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Making the Cauchy work

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Abstract. A truncated version of the Cauchy distribution is introduced. Unlike the Cauchy distribution, this possesses finite moments of all orders and could therefore be a better model for certain practical situations. More than 10 practical situations where the truncated distribution could be applied are discussed. Explicit expressions are derived for the moments, L moments, mean deviations, moment generating function, characteristic function, convolution properties, Bonferroni curve, Lorenz curve, entropies, order statistics and the asymptotic distribution of the extreme order statistics. Estimation procedures are detailed by the method of moments and the method of maximum likelihood and expressions derived for the associated Fisher information matrix. Simulation issues are discussed. Finally, an application is illustrated for consumer price indices from the six major economics.

1 Introduction

The Cauchy distribution given by the probability density function (pdf):

$$f(x) = \frac{1}{\pi\theta}\left\{1 + \left(\frac{x-\mu}{\theta}\right)^2\right\}^{-1} \tag{1.1}$$

(for $-\infty < x < \infty$, $\theta > 0$ and $-\infty < \mu < \infty$) has been studied in the mathematical world for over three centuries. An excellent historical account of the distribution has been prepared by Stigler (1974). As he points out, (1.1) seems to have appeared first in the works of Pierre de Fermat in the mid-17th century and was subsequently studied by many including Sir Issac Newton, Gottfried Leibniz, Christian Huygens, Guido Grandi, and Maria Agnesi. The parameters μ and θ are the location and scale parameters, respectively. The distribution is symmetrical about $x = \mu$. The median is μ; the upper and lower quartiles are $\mu \pm \theta$; and, the points of inflexion are at $\mu \pm \theta/\sqrt{3}$.

The main weakness of (1.1) is that it has no moments. In this paper, we overcome this weakness by introducing a truncated version. It has the pdf and cumulative distribution function (cdf) specified by

$$f(x; A, B) = \frac{1}{\theta D}\left\{1 + \left(\frac{x-\mu}{\theta}\right)^2\right\}^{-1} \tag{1.2}$$

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and

$$F(x; A, B) = \frac{1}{D}\left\{\arctan\left(\frac{x-\mu}{\theta}\right) - \arctan\left(\frac{A-\mu}{\theta}\right)\right\}, \tag{1.3}$$

respectively, for $-\infty \le A \le x \le B \le \infty$, $-\infty < \mu < \infty$ and $\theta > 0$, where

$$D(A, B) = \arctan(\beta) - \arctan(\alpha), \tag{1.4}$$

where $\alpha = (A - \mu)/\theta$ and $\beta = (B - \mu)/\theta$. This distribution originally appeared in Johnson and Kotz (1970) and Rohatgi (1976) in simpler forms. Johnson and Kotz (1970) derived the variance and discussed estimation issues for the symmetric standard case given by $A = -B$, $\mu = 0$ and $\theta = 1$. Rohatgi (1976) derived expressions for the first two moments for the standard case given by $\mu = 0$ and $\theta = 1$. Note that (1.2) is unimodal. The mode is at $x = \mu$ if $A \le \mu \le B$. If $B < \mu$ then the mode is at $x = B$. If $\mu < A$ then the mode is at $x = A$.

Because (1.2) is defined over a finite interval, the truncated Cauchy distribution has all its moments. So, (1.2) may prove to be a better model for certain practical situations than one based on just the Cauchy distribution. Below, we discuss more than 10 such situations.

The Cauchy distribution given by (1.1) has been applied in the past as models for depth map data, prices of speculative assets such as stock returns and the phase derivative (random frequency of a narrow-band mobile channel) of air components in an urban environment. For data of this kind, there is no reason to believe that empirical moments of any order should be infinite. So, the choice of the Cauchy distribution as a model is unrealistic since none of its moments are finite. The alternative truncated Cauchy distribution given by (1.2) will be a more appropriate model for the kind of data mentioned. The choice of the limits, A and B, could be easily based on historical records.

A main problem with characterizing employment productivity distributions is to find a reasonable measure of the minimal and maximal productivity. In both ends of the distribution one is likely to find accumulations of measurement errors due either to downright faulty data or time aggregation problems associated with, for example, plant closures and new plants. With respect to Swedish employment data, Forslund and Lindh (2004) took average wage costs as the measure of minimal sustainable productivity and 95th percentile productivity as a fairly reliable indicator of maximal sustainable productivity. Forslund and Lindh (2004) found that the empirical employment distribution between these two productivity values was well described by a truncated Cauchy distribution.

Consider the truncated Cauchy distribution in (1.2) for $A = 0$, $B = \infty$, $\mu = 0$ and $\theta = 1$. So, we have $f(x) = (2/\pi)(1 + x^2)^{-1}$ for $x > 0$ with the asymptotic tail $f(x) \sim (2/\pi)x^{-2}$ as $x \to \infty$. Yablonsky (1985) established that $(2/\pi)x^{-2}$ coincides with the classical *Lotka distribution of scientific productivity* (describing the frequency of publication by authors in any given field) up to a normalizing constant. So, Lotka's law is an approximate expression of the asymptotic form of the truncated Cauchy distribution.

Pdfs of surface velocity and surface velocity gradients in the ocean provide information about turbulence in a high-Reynolds-number geophysical flow. The theoretical velocity pdfs are usually Gaussian. The velocity gradient pdfs are usually truncated Cauchy. See Jimenez (1996) for theoretical arguments supporting these statements.

Mitra and Das (1989) use the truncated Cauchy distribution in the field of crystallographic statistics. They state "... the Cauchy or Lorentzian distribution, having no finite moment apart from above and the first—is looked upon with suspicion. But one never works with a distribution function ranging between $\pm\infty$; the function is cut off on the surface of the sphere of reflection. Thus we are actually dealing with a truncated Cauchy distribution function for which second and higher moments exist."

The ionosphere is the uppermost part of the atmosphere, distinguished because it is ionized by solar radiation. It plays an important part in atmospheric electricity and forms the inner edge of the magnetosphere. There are different layers of ionization. The most important of these is the E_s layer characterized by small clouds of intense ionization, which can support radio wave reflections from 25–225 MHz. The appearance of abnormally large fluctuations in the electron density (ED) of the E_s layer is relatively frequent. Moiseev (1997) determined that distribution of maximum values of the ED of the E_s layer is close to the Cauchy distribution. However, the Cauchy distribution permits negative values of ED and has no moments. To render the description more physical, one must impose boundary conditions prohibiting the appearance of ED values below the background level and above the maximum permissible level.

The *packet size* in many traffic models follows the truncated Cauchy distribution. For example, for the FUNET (Finnish University and Research Network) model the packet size is distributed according to (1.2) with $\mu = 0.8$, $\theta = 1$, $A = 0$ and $B = 10$ kilobytes. See Ni (2001).

Truncated Cauchy distributions can be applied in numerous industrial settings (Cho and Govindaluri, 2002; Jeang, 1997; Kapur and Cho, 1994, 1996; Phillips and Cho, 1998, 2000). Final products are often subject to screening inspection before being sent to the customer. The usual practice is that if a product's performance falls within certain tolerance limits, it is judged conforming and sent to the customer. If it fails, a product is rejected and thus scrapped or reworked. In this case, the actual distribution to the customer is truncated. Another example can be found in a multistage production process, in which inspection is performed at each production stage. If only conforming items are passed on to the next stage, the actual distribution is a truncated distribution. Accelerated life testing with samples censored is also a good example. In fact, the concept of a truncated distribution plays a significant role in analyzing a variety of production processes, process optimization and quality improvement.

Truncated Cauchy distributions can be used to model intensity statistics in the study of atomic heterogeneity (Bhowmick et al., 2000). The justification being that: (1) atomic heterogeneity led to the intensity statistics being modified from Gaussian to near Gaussian forms (Shmueli, 1979; Shmueli and Wilson, 1981); and (2) in reality, the structure factors or normalized structure factors do not range from $-\infty$ to ∞ but over a finite range.

Measurements on a high-performance Ethernet can match well a truncated Cauchy distribution, with a much better fit over smaller file/request sizes than the commonly used Pareto distribution. Field et al. (2004) showed that measured traffic from three locations on a state-of-the-art switched Ethernet fit closely various truncated Cauchy distributions.

Truncated Cauchy distributions have also been used to describe local magnetic fields due to susceptibility differences in porous media; see Borgia et al. (1996) and Fantazzini and Brown (2005).

Truncated Cauchy distributions also have use in Monte Carlo simulations. For example, consider simulating from the Poisson (λ) distribution by the rejection algorithm. One could choose the envelope density to be (1.2) with $\mu = \lambda$, $\theta = \sqrt{2\lambda}$, $A = 0$ and $B = \infty$.

Truncated Cauchy distributions are also popular priors for Bayesian models especially with respect to economic data; see, for example, Bauwens et al. (1999).

The aim of this paper is to provide a comprehensive account of the mathematical properties of (1.2). The following properties are derived: moment properties as well as a skewness-kurtosis chart (Section 2); L moments (Section 3); mean deviations (Section 4); moment generating and characteristic functions (Section 5); convolution (Section 6); Bonferroni and Lorenz curves (Section 7); entropy measures (Section 8); order statistics (Section 9); the asymptotic distribution of the extreme order statistics (Section 10); estimation procedures by the methods of moments and maximum likelihood as well as the associated Fisher information matrix (Section 11); and, simulation procedures (Section 12). Section 13 discusses an application to consumer price indices from the six major economies.

The calculations of this paper use several special functions, including Euler's psi function defined by

$$\psi(x) = \frac{d \log \Gamma(x)}{dx},$$

the exponential integral defined by

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{\exp(t)}{t}\, dt$$

and the Gauss hypergeometric function defined by

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},$$

where $(z)_k = z(z+1)\cdots(z+k-1)$ denotes the ascending factorial. The properties of these special functions can be found in Prudnikov et al. (1986) and Gradshteyn and Ryzhik (2000).

2 Moments

Here, we discuss moments of a random variable X having the pdf (1.2). Some of the results given have been reported earlier by Nadarajah and Kotz (2006). They are reproduced here for completeness. Note that one can write the nth moment of X as

$$E(X^n; A, B) = \frac{1}{\theta D} \int_A^B x^n \left\{ 1 + \left(\frac{x-\mu}{\theta} \right)^2 \right\}^{-1} dx. \tag{2.1}$$

Setting $y = (x-\mu)/\theta$ and using the binomial expansion

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,$$

one can rewrite (2.1) as

$$\begin{aligned} E(X^n; A, B) &= \frac{1}{D} \int_\alpha^\beta \frac{(\mu+\theta y)^n}{1+y^2} dy \\ &= \frac{1}{D} \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} \theta^k \int_\alpha^\beta \frac{y^k}{1+y^2} dy \\ &= \frac{1}{D} \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} \theta^k \{ I_k(\beta) - I_k(\alpha) \}, \end{aligned} \tag{2.2}$$

where

$$I_k(c) = \int_0^c \frac{y^k}{1+y^2} dy.$$

By equation (3.194.5) in Gradshteyn and Ryzhik (2000), one can calculate $I_k(c)$ as

$$I_k(c) = \frac{c^{k+1}}{k+1} {}_2F_1\left(1, \frac{k+1}{2}; \frac{k+3}{2}; -c^2\right). \tag{2.3}$$

By combining (2.2) and (2.3) it follows that the nth moment of X is given by

$$\begin{aligned} &E(X^n; A, B) \\ &= \frac{\mu^n}{D} \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} \left(\frac{\theta}{\mu} \right)^k \left\{ \beta^{k+1} {}_2F_1\left(1, \frac{k+1}{2}; 1 + \frac{k+1}{2}; -\beta^2\right) \right. \\ &\qquad \left. - \alpha^{k+1} {}_2F_1\left(1, \frac{k+1}{2}; 1 + \frac{k+1}{2}; -\alpha^2\right) \right\} \end{aligned} \tag{2.4}$$

for $n \geq 1$. In the standard case $\mu = 0$ and $\theta = 1$, using standard properties of the Gauss hypergeometric function, one can obtain the first four moments of X from (2.4) as:

$$E(X; A, B) = \{\log(1 + B^2) - \log(1 + A^2)\}/(2D), \quad (2.5)$$

$$E(X^2; A, B) = \{\arctan(A) - \arctan(B) - A + B\}/D, \quad (2.6)$$

$$E(X^3; A, B) = \{\log(1 + A^2) - \log(1 + B^2) - A^2 + B^2\}/(2D) \quad (2.7)$$

and

$$E(X^4; A, B) = \{3\arctan(B) - 3\arctan(A) - A^3 + B^3 + 3A - 3B\}/(3D). \quad (2.8)$$

The truncated Cauchy distribution given by (1.2) is much more flexible than the Cauchy distribution in (1.1). This is illustrated by the skewness-kurtosis charts [see, e.g., Dudewicz and Mishra (1988)] shown in Figure 1 for the case $\mu = 0$ and $\theta = 1$. The skewness and kurtosis were computed using (2.5)–(2.8) over the grid of values defined by $A = -10, -9.9, \ldots, 10$ and $B = A + 0.1, A + 0.2, \ldots, 10$. The boundaries of the chart on left hand side of the figure (i.e., left of the line where

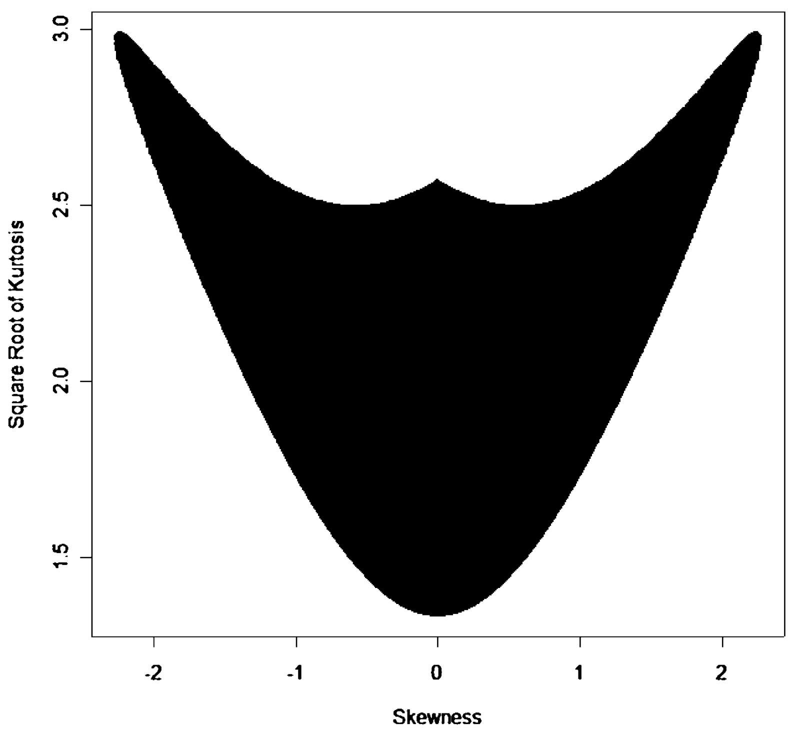

Figure 1 *Skewness-kurtosis chart for the truncated Cauchy distribution given by* (1.2) *with* $\mu = 0$ *and* $\theta = 1$.

skewness is zero) correspond to large negative values of A and those on the right of the figure correspond to large positive values of A.

3 L moments

L-moments are summary statistics for probability distributions and data samples (Hoskings, 1990). They are analogous to ordinary moments but are computed from linear combinations of the ordered data values. The nth L moment is defined by

$$\lambda_n = \sum_{j=0}^{n-1} (-1)^{n-1-j} \binom{n-1}{j} \binom{n-1+j}{j} \beta_j, \tag{3.1}$$

where

$$\beta_r = \int x \{F(x)\}^r f(x)\, dx.$$

The L moments have several advantages over ordinary moments: for example, they apply for any distribution having finite mean; no higher-order moments need be finite.

Suppose X is a truncated Cauchy random variable with its pdf specified by (1.2). Assume without loss of generality that $\mu = 0$ and $\theta = 1$. Then the nth L moment of X is given by (3.1), where

$$\beta_r = D^{-r-1} \sum_{i=0}^{r} \binom{r}{i} (-\arctan(A))^{r-i} \int_A^B x(\arctan(x))^i (1+x^2)^{-1}\, dx. \tag{3.2}$$

Using the series expansion

$$\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \tag{3.3}$$

one can calculate (3.2) as

$$\beta_r = D^{-r} \sum_{i=0}^{r} \binom{r}{i} (-\arctan(A))^{r-i}$$
$$\times \sum_{k_1=0}^{\infty} \cdots \sum_{k_i=0}^{\infty} \frac{(-1)^{k_1+\cdots+k_i}}{(2k_1+1)\cdots(2k_i+1)} E\left(X^{2(k_1+\cdots+k_i)+i+1}; A, B\right).$$

4 Mean deviations

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. These are known as the mean deviation about the mean and the mean deviation about the median—defined by

$$\delta_1(X) = \int_0^{\infty} |x-\mu| f(x)\, dx \tag{4.1}$$

and

$$\delta_2(X) = \int_0^\infty |x - M| f(x)\, dx, \tag{4.2}$$

respectively, where $\mu = \mathrm{E}(X)$ and M denotes the median. Let X be a random variable with its pdf given by (1.2). Then the measure in (4.1) can be calculated as

$$\begin{aligned}
\delta_1(X) &= \int_A^\mu (\mu - x) f(x; A, B)\, dx \\
&\quad + \int_\mu^B (x - \mu) f(x; A, B)\, dx \\
&= 2\left\{\mu F(\mu; A, B) - \int_A^\mu x f(x; A, B)\, dx\right\} \\
&= 2\left\{\mu F(\mu; A, B) - \frac{D(A, \mu) E(X; A, \mu)}{D(A, B)}\right\}.
\end{aligned}$$

Similarly, the measure in (4.2) can be calculated as

$$\begin{aligned}
\delta_2(X) &= \int_A^M (M - x) f(x; A, B)\, dx + \int_M^B (x - M) f(x; A, B)\, dx \\
&= \mathrm{E}(X; A, B) - 2\int_A^M x f(x; A, B)\, dx \\
&= \mathrm{E}(X; A, B) - 2\frac{D(A, M) E(X; A, M)}{D(A, B)}.
\end{aligned}$$

5 MGF and CHF

Let X be a random variable with its pdf given by (1.2). Let $i = \sqrt{-1}$ denote the imaginary unit. Then the mgf of X, $M(t) = E[\exp(tX)]$, can be expressed as

$$\begin{aligned}
M(t) &= \frac{1}{\theta D} \int_A^B \exp(tx) \left\{1 + \left(\frac{x - \mu}{\theta}\right)^2\right\}^{-1} dx \\
&= \frac{\exp(\mu t)}{\theta D} \int_\alpha^\beta \exp(\theta t y)(1 + y^2)^{-1}\, dy \\
&= \frac{\exp(\mu t)}{2\theta D i} \int_\alpha^\beta \exp(\theta t y)(y + 1/i)^{-1}\, dy - \frac{\exp(\mu t)}{2\theta D i} \\
&\quad \times \int_\alpha^\beta \exp(\theta t y)(y - 1/i)^{-1}\, dy \\
&= \frac{\exp(\mu t - \theta t/i)}{2\theta D i} \left\{\mathrm{Ei}\left(\beta\theta t + \frac{\theta t}{i}\right) - \mathrm{Ei}\left(\alpha\theta t + \frac{\theta t}{i}\right)\right\} \\
&\quad - \frac{\exp(\mu t + \theta t/i)}{2\theta D i} \left\{\mathrm{Ei}\left(\beta\theta t - \frac{\theta t}{i}\right) - \mathrm{Ei}\left(\alpha\theta t - \frac{\theta t}{i}\right)\right\},
\end{aligned}$$

where the last step follow by equation (3.352.1) in Gradshteyn and Ryzhik (2000). It follows that the chf of X, $\phi(t) = E[\exp(itX)]$, is

$$\begin{aligned}\phi(t) = \frac{\exp(\mu it - \theta t)}{2\theta Di}\{\mathrm{Ei}(\beta\theta it + \theta t) - \mathrm{Ei}(\alpha\theta it + \theta t)\} \\ - \frac{\exp(\mu it + \theta t)}{2\theta Di}\{\mathrm{Ei}(\beta\theta it - \theta t) - \mathrm{Ei}(\alpha\theta it - \theta t)\}.\end{aligned}$$

6 Convolution

If X_1 and X_2 are independent Cauchy random variables then it is well known that their convolution, $X_1 + X_2$, is also a Cauchy random variable. It is natural to ask whether this property holds for truncated Cauchy random variables. Suppose X_1 and X_2 are independent random variables with the pdf (1.2) for $(\mu, \theta, A, B) = (\mu_1, \theta_1, A_1, B_1)$ and $(\mu, \theta, A, B) = (\mu_2, \theta_2, A_2, B_2)$, respectively. Let D_1 and D_2 denote that corresponding normalizing constants given by (1.4). Let $S = X_1 + X_2$. Then the pdf of S can be written as

$$\begin{aligned}f_S(s) &= \frac{\theta_1\theta_2}{D_1D_2}\int \frac{dx}{\{(x-\mu_1)^2+\theta_1^2\}\{(s-x-\mu_2)^2+\theta_2^2\}} \\ &= \frac{1}{4D_1D_2}\int\left\{\frac{\alpha_1-\alpha_3}{x-\mu_1+i\theta_1} + \frac{\alpha_4-\alpha_2}{x-\mu_1-i\theta_1}\right. \\ &\qquad \left. + \frac{\alpha_2-\alpha_1}{x+\mu_2-s-i\theta_2}\frac{\alpha_3-\alpha_4}{x+\mu_2-s+i\theta_2}\right\}dx\end{aligned} \tag{6.1}$$

by partial fractions, where $i = \sqrt{-1}$, $\alpha_1 = (\mu_2 - s - i\theta_2 + \mu_1 - i\theta_1)^{-1}$, $\alpha_2 = (\mu_2 - s - i\theta_2 + \mu_1 + i\theta_1)^{-1}$, $\alpha_3 = (\mu_2 - s + i\theta_2 + \mu_1 - i\theta_1)^{-1}$ and $\alpha_4 = (\mu_2 - s + i\theta_2 + \mu_1 + i\theta_1)^{-1}$. If $B_1 + A_2 \le A_1 + B_2$ then (6.1) can be reduced to

$$f_S(s) = \begin{cases} \frac{1}{4D_1D_2}\{(\alpha_1-\alpha_3)\log\frac{s-A_2-\mu_1+i\theta_1}{A_1-\mu_1+i\theta_1} + (\alpha_4-\alpha_2)\log\frac{s-A_2-\mu_1-i\theta_1}{A_1-\mu_1-i\theta_1} \\ \quad + (\alpha_2-\alpha_1)\log\frac{s-A_2+\mu_2-s-i\theta_2}{A_1+\mu_2-s-i\theta_2} \\ \quad + (\alpha_3-\alpha_4)\log\frac{s-A_2+\mu_2-s+i\theta_2}{A_1+\mu_2-s+i\theta_2}\}, \\ \quad \text{if } A_1+A_2 \le s \le B_1+A_2, \\ \frac{1}{4D_1D_2}\{(\alpha_1-\alpha_3)\log\frac{B_1-\mu_1+i\theta_1}{A_1-\mu_1+i\theta_1} + (\alpha_4-\alpha_2)\log\frac{B_1-\mu_1-i\theta_1}{A_1-\mu_1-i\theta_1} \\ \quad + (\alpha_2-\alpha_1)\log\frac{B_1+\mu_2-s-i\theta_2}{A_1+\mu_2-s-i\theta_2} + (\alpha_3-\alpha_4)\log\frac{B_1+\mu_2-s+i\theta_2}{A_1+\mu_2-s+i\theta_2}\}, \\ \quad \text{if } B_1+A_2 \le s \le A_1+B_2, \\ \frac{1}{4D_1D_2}\{(\alpha_1-\alpha_3)\log\frac{B_1-\mu_1+i\theta_1}{s-B_2-\mu_1+i\theta_1} + (\alpha_4-\alpha_2)\log\frac{B_1-\mu_1-i\theta_1}{s-B_2-\mu_1-i\theta_1} \\ \quad + (\alpha_2-\alpha_1)\log\frac{B_1+\mu_2-s-i\theta_2}{s-B_2+\mu_2-s-i\theta_2} \\ \quad + (\alpha_3-\alpha_4)\log\frac{B_1+\mu_2-s+i\theta_2}{s-B_2+\mu_2-s+i\theta_2}\}, \\ \quad \text{if } A_1+B_2 \le s \le B_1+B_2. \end{cases}$$

If $B_1 + A_2 \geq A_1 + B_2$ then (6.1) can be reduced to

$$f_S(s) = \begin{cases} \frac{1}{4D_1D_2}\{(\alpha_1-\alpha_3)\log\frac{s-A_2-\mu_1+i\theta_1}{A_1-\mu_1+i\theta_1} + (\alpha_4-\alpha_2)\log\frac{s-A_2-\mu_1-i\theta_1}{A_1-\mu_1-i\theta_1} \\ \qquad + (\alpha_2-\alpha_1)\log\frac{s-A_2+\mu_2-s-i\theta_2}{A_1+\mu_2-s-i\theta_2} \\ \qquad + (\alpha_3-\alpha_4)\log\frac{s-A_2+\mu_2-s+i\theta_2}{A_1+\mu_2-s+i\theta_2}\}, \\ \quad \text{if } A_1+A_2 \leq s \leq A_1+B_2, \\ \frac{1}{4D_1D_2}\{(\alpha_1-\alpha_3)\log\frac{s-A_2-\mu_1+i\theta_1}{s-B_2-\mu_1+i\theta_1} + (\alpha_4-\alpha_2)\log\frac{s-A_2-\mu_1-i\theta_1}{s-B_2-\mu_1-i\theta_1} \\ \qquad + (\alpha_2-\alpha_1)\log\frac{s-A_2+\mu_2-s-i\theta_2}{s-B_2+\mu_2-s-i\theta_2} \\ \qquad + (\alpha_3-\alpha_4)\log\frac{s-A_2+\mu_2-s+i\theta_2}{s-B_2+\mu_2-s+i\theta_2}\}, \\ \quad \text{if } A_1+B_2 \leq s \leq A_2+B_1, \\ \frac{1}{4D_1D_2}\{(\alpha_1-\alpha_3)\log\frac{B_1-\mu_1+i\theta_1}{s-B_2-\mu_1+i\theta_1} + (\alpha_4-\alpha_2)\log\frac{B_1-\mu_1-i\theta_1}{s-B_2-\mu_1-i\theta_1} \\ \qquad + (\alpha_2-\alpha_1)\log\frac{B_1+\mu_2-s-i\theta_2}{s-B_2+\mu_2-s-i\theta_2} \\ \qquad + (\alpha_3-\alpha_4)\log\frac{B_1+\mu_2-s+i\theta_2}{s-B_2+\mu_2-s+i\theta_2}\}, \\ \quad \text{if } A_2+B_1 \leq s \leq B_1+B_2. \end{cases}$$

It is clear that the convolution, $X_1 + X_2$, is not a truncated Cauchy random variable of the type given by (1.2).

7 Bonferroni and Lorenz curves

Bonferroni and Lorenz curves (Bonferroni, 1930) have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. For a random variable X with quantile function $F^{-1}(\cdot)$, they are defined by

$$B(p) = \frac{1}{p\mu}\int_0^p F^{-1}(t)\,dt \tag{7.1}$$

and

$$L(p) = \frac{1}{\mu}\int_0^p F^{-1}(t)\,dt, \tag{7.2}$$

respectively, where $\mu = E(X)$. Suppose X is a truncated Cauchy random variable with its pdf specified by (1.2) for $0 \leq A < B < \infty$. Then $F^{-1}(t) = \mu + \theta\tan(Dt + \arctan(\alpha))$ and

$$\begin{aligned} \int_0^p F^{-1}(t)\,dt &= pE(X; A, B) + \theta\int_0^p \tan\bigl(Dt + \arctan(\alpha)\bigr)\,dt \\ &= pE(X; A, B) - \theta D\log\left|\frac{\cos(Dp + \arctan(\alpha))}{\cos(\arctan(\alpha))}\right|. \end{aligned}$$

So, (7.1) and (7.2) reduce to

$$B(p) = 1 - \frac{\theta D}{pE(X; A, B)}\log\left|\frac{\cos(Dp + \arctan(\alpha))}{\cos(\arctan(\alpha))}\right|$$

and

$$L(p) = p - \frac{\theta D}{E(X; A, B)} \log \left| \frac{\cos(Dp + \arctan(\alpha))}{\cos(\arctan(\alpha))} \right|,$$

respectively.

8 Entropies

An entropy of a random variable X is a measure of variation of the uncertainty. One of the popular entropy measure is the Rényi entropy defined by

$$\mathcal{J}_R(\gamma) = \frac{1}{1-\gamma} \log\left\{ \int f^{\gamma}(x)\, dx \right\}, \tag{8.1}$$

where $\gamma > 0$ and $\gamma \neq 1$ (Rényi, 1961). For the pdf (1.2), it can be seen that

$$\begin{aligned} \int f^{\gamma}(x)\, dx &= C^{-\gamma} \int_A^B \left\{ 1 + \left(\frac{x-\mu}{\theta} \right)^2 \right\}^{-\gamma} dx \\ &= 2^{-1} C^{-\gamma} \theta \{ J(\beta) - J(\alpha) \}, \end{aligned} \tag{8.2}$$

where $C = \theta D$, $D = \arctan(\beta) - \arctan(\alpha)$, $\alpha = (A-\mu)/\theta$, $\beta = (B-\mu)/\theta$ and

$$J(c) = \int_0^{c^2} \frac{z^{-1/2}}{(1+z)^{\gamma}}\, dz.$$

By equation (3.194.1) in Gradshteyn and Ryzhik (2000), one can calculate $J(c)$ as

$$J(c) = 2c\, {}_2F_1\left(\frac{1}{2}, \gamma; \frac{3}{2}; -c^2 \right). \tag{8.3}$$

By combining (8.2) and (8.3) it follows that the Rényi entropy (8.1) for the truncated Cauchy distribution is given by

$$\mathcal{J}_R(\gamma) = \frac{1}{1-\gamma} \log\{ C^{-\gamma} \theta H(\gamma) \}, \tag{8.4}$$

where

$$H(\gamma) = \beta\, {}_2F_1\left(\frac{1}{2}, \gamma; \frac{3}{2}; -\beta^2 \right) - \alpha\, {}_2F_1\left(\frac{1}{2}, \gamma; \frac{3}{2}; -\alpha^2 \right).$$

Shannon entropy defined by $\mathrm{E}[-\log f(X)]$ is the particular case of (8.1) for $\gamma \uparrow 1$. Limiting $\gamma \uparrow 1$ in (8.4) and using L'Hospital's rule and the facts

$${}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; z \right) = \frac{\operatorname{arctanh}(\sqrt{z})}{\sqrt{z}}$$

and

$$\frac{\partial}{\partial b}\, {}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k \psi(b+k)}{(c)_k} \frac{x^k}{k!} - \psi(b)\, {}_2F_1(a, b; c; x),$$

one obtains

$$\mathrm{E}[-\log f(X)] = \log C - \delta - \frac{\beta}{D}\sum_{k=0}^{\infty}\frac{(1/2)_k\psi(1+k)}{(3/2)_k}(-\beta^2)^k + \frac{\alpha}{D}\sum_{k=0}^{\infty}\frac{(1/2)_k\psi(1+k)}{(3/2)_k}(-\alpha^2)^k,$$

where δ denotes Euler's constant. Song (2001) observed that the gradient of the Rényi entropy $\mathcal{J}_R^{'}(\gamma) = (d/d\gamma)\mathcal{J}_R(\gamma)$ is related to the loglikelihood by $\mathcal{J}_R^{'}(1) = -(1/2)\mathrm{Var}[\log(f(X))]$. This equality and the fact that the quantity $-\mathcal{J}_R^{'}(1)$ remains invariant under location and scale transformations motivated Song to propose $-2\mathcal{J}_R^{'}(1)$ as a measure of the shape of a distribution. From (8.4), the first derivative is

$$\mathcal{J}_R^{'}(\gamma) = (1-\gamma)^{-1}\left\{-\log C + \frac{1}{H(\gamma)}\left[\beta\frac{\partial}{\partial\gamma}{}_2F_1\left(\frac{1}{2},\gamma;\frac{3}{2};-\beta^2\right) - \alpha\frac{\partial}{\partial\gamma}{}_2F_1\left(\frac{1}{2},\gamma;\frac{3}{2};-\alpha^2\right)\right]\right\} + (1-\gamma)^{-2}\{-\gamma\log C + \log\theta + \log H(\gamma)\}.$$

Using L'Hospital's rule again and the fact

$$\frac{\partial^2}{\partial b^2}{}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty}\frac{(a)_k\{\Gamma(b)\Gamma^{''}(b+k)-\Gamma^{'}(b)\Gamma^{'}(b+k)\}}{(c)_k\Gamma^2(b)}\frac{x^k}{k!} - \psi(b)\sum_{k=0}^{\infty}\frac{(a)_k(b)_k\psi(b+k)}{(c)_k}\frac{x^k}{k!} + \{\psi^2(b)-\psi^{'}(b)\}{}_2F_1(a,b;c;x),$$

one gets the expression

$$-2\mathcal{J}_R^{'}(1) = \delta^2 - \frac{\pi^2}{6} + \frac{\beta}{D}\sum_{k=0}^{\infty}\frac{(1/2)_k\{\Gamma^{''}(1+k)+\delta\Gamma^{'}(1+k)\}}{(3/2)_k}\frac{(-\beta^2)^k}{k!} - \frac{\alpha}{D}\sum_{k=0}^{\infty}\frac{(1/2)_k\{\Gamma^{''}(1+k)+\delta\Gamma^{'}(1+k)\}}{(3/2)_k}\frac{(-\alpha^2)^k}{k!} + \frac{\beta\delta}{D}\sum_{k=0}^{\infty}\frac{(1/2)_k\psi(1+k)}{(3/2)_k}(-\beta^2)^k - \frac{\alpha\delta}{D}\sum_{k=0}^{\infty}\frac{(1/2)_k\psi(1+k)}{(3/2)_k}(-\alpha^2)^k$$

$$
- \frac{1}{D^2} \left\{ \beta \sum_{k=0}^{\infty} \frac{(1/2)_k \psi(1+k)}{(3/2)_k} (-\beta^2)^k \right.
$$
$$
\left. - \alpha \sum_{k=0}^{\infty} \frac{(1/2)_k \psi(1+k)}{(3/2)_k} (-\alpha^2)^k + \delta D \right\}^2
$$

for the measure proposed by Song (2001).

9 Order statistics

Suppose $X_1, \ldots, X_n$ is a random sample from (1.2). Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the corresponding order statistics. It is well known that the pdf of the rth order statistic, say $Y = X_{r:n}$, is given by

$$
f_Y(y) = \frac{n!}{(r-1)!(n-r)!} F^{r-1}(y; A, B)\{1 - F(y; A, B)\}^{n-r} f(y; A, B) \quad (9.1)
$$

for $r = 1, 2, \ldots, n$. Using (1.2) and (1.3), one can express (9.1) as

$$
\begin{aligned}
f_Y(y) = {} & \frac{n!}{(r-1)!(n-r)!D^n} \\
& \times \frac{\{\arctan(y) - \arctan(A)\}^{r-1}\{\arctan(B) - \arctan(y)\}^{n-r}}{1+y^2},
\end{aligned} \quad (9.2)
$$

where we have assumed without loss of generality that $\mu = 0$ and $\theta = 1$. Using the binomial expansion, one can rewrite (9.2) as

$$
\begin{aligned}
f_Y(y) = {} & \frac{n!}{(r-1)!(n-r)!D^n} \\
& \times \sum_{i=0}^{r-1} \sum_{j=0}^{n-r} \binom{r-1}{i} \binom{n-r}{j} (-1)^{r-1-i+j} (\arctan(A))^{r-1-i} \\
& \times (\arctan(B))^{n-r-j} (\arctan(y))^{i+j} (1+y^2)^{-1}.
\end{aligned} \quad (9.3)
$$

Furthermore, using the series expansion, (3.3), one can rewrite (9.3) as

$$
\begin{aligned}
f_Y(y) = {} & \frac{n!}{(r-1)!(n-r)!D^n} \\
& \times \sum_{i=0}^{r-1} \sum_{j=0}^{n-r} \binom{r-1}{i} \binom{n-r}{j} (-1)^{r-1-i+j} (\arctan(A))^{r-1-i} \\
& \times (\arctan(B))^{n-r-j} \sum_{k_1=0}^{\infty} \cdots \sum_{k_{i+j}=0}^{\infty} \frac{(-1)^{k_1+\cdots+k_{i+j}} y^{2(k_1+\cdots+k_{i+j})+i+j}}{(2k_1+1)\cdots(2k_{i+j}+1)(1+y^2)}.
\end{aligned} \quad (9.4)
$$

Using the representation, (9.4), the lth moment of the rth order statistic can be expressed as

$$
\begin{aligned}
E(Y^l) = & \frac{n!}{(r-1)!(n-r)!D^{n-1}} \\
& \times \sum_{i=0}^{r-1} \sum_{j=0}^{n-r} \binom{r-1}{i} \binom{n-r}{j} (-1)^{r-1-i+j} (\arctan(A))^{r-1-i} \\
& \times (\arctan(B))^{n-r-j} \\
& \times \sum_{k_1=0}^{\infty} \cdots \sum_{k_{i+j}=0}^{\infty} \frac{(-1)^{k_1+\cdots+k_{i+j}} E(X^{2(k_1+\cdots+k_{i+j})+i+j+l}; A, B)}{(2k_1+1)\cdots(2k_{i+j}+1)}.
\end{aligned}
$$

Using the same representation, the cdf of the rth order statistic can be expressed as

$$
\begin{aligned}
F_Y(y) = & \frac{n!}{(r-1)!(n-r)!D^{n}} \\
& \times \sum_{i=0}^{r-1} \sum_{j=0}^{n-r} \binom{r-1}{i} \binom{n-r}{j} (-1)^{r-1-i+j} (\arctan(A))^{r-1-i} \\
& \times (\arctan(B))^{n-r-j} \\
& \times \sum_{k_1=0}^{\infty} \cdots \sum_{k_{i+j}=0}^{\infty} \frac{(-1)^{k_1+\cdots+k_{i+j}} K(2(k_1+\cdots+k_{i+j})+i+j, y)}{(2k_1+1)\cdots(2k_{i+j}+1)},
\end{aligned}
$$

where

$$
K(m, y) = \begin{cases} \displaystyle\sum_{k=0}^{m/2} \binom{m/2}{k} (-1)^{m/2-k} \{I(k-1, y) - I(k-1, A)\}, & \text{for } m \text{ even}, \\ \displaystyle\sum_{k=0}^{(m-1)/2} \binom{(m-1)/2}{k} (-1)^{(m-1)/2-k} & \\ \qquad \times \{J(k-1, y) - J(k-1, A)\}, & \text{for } m \text{ odd}, \end{cases}
$$

where

$$
I(k, y) = \sum_{l=0}^{k} \binom{k}{l} \frac{y^{2l+1}}{2l+1}, \qquad k \geq 0
$$

and

$$
J(k, y) = \frac{(1+y^2)^{k+1}}{2(k+1)}, \qquad k \geq 0,
$$

with the initial values $I(-1, y) = \arctan(y)$ and $J(-1, y) = (1/2)\log(1+y^2)$.

10 Asymptotic distributions

If $X_1, \ldots, X_n$ is a random sample from (1.2) and if $\bar{X} = (X_1 + \cdots + X_n)/n$ denotes the sample mean then by the usual central limit theorem $\sqrt{n}(\bar{X} - \mathrm{E}(X))/\sqrt{\mathrm{Var}(X)}$ approaches the standard normal distribution as $n \to \infty$. Sometimes one would be interested in the asymptotics of the extreme order statistics $M_n = \max(X_1, \ldots, X_n)$ and $m_n = \min(X_1, \ldots, X_n)$.

For the cdf (1.3), it can be seen using L'Hospital's rule that

$$\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} = \frac{1}{x}$$

if $B = \infty$,

$$\lim_{t \to 0} \frac{1 - F(B - tx)}{1 - F(B - t)} = x$$

if $B < \infty$,

$$\lim_{t \to -\infty} \frac{F(tx)}{F(t)} = \frac{1}{x}$$

if $A = -\infty$, and

$$\lim_{t \to 0} \frac{F(A + tx)}{F(A + t)} = x$$

if $A < \infty$. Hence, it follows from Theorem 1.6.2 in Leadbetter et al. (1987) that there must be norming constants $a_n > 0$, b_n, $c_n > 0$ and d_n such that

$$\Pr\{a_n(M_n - b_n) \le x\} \to \exp(-1/x)$$

if $B = \infty$ and

$$\Pr\{c_n(M_n - d_n) \le x\} \to \exp(x)$$

if $B < \infty$. The form of the norming constants can be determined using Corollary 1.6.3 in Leadbetter et al. (1987): one can see that $b_n = 0$, $d_n = B$,

$$\frac{1}{a_n} = \mu + \theta \cot\left(\frac{\pi}{2n} - \frac{1}{n} \arctan(\alpha)\right),$$

and

$$\frac{1}{c_n} = B - \mu - \theta \tan\left(\frac{1}{n} \arctan(\alpha) - \frac{n-1}{n} \arctan(\beta)\right).$$

Similarly, one can show that

$$\Pr\{a_n(m_n - b_n) \le x\} \to 1 - \exp(1/x)$$

if $A = -\infty$ and

$$\Pr\{c_n(m_n - d_n) \le x\} \to 1 - \exp(-x)$$

if $A > -\infty$, where $b_n = 0$, $d_n = A$,

$$\frac{1}{a_n} = \mu - \theta \cot\left(\frac{\pi}{2n} + \frac{1}{n}\arctan(\beta)\right),$$

and

$$\frac{1}{c_n} = \mu + \theta \tan\left(\frac{1}{n}\arctan(\beta) + \frac{n-1}{n}\arctan(\alpha)\right) - A.$$

11 Estimation

Here, we consider estimation by the method of moments and the method of maximum likelihood and provide expressions for the associated Fisher information matrix.

Suppose $X_1, \ldots, X_n$ is a random sample from (1.2). By equating the first four moments of (2.4) with the corresponding sample estimates, one can obtain the method of moments estimators as the simultaneous solutions of the four equations:

$$\frac{\mu^m}{D}\sum_{k=0}^{m}\frac{1}{k+1}\binom{m}{k}\left(\frac{\theta}{\mu}\right)^k\left\{\beta^{k+1}{}_2F_1\left(1, \frac{k+1}{2}; 1+\frac{k+1}{2}; -\beta^2\right)\right.$$
$$\left. - \alpha^{k+1}{}_2F_1\left(1, \frac{k+1}{2}; 1+\frac{k+1}{2}; -\alpha^2\right)\right\} = \frac{1}{n}\sum_{i=1}^{n}X_i^m$$

for $m = 1, 2, 3, 4$, where $D = \arctan(\beta) - \arctan(\alpha)$, $\alpha = (A - \mu)/\theta$ and $\beta = (B - \mu)/\theta$.

Now, consider the method of maximum likelihood. The log-likelihood for the random sample is

$$\log L(\mu, \theta, A, B) = -n \log C - \sum_{i=1}^{n}\log\left\{1 + \left(\frac{X_i - \mu}{\theta}\right)^2\right\}, \qquad (11.1)$$

where $C = \theta D$. The first-order derivatives of (11.1) with respect to the four parameters are:

$$\frac{\partial \log L}{\partial A} = -\frac{n}{C}\frac{\partial C}{\partial A},$$
$$\frac{\partial \log L}{\partial B} = -\frac{n}{C}\frac{\partial C}{\partial B},$$
$$\frac{\partial \log L}{\partial \mu} = -\frac{n}{C}\frac{\partial C}{\partial \mu} + \frac{2}{\theta^2}\sum_{i=1}^{n}(X_i - \mu)\left\{1 + \left(\frac{X_i - \mu}{\theta}\right)^2\right\}^{-1}$$

and

$$\frac{\partial \log L}{\partial \theta} = -\frac{n}{C}\frac{\partial C}{\partial \theta} + \frac{2}{\theta^3}\sum_{i=1}^{n}(X_i - \mu)^2\left\{1 + \left(\frac{X_i - \mu}{\theta}\right)^2\right\}^{-1}.$$

The maximum likelihood estimators of μ, θ, A and B are the simultaneous solutions of the equations $\partial \log L/\partial A = 0$, $\partial \log L/\partial B = 0$, $\partial \log L/\partial \mu = 0$ and $\partial \log L/\partial \theta = 0$. If the parameters A and B are assumed known then the maximum likelihood estimators of μ and θ are the simultaneous solutions of the equations

$$\frac{n}{C}\frac{\partial C}{\partial \mu} = \frac{2}{\theta^2}\sum_{i=1}^{n}(X_i - \mu)\left\{1 + \left(\frac{X_i - \mu}{\theta}\right)^2\right\}^{-1}$$

and

$$\frac{n}{C}\frac{\partial C}{\partial \theta} = \frac{2}{\theta^3}\sum_{i=1}^{n}(X_i - \mu)^2\left\{1 + \left(\frac{X_i - \mu}{\theta}\right)^2\right\}^{-1}.$$

For interval estimation of (A, B, μ, θ) and tests of hypothesis, one requires the Fisher information matrix. Standard calculations show that the elements of this matrix are

$$\begin{aligned}
E\left(-\frac{\partial^2 \log L}{\partial \lambda\, \partial \nu}\right) &= -\frac{n}{C^2}\frac{\partial C}{\partial \lambda}\frac{\partial C}{\partial \nu} + \frac{n}{C}\frac{\partial^2 C}{\partial \lambda\, \partial \nu},\\
E\left(-\frac{\partial^2 \log L}{\partial \mu^2}\right) &= -\frac{n}{C^2}\left(\frac{\partial C}{\partial \mu}\right)^2 + \frac{n}{C}\frac{\partial^2 C}{\partial \mu^2} + \frac{2n}{\theta^2}I(1,0) - \frac{4n}{\theta^2}I(2,2),\\
E\left(-\frac{\partial^2 \log L}{\partial \mu\, \partial \theta}\right) &= -\frac{n}{C^2}\frac{\partial C}{\partial \mu}\frac{\partial C}{\partial \theta} + \frac{n}{C}\frac{\partial^2 C}{\partial \mu\, \partial \theta} + \frac{4n}{\theta^2}I(1,1) - \frac{4n}{\theta^2}I(2,3)
\end{aligned}$$

and

$$E\left(-\frac{\partial^2 \log L}{\partial \theta^2}\right) = -\frac{n}{C^2}\left(\frac{\partial C}{\partial \theta}\right)^2 + \frac{n}{C}\frac{\partial^2 C}{\partial \theta^2} + \frac{6n}{\theta^2}I(1,2) - \frac{4n}{\theta^2}I(2,4),$$

where

$$I(m,n) = E\left\{\left[1 + \left(\frac{X - \mu}{\theta}\right)^2\right]^{-m}\left(\frac{X - \mu}{\theta}\right)^n\right\}. \tag{11.2}$$

By an easy application of equation (3.194.1) in Gradshteyn and Ryzhik (2000), one can calculate $I(m,n)$ in (11.2) as

$$\begin{aligned}
I(m,n) = \frac{\theta}{(n+1)C}\Bigg\{&\beta^{n+1}{}_2F_1\left(\frac{n+1}{2}, m+1; \frac{n+3}{2}; -\beta^2\right)\\
&- \alpha^{n+1}{}_2F_1\left(\frac{n+1}{2}, m+1; \frac{n+3}{2}; -\alpha^2\right)\Bigg\}
\end{aligned}$$

for $m \geq 0$ and $n \geq 0$.

12 Simulation

Simulation from the truncated Cauchy distribution in (1.2) is straightforward. Note that the inverse cdf corresponding to (1.2) is $F^{-1}(x) = \mu + \theta \tan(Dx + \arctan(\alpha))$. So, one can generate truncated Cauchy variates by $X = \mu + \theta \tan(DU + \arctan(\alpha))$, where U is a uniform random variate on the interval [0, 1].

13 Application

We now illustrate an application of the truncated Cauchy distribution to consumer price index data. We collected the data on this index for the six countries: United States, United Kingdom, Japan, Canada, Germany and Australia. The data were extracted from the website https://www.globalfinancialdata.com/ and the range of data for each country is shown in Table 1.

A distribution that is of interest to economists is the *positive consumer price distribution*, that is, the distribution of change in consumer price index when it increases from one year to the next. We propose the truncated Cauchy distribution in (1.2) with $A = 0$ and $B = \infty$ as a model for the positive consumer price distribution. The performance of this model is compared versus the truncated normal model given by the pdf $(1/\theta')\phi((x - \mu')/\theta')/\Phi(\mu'/\theta')$, where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the pdf and the cdf of the standard normal distribution.

Prior to fitting, following common practice, we transformed the data by first taking logarithms and then the lag 1 differences. A crucial assumption for the fitting is that the data are independent. We tested this by plotting the autocorrelation function. Figure 2 shows these plots for two of the six countries: the United Kingdom and Canada. The plots for the other countries are similar. It is clear that the autocorrelation is generally weak except for the first lag.

The method of maximum likelihood procedure in Section 11 was used to fit the two models. The maximum likelihood estimates of the parameters and the logarithms of the maximized likelihoods (L for the truncated Cauchy model and L' for the truncated normal model) are shown in Table 2. The numbers within

Table 1 *Datasets*

Country	Range of data
United Kingdom	1800 to 2006
United States	1820 to 2006
Japan	1868 to 2006
Canada	1910 to 2006
Germany	1923 to 2006
Australia	1901 to 2006

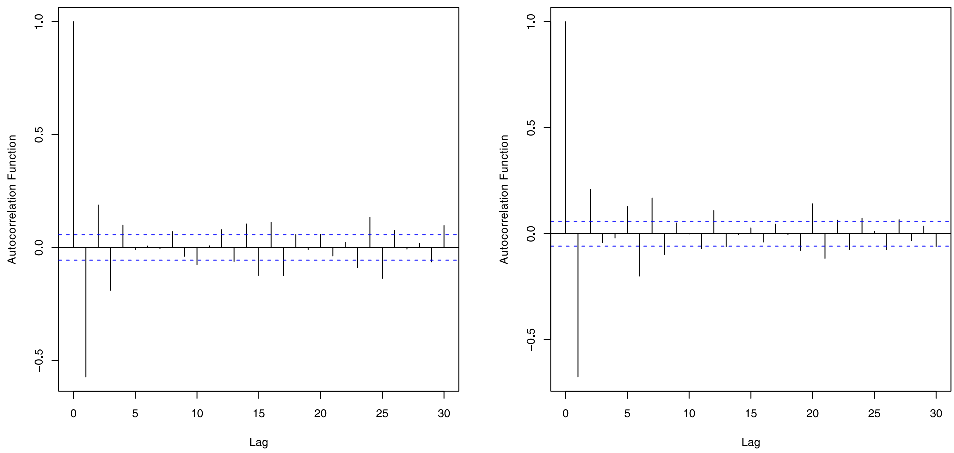

Figure 2 *Autocorrelation function plots for data from the United Kingdom (left) and Canada (right).*

Table 2 *Parameter estimates and standard errors of the truncated Cauchy and truncated normal models*

Country	$\widehat{\mu}$	$\widehat{\theta}$	$\log L$	$\widehat{\mu}'$	$\widehat{\theta}'$	$\log L'$
United Kingdom	0.00618	0.00324	2279.7	0.01245	0.01804	1786.5
	(0.00019)	(0.00018)		(0.00069)	(0.00049)	
United States	0.00480	0.00247	2844.6	0.00868	0.01458	2163.4
	(0.00014)	(0.00013)		(0.00053)	(0.00038)	
Japan	0.00609	0.00439	1782.0	0.02176	0.05410	927.8
	(0.00028)	(0.00026)		(0.00218)	(0.00154)	
Canada	0.00575	0.00261	2118.4	0.00744	0.00569	2096.2
	(0.00018)	(0.00015)		(0.00024)	(0.00018)	
Germany	0.00318	0.00166	2557.3	0.00530	0.00607	2293.2
	(0.00011)	(0.00010)		(0.00024)	(0.00018)	
Australia	0.01021	0.00643	871.6	0.01618	0.01442	863.2
	(0.00065)	(0.00052)		(0.00082)	(0.00059)	

brackets are the standard errors computed by inverting the expected information matrix.

The two fitted models are clearly not nested. In this case, testing can be based on Akaike information criterion (AIC) defined by $2k - \log L$, where k is the number of parameters in the model, and L is the maximized value of the likelihood function for the model (Akaike, 1974). However, the fitted models have the same number of parameters, so AIC reduces to the standard log likelihood ratio.

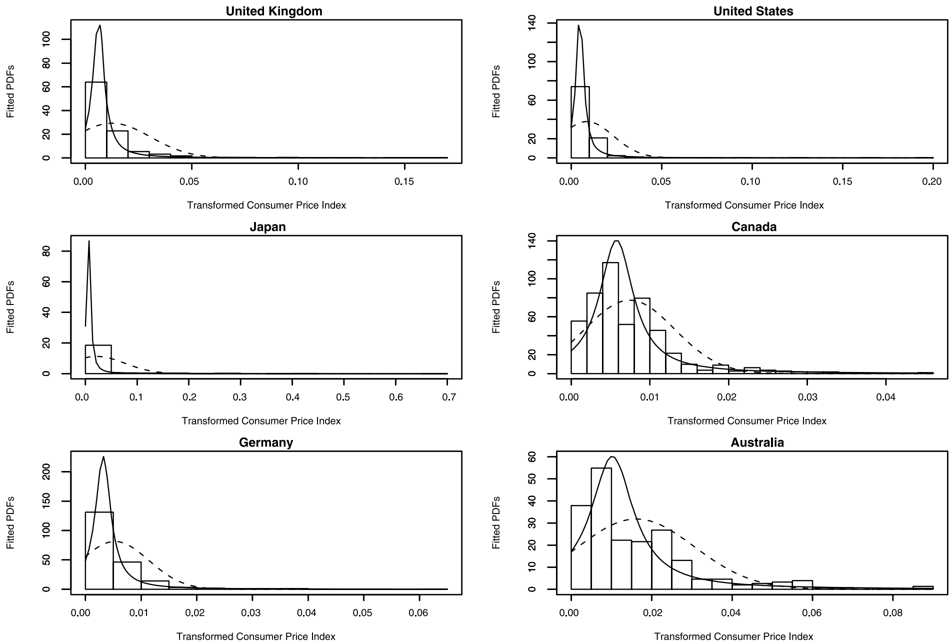

Figure 3 *Fitted pdfs of the truncated Cauchy model (solid curve) and the truncated normal model (broken curve) for the transformed consumer price index data from the six countries.*

It follows by the standard likelihood ratio test that the truncated Cauchy distribution provides the better fit for each of the six countries. This observation is confirmed by the fitted pdfs shown in Figure 3.

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