded away from zero.*

(B.2) *If $x \in E \setminus U$, the mean time τ at which a path issuing from x reaches the set U is finite, and $\sup_{x\in K} \mathbb{E}_x(\tau) < \infty$ for every compact subset $K \subset E$.*

Then the Markov process $\xi(\cdot)$ has a unique stationary distribution μ, and which satisfies the Strong Law of Large Numbers

$$\mathbb{P}_x\biggl(\lim_{T\to\infty} \frac{1}{T}\int_0^T f(\xi(t))\,dt = \int_E f(y)\mu(dy)\biggr) = 1, \qquad x \in E,$$

for any bounded, measurable function $f : E \to \mathbb{R}$.

Let us introduce the column vector $\imath := (1, \dots, 1)'$ and the subspace

$$\Pi := \{y \in \mathbb{R}^n | \imath' y = 0\}.$$

THEOREM 1. *In addition to (2.2) and (2.3), let us impose for every $\mathbf{p} \in \Sigma_n$ the following stability condition*:

$$\sum_{k=1}^{\ell} \bigl(g_k + \gamma_{\mathbf{p}(k)}\bigr) < 0, \qquad \ell = 1, \dots, n-1. \tag{3.3}$$

Then the deviations $\widetilde{Y}(\cdot) := (Y_1(\cdot) - \overline{Y}(\cdot), \dots, Y_n(\cdot) - \overline{Y}(\cdot))$ of the log-capitalizations $Y_1(\cdot), \dots, Y_n(\cdot)$ from their average are stable in distribution: *there exists a unique invariant probability measure μ for the Π-valued Markov process $\widetilde{Y}(\cdot)$, and for any bounded, measurable function $f : \Pi \to \mathbb{R}$ we have the Strong Law of Large Numbers*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(\widetilde{Y}(t))\,dt = \int_\Pi f(y)\mu(dy), \qquad a.s. \tag{3.4}$$

PROOF. From (2.1) and (3.1), we have

$$d\widetilde{Y}(t) = \widetilde{G}(\widetilde{Y}(t))\,dt + \widetilde{S}(\widetilde{Y}(t))\,dW(t), \qquad \widetilde{Y}(0) = \widetilde{y}, \tag{3.5}$$

where $\widetilde{y} := y - \imath' y \cdot \imath/n$, $\widetilde{G}(y) := G(y) - \gamma \cdot \imath$ and $\widetilde{S}(y) := S(y) - \imath\imath' S(y)/n$ for $y \in \mathbb{R}^n$. By (2.3) the covariance matrix in (3.5) is uniformly nondegenerate: for all $x, y \in \Pi$ we have

$$x'\widetilde{S}(y)x = x' S(y)x - x'\imath\imath' S(y)x/n = x' S(y)x = \sum_{\mathbf{p}\in\Sigma_n} \mathbf{1}_{\mathcal{R}_\mathbf{p}}(y) \cdot x'\mathfrak{s}_\mathbf{p} x$$

and

$$\lambda_0\|x\|^2 \le x'\widetilde{S}(y)x \le \lambda_1\|x\|^2, \tag{3.6}$$

where $\lambda_0(\lambda_1)$ are the minimum (maximum) of the smallest (largest) eigenvalues of the positive definite matrices $\mathfrak{s}_\mathbf{p}$ over $\mathbf{p} \in \Sigma_n$ in (2.3).

Summation-by-parts, along with (2.2) and (3.3), lead now to

$$
\begin{aligned}
y'\widetilde{G}(y) &= \sum_{i=1}^{n} y_i\big(g_{(\mathfrak{p}^y)^{-1}(i)} + \gamma_i\big) = \sum_{k=1}^{n} y_{\mathfrak{p}^y(k)}\big(g_k + \gamma_{\mathfrak{p}^y(k)}\big) \\
&= y_{\mathfrak{p}^y(n)} \underbrace{\sum_{k=1}^{n}\big(g_k + \gamma_{\mathfrak{p}^y(k)}\big)}_{=0} + \sum_{k=1}^{n-1}\big(y_{\mathfrak{p}^y(k)} - y_{\mathfrak{p}^y(k+1)}\big)\sum_{\ell=1}^{k}\big(g_\ell + \gamma_{\mathfrak{p}^y(\ell)}\big) \qquad (3.7) \\
&\le c\sqrt{n}\sum_{k=1}^{n}\big(y_{\mathfrak{p}^y(k)} - y_{\mathfrak{p}^y(k+1)}\big) \le c\|y\| < 0, \qquad y \in \Pi \cap \mathcal{R}_{\mathbf{p}},
\end{aligned}
$$

where $c := n^{-1/2}\max_{1\le \ell\le n-1, \mathbf{p}\in\Sigma_n}\sum_{k=1}^{\ell}(g_k + \gamma_{\mathbf{p}(k)}) < 0$. In the last inequality we have used for $\mathbf{p} \in \Sigma_n$ and $y \in \Pi \cap \mathcal{R}_{\mathbf{p}}$ the properties $y_{\mathbf{p}(1)} \ge y_{\mathbf{p}(2)} \ge \cdots \ge y_{\mathbf{p}(n)}$, thus also $y_{\mathbf{p}(1)} \ge 0 \ge y_{\mathbf{p}(n)}$ and

$$\|y\|^2 \le n\max\big(y_{\mathbf{p}(1)}^2, y_{\mathbf{p}(n)}^2\big) \le n\big(y_{\mathbf{p}(1)} - y_{\mathbf{p}(n)}\big)^2.$$

Now we consider the one-dimensional process $N(t) := f(\widetilde{Y}(t))$ with $f(y) = (\|y\|^2 + 1)^{1/2} > \|y\|$ for $y \in \Pi$. An application of Itô's rule gives

$$
\begin{aligned}
dN(t) &= \widetilde{f}(\widetilde{Y}(t))\,dt + [f(y)]^{-1}y'\widetilde{S}(y)\big|_{y=\widetilde{Y}(t)}\,dW(t), \qquad 0 \le t < \infty, \\
\widetilde{f}(y) &:= (f(y))^{-1}\big(y'\widetilde{G}(y) + \tfrac{1}{2}\operatorname{trace}(\widetilde{S}(y)\widetilde{S}(y)')\big) - \tfrac{1}{2}(f(y))^{-3}y'\widetilde{S}\widetilde{S}(y)'y
\end{aligned}
$$

for $y \in \Pi$. It follows from (3.6), (3.7) and the boundedness of $\widetilde{S}(\cdot)$ that there exists a constant $\kappa > 0$ such that $\widetilde{f}(y) \le c/2 < 0$ for $\|y\| > \kappa$. The diffusion coefficient $[f(y)]^{-1}y'\widetilde{S}(y)$ of $N(\cdot)$ is a vector whose entries are uniformly bounded by some constants from (3.6).

Thus $N(\cdot)$ is positive recurrent with respect to the interval $(0, \kappa)$, and hence so is $\widetilde{Y}(\cdot)$ with respect to $B \cap \Pi$ for some ball $B \subset \mathbb{R}^n$ centered at the origin.

Finally, we check the conditions (B.1) and (B.2) of Proposition 1. For our diffusion $\xi(\cdot) = \widetilde{Y}(\cdot)$ on $E = \Pi$ we have verified (B.1) in (3.6). Assumption (B.2) is verified from the positive recurrence of $\widetilde{Y}(\cdot)$ with respect to $U = B \cap \Pi$. Therefore, by Proposition 1, we obtain the existence of a unique invariant probability measure μ that satisfies (3.4). □

Condition (3.3) ensures that, if $y_1 < y_2 < \cdots < y_n$ and one subdivides at time $t = 0$ the "cloud" of n particles diffusing on the real line according to the dynamics of (2.1), into two "subclouds"—one consisting of the ℓ leftmost, and the other of the $n - \ell$ rightmost, particles—the two subclouds will eventually merge. They will not continue to evolve like separate galaxies, that never make contact with each other (cf. the Remark following Theorem 4 in Pal and Pitman [22] for an elaboration of this point in the case of the purely rank-based first-order model with equal variances).

COROLLARY 1. *Under the assumptions of Theorem 1, the* long-term average occupation time *that company i spends in the kth rank, that is,*

$$\theta_{k,i} := \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{1}_{Q_k^{(i)}}(\mathfrak{X}(t))\,dt, \qquad i,k=1,\dots,n, \tag{3.8}$$

exists almost surely in $[0,1]$.

The resulting array of numbers $\theta_{k,i}\in[0,1]$ satisfy $\sum_{j=1}^n \theta_{k,j} = \sum_{\ell=1}^n \theta_{\ell,i} = 1$ for each "name" $i=1,\dots,n$ and "rank" $k=1,\dots,n$; that is, $\vartheta := (\theta_{k,i})_{1\le k,i\le n}$ is a doubly stochastic matrix. Similarly, the average occupation time $\theta_{\mathbf{p}}$ of the market in the polyhedral chamber $\mathcal{R}_{\mathbf{p}}$, namely,

$$\theta_{\mathbf{p}} := \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{1}_{\mathcal{R}_{\mathbf{p}}}(\mathfrak{X}(t))\,dt \qquad \text{exists a.s. in } [0,1] \tag{3.9}$$

for every $\mathbf{p}\in\Sigma_n$, and we have $\theta_{k,i} = \sum \theta_{\mathbf{p}}$, where the summation is over the set $\{\mathbf{p}\in\Sigma_n \,|\, \mathbf{p}(k)=i\}$ of permutations for $1\le i,k\le n$.

Indeed, by Theorem 1 and in particular (3.4), the quantity of (3.8) satisfies

$$\theta_{k,i} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{1}_{Q_k^{(i)}}(\mathfrak{X}(t))\,dt = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{1}_{Q_k^{(i)}\cap\Pi}(\widetilde{Y}(t))\,dt = \mu\big(Q_k^{(i)}\big),$$

where μ is the unique invariant probability measure for the process $\widetilde{Y}(\cdot)$ of (3.5). Since $\bigcup_{\ell=1}^n Q_\ell^{(i)} = \mathbb{R}^n = \bigcup_{j=1}^n Q_k^{(j)}$, it is obvious that $\sum_{\ell=1}^n \theta_{\ell,i} = \sum_{j=1}^n \theta_{k,j} = 1$ for $1\le i,k\le n$. Equation (3.9), and the claim following it, are obtained similarly.

4. Rankings. Let us now look at the log-capitalizations of the various companies listed according to rank, namely

$$Z_k(t) := \sum_{i=1}^n \mathbf{1}_{Q_k^{(i)}}(Y(t))\cdot Y_i(t), \qquad k=1,\dots,n,\ 0\le t<\infty. \tag{4.1}$$

These are the order statistics $Z_1(\cdot)\ge\cdots\ge Z_n(\cdot)$ for the log-capitalizations $Y_1(\cdot) = \log X_1(\cdot),\dots,Y_n(\cdot)=\log X_n(\cdot)$, listed from largest down to smallest. We recall the indicator map $\mathfrak{p}^x$ introduced at the end of Section 1, and define the Σ_n-valued *index process*

$$\mathfrak{P}_t := \mathfrak{p}^{\mathfrak{X}(t)} = \mathfrak{p}^{Y(t)}, \qquad 0\le t<\infty,$$

so that $X_{\mathfrak{P}_t(1)}(t)\ge\cdots\ge X_{\mathfrak{P}_t(n)}(t)$. We may thus write $Z_k(\cdot) = Y_{\mathfrak{P}.(k)}(\cdot)$ from (4.1); loosely speaking, $\mathfrak{P}_t(k)$ is the index (name) of the company that occupies the kth rank, in terms of capitalization, at time t.

We shall also introduce the total market capitalization $X(\cdot) := \sum_{i=1}^n X_i(\cdot)$, as well as the market weights (relative capitalizations) for the individual companies and their ranked counterparts, respectively,

$$\mu_i(t) := \frac{X_i(t)}{X(t)}, \qquad i = 1, \dots, n, \quad \text{and}$$
$$\mu_{(k)}(t) := \frac{e^{Z_k(t)}}{X(t)}, \qquad k = 1, \dots, n. \tag{4.2}$$

COROLLARY 2. *Under* (2.2), (2.3) *and* (3.3), *the process of ranked deviations* $\widetilde{Z}(\cdot) := (Z_1(\cdot) - \overline{Y}(\cdot), \dots, Z_n(\cdot) - \overline{Y}(\cdot))'$ *of the log-capitalizations* $Y_1(\cdot), \dots, Y_n(\cdot)$ *from their average*, *is stable in distribution by Theorem* 1, *and so is the* $((\mathbb{R}_+)^{n-1} \times \Sigma_n)$*-valued process* $(\Xi(\cdot), \mathfrak{P}.)$, *where* $\Xi(\cdot) := (Z_1(\cdot) - Z_2(\cdot), \dots, Z_{n-1}(\cdot) - Z_n(\cdot))'$ *is the* rank-gap *process of* $Y(\cdot)$.

In fact, since $\widetilde{Z}(\cdot)$ is obtained by permuting the components of $\widetilde{Y}(\cdot)$, the stability in distribution of $\widetilde{Y}(\cdot)$ implies stability in distribution for $\widetilde{Z}(\cdot)$ from Theorem 1. Moreover, the components of the rank-gap process $\Xi(\cdot)$ can be written as linear combinations of those of $\widetilde{Z}(\cdot)$, and the index process $\mathfrak{P}.$ can be seen as $\mathfrak{P}. = \mathfrak{p}^{\widetilde{Z}(\cdot)}$, where the range Σ_n of the mapping $\mathfrak{p}$ is a finite set. Thus, the process $(\Xi(\cdot), \mathfrak{P}.)$ is stable in distribution.

We shall denote by $\Lambda^{k,j}(t) := \Lambda_{Z_k - Z_j}(t)$ the local time accumulated at the origin by the nonnegative semimartingale $Z_k(\cdot) - Z_j(\cdot)$ up to time t for $1 \le k < j \le n$, and set $\Lambda^{0,1}(\cdot) \equiv 0 \equiv \Lambda^{n,n+1}(\cdot)$. Then from Theorem 2.5 of Banner and Ghomrasni [4] it can be shown that we have for $k = 1, \dots, n$, $0 \le t < \infty$ the dynamics

$$dZ_k(t) = \sum_{i=1}^{n} \mathbf{1}_{Q_k^{(i)}}(Y(t))\, dY_i(t) + (N_k(t))^{-1} \Bigg[\sum_{\ell=k+1}^{n} d\Lambda^{k,\ell}(t) - \sum_{\ell=1}^{k-1} d\Lambda^{\ell,k}(t) \Bigg]. \tag{4.3}$$

Here $N_k(t)$ is the cardinality of the set of indices of those random variables among $Y_1(t), \dots, Y_n(t)$ which have the same value as $Z_k(t)$, that is, $N_k(t) := |\{i : Y_i(t) = Z_k(t)\}|$. Note that under the assumptions on the coefficients, the finite variation part of the continuous semimartingale $Y(\cdot)$ in (2.1) is absolutely continuous with respect to Lebesgue measure a.s., and it follows from an application of Fubini's theorem and an estimate of Krylov [21] that the Lebesgue measure of the set $\{t : Y_i(t) = Y_j(t)\}$ is zero a.s. for $1 \le i \ne j \le n$. Thus we can verify the sufficient conditions (2.11 and 2.12) of Theorem 2.5 in [4].

Each local time $\Lambda^{k,\ell}(\cdot)$ is flat away from the set $\{0 \le t < \infty | Z_k(t) = \cdots = Z_\ell(t)\}$; it increases only when the corresponding coordinate processes collide with each other. Examples in [5, 18] study such multiple collisions of order three or higher and use comparisons with Bessel processes in a crucial manner. Here again, the nonnegative semimartingale $Z_k(\cdot) - Z_\ell(\cdot)$ is compared to an appropriate Bessel process. Since a Bessel process with dimension $\delta > 1$ does not accumulate any local time at the origin (a consequence of Proposition XI.1.11 of [23] and of Theorem V.48.6 in [24]), appropriate comparison arguments yield the following result; its proof is in Section A.2.

LEMMA 1. *Under* (2.3), *the local times* $\Lambda^{k,\ell}(\cdot)$ *generated by triple or higher-order collisions are identically equal to zero*, *that is*, $\Lambda^{k,\ell}(\cdot) \equiv 0$ *for* $1 \le k, \ell \le n$ *and* $|k-\ell| \ge 2$, *and* (4.3) *takes for* $k = 1, \dots, n, 0 \le t < \infty$ *the form*

$$dZ_k(t) = \sum_{i=1}^n \mathbf{1}_{Q_k^{(i)}}(Y(t))\, dY_i(t) + \frac{1}{2}\big(d\Lambda^{k,k+1}(t) - d\Lambda^{k-1,k}(t)\big). \tag{4.4}$$

PROPOSITION 2. *Under the convention* (2.2) *and the assumptions* (2.3) *and* (3.3), *we obtain a Strong Law of Large Numbers for local times*

$$\lim_{T\to\infty} \frac{1}{T}\Lambda^{k,k+1}(T) = -2\sum_{\ell=1}^k \left(g_\ell + \sum_{i=1}^n \gamma_i \theta_{\ell,i}\right), \qquad k = 1, \dots, n-1, \tag{4.5}$$

almost surely. Moreover, *we obtain the following long-term growth relations*, *in addition to those of* (3.2): all log-capitalizations grow at the same rate

$$\lim_{T\to\infty} \frac{Y_i(T)}{T} = \lim_{T\to\infty} \frac{\log X_i(T)}{T} = \gamma, \qquad i = 1, \dots, n, \tag{4.6}$$

almost surely. This holds also for the total market capitalization

$$\lim_{T\to\infty} \frac{1}{T}\log X(T) = \lim_{T\to\infty} \frac{1}{T}\log\left(\sum_{i=1}^n X_i(T)\right) = \gamma, \qquad a.s., \tag{4.7}$$

and thus the model is coherent; *that is*, *in the notation of* (4.2) *we have*

$$\lim_{T\to\infty} \frac{1}{T}\log \mu_i(T) = 0, \qquad a.s.;\ i = 1, \dots, n. \tag{4.8}$$

PROOF. It follows from Corollary 2 that

$$\lim_{T\to\infty} \frac{1}{T}\big(Z_k(T) - Z_{k+1}(T)\big) = 0, \qquad \text{a.s.};\ k = 1, \dots, n-1.$$

Combining this with (2.1), (3.8) and (4.4), we observe

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{2T}&\big(\Lambda^{k-1,k}(T) + \Lambda^{k+1,k+2}(T) - 2\Lambda^{k,k+1}(T)\big)\\ &= g_k + \sum_{i=1}^n \gamma_i\theta_{k,i} - \left(g_{k+1} + \sum_{i=1}^n \gamma_i \theta_{k+1,i}\right) = \mathfrak{g}_k - \mathfrak{g}_{k+1}, \qquad \text{a.s.},\end{aligned}$$

where we have set $\mathfrak{g}_k := g_k + \sum_{i=1}^n \gamma_i \theta_{k,i}$ for $k = 1, \dots, n-1$. Adding up these equations over $k = \ell, \dots, n-1$ yields

$$\lim_{T\to\infty} \frac{1}{2T}\big(\Lambda^{\ell-1,\ell}(T) - \Lambda^{\ell,\ell+1}(T) - \Lambda^{n-1,n}(T)\big) = \mathfrak{g}_\ell - \mathfrak{g}_n, \qquad \text{a.s.} \tag{4.9}$$

for each $\ell = 1, \dots, n$; adding up over all these values of ℓ and using the convention (2.2) for clarity, we obtain

$$\lim_{T\to\infty} \frac{1}{2T}\Lambda^{n-1,n}(T) = \mathfrak{g}_n, \qquad \text{a.s.} \tag{4.10}$$

In conjunction with (4.9), we obtain from (4.10) that for $k = 1, \dots, n$

$$\lim_{T\to\infty} \frac{1}{2T}\big(\Lambda^{k-1,k}(T) - \Lambda^{k,k+1}(T)\big) = \mathfrak{g}_k = g_k + \sum_{i=1}^n \gamma_i \theta_{k,i}, \qquad \text{a.s.} \tag{4.11}$$

Since $\sum_{k=1}^n \mathfrak{g}_k = 0$ from (2.2) and Corollary 1, we obtain (4.5) from (4.10) and (4.11). From this, (4.4), and the strong law of large numbers for Brownian motion, we get the long-term average growth rate of ranked log-capitalizations,

$$\lim_{T\to\infty} \frac{Z_k(T)}{T} = \gamma, \qquad \text{a.s.; } k = 1, \dots, n.$$

This yields (4.6); the elementary inequality $\exp\{y_{\mathfrak{p}^y(1)}\} \le \sum_{i=1}^n \exp\{y_i\} \le n \times \exp\{y_{\mathfrak{p}^y(1)}\}$ for $y \in \mathbb{R}^n$ then implies (4.7), and equation (4.8) is a direct consequence of (4.6) and (4.7). $\square$

COROLLARY 3. *Under* (2.2), (2.3) *and* (3.3), *the long-term average occupation times* $\theta_{k,i}$ *of* (3.8) *satisfy the equilibrium identity*

$$\sum_{k=1}^n \theta_{k,i} g_k + \gamma_i = 0, \qquad i = 1, \dots, n. \tag{4.12}$$

Indeed, by substituting (4.6) into (2.1), and using the strong law of large numbers for Brownian motion, we obtain the a.s. identities

$$\lim_{T\to\infty} \frac{1}{T}\sum_{k=1}^n g_k \int_0^T \mathbf{1}_{Q_k^{(i)}}(Y(t))\,dt = -\gamma_i, \qquad i = 1, \dots, n,$$

and so in conjunction with (3.8) we deduce (4.12).

EXAMPLE 1. Suppose that the rank-based growth parameters are given as $g_n = (n-1)g$, $g_1 = \cdots = g_{n-1} = -g < 0$ for some $g > 0$. This is the "Atlas configuration," in which the company at the lowest capitalization rank provides all the growth (or support, as with the Titan of mythical lore) for the entire structure. Suppose also that the name-based growth rates $\gamma_1, \dots, \gamma_n$ satisfy $\sum_{i=1}^n \gamma_i = 0$ and $\max_{1\le i\le n} \gamma_i < g$.

It is then checked easily that conditions (2.2) and (3.3) are satisfied. By Corollary 1, the average occupation times $\{\theta_{k,i}\}$ exist a.s. We shall provide an explicit expression for the $\theta_{k,i}$ under an additional condition (5.7) on the correlation structure, in Section 5.2. For the time being, let us just remark that in this case we get directly from (4.12) the long-term proportions of time

$$\theta_{n,i} = \frac{1}{n}\left(1 - \frac{\gamma_i}{g}\right), \qquad i = 1, \dots, n,$$

with which the various companies occupy the lowest ("Atlas") rank.

5. Invariant measure.

5.1. *Reflected Brownian motions*. Observe now from (4.4) the following representation for the vector $\Xi(\cdot) = (\Xi_1(\cdot), \dots, \Xi_{n-1}(\cdot))'$ of gaps in the ranked log-capitalizations $\Xi_k(\cdot) := Z_k(\cdot) - Z_{k+1}(\cdot) = \log(X_{(k)}(\cdot)/X_{(k+1)}(\cdot)) \geq 0$, $k = 1, \dots, n-1$,

$$\Xi(t) = \Xi(0) + \zeta(t) + \mathfrak{R}\Lambda(t), \qquad 0 \leq t < \infty. \tag{5.1}$$

Here we have set $\zeta(\cdot) := (\zeta_1(\cdot), \dots, \zeta_{n-1}(\cdot))'$ with

$$\zeta_k(\cdot) = \sum_{i=1}^{n} \int_0^{\cdot} \mathbf{1}_{Q_k^{(i)}}(Y(s))\, dY(s) - \sum_{i=1}^{n} \int_0^{\cdot} \mathbf{1}_{Q_{k+1}^{(i)}}(Y(s))\, dY(s);$$

and we have introduced the vector $\Lambda(\cdot) := (\Lambda^{1,2}(\cdot), \dots, \Lambda^{n-1,n}(\cdot))' = (\Lambda_{\Xi_1}(\cdot), \dots, \Lambda_{\Xi_{n-1}}(\cdot))'$ of local times, as well as the $((n-1) \times (n-1))$ matrix

$$\mathfrak{R} := \begin{pmatrix} 1 & -1/2 & & & \\ -1/2 & 1 & -1/2 & & \\ & \ddots & \ddots & \ddots & \\ & & -1/2 & 1 & -1/2 \\ & & & -1/2 & 1 \end{pmatrix}. \tag{5.2}$$

This rank-gap process $\Xi(\cdot)$ in (5.1) belongs to a class of processes which Harrison and Williams [16, 17], Williams [28] and Dai and Williams [10] call "semimartingale reflected (or regulated) Brownian motions" (SRBM) in polyhedral domains.

The process $\Xi(\cdot)$ has state-space $(\mathbb{R}_+)^{n-1}$ and behaves like the $(n-1)$-dimensional continuous semimartingale $\zeta(\cdot)$ on the interior of $(\mathbb{R}_+)^{n-1}$. When the face $\mathfrak{F}_k := \{(z_1, \dots, z_{n-1})' \in (\mathbb{R}_+)^{n-1} | z_k = 0\}$, $k = 1, \dots, n-1$, of the boundary is hit, the kth component of $\Lambda(\cdot)$ increases, which causes an instantaneous displacement (reflection) in a continuous fashion. The directions of this reflection are given by the entries in $\mathfrak{r}_k$, the kth column of the matrix $\mathfrak{R}$. For every principal submatrix $\widetilde{\mathfrak{R}}$ of $\mathfrak{R}$, there exists a nonzero vector y such that $\widetilde{\mathfrak{R}}y > 0$, and so the reflection matrix $\mathfrak{R}$ satisfies the so-called *completely-$\mathcal{S}$* (or "strictly semi-monotone") (see Dai and Williams [10] for details) condition for $\mathcal{S} = (\mathbb{R}_+)^{n-1}$.

Let us define the differential operators $\mathcal{A}$ and $\mathcal{D}_k$, acting on $C^2((\mathbb{R}_+)^{n-1})$ functions

$$[\mathcal{A}f](z,\mathbf{p}) := \frac{1}{2}\sum_{k,\ell=1}^{n-1} a_{k,\ell}(\mathbf{p})\frac{\partial^2 f(z)}{\partial z_k\,\partial z_\ell} + \sum_{k=1}^{n-1} b_k(\mathbf{p})\frac{\partial f}{\partial z_k}(z),$$

$$(z,\mathbf{p}) \in (\mathbb{R}_+)^{n-1}\times\Sigma_n, \tag{5.3}$$

$$[\mathcal{D}_k f](z) := \langle \mathfrak{r}_k, \nabla f(z)\rangle, \qquad z\in\mathfrak{F}_k, k=1,\dots,n-1.$$

Here $(a_{k,\ell}(\cdot))_{1\le k,\ell\le n-1}$ is the covariance matrix corresponding to the semimartingale $\zeta(\cdot)$ with entries

$$\begin{aligned} a_{k,\ell}(\mathbf{p}) :=& (\sigma_k^2+\sigma_{k+1}^2)\cdot\mathbf{1}_{\{k=\ell\}} - \sigma_k^2\cdot\mathbf{1}_{\{k=\ell+1\}} - \sigma_{k+1}^2\cdot\mathbf{1}_{\{k=\ell-1\}} \\ &+ \sum_{m=1}^{n}\bigl(\rho_{\mathbf{p}(k),m}-\rho_{\mathbf{p}(k+1),m}\bigr)\bigl(\rho_{\mathbf{p}(\ell),m}-\rho_{\mathbf{p}(\ell+1),m}\bigr) \\ &+ \sum_{(\alpha,\beta)\in\{(k,\ell),(\ell,k)\}}\bigl\{\sigma_\alpha\bigl(\rho_{\mathbf{p}(\beta),\alpha}-\rho_{\mathbf{p}(\beta+1),\alpha}\bigr) \\ &\qquad + \sigma_{\alpha+1}\bigl(\rho_{\mathbf{p}(\beta+1),\alpha+1}-\rho_{\mathbf{p}(\beta),\alpha+1}\bigr)\bigr\} \end{aligned} \tag{5.4}$$

for $k,\ell=1,\dots,n-1$, $\mathbf{p}\in\Sigma_n$; whereas the $((n-1)\times 1)$ vector $\mathfrak{r}_k$ is the kth column of the reflection matrix $\mathfrak{R}$. We also define the $((n-1)\times 1)$ drift coefficient vector $b(\cdot) := (b_1(\cdot),\dots,b_{n-1}(\cdot))'$ for the semimartingale $\zeta(\cdot)$, with components

$$b_k(\mathbf{p}) := g_k + \gamma_{\mathbf{p}^{-1}(k)} - g_{k+1} - \gamma_{\mathbf{p}^{-1}(k+1)}, \qquad k=1,\dots,n-1, \mathbf{p}\in\Sigma_n. \tag{5.5}$$

From Corollary 2 we know that there exists an invariant measure $\nu(\cdot,\cdot)$ for the $((\mathbb{R}_+)^{n-1}\times\Sigma_n)$-valued process $(\Xi(\cdot),\mathfrak{P}_\cdot)$. Let us denote by $\nu_0(\cdot)$ the marginal invariant distribution of $\Xi(\cdot)$. As a consequence of Itô's rule and the formulation of the *submartingale problem* studied by Stroock and Varadhan [25] and Harrison and Williams [16], we obtain a characterization of the invariant distribution $\nu(\cdot,\cdot)$ for $(\Xi(\cdot),\mathfrak{P}_\cdot)$.

LEMMA 2. *Recall convention* (2.2), *and conditions* (2.3) *and* (3.3). *For each* $k=1,\dots,n-1$ *there is a finite measure* $\nu_{0k}(\cdot)$, *absolutely continuous with respect to Lebesgue measure on the kth face* $\mathfrak{F}_k$, *such that the so-called* Basic Adjoint Relationship (BAR) *holds for any* C_b^2*-function* $f:(\mathbb{R}_+)^{n-1}\to\mathbb{R}$, *namely*

$$\int_{(\mathbb{R}_+)^{n-1}\times\Sigma_n}[\mathcal{A}f](z,\mathbf{p})\,d\nu(z,\mathbf{p}) + \frac{1}{2}\sum_{k=1}^{n-1}\int_{\mathfrak{F}_k}[\mathcal{D}_k f](z)\,d\nu_{0k}(z) = 0. \tag{5.6}$$

This condition is necessary for the stationarity of $\nu(\cdot,\cdot)$. A proof of Lemma 2 is given in Section A.3. It is not easy to solve (5.6) in general; however, following Harrison and Williams [17], we may obtain an explicit formula for the invariant joint distribution $\nu(\cdot,\cdot)$ under the so-called *skew symmetry condition* between the covariance and reflection matrices (see Theorem 2 and Corollaries 4 and 5).

LEMMA 3. *Assume that the rank-based variances* $\{\sigma_k^2\}$ *grow linearly, and that there are* no name-based correlations *in* (2.1), *that is*,

$$\begin{aligned} &\sigma_2^2 - \sigma_1^2 = \sigma_3^2 - \sigma_2^2 = \cdots = \sigma_n^2 - \sigma_{n-1}^2, \\ &\rho_{i,j} = 0, \qquad 1 \le i, j \le n. \end{aligned} \tag{5.7}$$

Then the components of the covariance matrix $\mathfrak{A} \equiv (\mathfrak{a}_{k,\ell})_{1\le k,\ell\le n-1}$ *from* (5.4) *become*

$$\mathfrak{a}_{k,\ell} = (\sigma_k^2 + \sigma_{k+1}^2) \cdot \mathbf{1}_{\{k=\ell\}} - \sigma_k^2 \cdot \mathbf{1}_{\{k=\ell+1\}} - \sigma_{k+1}^2 \cdot \mathbf{1}_{\{k=\ell-1\}}$$

and do not *depend on the permutation* $\mathbf{p} \in \Sigma_n$. *Moreover, the matrix* $\mathfrak{A}$ *satisfies the so-called* skew symmetry *condition*,

$$(2\mathfrak{D} - \mathfrak{H}\mathfrak{D} - \mathfrak{D}\mathfrak{H} - 2\mathfrak{A})_{k,\ell} = 0, \qquad 1 \le k, \ell \le n-1. \tag{5.8}$$

Here we have introduced the diagonal matrix $\mathfrak{D} := \operatorname{diag}(\mathfrak{A})$, *and the* $((n-1) \times (n-1))$ *matrix* $\mathfrak{H} := I - \mathfrak{R}$ *from the reflection matrix* $\mathfrak{R}$ *in* (5.2).

Lemma 3 is proved by straightforward computation; details are in Section 5.5 of [18]. Note that, even under (5.7), the operator (5.3) still depends on the permutation $\mathbf{p}$ through the drift component $b(\mathbf{p})$ for $\mathbf{p} \in \Sigma_n$ in (5.5).

THEOREM 2. *Under* (2.2), (2.3), (3.3) *and* (5.7), *the invariant joint distribution* $\nu(\cdot,\cdot)$ *of the* $((\mathbb{R}_+)^{n-1} \times \Sigma_n)$*-valued process* $(\Xi(\cdot), \mathfrak{P}_\cdot)$ *is*

$$\nu(A \times B) := \left(\sum_{\mathbf{q}\in\Sigma_n} \prod_{k=1}^{n-1} \lambda_{\mathbf{q},k}^{-1} \right)^{-1} \sum_{\mathbf{p}\in B} \int_A \exp(-\langle \lambda_{\mathbf{p}}, z\rangle)\, dz \tag{5.9}$$

for any measurable sets $A \subset (\mathbb{R}_+)^{n-1}$ *and* $B \subset \Sigma_n$, *where* $\lambda_{\mathbf{p}} := (\lambda_{\mathbf{p},1}, \dots, \lambda_{\mathbf{p},n-1})'$ *is the vector with components*

$$\lambda_{\mathbf{p},k} := \frac{-4\sum_{\ell=1}^k (g_\ell + \gamma_{\mathbf{p}(\ell)})}{\sigma_k^2 + \sigma_{k+1}^2}, \qquad \mathbf{p} \in \Sigma_n, 1 \le k \le n-1. \tag{5.10}$$

In particular, the density $\wp(\cdot)$ *of the marginal invariant distribution* $\nu_0(\cdot)$ *of* $\Xi(\cdot)$ *has the* sum-of-products-of-exponentials *form*

$$\wp(z) := \left(\sum_{\mathbf{q}\in\Sigma_n} \prod_{k=1}^{n-1} \lambda_{\mathbf{q},k}^{-1} \right)^{-1} \sum_{\mathbf{p}\in\Sigma_n} \exp(-\langle \lambda_{\mathbf{p}}, z\rangle), \qquad z \in (\mathbb{R}_+)^{n-1}. \tag{5.11}$$

PROOF. First, we carry out a linear transformation of the state space to remove the correlation between the components of $\Xi(\cdot)$; this is possible, because the covariance matrix $\mathfrak{A}$ does *not* depend on the index process $\mathfrak{P}_\cdot$, under (5.7) from Lemma 3. Let $\mathfrak{U}$ be the matrix whose columns are the orthogonal eigenvectors of the covariance $\mathfrak{A}$, and let $\mathfrak{L}$ be the corresponding diagonal matrix of eigenvalues such that $\mathfrak{L} = \mathfrak{U}'\mathfrak{A}\mathfrak{U}$. Define $\widetilde{\Xi}(\cdot) := \mathfrak{L}^{-1/2}\mathfrak{U}\Xi(\cdot)$. By this deterministic rotation and scaling, we obtain

$$\widetilde{\Xi}(t) = \widetilde{\Xi}(0) + \widetilde{\zeta}(t) + \widetilde{\mathfrak{R}}\Lambda(t), \qquad 0 \le t < \infty, \tag{5.12}$$

from (5.1) where $\widetilde{\zeta}(\cdot) = \mathfrak{L}^{-1/2}\mathfrak{U}\zeta(\cdot)$ is a Brownian motion with drift coefficient $\widetilde{b}(\cdot) := \mathfrak{L}^{-1/2}\mathfrak{U}b(\cdot)$ and $b(\cdot)$ is defined in (5.5). We may regard $\widetilde{\Xi}(\cdot)$ as a reflected Brownian motion in a new state space $\mathfrak{S} := \mathfrak{L}^{-1/2}\mathfrak{U}(\mathbb{R}_+)^{n-1}$ with faces $\widetilde{\mathfrak{F}}_k := \mathfrak{L}^{-1/2}\mathfrak{U}\mathfrak{F}_k$, $k = 1, \dots, n-1$. The transformed reflection matrix $\widetilde{\mathfrak{R}} := \mathfrak{L}^{-1/2}\mathfrak{U}\mathfrak{R}$ can be written $\widetilde{\mathfrak{R}} = (\widetilde{\mathfrak{N}} + \widetilde{\mathfrak{Q}})\widetilde{\mathfrak{C}} = (\widetilde{\mathfrak{r}}_1, \dots, \widetilde{\mathfrak{r}}_{n-1})$, where $\widetilde{\mathfrak{C}} := \mathfrak{D}^{-1/2}$, $\mathfrak{D} := \operatorname{diag}(\mathfrak{A})$, $\widetilde{\mathfrak{N}} := \mathfrak{L}^{1/2}\mathfrak{U}\widetilde{\mathfrak{C}} = (\widetilde{\mathfrak{n}}_1, \dots, \widetilde{\mathfrak{n}}_{n-1})$, $\widetilde{\mathfrak{Q}} := \mathfrak{L}^{-1/2}\mathfrak{U}\mathfrak{R}\widetilde{\mathfrak{C}}^{-1} - \widetilde{\mathfrak{N}} = (\widetilde{\mathfrak{q}}_1, \dots, \widetilde{\mathfrak{q}}_{n-1})$. The constant vectors $\widetilde{\mathfrak{r}}_k, \widetilde{\mathfrak{q}}_k, \widetilde{\mathfrak{n}}_k$, $k = 1, \dots, n-1$, are $((n-1) \times 1)$ column vectors.

The corresponding differential operators $\widetilde{\mathcal{A}}, \widetilde{\mathcal{D}}_k$ and their adjoints $\widetilde{\mathcal{A}}^*, \widetilde{\mathcal{D}}_k^*$ are defined by

$$\begin{aligned}
[\widetilde{\mathcal{A}} f](z, \mathbf{p}) &:= \tfrac{1}{2}\Delta f(z) + \langle \widetilde{b}(\mathbf{p}), \nabla f(z)\rangle, \\
[\widetilde{\mathcal{D}}_k f](z) &:= \langle \widetilde{\mathfrak{r}}_k, \nabla f(z)\rangle, \\
[\widetilde{\mathcal{A}}^* f](z, \mathbf{p}) &:= \tfrac{1}{2}\Delta f(z) - \langle \widetilde{b}(\mathbf{p}), \nabla f(z)\rangle, \\
[\widetilde{\mathcal{D}}_k^* f](z) &:= \langle \widetilde{\mathfrak{r}}_k^*, \nabla f(z)\rangle,
\end{aligned} \tag{5.13}$$

where we define the adjoint direction $\widetilde{\mathfrak{r}}_k^* := \widetilde{\mathfrak{n}}_k - \widetilde{\mathfrak{q}}_k + \langle \widetilde{\mathfrak{n}}_k, \widetilde{\mathfrak{q}}_k\rangle \widetilde{\mathfrak{n}}_k$ of reflection to $\widetilde{\mathfrak{r}}_k$ for $k = 1, \dots, n-1$, $z \in (\mathbb{R}_+)^{n-1}$, $\mathbf{p} \in \Sigma_n$.

With these differential operators as in Lemma 2, we obtain the (BAR) for the process $(\widetilde{\Xi}(\cdot), \mathfrak{P}.)$ and its invariant distribution $\widetilde{\nu}(\cdot, \cdot)$; that is, for every $k = 1, \dots, n-1$, there exists a finite measure $\{\widetilde{\nu}_{0k}(\cdot)\}$ which is absolutely continuous with respect to the $(n-2)$-dimensional Lebesgue measure on $\widetilde{\mathfrak{F}}_k$ and such that for any C_b^2-function $f : \mathfrak{S} \mapsto \mathbb{R}$ we have

$$\int_{\mathfrak{S} \times \Sigma_n} [\widetilde{\mathcal{A}} f](z, \mathbf{p})\, d\widetilde{\nu}(z, \mathbf{p}) + \frac{1}{2}\sum_{k=1}^{n-1} \int_{\widetilde{\mathfrak{F}}_k} [\widetilde{\mathcal{D}}_k f](z)\, d\widetilde{\nu}_{0k}(z) = 0. \tag{5.14}$$

Our argument, especially from here onward, relies heavily on the elaborate analysis given by Harrison and Williams [16, 17]. The main distinction between their setting and ours is in the drift coefficient $b(\cdot)$, which here varies from chamber to chamber as well as within each chamber, and is evaluated along the path of the index process $\mathfrak{P}.$. Here, however, we can use the following observation.

LEMMA 4. *The following two conditions are equivalent*:

(i) *For each collection of constants* $\{g_k, \gamma_i; 1 \le i, k \le n\}$, *there are* $(n-1)$-*dimensional vectors* $\widetilde{\lambda}_{\mathbf{p}} := (\widetilde{\lambda}_{\mathbf{p},1}, \dots, \widetilde{\lambda}_{\mathbf{p},n-1})'$ *for* $\mathbf{p} \in \Sigma_n$, *such that a probability measure in the form of sum of products of exponentials*

$$\widetilde{\nu}(A \times B) := c \sum_{\mathbf{p} \in B} \int_A \exp(\langle \widetilde{\lambda}_{\mathbf{p}}, z\rangle)\, dz =: \sum_{\mathbf{p} \in B} \int_A \widetilde{\wp}_{\mathbf{p}}(z)\, dz \tag{5.15}$$

for measurable sets $A \subset \mathfrak{S}$ *and* $B \subset \Sigma_n$, *satisfies* (5.14) *for* $f(\cdot) \in C_c^2(\mathfrak{S})$, *where* c *in* (5.15) *is a normalizing constant.*

(ii) *The covariance and the direction of reflection satisfy the skew symmetry condition* (5.8).

Indeed, substituting (5.15) into (5.14) and combining the summation over $\mathbf{p} \in \Sigma_n$, we observe that the left-hand side of (5.14) becomes

$$\sum_{\mathbf{p}\in\Sigma_n}\left\{\int_{\mathfrak{S}}[\widetilde{\mathcal{A}}f](z,\mathbf{p})\cdot\widetilde{\wp}_{\mathbf{p}}(z)\,dz+\frac{1}{2}\sum_{k=1}^{n-1}\int_{\widetilde{\mathfrak{F}}_k}[\widetilde{\mathcal{D}}_k f](z)\cdot\widetilde{\wp}_{\mathbf{p}}(z)\,dz\right\}$$

for $f \in C_c^2(\mathfrak{S})$, where the expression in the curly bracket corresponds exactly to the BAR condition studied in [17] with some differences in notation. This way, we may reduce our problem to the case of [17]. Following the proof of Lemma 7.1 in [17], we observe that condition (i) in Lemma 4 is equivalent to the following conditions (iii) and (iv), where:

(iii) $[\widetilde{\mathcal{A}}^*\widetilde{\wp}_\cdot](\cdot,\cdot) = 0$ in $\mathfrak{S} \times \Sigma_n$, and

(iv) $[\widetilde{\mathcal{D}}_k^*\widetilde{\wp}_{\mathbf{p}}](\cdot) = 2b_k(\cdot)\widetilde{\wp}_{\mathbf{p}}(\cdot)$ on $\widetilde{\mathfrak{F}}_k$ for $k = 1,\dots,n-1$, $\mathbf{p} \in \Sigma_n$.

Here the adjoint operators $\widetilde{\mathcal{A}}^*$, $\widetilde{\mathcal{D}}_k^*$ are defined in (5.13).

Then the same reasoning as in the proof of Theorem 2.1 in [17] yields our Lemma 4, and we obtain $\widetilde{\lambda}_{\mathbf{p}} = 2(I - \widetilde{\mathfrak{N}}\widetilde{\mathfrak{Q}})^{-1}b(\mathbf{p})$ for $\mathbf{p} \in \Sigma_n$ along the way. This gives the invariant distribution $\widetilde{\nu}(\cdot)$ of $\widetilde{\Xi}(\cdot)$ in (5.12). Now transforming back to $\Xi(\cdot)$, we obtain (5.10), (5.9) and then (5.11). □

EXAMPLE 2. With $\gamma_i = 0$, $\rho_{i,j} = 0$, $1 \le i, j \le n$ and $\sigma_1^2 = \cdots = \sigma_n^2$, we recover the case studied by Banner, Fernholz and Karatzas [3] and Pitman and Pal [22]. Our Theorem 2 is an extension of their results, to the case of variances that are not necessarily equal and, as far as the second of these papers is concerned, to a finite number of particles.

5.2. *Average occupation times.* The long-term average occupation time $\theta_{\mathbf{p}}$ of the vector process $\mathfrak{X}(\cdot)$ in the polyhedral chamber $\mathcal{R}_{\mathbf{p}}$ of (3.9) is the probability mass $\nu_1(\mathbf{p}) := \nu((\mathbb{R}_+)^{n-1}, \mathbf{p})$ assigned to such a particular chamber by the marginal invariant distribution of the index process $\mathfrak{P}_\cdot$, which we can compute directly from (5.9) for $\mathbf{p} \in \Sigma_n$.

COROLLARY 4. *Under the assumptions of Theorem* 2, *the long-term average occupation time* $\theta_{\mathbf{p}}$ *of* $\mathfrak{X}(\cdot)$ *in the chamber* $\mathcal{R}_{\mathbf{p}}$ *for* $\mathbf{p} \in \Sigma_n$, *and the long-term proportion* $\theta_{k,i}$ *of time spent by company* i *in the* k*th rank as in* (3.8), *are explicitly given by the respective formulae*

$$\theta_{\mathbf{p}} = \left(\sum_{\mathbf{q}\in\Sigma_n}\prod_{j=1}^{n-1}\lambda_{\mathbf{q},j}^{-1}\right)^{-1}\prod_{j=1}^{n-1}\lambda_{\mathbf{p},j}^{-1} \quad \textit{and} \quad \theta_{k,i} = \sum \theta_{\mathbf{p}}. \tag{5.16}$$

Here $\lambda_{\mathbf{p}}$ *is as in* (5.10), *and the summation for* $\theta_{k,i}$ *is taken over the set* $\{\mathbf{p} \in \Sigma_n | \mathbf{p}(k) = i\}$ *for* $1 \le i, k \le n$.

From Corollary 3, the average occupation times $(\theta_{k,i})$ satisfy the equilibrium identity (4.12). As a sanity check, we verify this identity for the expressions of (5.16), through some algebraic computations in Section A.4.

EXAMPLE 3. It should be noted that in the presence of name-based variances, (5.16) can fail significantly. Consider the case where $n = 3$, with $\gamma_i = 0$, for $i = 1, 2, 3$; $\sigma_k = \sigma > 0$, for $k = 1, 2, 3$; $g_3 = g > 0$, $g_2 = 0$ and $g_1 = -g$; all $\rho_{i,j}$ is zero for $i, j = 1, 2, 3$ except $\rho_{3,3} = \rho \gg \sigma$. In this case, $Y_1(\cdot)$ and $Y_2(\cdot)$ will vibrate quietly in the middle with variance rate σ^2, while $Y_3(\cdot)$, with much greater variance rate $(\sigma + \rho)^2$, will be wandering far and wide. From Corollary 1 and (4.12) we obtain

$$\vartheta = (\theta_{k,i})_{1 \le i,k \le 3} = \begin{pmatrix} \frac{1-\alpha}{2} & \frac{1-\alpha}{2} & \alpha \\ \alpha & \alpha & 1-2\alpha \\ \frac{1-\alpha}{2} & \frac{1-\alpha}{2} & \alpha \end{pmatrix}, \tag{5.17}$$

where the parameter α is in the interval $(1/3, 1/2)$ for $\rho > 0$. The upper bound $1/2$ is obtained as $\lim_{\rho \to \infty} \theta_{1,3}$. Without name-based variances, that is, if the $\rho_{i,j}$ were all zero, the $Y_i(\cdot)$ would each spend the same proportion of time in every rank, yielding a matrix ϑ in (5.17) with all entries equal to $1/3$ from Corollary 4. This gives the lower bound $1/3$.

EXAMPLE 4. Let us consider a numerical computation of $(\theta_{k,i})$ for descending name-based drifts γ_i and ascending rank-based drifts g_k, for example, $n = 10$ and $\sigma_k^2 = 1 + k$, as well as $g_k = -1$ for $k = 1, \dots, 9$, $g_{10} = 9$, $\gamma_i = 1 - (2i)/(n+1)$ for $i = 1, \dots, n$. This is a rather extreme case of Example 1, with $g = 1$. The overall maximum is $\theta_{1,1} = 0.5184$, and the overall minimum is $\theta_{1,10} = 0.00485$. The company "$i = 1$" stays at the first rank longer than any other companies because of its relatively strong name-based drift; whereas the company "$i = 10$" stays at the first rank only for a tiny amount of time because of its relatively poor name-based drift.

Figure 1 shows a gray scale heat map for the different values of $\{\theta_{k,i}\}$; of course we know from Example 1 that $\theta_{10,i} = i/55$, $i = 1, \dots, 10$.

For a larger number of companies, say $n \sim 5000$, it seems rather hopeless for the current computational environment to perform direct computations of $\theta_{k,i}$ via the sum of (5.16) over $(n-1)!$ permutations in general.

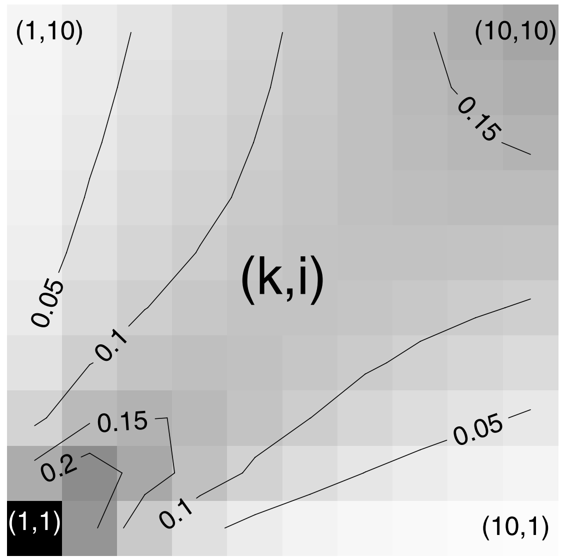

FIG. 1. *Different values of* $\{\theta_{k,i}\}$ *for* (k,i), *when the parameters are specified for an extreme case in Example* 4.

5.3. *Capital distribution curve*. The capital distribution curve is the log-log plot of market weights in descending order, as in (1.1). The empirical capital distribution curves, for the U.S. stock market over the seven decades 1929–1999, are shown in [13] (Figure 5.1 on page 95). Our next result computes the capital distribution curves directly from Theorem 2, from the gaps $\Xi_k(\cdot) = \log(\mu_{(k)}(\cdot)/\mu_{(k+1)}(\cdot))$ in the ranked log-market-weights to the ranked log-market-weights $\mathfrak{c}_k(\cdot) := \log \mu_{(k)}(\cdot)$ themselves.

COROLLARY 5. *Under the assumptions of Theorem* 2, *the ranked market weights* $\mu_{(1)}(\cdot), \dots, \mu_{(n)}(\cdot)$ *in* (1.1), (4.2) *have invariant distribution with*

$$\wp(m_1, \dots, m_{n-1}) = \sum_{\mathbf{p} \in \Sigma_n} \left[\theta_{\mathbf{p}} \cdot \prod_{k=1}^{n-1} \lambda_{\mathbf{p},k} \cdot \left(\prod_{j=1}^{n} m_j^{\lambda_{\mathbf{p},j} - \lambda_{\mathbf{p},j-1} + 1} \right)^{-1} \right] \tag{5.18}$$

as its density, for $0 < m_n \le m_{n-1} \le \cdots \le m_1 < 1$ *and* $m_n = 1 - m_1 - \cdots - m_{n-1}$. *Here we set* $\lambda_{\mathbf{p},0} = 0 = \lambda_{\mathbf{p},n}$, $\mathbf{p} \in \Sigma_n$, *for notational simplicity.*

Moreover, the log-ranked market weights $\mathfrak{c}_k(\cdot) = \log \mu_{(k)}(\cdot)$ *have invariant distribution with density*

$$\wp(c_1, \dots, c_{n-1}) = \sum_{\mathbf{p} \in \Sigma_n} \left[\theta_{\mathbf{p}} \cdot \prod_{j=1}^{n-1} \left(\lambda_{\mathbf{p},j} \cdot e^{-(\lambda_{\mathbf{p},j} - \lambda_{\mathbf{p},j+1}) c_j} \right) \cdot e^{\lambda_{\mathbf{p},n-1} c_n} \right] \tag{5.19}$$

for $-\infty < c_n \le \cdots \le c_2 \le c_1 < 0$, $c_n = \log(1 - \sum_{j=1}^{n-1} e^{c_j})$.

From the invariant density functions given by (5.11) and (5.18), (5.19) [or simply (5.16)], the piecewise linear capital distribution curve (1.1) has the expected slope

$$\mathbb{E}^{\nu}\left[\frac{\log\mu_{(k+1)}-\log\mu_{(k)}}{\log(k+1)-\log k}\right]=-\frac{\mathbb{E}^{\nu}(\Xi_k)}{\log(1+k^{-1})}=-\frac{\sum_{\mathbf{p}\in\Sigma_n}\theta_{\mathbf{p}}\lambda_{\mathbf{p},k}^{-1}}{\log(1+k^{-1})}\tag{5.20}$$

between the kth and the $(k+1)$st ranked stocks for $k=1,\dots,n-1$, and the initial value

$$\mathbb{E}^{\nu}(\log\mu_{(1)})=\mathbb{E}^{\nu}(\mathfrak{c}_1)=\mathbb{E}^{\nu}[-\log(1+e^{-\Xi_1}+e^{-(\Xi_1+\Xi_2)}+\cdots+e^{-(\Xi_1+\cdots+\Xi_{n-1})})]$$

for the first rank. From (5.9) this expected initial value may be obtained through a Monte Carlo simulation of generating $(n-1)$ independent exponential random variables with intensities $\lambda_{\mathbf{p},j}$ for $j=1,\dots,n-1$, $\mathbf{p}\in\Sigma_n$. From (5.20) we obtain the following simple criterion for convexity (or concavity) of the expected capital distribution curves.

COROLLARY 6. *Under the assumptions of Theorem 2, a sufficient condition for the expected capital distribution curve* $\log k\mapsto\mathbb{E}^{\nu}(\log\mu_{(k)})$ *under the invariant distribution* ν *to be* convex (*resp*., concave), *is that*

$$\lambda_{\mathbf{p},k+1}\log\left(1+\frac{1}{k+1}\right)-\lambda_{\mathbf{p},k}\log\left(1+\frac{1}{k}\right)\geq 0\qquad\forall\mathbf{p}\in\Sigma_n\tag{5.21}$$

(*resp*., $\leq$) *hold for each* $k=1,\dots,n-2$, *where* $\lambda_{\mathbf{p},k}$ *is given in* (5.10).

EXAMPLE 5. Let us consider the first-order Atlas model which is a combination of the "Atlas configuration" in Example 1 with the further restrictions of Example 2; to wit, $g_n=(n-1)g$, $g_1=\cdots=g_{n-1}=-g<0$ for some $g>0$, as well as $\gamma_i=0$, $\rho_{i,j}=0$, $1\leq i,j\leq n$, and $\sigma_1^2=\cdots=\sigma_n^2=\sigma^2>0$ for some $\sigma^2>0$. From Corollary 6, the expected capital distribution curve is *convex* but *almost linear* for larger k. Indeed, the quantity $\lambda_{\mathbf{p},k}\log(1+k^{-1})=2(gk/\sigma^2)\cdot\log(1+k^{-1})$ increases in $k\geq 1$, and converges to $2g/\sigma^2$, as $k\uparrow\infty$, for all $\mathbf{p}\in\Sigma_n$, and so the difference in (5.21) is positive for each $k=1,\dots,n-2$ but decreases to zero quite rapidly in the order of $O(k^{-2})$, as $k\uparrow\infty$. Another explanation of such linearity ("Pareto line") of the capital distribution curves from an application of Poisson point processes can be found in Example 5.1.1 on page 94 of [13].

EXAMPLE 6. Suppose now that we change only the rank-based variances in Example 5; namely, we take linearly growing variances $\sigma_k^2=k\sigma^2$ for some $\sigma^2>0$, $k=1,\dots,n$. Then

$$\lambda_{\mathbf{p},k}\log\left(1+\frac{1}{k}\right)=\frac{4kg}{(2k+1)\sigma^2}\cdot\log\left(1+\frac{1}{k}\right)$$

is decreasing in $k\geq 1$ for every $\mathbf{p}\in\Sigma_n$, and so the difference in (5.21) is negative for each $k=1,\dots,n-2$. Thus, from Corollary 6, the expected capital distribution curve becomes *concave*.

EXAMPLE 7 ("Pure" hybrid market conjecture). A pure hybrid market is one in which all the parameters are determined by the "name" of the stock, with the exception of the growth rate of the smallest stock. The log-capitalization $Z_n(\cdot)$ of the smallest stock has its growth rate incremented by $g > 0$, as in the Atlas model. Hence, this market will look like

$$dY_i(t) = \begin{cases} -\gamma_i\,dt + \sigma_i\,dW_i(t), & \text{if } Y_i(t) \neq Z_n(t), \\ (g - \gamma_i)\,dt + \sigma_i\,dW_i(t), & \text{if } Y_i(t) = Z_n(t), \end{cases}$$

for $i = 1, \dots, n$ and $t \in [0, \infty)$, where $\gamma_i > 0$, $\sigma_i > 0$, and $g = \sum_{i=1}^n \gamma_i$. We conjecture that the capital distribution curve for this market is convex.

This conjecture is based on the following reasoning: The Atlas stock $Z_n(\cdot)$ performs a role similar to a local time process, reflecting each stock away from the bottom position. Hence, outside the set where $Y_i(\cdot) = Z_n(\cdot)$, the distance $Y_i(\cdot) - Z_n(\cdot)$ will be approximately exponentially distributed. Accordingly, suppose we replace $Y_i(\cdot) - Z_n(\cdot)$ by an exponentially distributed random variable $\mathbf{Z}_i$ with rate parameter $\alpha_i = \sigma_i^2/(2\gamma_i)$

$$P\{\mathbf{Z}_i > x\} = e^{-\alpha_i x}, \qquad x > 0, i = 1, \dots, n.$$

Let $\mathbf{Z}$ represent a generic member of such random variables $(\mathbf{Z}_i, i = 1, \dots, n)$ as a mixed exponential distribution

$$P\{\mathbf{Z} > x\} = \frac{1}{n}\sum_{i=1}^n e^{-\alpha_i x}, \qquad x > 0,$$

and define $z_{(k)}$ as $P\{\mathbf{Z} > z_{(k)}\} = k/n$ for $k = 1, \dots, n$. In this case, the capital distribution curve is approximately proportional to the graph of $z_{(k)}$ versus $\log k$, and this graph, $\log k \mapsto z_{(k)}$, $k = 1, \dots, n$, will be convex on average. In fact, the graph of $\log(k/n) = \log(\sum_{i=1}^n e^{-\alpha_i x}/n)$, where $\log k$ is considered to be a function of x, is convex, because with $\phi(x) := \sum_{i=1}^n e^{-\alpha_i x}$,

$$\frac{d^2}{dx^2}\log k = \frac{\phi''(x)\phi(x) - (\phi'(x))^2}{(\phi(x))^2} = \frac{\sum_{i,j=1}^n (\alpha_i - \alpha_j)^2 e^{-(\alpha_i+\alpha_j)x}}{2(\phi(x))^2} \geq 0.$$

Note, of course, that this holds for the random variables $(\mathbf{Z}_i, i = 1, \dots, n)$ and $\mathbf{Z}$, but that it holds for the process $Y_i(\cdot)$ is only a conjecture. This conjecture is of interest because, historically, capital distribution curves appear to be concave which could imply that rank-based parameters as well as name-based parameters are needed to explain stock market behavior.

EXAMPLE 8. To see different shapes of the expected capital distribution curve under different parameter configurations apart from Examples 5 and 6, let us consider a pure hybrid market whose drift and volatility coefficients do not depend on ranks, except for the smallest (Atlas) stock. For example, take $n = 5000$, $g_k = 0$, $1 \leq k \leq n-1$, $g_n = c_*(2n-1)$, $\gamma_1 = -c_*$, $\gamma_i = -2c_*$, $2 \leq i \leq n$, $\sigma_k^2 = 0.075$,

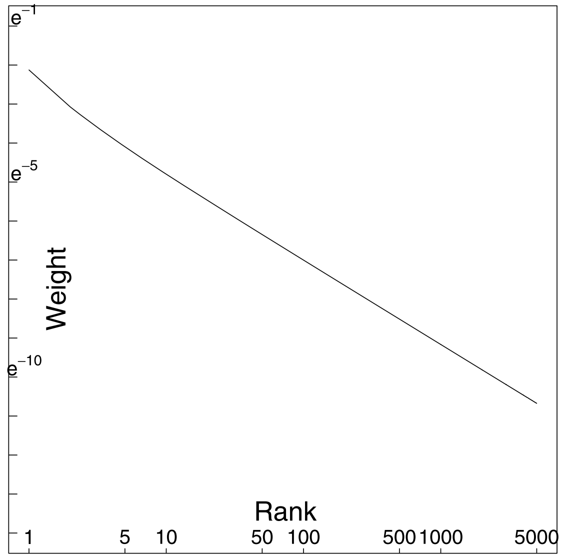

FIG. 2. *Expected capital distribution curve for the pure hybrid model in Example 8.*

$1 \le k \le n$ and $\rho_{i,j} = 0$ for $1 \le i, j \le n$ with a parameter $c_* = 0.02$. These parameters satisfy the assumptions of Theorem 2. We cannot apply Corollary 6 because the difference in (5.21) is positive on $\{\mathbf{p} \in \Sigma_n : \mathbf{p}(k+1) \neq 1\}$ but negative on its (smaller) complement. The resulting expected capital distribution curve is *convex*; it is depicted in Figure 2.

EXAMPLE 9. Let us consider now a variant of this pure hybrid model, with a variance structure that is observed in practice. The parameters are the same as in Example 8, except for the different choices of the parameter c_* and for the rank-based variances $\sigma_k^2 := 0.075 + 6k \times 10^{-5}$ which are obtained from the smoothed annualized values for 1990–1999 data as in Section 5.4, page 109 of [13] (see page 2319 of [3]). The criterion from Corollary 6 cannot apply directly to this case because the inequalities (5.21) do not hold for all $\mathbf{p} \in \Sigma_n$. The expected capital distribution curves under these parameters with (i) $c_* = 0.02$, (ii) $c_* = 0.03$, (iii) $c_* = 0.04$ are shown in Figure 3. The curve (i) is convex from the top rank to about the 25th rank, then turns concave until the lowest rank. The other curves (ii) and (iii) behave similarly.

EXAMPLE 10. Adopting the same parameter specifications in Example 9(i) $c_* = 0.02$, except the rank-based drift, that is, (iv) the upwind first ranked stock $g_1 = -0.016$, $g_k = 0, 2 \le k \le n-1$, $g_n = (0.02)(2n-1) + 0.016$ and (v) the windward top 50 stocks $g_1 = g_2 = \cdots = g_{50} = -0.016$, $g_k = 0, 51 \le k \le n-1$, $g_n = (0.02)(2n-1) + 0.8$, we obtain concave curves as in Figure 4. The observed average curve and the estimated curve of the first-order Atlas model for 1990–1999

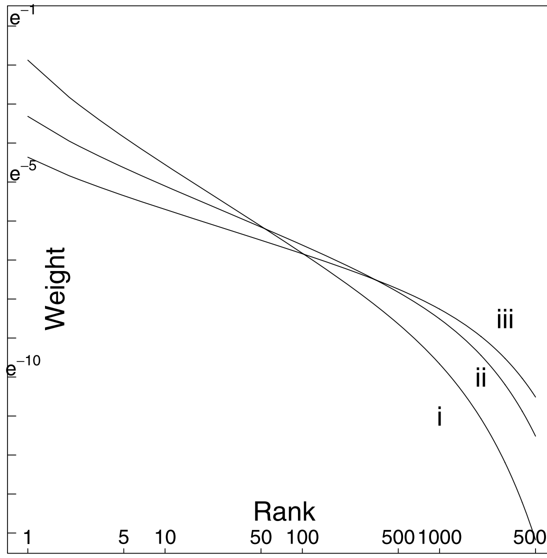

FIG. 3. *Expected capital distribution curves for the hybrid model in Example 9.*

(Figure 3 of [3], page 2320) are concave. The statistical inference for the capital distribution curves is an interesting problem that we do not discuss here.

6. Portfolio analysis. Let us consider investing in the market of (2.1) according to a portfolio rule $\Pi(\cdot) = (\Pi_1(\cdot), \dots, \Pi_n(\cdot))'$. This is an $\{\mathcal{F}_t\}$-adapted, locally

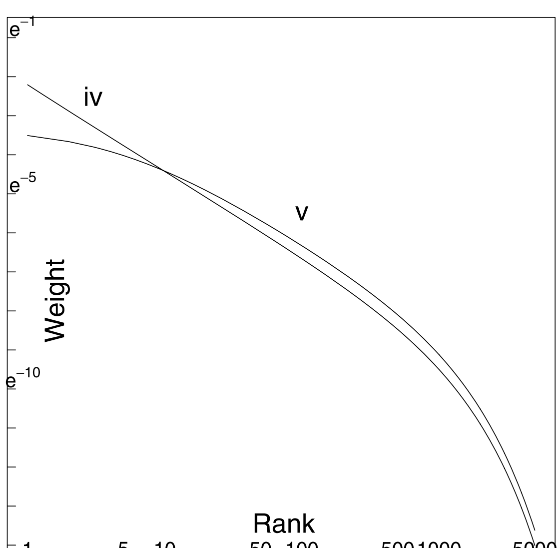

FIG. 4. *Expected capital distribution curves for the hybrid model in Example 10.*

square-integrable process with $\sum_{i=1}^n \Pi_i(\cdot) = 1$. Each $\Pi_i(t)$ represents the proportion of the portfolio's wealth $V^\Pi(t)$ invested in stock i at time t, so

$$\text{(6.1)} \qquad \frac{dV^\Pi(t)}{V^\Pi(t)} = \sum_{i=1}^n \Pi_i(t) \cdot \frac{dX_i(t)}{X_i(t)}, \qquad V^\Pi(0) = w > 0.$$

For example, we may choose for every $t \in [0, \infty)$ the vector of market weights $\mu_i(t)$, $i = 1, \dots, n$, as in (4.2). We shall call the resulting $\Pi(\cdot) \equiv \mu(\cdot)$ the *market portfolio* and note $V^\mu(\cdot) = wX(\cdot)/X(0)$, thus from Proposition 2: $\lim_{T\to\infty}(1/T)\log V^\mu(T) \equiv \gamma$, a.s.

For a constant-proportion portfolio $\Pi(\cdot) \equiv \pi \in \Gamma^n := \{(\pi_1, \dots, \pi_n)' \in \mathbb{R}^n | \sum_{i=1}^n \pi_i = 1\}$ (which of course the market portfolio is not), the solution of (6.1) is given by

$$\text{(6.2)} \qquad d\log V^\pi(t) = \gamma_\pi^*(t)\,dt + \sum_{i=1}^n \pi_i d\log X_i(t), \qquad 0 \le t < \infty.$$

Here we shall denote by $(a_{ij}(t))_{1\le i,j\le n} = S(Y(t))S(Y(t))'$ the covariance process from (2.4), and introduce

$$\text{(6.3)} \qquad \gamma_\pi^*(t) := \frac{1}{2}\left(\sum_{i=1}^n \pi_i a_{ii}(t) - \sum_{i,j=1}^n \pi_i a_{ij}(t)\pi_j\right), \qquad 0 \le t < \infty,$$

the *excess growth rate* of the constant-proportion $\Pi(\cdot) \equiv \pi \in \Gamma^n$. Thus, for a constant-proportion portfolio we can write the solution of (6.1), namely

$$\text{(6.4)} \qquad V^\pi(t) = w \cdot \exp\left[\sum_{i=1}^n \pi_i \left\{\frac{A_{ii}(t)}{2} + \log\left(\frac{X_i(t)}{X_i(0)}\right)\right\} - \frac{1}{2}\sum_{i,j=1}^n \pi_i A_{ij}(t)\pi_j\right]$$

as in (2.4) of [19], where $A_{ij}(\cdot) = \int_0^\cdot a_{ij}(t)\,dt$; we set $A(\cdot) := (A_{ij}(\cdot))_{1\le i,j\le n}$.

6.1. *Target portfolio.* Let us assume that, for every $(t, \omega) \in [0, \infty) \times \Omega$, there exists a vector $\Pi^*(t, \omega) := (\Pi_1^*(t, \omega), \dots, \Pi_n^*(t, \omega))' \in \Gamma^n$ that attains the maximum of the wealth $V^\pi(t, \omega)$ over vectors $\pi \in \Gamma^n$; and that the resulting process $\Pi^*(\cdot)$ defines a portfolio. Along with Cover [9] and Jamshidian [19], we shall call this $\Pi^*(\cdot)$ a *Target Portfolio*, and

$$\text{(6.5)} \qquad V_*(t) := \max_{\pi\in\Gamma^n} V^\pi(t), \qquad 0 \le t < \infty,$$

the *Target Performance* for the model. [The quantity of (6.5) is not necessarily equal to the performance $V^{\Pi^*}(\cdot)$ of the portfolio Π^*.]

The Target Performance $V_*(\cdot)$ exceeds the performance of the leading stock, of the value-line index (the geometric mean), and of any arithmetic average (such as the Dow Jones Industrial Average): to wit, taking $X_1(0) = \cdots = X_n(0) = 1$, we have for every vector $(\alpha_1, \dots, \alpha_n)' \in \Gamma_+^n := \{(\pi_1, \dots, \pi_n)' \in \Gamma^n | \pi_i \geq 0, i = 1, \dots, n\}$ the almost sure comparisons

$$V_*(\cdot) \geq \max\left[\max_{1\leq i\leq n} X_i(\cdot), \left(\prod_{j=1}^n X_j(\cdot)\right)^{1/n}, \sum_{j=1}^n \alpha_j X_j(\cdot)\right]. \tag{6.6}$$

Under the assumptions of Theorem 1, the limits $\theta_{\mathbf{p}}$ of the average occupation times in (3.9) exist almost surely, and so do the limits of the average covariance rate $\mathfrak{a}_{ij}^\infty := \lim_{T\to\infty} A_{ij}(T)/T$; therefore, $\mathfrak{a}^\infty := (\mathfrak{a}_{ij}^\infty)_{1\leq i,j\leq n}$ is

$$\begin{aligned}\mathfrak{a}^\infty &= \lim_{T\to\infty} \frac{1}{T}\int_0^T (a_{ij}(t))_{1\leq i,j\leq n}\,dt \\ &= \lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{\mathbf{p}\in\Sigma_n} \mathbf{1}_{\mathcal{R}_{\mathbf{p}}}(Y(s)) \cdot \mathfrak{s}_{\mathbf{p}}\mathfrak{s}_{\mathbf{p}}'\,dt = \sum_{\mathbf{p}\in\Sigma_n} \theta_{\mathbf{p}}\mathfrak{s}_{\mathbf{p}}\mathfrak{s}_{\mathbf{p}}',\end{aligned} \tag{6.7}$$

with $\mathfrak{s}_{\mathbf{p}}$ defined in (2.3). It follows from (6.4) and Proposition 2 that the asymptotic long-term-average growth rate of a constant-proportion portfolio $\pi \in \Gamma^n$ is

$$\lim_{T\to\infty} \frac{1}{T}\log V^\pi(T) = \gamma + \frac{1}{2}\left(\sum_{i=1}^n \pi_i \mathfrak{a}_{ii}^\infty - \sum_{i,j=1}^n \pi_i \mathfrak{a}_{ij}^\infty \pi_j\right) =: \gamma + \gamma_\pi^\infty. \tag{6.8}$$

Maximizing this expression over $\pi \in \Gamma^n$ amounts to maximizing, over constant-proportion portfolios, the excess growth rate

$$\gamma_\pi^\infty = \frac{1}{2}\left(\sum_{i=1}^n \pi_i \mathfrak{a}_{ii}^\infty - \sum_{i,j=1}^n \pi_i \mathfrak{a}_{ij}^\infty \pi_j\right)$$

that corresponds to the asymptotic covariance structure.

We shall call *Asymptotic Target Portfolio* a vector $\bar{\pi} = (\bar{\pi}_1, \dots, \bar{\pi}_n)' \in \Gamma^n$ that attains $\max_{\pi\in\Gamma^n} \gamma_\pi^\infty$. We can regard this portfolio as *asymptotic growth-optimal* over all constant-proportion portfolios, in the sense that $\lim_{T\to\infty}(1/T) \times \log(V^\pi(T)/V^{\bar{\pi}}(T)) \leq 0$ holds a.s. for every $\pi \in \Gamma^n$.

EXAMPLE 11. When there is no covariance structure by name, that is, $\rho_{i,j} \equiv 0$ for every $1 \leq i, j \leq n$, we have $A_{ij}(\cdot) \equiv 0$ for $i \neq j$ in accordance with (2.3), (2.4). In this case, we compute a target portfolio $\Pi^*(\cdot)$ as

$$\begin{aligned}\Pi_i^*(t) = {}&\left(2A_{ii}(t)\sum_{j=1}^n \frac{1}{A_{jj}(t)}\right)^{-1}\left[2 - n - 2\sum_{j=1}^n \frac{1}{A_{jj}(t)}\log\left(\frac{X_j(t)}{X_j(0)}\right)\right] \\ &+ \frac{1}{2} + \frac{1}{A_{ii}(t)}\log\left(\frac{X_i(t)}{X_i(0)}\right), \qquad i = 1, \dots, n,\end{aligned} \tag{6.9}$$

and an asymptotic target portfolio by

$$\bar{\pi}_i = \frac{1}{2}\left[1 - \frac{n-2}{\mathfrak{a}_{ii}^{\infty}}\left(\sum_{j=1}^{n}\frac{1}{\mathfrak{a}_{jj}^{\infty}}\right)^{-1}\right] = \lim_{t\to\infty}\Pi_i^*(t), \qquad i=1,\dots,n, \text{ a.s.} \tag{6.10}$$

This constant portfolio $\bar{\pi}$ has exactly the same long-term growth rate as the target performance in (6.5), in particular

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\log V_*(T) &= \lim_{T\to\infty}\frac{1}{T}\log V^{\bar{\pi}}(T)\\ &= \gamma + \sum_{i=1}^{n}\frac{\mathfrak{a}_{ii}^{\infty}}{2}\bar{\pi}_i(1-\bar{\pi}_i) \qquad \text{a.s.;}\end{aligned} \tag{6.11}$$

on the other hand, we see from (6.8) that it outperforms the overall market rather significantly over long time horizons, namely

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\log\left(\frac{V^{\bar{\pi}}(T)}{V^{\mu}(T)}\right) &= \frac{1}{2}\sum_{i=1}^{n}\bar{\pi}_i(1-\bar{\pi}_i)\mathfrak{a}_{ii}^{\infty}\\ &= \frac{1}{8}\left[\sum_{i=1}^{n}\mathfrak{a}_{ii}^{\infty} - (n-2)^2\left(\sum_{j=1}^{n}\frac{1}{\mathfrak{a}_{jj}^{\infty}}\right)^{-1}\right]\\ &\geq \frac{n-1}{2}\left(\sum_{i=1}^{n}\frac{1}{\mathfrak{a}_{ii}^{\infty}}\right)^{-1}\end{aligned} \tag{6.12}$$

a.s., from the arithmetic mean–harmonic mean inequality.

With Cover [9] and Jamshidian [19], we shall say that stock i is *asymptotically active*, if for the expression of (6.10) we have $\bar{\pi}_i > 0$; and that the entire *market is asymptotically active*, if all its stocks are asymptotically active, that is, if $\bar{\pi} \in \Gamma_{++}^n := \{(\pi_1,\dots,\pi_n)' \in \Gamma^n | \pi_i > 0, i=1,\dots,n\}$.

EXAMPLE 12. A sufficient condition for asymptotic activity of the model with $n \geq 3$ under the condition of Theorem 2, is obtained from (6.10) as

$$\frac{1}{\mathfrak{a}_{ii}^{\infty}} < \frac{1}{n-2}\left(\sum_{\ell=1}^{n}\frac{1}{\mathfrak{a}_{\ell\ell}^{\infty}}\right), \qquad \text{or equivalently} \tag{6.13}$$

$$\left(\sum_{\mathbf{p}\in\Sigma_n}\sigma_{\mathbf{p}^{-1}(i)}^2\prod_{j=1}^{n-1}\lambda_{\mathbf{p},j}^{-1}\right)^{-1} < \frac{1}{n-2}\left[\sum_{\ell=1}^{n}\left(\sum_{\mathbf{p}\in\Sigma_n}\sigma_{\mathbf{p}^{-1}(\ell)}^2\prod_{j=1}^{n-1}\lambda_{\mathbf{p},j}^{-1}\right)^{-1}\right] \tag{6.14}$$

for every $i=1,\dots,n$, with $\lambda_{\mathbf{p},j}$ defined in (5.10); recall (6.7), (5.16) and (2.3). This is the case in the constant variance model $\sigma_1^2 = \cdots = \sigma_n^2$. In general, it seems that the drift and volatility coefficients have nontrivial effects on the condition (6.14).

6.2. *Universal portfolio*. The *universal portfolio* of Cover [9] and Jamshidian [19] is defined as

$$\widehat{\Pi}_i(t) := \frac{\int_{\Gamma_+^n} \pi_i V^{\pi}(t)\, d\pi}{\int_{\Gamma_+^n} V^{\pi}(t)\, d\pi}, \qquad 0 \le t < \infty, 1 \le i \le n.$$

It is constructed completely in terms of quantities, such as the $V^{\pi}(\cdot)$ for constant-proportion portfolios π, that are observable: no model-specific knowledge is required for its construction. As can be checked easily, the wealth process of this portfolio is given by the "performance-weighting"

$$V^{\widehat{\Pi}}(t) = \frac{\int_{\Gamma_+^n} V^{\pi}(t)\, d\pi}{\int_{\Gamma_+^n} d\pi}, \qquad 0 \le t < \infty,$$

yet another observable quantity. It follows from Theorem 2.4 of Jamshidian [19] that the universal portfolio does not lag significantly behind the target portfolio: its performance lag is only polynomial in time under an asymptotically active model. To wit, there exists then a positive constant C, such that

$$\lim_{T\to\infty} \left(\frac{V^{\widehat{\Pi}}(T)}{V_*(T)} \cdot T^{(n-1)/2} \right) = C$$

holds almost surely, thus also

$$\lim_{T\to\infty} \frac{1}{T} \log\left(\frac{V^{\widehat{\Pi}}(T)}{V^{\bar{\pi}}(T)} \right) = \lim_{T\to\infty} \frac{1}{T} \log\left(\frac{V^{\widehat{\Pi}}(T)}{V_*(T)} \right) = 0. \tag{6.15}$$

In the context of the hybrid model, under the assumptions of Theorem 2 and of Example 12, the universal portfolio attains the long-term growth rate of the target portfolio Π^* and of the asymptotic target portfolio $\bar{\pi}$. These are precisely the characteristics that make the universal portfolio interesting: it is constructed based entirely on quantities which are completely observable, yet its long-term performance matches that of $V_*(\cdot)$ in (6.5), and thus exceeds the performance of any constant-proportion portfolio.

6.3. *Growth-optimal portfolio*. We shall call *growth-optimal* a portfolio $\varpi(\cdot)$ that satisfies the inequality $\lim_{T\to\infty}(1/T)\log(V^{\Pi}(T)/V^{\varpi}(T)) \le 0$ almost surely, for any portfolio $\Pi(\cdot)$.

In order to find such a growth-optimal portfolio under no-name based correlation $\rho_{i,j} \equiv 0$ for $1 \le i, j \le n$, we need to maximize over $\pi \in \Gamma^n$ the quantity (growth rate)

$$\Gamma(t;\pi) := \sum_{i=1}^{n} \left(\widetilde{\gamma}_i(t) + \frac{1}{2} a_{ii}(t) \right) \pi_i - \frac{1}{2} \sum_{i=1}^{n} a_{ii}(t) \pi_i^2, \tag{6.16}$$

where $\widetilde{\gamma}_i(t) = \sum_{\mathbf{p}\in\Sigma_n} \mathbf{1}_{\mathcal{R}_\mathbf{p}}(Y(t)) g_{\mathbf{p}^{-1}(i)} + \gamma_i + \gamma$ is the ith element of $G(Y(t))$ of (2.4) (cf. Problem 4.6, page 108 in Fernholz and Karatzas [14]). By the Lagrange multiplier method, we obtain a vector that attains this maximum, as

$$\varpi_i(t) = \frac{1}{2} + \frac{\widetilde{\gamma}_i(t) + \overline{\gamma}(t)}{a_{ii}(t)}, \qquad i = 1, \dots, n, 0 \le t < \infty, \tag{6.17}$$

where the constraint $\sum_{i=1}^n \varpi_i(t) = 1$ is enforced by the multiplier

$$\overline{\gamma}(t) = \left(\sum_{i=1}^n \frac{1}{a_{ii}(t)}\right)^{-1} \left(1 - \frac{n}{2} - \sum_{j=1}^n \frac{\widetilde{\gamma}_j(t)}{a_{jj}(t)}\right).$$

The growth rate $\Gamma(t; \varpi)$ of this portfolio $\varpi(\cdot)$, in the notation of (6.16), (6.17) and using (2.2), is

$$\Gamma(t; \varpi) = \frac{n\gamma}{2} + \frac{1}{2}\sum_{i=1}^n \frac{\widetilde{\gamma}_i^2(t)}{a_{ii}(t)} - \frac{\overline{\gamma}^2(t)}{2}\sum_{i=1}^n \frac{1}{a_{ii}(t)} + \frac{1}{8}\sum_{i=1}^n a_{ii}(t).$$

• In order to make some comparisons, let us specialize to the equal-variance case, that is, $\sigma_1^2 = \cdots = \sigma_n^2 = \sigma^2$ with no name-based correlations $\rho_{i,j} \equiv 0$; we obtain under these assumptions the expression

$$\varpi_i(t) = \frac{1}{n} + \frac{1}{\sigma^2}\left(\gamma_i + \sum_{k=1}^n g_k \mathbf{1}_{Q_k^{(i)}}(Y(t))\right), \qquad i = 1, \dots, n \tag{6.18}$$

for the growth-optimal portfolio, and

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{T} \log V^{\varpi}(T) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T \Gamma(t; \varpi)\, dt \\ &= \gamma + \frac{\sigma^2}{2}\left(1 - \frac{1}{n}\right) + \frac{1}{2\sigma^2}\left(\sum_{k=1}^n g_k^2 - \sum_{i=1}^n \gamma_i^2\right), \end{aligned} \tag{6.19}$$

and from (3.8), (4.12) we obtain

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \varpi_i(t)\, dt = \frac{1}{n} + \frac{1}{\sigma^2}\left(\gamma_i + \sum_{k=1}^n g_k \theta_{k,i}\right) = \frac{1}{n} = \bar{\pi}_i, \qquad i = 1, \dots, n,$$

almost surely. On the other hand, from (6.15), (6.10) and (6.8) we see that the universal portfolio $\widehat{\Pi}(\cdot)$ and the asymptotic target portfolio $\overline{\pi}_i = \frac{1}{n}$, $i = 1, \dots, n$, have the same long-term growth rate, namely

$$\lim_{T\to\infty} \frac{1}{T}\log V^{\bar{\pi}}(T) = \lim_{T\to\infty}\frac{1}{T}\log V^{\widehat{\Pi}}(T) = \gamma + \frac{\sigma^2}{2}\left(1 - \frac{1}{n}\right). \tag{6.20}$$

Under the conditions of (2.2) and (3.3), we can verify

$$\sum_{k=1}^n g_k^2 > \sum_{i=1}^n \gamma_i^2. \tag{6.21}$$

To show (6.21), we may assume without loss of generality $\gamma_1 \geq \cdots \geq \gamma_n$ and hence that there exists $(\delta_1, \ldots, \delta_{n-1})' \in (\mathbb{R}_+)^{n-1} \setminus \{0\}$ such that $g_k = -(\gamma_k + \delta_k)$ for $k = 1, \ldots, n-1$, and $g_n = -\gamma_n + (\delta_1 + \cdots + \delta_{n-1})$ for (2.2) and (3.3). Then we obtain

$$\sum_{k=1}^{n} g_k^2 = \sum_{i=1}^{n-1} (\gamma_i + \delta_i)^2 + \bigl(-\gamma_n + (\delta_1 + \cdots + \delta_{n-1})\bigr)^2$$
$$= \sum_{i=1}^{n} \gamma_i^2 + \sum_{i=1}^{n-1} \bigl(\delta_i^2 + 2\delta_i(\gamma_i - \gamma_n)\bigr) + \left(\sum_{i=1}^{n-1} \delta_i\right)^2 > \sum_{i=1}^{n} \gamma_i^2.$$

We observe from (6.18)–(6.21) that the growth-optimal portfolio $\varpi(\cdot)$ dominates in the long run both the universal portfolio $\widehat{\Pi}(\cdot)$ and the asymptotic target portfolio $\bar{\pi}$, a.s. The advantage of the universal portfolio is that it can be constructed with total oblivion as to what the actual values of the parameters of the model might be; some of these may be quite hard to estimate in practice. By contrast, constructing the growth-optimal portfolio $\varpi(\cdot)$ as in (6.18) requires knowledge of all the model parameters, and keeping track of the positions of all stocks in all ranks at all times.

APPENDIX

A.1. Preparations for the proof of Lemma 1. The stochastic exponential

$$\zeta(t) = \exp\left[-\int_0^t \langle \xi(u), dW(u)\rangle - \frac{1}{2}\int_0^t \|\xi(u)\|^2\, du\right], \qquad 0 \leq t < \infty,$$

is a continuous martingale, where $\xi(t) := S^{-1}(Y(t))G(Y(t))$ for $0 \leq t < \infty$ and $\|x\|^2 := \sum_{j=1}^n x_j^2$, $x \in \mathbb{R}^n$, and $\langle x, y\rangle = \sum_{j=1}^n x_j y_j$, $x, y \in \mathbb{R}^n$. Recall that $S(\cdot)$, $S^{-1}(\cdot)$ and $G(\cdot)$ in (2.4) are bounded. By Girsanov's theorem

$$\widetilde{W}(t) := W(t) + \int_0^t S^{-1}(Y(u))G(Y(u))\, du, \qquad 0 \leq t < \infty,$$

is an n-dimensional Brownian motion under the new probability measure $\mathbb{Q}$, locally equivalent to $\mathbb{P}$, that satisfies

$$\mathbb{Q}(C) = \mathbb{E}^{\mathbb{P}}(\zeta(T)\mathbf{1}_C), \qquad C \in \mathcal{F}_T, 0 \leq T < \infty. \tag{A.1}$$

Thus, equation (2.4) under $\mathbb{P}$ is reduced to

$$dY(t) = S(Y(t))\, d\widetilde{W}(t), \qquad 0 \leq t < T, \text{ under } \mathbb{Q}. \tag{A.2}$$

A.1.1. *Local time of Bessel processes.* Let us denote the δ-dimensional Bessel process by $\mathfrak{r}^{(\delta)}(\cdot)$ for $\delta > 1$

$$\mathfrak{r}^{(\delta)}(t) = \mathfrak{r}^{(\delta)}(0) + \int_0^t \frac{\delta - 1}{2\mathfrak{r}^{(\delta)}(s)}\, ds + \widetilde{B}(t), \qquad 0 \leq t < \infty,$$

where $\widetilde{B}(\cdot)$ is the standard Brownian motion. Since it is a continuous semimartingale, there is a modification $\Lambda_{\mathfrak{r}^{(\delta)}}(\cdot)$ of its local time accumulated at the origin, defined by

$$\Lambda_{\mathfrak{r}^{(\delta)}}(t) = \frac{1}{2}\Big(\mathfrak{r}^{(\delta)}(t) - \mathfrak{r}^{(\delta)}(0) - \int_0^t \operatorname{sgn}(\mathfrak{r}^{(\delta)}(s))\, d\mathfrak{r}^{(\delta)}(s)\Big), \qquad 0 \le t < \infty,$$

where the function sgn is defined by $\operatorname{sgn}(x) = 1$ if $x > 0$ and $\operatorname{sgn}(x) = -1$ if $x \le 0$. When $\delta \ge 2$, $\mathfrak{r}^{(\delta)}(\cdot)$ never hits the origin, and its local time at the origin is identically equal to zero. Thus let us consider the case $1 < \delta < 2$. By the occupation times formula and the right continuity of the semimartingale local time, we obtain

$$\text{(A.3)} \qquad \Lambda_{\mathfrak{r}^{(\delta)}}(t) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{0 \le \mathfrak{r}^{(\delta)}(s) \le \varepsilon\}}\, ds \qquad \text{almost surely for } 0 \le t < \infty.$$

On the other hand, it can be shown from Lemma 3.1 and equation (3f) of Biane and Yor [6], and also form pages 285–289 of Rogers and Williams [24] that there exists a finite limit

$$\text{(A.4)} \qquad \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon^\delta} \int_0^t \mathbf{1}_{\{0 \le \mathfrak{r}^{(\delta)}(s) \le \varepsilon\}}\, ds \qquad \text{almost surely for } 0 \le t < \infty$$

[see (A.8) below]. Combining this fact with (A.3), there is no accumulation of local time at the origin for the case $1 < \delta < 2$. Therefore, we conclude that the local time $\Lambda_{\mathfrak{r}^{(\delta)}}(\cdot)$of the δ-dimensional Bessel process $\mathfrak{r}^{(\delta)}(\cdot)$ accumulated at the origin is *identically equal to zero*,

$$\text{(A.5)} \qquad \Lambda_{\mathfrak{r}^{(\delta)}}(t) \equiv 0, \qquad 0 \le t < \infty, \delta > 1.$$

PROOF OF (A.4) (Abridged from [6, 24]). Given the δ-dimensional Bessel processes $\mathfrak{r}^{(\delta)}(\cdot)$, there is a one-dimensional Bessel process $\mathfrak{r}^{(1)}(\cdot)$ which starts at $\mathfrak{r}^{(1)}(0) = (2-\delta)^{-(2-\delta)}(\mathfrak{r}^{(\delta)}(0))^{2-\delta}$ and satisfies the following pathwise relation:

$$\text{(A.6)} \qquad \begin{aligned} &\mathfrak{r}^{(\delta)}(t) = (2-\delta)\big(\mathfrak{r}^{(1)}(A_t)\big)^{1/(2-\delta)}, \qquad A_t := \inf\{s \ge 0 : C_s \ge t\}, \\ &C_t := \int_0^t \big(\mathfrak{r}^{(1)}(s)\big)^{(2\delta-2)/(2-\delta)}\, ds, \qquad 0 \le t < \infty. \end{aligned}$$

(This time-change formula is obtained with the parameters $\nu = -1/2$, $q = 2 - \delta$, $p = \frac{2-\delta}{1-\delta}$, $-\frac{2}{p} = \frac{2\delta-2}{2-\delta} > 0$ in Proposition XI.1.11 of [23], which is originally from Lemma 3.1 of [6]. The index $\nu = \frac{1}{2} - 1$ corresponds to the one-dimensional Bessel process and the index $\nu q = \frac{\delta}{2} - 1$ corresponds to the δ-dimensional Bessel process.) The stochastic clocks $C_\cdot$ and $A_\cdot$ in (A.6) do not explode in a finite time because of the instantaneous reflection of $\mathfrak{r}^{(1)}(\cdot)$. Substituting this relation, we compute the occupation time

$$\text{(A.7)} \qquad \begin{aligned} \int_0^t \mathbf{1}_{\{0 \le \mathfrak{r}^{(\delta)}(s) \le \varepsilon\}}\, ds &= \int_0^t \mathbf{1}_{\{0 \le (2-\delta)(\mathfrak{r}^{(1)}(A_s))^{1/(2-\delta)} \le \varepsilon\}}\, ds \\ &= \int_0^{A_t} \mathbf{1}_{\{0 \le (2-\delta)(\mathfrak{r}^{(1)}(s))^{1/(2-\delta)} \le \varepsilon\}}\, dC_s. \end{aligned}$$

It follows from (A.6) that $\frac{dC_t}{dt} = (\mathfrak{r}^{(1)}(t))^{(2\delta-2)/(2-\delta)}$ and hence the right-hand side of (A.7) becomes

$$\int_0^{A_t} \mathbf{1}_{\{0 \le (2-\delta)(\mathfrak{r}^{(1)}(s))^{1/(2-\delta)} \le \varepsilon\}} \cdot \big(\mathfrak{r}^{(1)}(s)\big)^{(2\delta-2)/(2-\delta)}\, ds, \qquad 0 \le t < \infty.$$

By the occupation time formula for the one-dimensional Bessel process $\mathfrak{r}^{(1)}(\cdot)$, this expression becomes

$$2 \int_{(0,\eta)} y^{(2\delta-2)/(2-\delta)} \Lambda_{A_t}^{\mathfrak{r}^{(1)}}(y)\, dy,$$

where $\eta := (\frac{\varepsilon}{2-\delta})^{2-\delta}$ and $\Lambda_t^{\mathfrak{r}^{(1)}}(y)$ is the local time accumulated by $\mathfrak{r}^{(1)}(\cdot)$ at the level $y \in [0, \infty)$ over the time interval $[0, t]$. Changing the variable from y to $x = (2-\delta)y^{1/(2-\delta)}$ with $dy = \frac{x^{1-\delta}}{(2-\delta)^{1-\delta}}\, dx$, we obtain

$$\begin{aligned}
&\int_0^t \mathbf{1}_{\{0 \le \mathfrak{r}^{(\delta)}(s) \le \varepsilon\}}\, ds \\
&\quad = 2 \int_0^\infty \mathbf{1}_{\{0 \le x \le \varepsilon\}} \cdot \frac{x^{2\delta-2}}{(2-\delta)^{2\delta-2}} \cdot \frac{x^{1-\delta}}{(2-\delta)^{1-\delta}} \cdot \Lambda_{A_t}^{\mathfrak{r}^{(1)}}\left(\frac{x^{2-\delta}}{(2-\delta)^{2-\delta}}\right) dx \\
&\quad = 2 \int_0^\varepsilon \frac{x^{\delta-1}}{(2-\delta)^{\delta-1}} \cdot \Lambda_{A_t}^{\mathfrak{r}^{(1)}}\left(\frac{x^{2-\delta}}{(2-\delta)^{2-\delta}}\right) dx, \qquad 0 \le t < \infty.
\end{aligned}$$

Now by $A_t < \infty$, $0 \le t < \infty$, and by the right continuity of $y \mapsto \Lambda_\cdot^{\mathfrak{r}^{(1)}}(y)$, we obtain

$$\Lambda_{A_t}^{\mathfrak{r}^{(1)}}(0) = \lim_{\varepsilon \downarrow 0} \frac{\delta(2-\delta)^{\delta-1}}{2\varepsilon^\delta} \int_0^t \mathbf{1}_{\{0 \le \mathfrak{r}^{(\delta)}(s) \le \varepsilon\}}\, ds < \infty, \qquad 0 \le t < \infty. \tag{A.8}$$

Therefore, we conclude that (A.4) holds for $1 < \delta < 2$. □

A.1.2. *Comparisons with Bessel processes.* Now let us fix integers $1 \le i < j < k \le n$. Under $\mathbb{Q}$ in (A.1) we shall compare the rank gap process

$$\eta(t) := \max_{\ell=i,j,k} Y_\ell(t) - \min_{m=i,j,k} Y_m(t)$$

with a Bessel process of dimension $\delta > 1$, using Lemmata 5 and 6 below.

We introduce the function $g(y) := [(y_i - y_j)^2 + (y_j - y_k)^2 + (y_k - y_i)^2]^{1/2}$ for $y \in \mathbb{R}^n$ and note the comparison $\sqrt{3}\eta(\cdot) \ge g(Y(\cdot))$. An application of Itô's rule to $g(Y(\cdot))$ yields the semimartingale decomposition

$$dg(Y(t)) = h(Y(t))\, dt + d\Theta(t), \qquad 0 \le t < \infty, \tag{A.9}$$

where we introduce the $(n \times 3)$ matrix $D_{ijk} := (d_i, d_j, d_k)$ with $(n \times 1)$ vectors $d_i := \mathfrak{e}_i - \mathfrak{e}_j$, $d_j := \mathfrak{e}_j - \mathfrak{e}_k$, $d_k := \mathfrak{e}_k - \mathfrak{e}_i$, we denote by $\mathfrak{e}_i$, $i = 1, \dots, n$, the ith unit vector in $\mathbb{R}^n$, and

$$h(y) := \frac{(R(y)-1)Q(y)}{2g(y)}, \qquad R(y) := \frac{\operatorname{Tr}(D'_{ijk}S(y)S'(y)D_{ijk})}{Q(y)},$$

$$Q(y) := \frac{y'D_{ijk}D'_{ijk}S(y)S(y)'D_{ijk}D'_{ijk}y}{y'D_{ijk}D'_{ijk}y}, \qquad y \in \mathbb{R}^n \setminus \mathcal{Z},$$

$$\text{(A.10)} \qquad \mathcal{Z} := \{y \in \mathbb{R}^n | g(y) = (y'D_{ijk}D'_{ijk}y) = 0\},$$

$$\Theta(t) := \int_0^t \biggl(\sum_{\ell=i,j,k} \frac{S'(y)d_\ell d'_\ell y}{g(y)}\bigg|_{y=Y(s)} \biggr) d\widetilde{W}(s),$$

$$\langle\Theta\rangle(t) = \int_0^t Q(Y(s))\,ds, \qquad 0 \le t < \infty.$$

Here note that under the assumption on (2.3), and because $3D_{ijk}D'_{ijk} = D_{ijk} \times D'_{ijk}D_{ijk}D'_{ijk}$, we have

$$\text{(A.11)} \qquad Q(\cdot) = \frac{3y'D_{ijk}D'_{ijk}S(\cdot)S(\cdot)'D_{ijk}D'_{ijk}y}{y'D_{ijk}D'_{ijk}D_{ijk}D'_{ijk}y} \ge 3 \min_{\mathbf{p}\in\Sigma_n} \min_{\ell=1,\dots,n} \tilde{\lambda}_{\ell,\mathbf{p}} > 0$$

in $\mathbb{R}^n \setminus \mathcal{Z}$, where $\tilde{\lambda}_{\ell,\mathbf{p}}, \ell = 1, \dots, n$, are the eigenvalues of the positive-definite matrices $\mathfrak{s}_\mathbf{p}\mathfrak{s}'_\mathbf{p}$ for $\mathbf{p} \in \Sigma_n$, and so $\langle\Theta\rangle(\cdot)$ is strictly increasing when $Y(\cdot) \in \mathbb{R}^n \setminus \mathcal{Z}$. Now define the stopping time $\tau_u := \inf\{t \ge 0 | \langle\Theta\rangle(t) \ge u\}$, and note

$$\mathfrak{G}(u) := g(Y(\tau_u)) = g(Y(0)) + \int_0^{\tau_u} h(Y(t))\,dt + \widetilde{B}(u), \qquad 0 \le u < \infty,$$

where $\widetilde{B}(u) := \Theta(\tau_u)$, $0 \le u < \infty$, is a standard Brownian motion, by the Dambis–Dubins–Schwarz theorem of time-change for martingales. Note that $1/[Q(Y(\tau_u))] = d\tau_u/du$, when $Y(\tau_u) \in \mathbb{R}^n \setminus \mathcal{Z}$. Thus, with $\mathfrak{d}(u) := R(Y(\tau_u))$, we can write

$$d\mathfrak{G}(u) = \frac{\mathfrak{d}(u)-1}{2\mathfrak{G}(u)}\,du + d\widetilde{B}(u), \qquad 0 \le u < \infty, \mathfrak{G}(0) = g(Y(0)).$$

The dynamics of the process $\mathfrak{G}(\cdot)$ are comparable to those of a Bessel process $\mathfrak{r}^{(\delta)}(\cdot)$ with dimension δ, generated by the same $\widetilde{B}(\cdot)$ and started at the same initial point $g(Y(0))$. Since $S(\cdot)S(\cdot)'$ is positive definite under (2.3) and rank $(D_{ijk}) = 2$, the (3×3) matrix $D'_{ijk}S(\cdot)S(\cdot)'D_{ijk}$ is nonnegative definite and the number of its nonzero eigenvalues is equal to $\operatorname{rank}(D'_{ijk}S(\cdot)S(\cdot)'D_{ijk}) = 2$. Let us denote by $\bar{\lambda}_{\ell,\mathbf{p}}$, $\ell = 1, 2, 3$, the eigenvalues of $D'_{ijk}\mathfrak{s}_\mathbf{p}\mathfrak{s}_{\mathbf{p}'}D_{ijk}$ for $\mathbf{p} \in \Sigma_n$. Then for $R(\cdot)$ in (A.10) we obtain

$$\text{(A.12)} \qquad R(\cdot) \ge \delta_0 := \min_{\mathbf{p}\in\Sigma_n} \biggl(\frac{\sum_{\ell=1}^3 \bar{\lambda}_{\ell,\mathbf{p}}}{\max_{1\le\ell\le3} \bar{\lambda}_{\ell,\mathbf{p}}} \biggr) > 1 \qquad \text{in } \mathbb{R}^n \setminus \mathcal{Z},$$

and so $\mathfrak{d}(\cdot) \geq \delta_0 > 1$ when $Y(\tau.) \in \mathbb{R}^n \setminus \mathcal{Z}$. By a comparison argument similar to that in the proof of Lemma 2.1 of [18], we may show that $\mathfrak{G}(t) \geq \mathfrak{r}^{(\delta_0)}(t)$ for $0 \leq t < \infty$ a.s. Since $\sqrt{3}\eta(t) \geq g(Y(t)) = \mathfrak{G}(\langle\Theta\rangle(t))$ implies $\sqrt{3}\eta(t) \geq \mathfrak{r}^{(\delta_0)}(\langle\Theta\rangle(t))$ for $0 \leq t < \infty$, a.s., we obtain the following result.

LEMMA 5. *For the process $Y(\cdot)$ of (A.2) with (2.3), the multiple $\sqrt{3}\eta(\cdot)$ of the rank-gap process dominates, a.s. under $\mathbb{Q}$, a time-changed Bessel process $\tilde{\mathfrak{r}}(\cdot) := \mathfrak{r}^{(\delta_0)}(\langle\Theta\rangle(\cdot))$ with dimension δ_0 as in (A.12)*

$$\mathbb{Q}\big(\sqrt{3}\eta(t) \geq \tilde{\mathfrak{r}}(t), 0 \leq t < \infty\big) = 1.$$

LEMMA 6. *Under $\mathbb{Q}$, the rank-gap process $\eta(\cdot)$ satisfies $\langle\eta\rangle(t) \leq c_1 t$, $0 \leq t < \infty$ a.s. for some constant $c_1 > 0$ and the local time $\Lambda_\eta(\cdot)$ of $\eta(\cdot)$ at the origin is identically equal to zero, that is, $\Lambda_\eta(\cdot) \equiv 0$, a.s.*

PROOF. In fact, since the diffusion coefficient matrix $S(\cdot)$ of $Y(\cdot)$ in (A.2) is bounded and positive definite under (2.3), there exists a constant c_1 such that $\langle\eta\rangle(t) \leq c_1 t$ for $0 \leq t < \infty$ a.s. Moreover, from (A.11) and Lemma 5, there exists a constant $c_2 := \min_{\mathbf{p}\in\Sigma_n, \ell=1,\dots,n} \tilde{\lambda}_{\ell,\mathbf{p}} > 0$, such that $\langle\Theta\rangle(t) \geq c_2 t$ holds for $0 \leq t < \infty$ a.s. It follows from the representation of local times (Theorem VI. 1.7 of [23]) and (A.5) with Lemma 5 that

$$\begin{aligned}
\Lambda_\eta(t) &= \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{0\leq\eta(s)<\varepsilon\}}\, d\langle\eta\rangle(s) \leq \lim_{\varepsilon\downarrow 0} \frac{\sqrt{3}c_1}{2\varepsilon} \int_0^t \mathbf{1}_{\{0\leq\sqrt{3}\eta(s)<\varepsilon\}}\, ds \\
&\leq \lim_{\varepsilon\downarrow 0} \frac{\sqrt{3}c_1}{2\varepsilon} \int_0^t \mathbf{1}_{\{0\leq\tilde{\mathfrak{r}}(s)<\varepsilon\}}\, ds \leq \lim_{\varepsilon\downarrow 0} \frac{\sqrt{3}c_1}{2c_2\varepsilon} \int_0^{\langle\Theta\rangle(t)} \mathbf{1}_{\{0\leq\mathfrak{r}^{(\delta)}(u)<\varepsilon\}}\, du \qquad \text{(A.13)} \\
&\leq \sqrt{3}c_1 c_2^{-1} \Lambda_{\mathfrak{r}^{(\delta)}}(\langle\Theta\rangle(t)) \equiv 0, \qquad 0 \leq t < \infty.
\end{aligned}$$

□

A.2. Proof of Lemma 1. Define an increasing family of events $C_T := \{\Lambda_\eta(t) > 0 \text{ for some } t \in [0, T]\}$, $T \geq 0$. By Lemma 6 we obtain $\mathbb{Q}(C_\infty) = 0$ and $0 = \mathbb{Q}(C_\ell) = \mathbb{P}(C_\ell)$ for $\ell \geq 1$. Then $\mathbb{P}(\Lambda_\eta(t) > 0 \text{ for some } t \geq 0) = \mathbb{P}(\bigcup_{\ell=1}^\infty C_\ell) = \lim_{\ell=\infty} \mathbb{P}(C_\ell) = 0$. Thus the local time $\Lambda_\eta(t)$ of the rank gap process $\eta(\cdot)$ for $(Y_i(\cdot), Y_j(\cdot), Y_k(\cdot))$ is zero for $0 \leq t < \infty$ a.s. under $\mathbb{P}$.

Since the choice of i, j, k is arbitrary, there is no local time generated by the rank gap process of any three coordinates. The rank gap process of more than three coordinates [e.g., $\max_{\ell=h,i,j,k} Y_\ell(\cdot) - \min_{m=h,i,j,k} Y_m(\cdot)$] dominates that of any three sub-coordinates. Therefore, by a similar argument as (A.13) and its consequence, any local time of rank gap process of more than three coordinates is zero for $0 \leq t < \infty$ a.s. *under* $\mathbb{P}$.

To establish (4.4) from this and (4.3), and thus complete the proof of Lemma 1, consider any integers (ranks) $1 \leq a \leq \ell < m \leq b \leq n$ with $b - a \geq 2$, and observe that we have almost surely

$$
\begin{aligned}
0 \equiv \Lambda^{a,b}(t) &= \int_0^t \mathbf{1}_{\{Z_a(s)=Z_b(s)\}}\, d\big(Z_a(s) - Z_b(s)\big) \\
&= \int_0^t \mathbf{1}_{\{Z_a(s)=Z_b(s)\}}\, d\big(Z_a(s) - Z_\ell(s)\big) + \int_0^t \mathbf{1}_{\{Z_a(s)=Z_b(s)\}}\, d\big(Z_\ell(s) - Z_m(s)\big) \\
&\quad + \int_0^t \mathbf{1}_{\{Z_a(s)=Z_b(s)\}}\, d\big(Z_m(s) - Z_b(s)\big) \\
&= \int_0^t \mathbf{1}_{\{Z_a(s)=Z_b(s)\}}\, d\big(\Lambda^{a,\ell}(s) + \Lambda^{\ell,m}(s) + \Lambda^{m,b}(s)\big) \\
&\geq \int_0^t \mathbf{1}_{\{Z_a(s)=Z_b(s)\}}\, d\Lambda^{\ell,m}(s) \geq 0.
\end{aligned}
$$

The a.s. equality $\int_0^t \mathbf{1}_{\{Z_a(s)=Z_b(s)\}}\, d\Lambda^{\ell,m}(s) = 0$ follows readily from this, as does

$$
\int_0^t \mathbf{1}_{\{N_k(t)\geq 3\}} \left(\sum_{\ell=k+1}^{n} d\Lambda^{k,\ell}(s) - \sum_{\ell=1}^{k-1} d\Lambda^{\ell,k}(s) \right) = 0
$$

and thus (4.4) as well.

A.3. Proof of Lemma 2. For each $k = 1, \dots, n-1$ the local time $\Lambda^{k,k+1}(\cdot)$ is a continuous additive functional of $(\Xi(\cdot), \mathfrak{P}.)$ with support in $\mathfrak{F}_k$, and the expectation of $\Lambda^{k,k+1}(t)$ with respect to the invariant distribution $\nu(\cdot,\cdot)$ is finite for $t \geq 0$.

It follows from the theory of additive functionals [2] that there is a finite measure $\nu_k(\cdot,\cdot)$ on $\mathfrak{F}_k \times \Sigma_n$ such that

$$
\text{(A.14)} \qquad \frac{1}{T}\mathbb{E}_\nu\left[\int_0^T g(\Xi(s), \mathfrak{P}_s)\, d\Lambda^{k,k+1}(s)\right] = \frac{1}{2}\int_{\mathfrak{F}_k \times \Sigma_n} g(z, \mathbf{p})\, d\nu_k(z, \mathbf{p})
$$

for every bounded measurable function $g : \mathfrak{F}_k \times \Sigma_n \mapsto \mathbb{R}$. Let us denote by $\nu_{0k}(\cdot) = \nu_k(\cdot, \Sigma_n)$ the marginal distribution on $\mathfrak{F}_k$. The absolute continuity of $\nu_{0k}(\cdot)$ with respect to $(n-1)$-dimensional Lebesgue measure is argued by localization and the properties of Reflected Brownian motion as in Theorem 7.1, Lemmata 7.7 and 7.9 of [16].

Now, by an application of Itô's rule, for $f \in C_b^2((\mathbb{R}_+)^{n-1})$ we obtain

$$
\begin{aligned}
f(\Xi(T)) = f(\Xi(0)) &+ \int_0^T \langle \nabla f(\Xi(s)), d\zeta^{\mathrm{mart}}(s)\rangle \\
&+ \sum_{k=1}^{n-1} \int_0^T [\mathcal{D}_k f](\Xi(s))\, d\Lambda^{k,k+1}(s) \\
&+ \int_0^T [\mathcal{A} f](\Xi(s), \mathfrak{P}_s)\, ds, \qquad T \geq 0,
\end{aligned}
$$

where $\zeta^{\text{mart}}(\cdot)$ is the martingale part of $\zeta(\cdot)$ and $\mathcal{D}_k$ and $\mathcal{A}$ are differential operators defined in (5.3). Taking expectations with respect to $\mathbb{P}$ and then integrating for the initial values with respect to the stationary distribution $\nu(\cdot,\cdot)$ with Fubini's theorem and (A.14), we obtain

$$0 = \frac{T}{2}\sum_{k=1}^{n-1}\int_{\mathfrak{F}_k}[\mathcal{D}_k f](z)\,d\nu_{0k}(z) + T\int_{(\mathbb{R}_+)^{n-1}\times\Sigma_n}[\mathcal{A}f](z,\mathbf{p})\,d\nu(z,\mathbf{p}).$$

Dividing by $T > 0$, we obtain the basic adjoint relationship (5.6).

A.4. A sanity check of Corollary 4. In this section we verify that the entities $(\theta_{k,i})_{1\le i,k\le n}$ in (5.16) satisfy (4.12). Since $\theta_{k,i}$ is homogeneous in the product $\prod_{j=1}^{n-1}[-4(\sigma_j^2+\sigma_{j+1}^2)^{-1}]$, it suffices to show $\sum_{k=1}^{n}\widetilde{\theta}_{k,i}(g_k+\gamma_i)=0$ where we use the modifications $\widetilde{\theta}_{k,i} := \sum_{\{\mathbf{p}(k)=i\}}\widetilde{\theta}_{\mathbf{p}}$,

$$\widetilde{\theta}_{\mathbf{p}} := \Bigg(\sum_{\mathbf{q}\in\Sigma_n}\prod_{j=1}^{n-1}\widetilde{\lambda}_{\mathbf{q},j}^{-1}\Bigg)^{-1}\prod_{j=1}^{n-1}\widetilde{\lambda}_{\mathbf{p},j}^{-1}, \qquad \widetilde{\lambda}_{\mathbf{p},j} := \sum_{\ell=1}^{j}(g_\ell+\gamma_{\mathbf{p}(\ell)})$$

of $(\theta_{k,i},\theta_{\mathbf{p}},\lambda_{\mathbf{p},j})$, $1\le i,j,k\le n$, $\mathbf{p}\in\Sigma_n$, for notational simplicity. Note that $\widetilde{\lambda}_{\mathbf{p},n}=0$ from (2.2) for $\mathbf{p}\in\Sigma_n$.

First, observe for $\ell = 2,\dots,n$ and $i=1,\dots,n$,

$$\text{(A.15)}\qquad \sum_{\{\mathbf{p}:\,\mathbf{p}(\ell-1)=i\}}\widetilde{\lambda}_{\mathbf{p},\ell-1}\widetilde{\theta}_{\mathbf{p}} + \sum_{\{\mathbf{p}:\,\mathbf{p}(\ell)=i\}}(g_\ell+\gamma_i)\widetilde{\theta}_{\mathbf{p}} = \sum_{\{\mathbf{p}:\,\mathbf{p}(\ell)=i\}}\widetilde{\lambda}_{\mathbf{p},\ell}\widetilde{\theta}_{\mathbf{p}}.$$

In fact, for every i,ℓ define another permutation $\widetilde{\mathbf{p}}$ from a (fixed) permutation $\mathbf{p}\in\{\mathbf{q}\in\Sigma_n:\mathbf{q}(\ell-1)=i\}$ by

$$\widetilde{\mathbf{p}}(k) := \widetilde{\mathbf{p}}(k;\mathbf{p}) = \begin{cases}\mathbf{p}(k), & k=1,\dots,\ell-2,\ell+1,\dots,n,\\ \mathbf{p}(\ell), & k=\ell-1,\\ i, & k=\ell,\end{cases}$$

which is obtained by exchanging $(\ell-1)$st and ℓth elements of $\mathbf{p}\in\{\mathbf{q}\in\Sigma_n:\mathbf{q}(\ell-1)=i\}$, and also define $M := (\sum_{\mathbf{q}\in\Sigma_n}\prod_{j=1}^{n-1}\widetilde{\lambda}_{\mathbf{q},j}^{-1})^{-1}$ here. Then $\widetilde{\lambda}_{\mathbf{p},j}=\widetilde{\lambda}_{\widetilde{\mathbf{p}},j}$ for $j\neq\ell-1$ and hence the left-hand side of (A.15) is

$$\begin{aligned}
&\sum_{\{\mathbf{p}:\,\mathbf{p}(\ell-1)=i\}}\widetilde{\lambda}_{\mathbf{p},\ell-1}\cdot M\prod_{j=1}^{n-1}\widetilde{\lambda}_{\mathbf{p},j}^{-1} + \sum_{\{\mathbf{p}:\,\mathbf{p}(\ell)=i\}}(g_\ell+\gamma_i)M\prod_{j=1}^{n-1}\widetilde{\lambda}_{\mathbf{p},j}^{-1}\\
&\quad= \sum_{\{\widetilde{\mathbf{p}}:\,\widetilde{\mathbf{p}}(\ell)=i\}}M\prod_{j\neq\ell-1}^{n-1}\widetilde{\lambda}_{\widetilde{\mathbf{p}},j}^{-1} + \sum_{\{\widetilde{\mathbf{p}}:\,\widetilde{\mathbf{p}}(\ell)=i\}}(g_\ell+\gamma_{\widetilde{p}(\ell)})M\prod_{j=1}^{n-1}\widetilde{\lambda}_{\widetilde{\mathbf{p}},j}^{-1}\\
&\quad= \sum_{\{\widetilde{\mathbf{p}}:\,\widetilde{\mathbf{p}}(\ell)=i\}}\big[\widetilde{\lambda}_{\widetilde{\mathbf{p}},\ell-1}+g_\ell+\gamma_{\widetilde{\mathbf{p}}(\ell)}\big]\cdot M\prod_{j=1}^{n-1}\widetilde{\lambda}_{\widetilde{\mathbf{p}},j}^{-1} = \sum_{\{\mathbf{p}:\,\mathbf{p}(\ell)=i\}}\widetilde{\lambda}_{\mathbf{p},\ell}\widetilde{\theta}_{\mathbf{p}},
\end{aligned}$$

which is the right-hand side of (A.15). Now applying (A.15) for $\ell = 2, \dots, n$, we obtain

$$\begin{aligned}\sum_{k=1}^{n}(g_k+\gamma_i)\widetilde{\theta}_{k,i} &= (g_1+\gamma_i)\widetilde{\theta}_{1,i} + (g_2+\gamma_i)\widetilde{\theta}_{2,i} + \sum_{k=3}^{n}(g_k+\gamma_i)\widetilde{\theta}_{k,i} \\ &= \sum_{\{\mathbf{p}:\,\mathbf{p}(2)=i\}} \widetilde{\lambda}_{\mathbf{p},2}\widetilde{\theta}_{\mathbf{p}} + \sum_{k=3}^{n}(g_k+\gamma_i)\widetilde{\theta}_{k,i} \\ &= \cdots = \sum_{\{\mathbf{p}:\,\mathbf{p}(n)=i\}} \widetilde{\lambda}_{\mathbf{p},n}\widetilde{\theta}_{\mathbf{p}} = 0\end{aligned}$$

for $i = 1, \dots, n$, because $\widetilde{\lambda}_{\mathbf{p},n} = 0$ for $\mathbf{p} \in \Sigma_n$. Therefore, (4.12) is satisfied.

Acknowledgments. We are thankful to Professors Toshio Yamada, Peter Bank, Constantinos Kardaras, Erhan Bayraktar and also the participants for their helpful comments and discussions at the 8th Ritsumeikan–Columbia–JAFEE International Symposium on Stochastic Processes/Mathematical Finance, in the seminars at Quantitative Products Laboratory in Berlin, at Boston University, at Columbia University and at the University of Michigan. We are also thankful to the referee for valuable suggestions, and to the Associate Editor for a remark on the proof of Lemma 1.

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