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The log-generalized modified Weibull regression model

Edwin M. M. Ortega[a], **Gauss M. Cordeiro**[b] **and Jalmar M. F. Carrasco**[c]

[a]*Departamento de Ciências Exatas, ESALQ—USP*
[b]*Departamento de Estatística e Informática, DEINFO—UFRPE*
[c]*Departamento de Estatística, IME—USP*

Abstract. For the first time, we introduce the log-generalized modified Weibull regression model based on the modified Weibull distribution [Carrasco, Ortega and Cordeiro *Comput. Statist. Data Anal.* **53** (2008) 450–462]. This distribution can accommodate increasing, decreasing, bathtub and unimodal shaped hazard functions. A second advantage is that it includes classical distributions reported in lifetime literature as special cases. We also show that the new regression model can be applied to censored data since it represents a parametric family of models that includes as submodels several widely known regression models and therefore can be used more effectively in the analysis of survival data. We obtain maximum likelihood estimates for the model parameters by considering censored data and evaluate local influence on the estimates of the parameters by taking different perturbation schemes. Some global-influence measurements are also investigated. In addition, we define martingale and deviance residuals to detect outliers and evaluate the model assumptions. We demonstrate that our extended regression model is very useful to the analysis of real data and may give more realistic fits than other special regression models.

1 Introduction

Standard lifetime distributions usually present very strong restrictions to produce bathtub curves, and thus appear to be inappropriate for interpreting data with this characteristic. Some distributions were introduced to model this kind of data, as the generalized gamma distribution proposed by Stacy (1962), the exponential power family introduced by Smith and Bain (1975), the beta-integrated model defined by Hjorth (1980), the generalized log-gamma distribution investigated by Lawless (2003), among others. A good review of these models is presented, for instance, in Rajarshi and Rajarshi (1988). In the last decade, new classes of distributions for modeling this kind of data based on extensions of the Weibull distribution were developed. Mudholkar, Srivastava, and Friemer (1995) introduced the exponentiated Weibull (EW) distribution, Xie and Lai (1995) presented the additive Weibull distribution, Lai, Xie, and Murthy (2003) proposed the modified Weibull (MW) distribution and Carrasco, Ortega, and Cordeiro (2008) defined the generalized

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modified Weibull (GMW) distribution. The GMW distribution, due to its flexibility in accommodating many forms of the risk function, seems to be an important distribution that can be used in a variety of problems in modeling survival data. Furthermore, the main motivation for its use is that it contains as special submodels several distributions such as the EW, exponentiated exponential (EE) [Gupta and Kundu (1999)], MW [Lai, Xie and Murthy (2003)] and generalized Rayleigh (GR) [Kundu and Rakab (2005)] distributions. The new distribution can model four types of failure rate function (i.e., increasing, decreasing, unimodal and bathtub) depending on its parameters. It is also suitable for testing goodness of fit of some special submodels such as the EW, MW and GR distributions.

Different forms of regression models have been proposed in survival analysis. Among them, the location-scale regression model [Lawless (2003)] is distinguished since it is frequently used in clinical trials. In this paper, we propose a location-scale regression model based on the GMW distribution [Carrasco, Ortega, and Cordeiro (2008)], referred to as the log-generalized modified Weibull (LGMW) regression model, which is a feasible alternative for modeling the four existing types of failure rate functions.

For the assessment of model adequacy, we develop diagnostic studies to detect possible influential or extreme observations that can cause distortions on the results of the analysis. We discuss the influence diagnostics based on case deletion [Cook (1977)] in which the influence of the ith observation on the parameter estimates is studied by removing this observation from the analysis. We propose diagnostic measures based on case deletion to determine which observations might be influential in the analysis. This methodology has being applied to various statistical models [Davison and Tsai (1992); Xie and Wei (2007)].

Nevertheless, when case deletion is used, all information from a single subject is deleted at once and therefore it is hard to say whether an observation has some influence on a specific aspect of the model. A solution for this problem can be found in the local influence approach where we again investigate how the results of the analysis are changed under small perturbations in the model or data. Cook (1986) proposed a general framework to detect influential observations which indicate how sensitive is the analysis when small perturbations are provoked on the data or in the model. Some authors have investigated the assessment of local influence in survival analysis models. For example, Pettitt and Bin Daud (1989) investigated local influence in proportional hazard regression models, Escobar and Meeker (1992) adapted local influence methods to regression analysis under censoring scheme and Ortega, Bolfarine, and Paula (2003) considered the problem of assessing local influence in generalized log-gamma regression models with censored observations. Recently, Ortega, Cancho and Bolfarine (2006) derived curvature calculations under various perturbation schemes in log-exponentiated Weibull regression models with censored data. Xie and Wei (2007) developed the application of influence diagnostics in censored generalized Poisson regression models based on a case-deletion method and local influence analysis. Fachini, Ortega, and

Louzada-Neto (2008) considered local influence methods to polyhazard models under the presence of explanatory variables. Silva et al. (2008) adapted local influence methods to the log-Burr XII regression analysis with censoring. Carrasco, Ortega and Paula (2008) investigated local influence in log-modified Weibull (LMW) regression models with censored data and Ortega, Cancho and Paula (2009) derived curvature calculations under various perturbation schemes in generalized log-gamma regression models with cure fraction. We propose a similar methodology to detect influential subjects in LGMW regression models with censored data.

The paper is organized as follows. In Section 2, we define the LGMW distribution and derive an expansion for its moments. In Section 3, we propose a LGMW regression model, estimate the parameters by the method of maximum likelihood and derive the observed information matrix. Several diagnostic measures are presented in Section 4 by considering case deletion and normal curvatures of local influence under various perturbation schemes with censored observations. In Section 5, a kind of deviance residual is proposed to assess departures from the underlying LGMW distribution as well as outlying observations. We also present and discuss some simulation studies. In Section 6, a real dataset is analyzed which shows the flexibility, practical relevance and applicability of our regression model. Section 7 ends with some concluding remarks.

2 The log-generalized modified Weibull distribution

Most generalized Weibull distributions have been proposed in reliability literature to provide a better fitting of certain datasets than the traditional two and three-parameter Weibull models. The GMW distribution with four parameters $\alpha > 0$, $\gamma \geq 0$, $\lambda \geq 0$ and $\varphi > 0$, introduced by Carrasco, Ortega and Cordeiro (2008), extends the MW distribution [Lai, Xie and Murthy (2003)] and should be able to fit various types of data. Its density function for $t > 0$ is given by

$$f(t) = \frac{\alpha\varphi(\gamma + \lambda t)t^{\gamma-1}\exp[\lambda t - \alpha t^{\gamma}\exp(\lambda t)]}{\{1 - \exp[-\alpha t^{\gamma}\exp(\lambda t)]\}^{1-\varphi}}. \tag{2.1}$$

The parameter α controls the scale of the distribution, whereas the parameters γ and φ control its shape. The parameter λ is a kind of accelerating factor in the imperfection time and thus it works as a factor of fragility in the survival of the individual when the time increases.

Another important characteristic of the distribution is that it contains, as special submodels, the EE distribution [Gupta and Kundu (1999)], the EW distribution [Mudholkar, Srivastava and Friemer (1995)], the MW distribution [Lai, Xie and Murthy (2003)], the GR distribution [Kundu and Rakab (2005)], and some other distributions [see, e.g., Carrasco, Ortega and Cordeiro (2008)]. The survival and hazard rate functions corresponding to (2.1) are given by $S(t) = 1 - \{1 - \exp[-\alpha t^{\gamma}\exp(\lambda t)]\}^{\varphi}$ and

$$h(t) = \frac{\alpha\varphi(\gamma + \lambda t)t^{\gamma-1}\exp[\lambda t - \alpha t^{\gamma}\exp(\lambda t)]\{1 - \exp[-\alpha t^{\gamma}\exp(\lambda t)]\}^{\varphi-1}}{1 - \{1 - \exp[-\alpha t^{\gamma}\exp(\lambda t)]\}^{\varphi}},$$

respectively. A characteristic of the GMW distribution is that its failure rate function accommodates four shapes of the hazard rate functions that depend basically on the values of the parameters γ and β [Carrasco, Ortega and Cordeiro (2008)]. For $\gamma \geq 1$, $0 < \varphi < 1$ and $\forall t > 0$, $h'(t) > 0$, $h(t)$ is increasing. For $0 < \gamma < 1$, $\varphi > 1$ and $\forall t > 0$, $h'(t) < 0$, $h(t)$ is decreasing. For $0 < \gamma < 1$ and $\varphi \to \infty$, $h(t)$ is unimodal. If $\lambda = 0$, $\gamma > 1$ and $\gamma\varphi < 1$, $h(t)$ is bathtub shaped; if $\varphi = 1$, we have $h'(t) = \alpha t^{\gamma-1}\exp(\lambda t)[(\gamma + \lambda t)\{(\gamma - 1)t^{-1} + \lambda\} + \lambda] = 0$, and solving this equation yields a change point $t^* = (-\gamma + \sqrt{\gamma})/\lambda$. When $0 < \gamma < 1$, we can show that t^* exists and is finite. When $t < t^*$, $h'(t^*) < 0$, the hazard rate function is decreasing; when $t > t^*$, $h'(t^*) > 0$, the hazard rate function is increasing. Hence, the hazard rate function can be of bathtub shape.

Henceforth, T is a random variable following the GMW density function (2.1) and Y is defined by $Y = \log(T)$. It is easy to verify that the density function of Y obtained by replacing $\gamma = 1/\sigma$ and $\alpha = \exp(-\mu/\sigma)$ reduces to

$$\begin{aligned} f(y) = {} & \varphi[\sigma^{-1} + \lambda\exp(y)] \\ & \times \exp\left\{\left(\frac{y-\mu}{\sigma}\right) + \lambda\exp(y) - \exp\left[\left(\frac{y-\mu}{\sigma}\right) + \lambda\exp(y)\right]\right\} \\ & \times \left\{1 - \exp\left[-\exp\left\{\left(\frac{y-\mu}{\sigma}\right) + \lambda\exp(y)\right\}\right]\right\}^{\varphi-1}, \\ & -\infty < y < \infty, \end{aligned} \tag{2.2}$$

where $-\infty < \mu < \infty$, $\sigma > 0$, $\lambda \geq 0$ and $\varphi > 0$. We refer to equation (2.2) as the LGMW distribution, say $Y \sim \mathrm{LGMW}(\lambda, \varphi, \sigma, \mu)$, where $\mu \in \Re$ is the location parameter, $\sigma > 0$ is the scale parameter and λ and φ are shape parameters. Figure 1 plots this density function for selected values of the parameters σ and φ showing that the LGMW distribution could be very flexible for modeling its kurtosis. The corresponding survival function is

$$S(y) = 1 - \left\{1 - \exp\left[-\exp\left\{\left(\frac{y-\mu}{\sigma}\right) + \lambda\exp\left[\left(\frac{y-\mu}{\sigma}\right)\sigma\right]\exp(\mu)\right\}\right]\right\}^{\varphi}, \tag{2.3}$$

and the hazard rate function is simply $h(y) = f(y)/S(y)$. The random variable $Z = (Y - \mu)/\sigma$ has density function

$$f(z) = \varphi\sigma(\sigma^{-1} + v)\exp[z + v - \exp(z + v)]\{1 - \exp[-\exp(z + v)]\}^{\varphi-1}, \tag{2.4}$$

where $v = \lambda\exp(\mu + \sigma z)$.

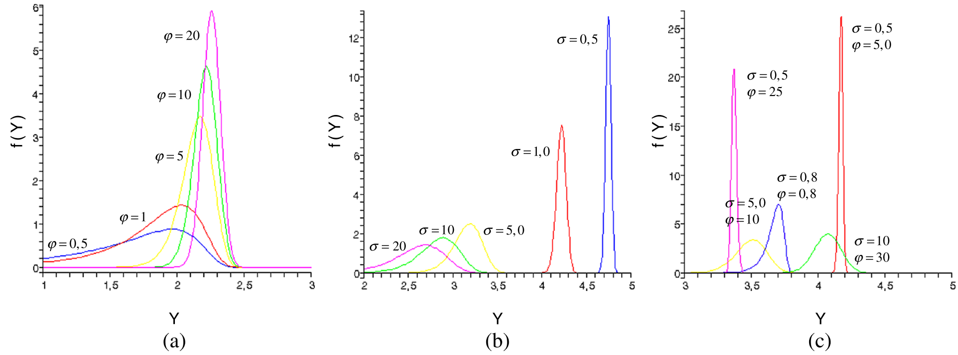

Figure 1 *The LGMW density curves:* (a) *For some values of* φ *with* $\sigma = 5$, $\lambda = 0.5$ *and* $\mu = 20$. (b) *For some values of* σ *with* $\lambda = 0.1$, $\mu = 10$ *and* $\varphi = 10$. (c) *For some values of* φ *and* σ *with* $\lambda = 0.5$ *and* $\mu = 10$.

The rth ordinary moment $\mu'_r = E(T^r)$ of the GMW density function (2.1) can be expressed parameterized in terms of λ, φ, σ and μ as

$$\begin{aligned}\mu'_r = \exp(-\mu/\sigma)\frac{\varphi}{\sigma}\int_0^\infty & t^{r+1/\sigma-1}(1+t)\exp\{\lambda t - \exp(-\mu/\sigma)t^{1/\sigma}\exp(\lambda t)\} \\ & \times [1-\exp\{-\exp(-\mu/\sigma)t^{1/\sigma}\exp(\lambda t)\}]^{\varphi-1}\,dt.\end{aligned} \tag{2.5}$$

Carrasco, Ortega and Cordeiro (2008) derived an infinite sum representation for μ'_r given by

$$\mu'_r = \exp(-\mu/\sigma)\varphi\sum_{j=0}^{\infty}\frac{(1-\varphi)_j}{j!}\sum_{i_1,\ldots,i_r=1}^{\infty}\frac{A_{i_1,\ldots,i_r}\Gamma(s_r/\gamma)}{\{\exp(-\mu/\sigma)(j+1)\}^{s_r/\gamma+1}}. \tag{2.6}$$

Here, $(1-\varphi)_j = (1-\varphi)(1-\varphi+1)\cdots(j-\varphi)$ is the ascending factorial, $s_r = i_1+\cdots+i_r$ and the product $A_{i_1,\ldots,i_r} = a_{i_1}\cdots a_{i_r}$ can be easily computed from the quantities

$$a_i = \frac{(-1)^{i+1}i^{i-2}}{(i-1)!}(\lambda\sigma)^{i-1}.$$

When φ is real noninteger, we can use the formula $(1-\varphi)_j = (-1)^j\Gamma(\varphi)/\Gamma(\varphi - j)$ in terms of gamma functions.

Formula (2.6) for the rth moment of the GMW distribution is quite general and holds when both parameters λ and γ are positive and $\varphi \neq 1$. By expanding $Y^s = \log(T)^s$ in Taylor series around μ'_1, the sth moment of Y can be written as

$$E(Y^s) = \log(\mu'_1)^s + \sum_{i=2}^{\infty}\frac{G^{(i)}(\mu'_1)\mu_i}{i!},$$

where $G^{(i)}(\mu'_1)$ is the ith derivative of $G(\mu'_1) = \log(\mu'_1)^s$ with respect to μ'_1 and $\mu_i = E(T-\mu'_1)^i$ is the ith central moment of T.

Expressing the central moments of T in terms of the ordinary moments, $E(Y^s)$ can be written as an infinite sum of products of two ordinary moments of T

$$E(Y^s) = \log(\mu_1')^r + \sum_{i=2}^{\infty}\sum_{k=0}^{i} \frac{(-1)^k G^{(i)}(\mu_1')\mu_{i-k}'\mu_1'^k}{(i-k)!k!}, \tag{2.7}$$

where the moments μ_{i-k}' and μ_1' come directly from equation (2.6). Formula (2.7) is the main result of this section. The derivatives of $G(\mu_1') = \log(\mu_1')^s$ are easily obtained in Maple up to any order. Hence, the ordinary moments of the LGMW distribution are functions of the parameters λ, φ, σ and μ. A further research could be addressed to study the finiteness of the moments of Y. Clearly, the moments of Z are easily obtained from the moments of Y.

3 The log-generalized modified Weibull regression model

In many practical applications, the lifetimes are affected by explanatory variables such as the cholesterol level, blood pressure, weight and many others. Parametric models to estimate univariate survival functions and for censored data regression problems are widely used. A parametric model that provides a good fit to lifetime data tends to yield more precise estimates of the quantities of interest. Based on the LGMW density, we propose a linear location-scale regression model linking the response variable y_i and the explanatory variable vector $\mathbf{x}_i^T = (x_{i1}, \dots, x_{ip})$ as follows:

$$y_i = \mathbf{x}_i^T \boldsymbol{\beta} + \sigma z_i, \qquad i = 1, \dots, n, \tag{3.1}$$

where the random error z_i has density function (2.4), $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)^T$, $\sigma > 0$, $\lambda \geq 0$ and $\varphi > 0$ are unknown parameters. The parameter $\mu_i = \mathbf{x}_i^T \boldsymbol{\beta}$ is the location of y_i. The location parameter vector $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)^T$ is represented by a linear model $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$, where $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_n)^T$ is a known model matrix. The LGMW model (3.1) opens new possibilities for fitted many different types of data. It contains as special submodels the following well-known regression models:

- *Log-Weibull (LW) or extreme value regression model*
  For $\lambda = 0$ and $\varphi = 1$, the survival function is
  $$S(y) = \exp\left[-\exp\left(\frac{y - \mathbf{x}^T\boldsymbol{\beta}}{\sigma}\right)\right],$$
  which is the classical Weibull regression model [see, e.g., Lawless (2003)]. If $\sigma = 1$ and $\sigma = 0.5$ in addition to $\lambda = 0, \varphi = 1$, it coincides with the exponential and Rayleigh regression models, respectively.
- *Log-Exponentiated Weibull (LEW) regression model*
  For $\lambda = 0$, the survival function is
  $$S(y) = 1 - \left\{1 - \exp\left[-\exp\left(\frac{y - \mathbf{x}^T\boldsymbol{\beta}}{\sigma}\right)\right]\right\}^{\varphi},$$

which is the log-exponentiated Weibull regression model introduced by Mudholkar, Srivastava and Friemer (1995), Cancho, Bolfarine and Achcar (1999), Ortega, Cancho and Bolfarine (2006) and Cancho, Ortega and Bolfarine (2009). If $\sigma = 1$ in addition to $\lambda = 0$, the LGMW regression model becomes the log-exponentiated exponential regression model. If $\sigma = 0.5$ in addition to $\lambda = 0$, the LGMW model becomes the log-generalized Rayleigh regression model.

- *Log-Modified Weibull (LMW) distribution*

  For $\varphi = 1$, the survival function becomes

$$S(y) = \exp\left\{-\exp\left[\left(\frac{y - \mathbf{x}^T\boldsymbol{\beta}}{\sigma}\right) + \lambda \exp\left[\left(\frac{y - \mathbf{x}^T\boldsymbol{\beta}}{\sigma}\right)\sigma\right]\exp(\mathbf{x}^T\boldsymbol{\beta})\right]\right\},$$

  which is the LMW regression model introduced by Carrasco, Ortega and Paula (2008).

Consider a sample $(y_1, \mathbf{x}_1), \ldots, (y_n, \mathbf{x}_n)$ of n independent observations, where each random response is defined by $y_i = \min\{\log(t_i), \log(c_i)\}$. We assume noninformative censoring such that the observed lifetimes and censoring times are independent. Let F and C be the sets of individuals for which y_i is the log-lifetime or log-censoring, respectively. Conventional likelihood estimation techniques can be applied here. The log-likelihood function for the vector of parameters $\boldsymbol{\theta} = (\lambda, \varphi, \sigma, \boldsymbol{\beta}^T)^T$ from model (3.1) has the form $l(\boldsymbol{\theta}) = \sum_{i \in F} l_i(\boldsymbol{\theta}) + \sum_{i \in C} l_i^{(c)}(\boldsymbol{\theta})$, where $l_i(\boldsymbol{\theta}) = \log[f(y_i)]$, $l_i^{(c)}(\boldsymbol{\theta}) = \log[S(y_i)]$, $f(y_i)$ is the density (2.2) and $S(y_i)$ is survival function (2.3) of Y_i. The total log-likelihood function for $\boldsymbol{\theta}$ reduces to

$$l(\boldsymbol{\theta}) = \sum_{i \in F} l_1(\lambda, \varphi, z_i, u_i) + \sum_{i \in C} l_2(\lambda, \varphi, z_i, u_i), \tag{3.2}$$

where

$$\begin{aligned} l_1(\lambda, \varphi, z_i, u_i) &= \log[\varphi(\sigma^{-1} + u_i)] + [z_i + u_i - \exp(z_i + u_i)] \\ &\quad + (\varphi - 1)\log\{1 - \exp[-\exp(z_i + u_i)]\}, \\ l_2(\lambda, \varphi, z_i, u_i) &= \log\{1 - [1 - \exp\{-\exp(z_i + u_i)\}]^{\varphi}\}, \end{aligned}$$

$u_i = \lambda \exp(\sigma z_i + \mathbf{x}_i^T\boldsymbol{\beta})$, $z_i = (y_i - \mathbf{x}_i^T\boldsymbol{\beta})/\sigma$ and r is the number of uncensored observations (failures). The maximum likelihood estimate (MLE) $\widehat{\boldsymbol{\theta}}$ of the vector of unknown parameters can be calculated by maximizing the log-likelihood (3.2). We use the matrix programming language Ox (MaxBFGS function) [see Doornik (2007)] to calculate the estimate $\widehat{\boldsymbol{\theta}}$. Initial values for $\boldsymbol{\beta}$ and σ are taken from the fit of the LW regression model with $\lambda = 0$ and $\varphi = 1$. The fit of the LGMW model produces the estimated survival function for y_i ($\hat{z}_i = (y_i - \mathbf{x}_i^T\hat{\boldsymbol{\beta}})/\hat{\sigma}$) given by

$$S(y_i; \hat{\lambda}, \hat{\varphi}, \hat{\sigma}, \widehat{\boldsymbol{\beta}}^T) = 1 - \{1 - \exp[-\exp\{\hat{z}_i + \hat{\lambda}\exp(\hat{\sigma}\hat{z}_i)\exp(\mathbf{x}_i^T\hat{\boldsymbol{\beta}})\}]\}^{\hat{\varphi}}.$$

Under conditions that are fulfilled for the parameter vector $\boldsymbol{\theta}$ in the interior of the parameter space but not on the boundary, the asymptotic distribution of

$\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ is multivariate normal $N_{p+3}(0, K(\boldsymbol{\theta})^{-1})$, where $K(\boldsymbol{\theta})$ is the information matrix. The asymptotic covariance matrix $K(\boldsymbol{\theta})^{-1}$ of $\widehat{\boldsymbol{\theta}}$ can be approximated by the inverse of the $(p + 3) \times (p + 3)$ observed information matrix $-\ddot{\mathbf{L}}(\boldsymbol{\theta})$. The elements of the observed information matrix $-\ddot{\mathbf{L}}(\boldsymbol{\theta})$, namely $-\mathbf{L}_{\lambda\lambda}, -\mathbf{L}_{\lambda\varphi}, -\mathbf{L}_{\lambda\sigma}, -\mathbf{L}_{\lambda\beta_j}, -\mathbf{L}_{\varphi\varphi}, -\mathbf{L}_{\varphi\sigma}, -\mathbf{L}_{\varphi\beta_j}, -\mathbf{L}_{\sigma\sigma}, -\mathbf{L}_{\sigma\beta_j}$ and $-\mathbf{L}_{\beta_j\beta_s}$ for $j, s = 1, \dots, p$, are given in Appendix A. The approximate multivariate normal distribution $N_{p+3}(0, -\ddot{\mathbf{L}}(\boldsymbol{\theta})^{-1})$ for $\widehat{\boldsymbol{\theta}}$ can be used in the classical way to construct approximate confidence regions for some parameters in $\boldsymbol{\theta}$.

We can use the likelihood ratio (LR) statistic for comparing some special submodels with the LGMW model. We consider the partition $\boldsymbol{\theta} = (\boldsymbol{\theta}_1^T, \boldsymbol{\theta}_2^T)^T$, where $\boldsymbol{\theta}_1$ is a subset of parameters of interest and $\boldsymbol{\theta}_2$ is a subset of remaining parameters. The LR statistic for testing the null hypothesis $H_0 : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_1^{(0)}$ versus the alternative hypothesis $H_1 : \boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_1^{(0)}$ is given by $w = 2\{\ell(\widehat{\boldsymbol{\theta}}) - \ell(\widetilde{\boldsymbol{\theta}})\}$, where $\widetilde{\boldsymbol{\theta}}$ and $\widehat{\boldsymbol{\theta}}$ are the estimates under the null and alternative hypotheses, respectively. The statistic w is asymptotically (as $n \to \infty$) distributed as χ_k^2, where k is the dimension of the subset of parameters $\boldsymbol{\theta}_1$ of interest.

4 Sensitivity analysis

In order to assess the sensitivity of the MLEs, global influence and local influence [Cook (1986)] under three perturbation schemes are now carried out.

4.1 Global influence

The first tool to perform sensitivity analysis is the global influence starting from case deletion [see Cook (1977)]. Case deletion is a common approach to study the effect of dropping the ith observation from the dataset. The case deletion for model (3.1) is given by

$$Y_l = \mathbf{x}_l^T \boldsymbol{\beta} + \sigma Z_l, \qquad l = 1, \dots, n, l \neq i. \tag{4.1}$$

In the following, a quantity with subscript "(i)" means the original quantity with the ith observation deleted. The log-likelihood function for the model (4.1) is $l_{(i)}(\boldsymbol{\theta})$ and let $\hat{\boldsymbol{\theta}}_{(i)} = (\hat{\lambda}_{(i)}, \hat{\varphi}_{(i)}, \hat{\sigma}_{(i)}, \hat{\boldsymbol{\beta}}_{(i)}^T)^T$ be the corresponding estimate of $\boldsymbol{\theta}$. The basic idea to assess the influence of the ith observation on the MLE $\hat{\boldsymbol{\theta}} = (\hat{\lambda}, \hat{\varphi}, \hat{\sigma}, \hat{\boldsymbol{\beta}}^T)^T$ is to compare the difference between $\hat{\boldsymbol{\theta}}_{(i)}$ and $\hat{\boldsymbol{\theta}}$. If deletion of an observation seriously influences the estimates, more attention should be paid to that observation. Hence, if $\hat{\boldsymbol{\theta}}_{(i)}$ is far away from $\hat{\boldsymbol{\theta}}$, then the case can be regarded as an influential observation. A first measure of global influence is the well-known generalized Cook distance defined by $GD_i(\hat{\boldsymbol{\theta}}) = (\hat{\boldsymbol{\theta}}_{(i)} - \hat{\boldsymbol{\theta}})^T\{-\ddot{\mathbf{L}}(\hat{\boldsymbol{\theta}})\}(\hat{\boldsymbol{\theta}}_{(i)} - \hat{\boldsymbol{\theta}})$. Other alternative is to assess the values $GD_i(\boldsymbol{\beta})$ and $GD_i(\lambda, \varphi, \sigma)$ which reveal the impact of the ith observation on the estimates of $\boldsymbol{\beta}$ and $(\lambda, \varphi, \sigma)$, respectively.

Another well-known measure of the difference between $\hat{\boldsymbol{\theta}}_{(i)}$ and $\hat{\boldsymbol{\theta}}$ is the likelihood displacement given by $LD_i(\hat{\boldsymbol{\theta}}) = 2\{l(\hat{\boldsymbol{\theta}}) - l(\hat{\boldsymbol{\theta}}_{(i)})\}$.

Further, we can also compute $\hat{\beta}_j - \hat{\beta}_{j(i)}(j = 1, \ldots, p)$ to detect the difference between $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\beta}}_{(i)}$. Alternative global influence measures are possible. We study the behavior of a test statistic, such as a Wald test for an explanatory variable or censoring effect, under a case deletion scheme. We can avoid the direct estimation without the ith observation using the one-step approximation $\hat{\boldsymbol{\theta}}_{(i)} = \hat{\boldsymbol{\theta}} - \ddot{\mathbf{L}}(\hat{\boldsymbol{\theta}})^{-1}\dot{l}_{(i)}(\hat{\boldsymbol{\theta}})$, where $\dot{l}_{(i)}(\hat{\boldsymbol{\theta}})$ is equal to $\frac{\partial l_{(i)}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$ evaluated at $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$ [see Cook, Peña and Weisberg (1988)].

4.2 Local influence

Another approach suggested by Cook (1986) considers small perturbations represented by the vector $\boldsymbol{\omega}$ instead of removing observations and is related to a particular perturbation scheme. Local influence calculation can be carried out for model (4.1). If likelihood displacement $LD(\boldsymbol{\omega}) = 2\{l(\hat{\boldsymbol{\theta}}) - l(\hat{\boldsymbol{\theta}}_{\boldsymbol{\omega}})\}$ is used, where $\hat{\boldsymbol{\theta}}_{\boldsymbol{\omega}}$ is the MLE under the perturbed model, the normal curvature for $\boldsymbol{\theta}$ at the direction $\mathbf{d}$, where $\|\mathbf{d}\| = 1$, is given by $C_{\mathbf{d}}(\boldsymbol{\theta}) = 2|\mathbf{d}^T \boldsymbol{\Delta}^T [\ddot{\mathbf{L}}(\boldsymbol{\theta})]^{-1} \boldsymbol{\Delta}\mathbf{d}|$, where $\boldsymbol{\Delta}$ is a $(p+3) \times n$ matrix which depends on the perturbation scheme, and whose elements are given by $\Delta_{ji} = \partial^2 l(\boldsymbol{\theta}|\boldsymbol{\omega})/\partial\theta_j \partial\omega_i$, $i = 1, \ldots, n$ and $j = 1, \ldots, p+3$ evaluated at $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{\omega}_0$, where $\boldsymbol{\omega}_0$ is the no perturbation vector [see, e.g., Cook (1986); Zhu et al. (2007); Jung (2008)]. For the LGMW regression model with censored data, the elements of $\ddot{\mathbf{L}}(\boldsymbol{\theta})$ are given in Appendix A. We can also calculate normal curvatures $C_{\mathbf{d}}(\lambda)$, $C_{\mathbf{d}}(\varphi)$, $C_{\mathbf{d}}(\sigma)$ and $C_{\mathbf{d}}(\boldsymbol{\beta})$ to perform various index plots, for instance, the index plot of the eigenvector $\mathbf{d}_{\max}$ corresponding to the largest eigenvalue $C_{\mathbf{d}_{\max}}$ of the matrix $\mathbf{B} = -\boldsymbol{\Delta}^T [\ddot{\mathbf{L}}(\boldsymbol{\theta})]^{-1} \boldsymbol{\Delta}$, and the index plots of $C_{\mathbf{d}_i}(\lambda)$, $C_{\mathbf{d}_i}(\varphi)$, $C_{\mathbf{d}_i}(\sigma)$ and $C_{\mathbf{d}_i}(\boldsymbol{\beta})$, the so-called total local influence [see, e.g., Lesaffre and Verbeke (1998)], where $\mathbf{d}_i$ is an $n \times 1$ vector of zeros with one at the ith position. Thus, the curvature at direction $\mathbf{d}_i$ takes the form $C_i = 2|\boldsymbol{\Delta}_i^T [\ddot{\mathbf{L}}(\boldsymbol{\theta})]^{-1} \boldsymbol{\Delta}_i|$, where $\boldsymbol{\Delta}_i^T$ denotes the ith row of $\boldsymbol{\Delta}$. It is usual to point out those cases such that $C_i \geq 2\bar{C}$, where $\bar{C} = \frac{1}{n}\sum_{i=1}^n C_i$.

Consider the vector of weights $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)^T$. From the log-likelihood (3.2), under three perturbation schemes, we derive the matrix

$$\boldsymbol{\Delta} = (\boldsymbol{\Delta}_{ji})_{(p+3)\times n} = \left(\frac{\partial^2 l(\boldsymbol{\theta}|\boldsymbol{\omega})}{\partial\theta_i \boldsymbol{\omega}_j}\right)_{(p+3)\times n}, \qquad j = 1, \ldots, p+3 \text{ and } i = 1, \ldots, n.$$

- *Case-weight perturbation*

  In this case, the log-likelihood function has the form

$$l(\boldsymbol{\theta}|\boldsymbol{\omega}) = \sum_{i\in F} \omega_i l_1(\lambda, \varphi, z_i, u_i) + \sum_{i\in C} \omega_i l_2(\lambda, \varphi, z_i, u_i),$$

  where $0 \leq \omega_i \leq 1$, $\boldsymbol{\omega}_0 = (1, \ldots, 1)^T$ and $l_m(\cdot)$ is defined in equation (3.2) for $m = 1, 2$. The matrix $\boldsymbol{\Delta} = (\boldsymbol{\Delta}_\lambda^T, \boldsymbol{\Delta}_\varphi^T, \boldsymbol{\Delta}_\sigma^T, \boldsymbol{\Delta}_{\boldsymbol{\beta}}^T)^T$ is given in Appendix B.

- *Response perturbation*

  Here, we consider that each y_i is perturbed as $y_{iw} = y_i + \omega_i S_y$, where S_y is a scale factor that may be estimated by the standard deviation of the observed response y and $\omega_i \in \Re$. The perturbed log-likelihood function can be expressed as

$$l(\boldsymbol{\theta}|\boldsymbol{\omega}) = \sum_{i \in F} l_1(\lambda, \varphi, z_i^*, u_i^*) + \sum_{i \in C} l_2(\lambda, \varphi, z_i^*, u_i^*),$$

  where $z_i^* = [(y_i + \omega_i S_y) - \mathbf{x}_i^T \boldsymbol{\beta}]/\sigma$, $u_i^* = \lambda \exp(\sigma z_i^* + \mathbf{x}_i^T \boldsymbol{\beta})$, $\boldsymbol{\omega}_0 = (0, \ldots, 0)^T$ and $l_m(\cdot)$ is defined in equation (3.2) for $m = 1, 2$. The matrix $\boldsymbol{\Delta} = (\boldsymbol{\Delta}_\lambda^T, \boldsymbol{\Delta}_\varphi^T, \boldsymbol{\Delta}_\sigma^T, \boldsymbol{\Delta}_{\boldsymbol{\beta}}^T)^T$ is given in Appendix C.

- *Explanatory variable perturbation*

  Consider now an additive perturbation on a particular continuous explanatory variable, say X_q, by setting $x_{iq\omega} = x_{iq} + \omega_i S_q$, where S_q is a scale factor and $\omega_i \in \Re$. The perturbed log-likelihood function has the form

$$l(\boldsymbol{\theta}|\boldsymbol{\omega}) = \sum_{i \in F} l_1(\lambda, \varphi, z_i^{**}, u_i^{**}) + \sum_{i \in C} l_2(\lambda, \varphi, z_i^{**}, u_i^{**}),$$

  where $z_i^{**} = (y_i - x_i^{*T} \boldsymbol{\beta})/\sigma$, $\mathbf{x}_i^{*T} \boldsymbol{\beta} = \beta_1 + \beta_2 x_{i2} + \cdots + \beta_q(x_{iq} + \omega_i S_q) + \cdots + \beta_p x_{ip}$, $u_i^{**} = \lambda \exp(\sigma z_i^{**} + \mathbf{x}_i^{*T} \boldsymbol{\beta})$, $\boldsymbol{\omega}_0 = (0, \ldots, 0)^T$ and $l_m(\cdot)$ is defined in equation (3.2) for $m = 1, 2$. The matrix $\boldsymbol{\Delta} = (\boldsymbol{\Delta}_\lambda^T, \boldsymbol{\Delta}_\varphi^T, \boldsymbol{\Delta}_\sigma^T, \boldsymbol{\Delta}_{\boldsymbol{\beta}}^T)^T$ is given in Appendix D.

Previous works for which local influence curvatures are derived in regression models with censored data are due to Escobar and Meeker (1992), Ortega, Bolfarine, and Paula (2003), Silva et al. (2008) and Ortega, Cancho and Paula (2009). The interplay between local and global influence could be further elaborated following the proposal of Wu and Luo (1993). However, this approach will be addressed in a future research.

5 Residual analysis

For studying departures from error assumptions as well as the presence of outliers, we consider two types of residuals: a deviance component residual [McCullagh and Nelder (1989)] and a martingale-type residual [Therneau, Grambsch, and Fleming (1990)]. Therneau, Grambsch and Fleming (1990) introduced the deviance component residual in counting process by using basically martingale residuals. The martingale residuals are skew, have maximum value $+1$ and minimum value $-\infty$. In parametric lifetime models, the martingale residual can be expressed as $r_{M_i} = \delta_i + \log[S_Y(y_i; \hat{\boldsymbol{\theta}})]$, where $\delta_i = 0$ if the ith observation is censored and $\delta_i = 1$ if the ith observation is uncensored [see, e.g., Klein and Moeschberger

(1997); Ortega, Bolfarine and Paula (2003, 2008)]. Hence, the martingale residual for the LGMW model takes the form

$$r_{M_i} = \begin{cases} 1 + \log\{1 - [1 - \exp(-\exp[\hat{z}_i + \hat{\lambda}\exp(\hat{z}_i\hat{\sigma})\exp(\mathbf{x}_i^T\hat{\boldsymbol{\beta}})])]^{\hat{\varphi}}\}, & \text{if } i \in F, \\ \log\{1 - [1 - \exp(-\exp[\hat{z}_i + \hat{\lambda}\exp(\hat{z}_i\hat{\sigma})\exp(\mathbf{x}_i^T\hat{\boldsymbol{\beta}})])]^{\hat{\varphi}}\}, & \text{if } i \in C, \end{cases}$$

where the sets F and C are defined in Section 3.

The deviance component residual proposed by Therneau, Grambsch and Fleming (1990) is a transformation of the martingale residual to attenuate the skewness which was motivated by the deviance component residual in generalized linear models. In particular, the deviance component residual for the Cox's proportional hazards model with no time-dependent explanatory variables can be written as

$$r_{D_i} = \mathrm{sinal}(\mathrm{r_{M_i}})\{-2[\mathrm{r_{M_i}} + \delta_\mathrm{i}\log(\delta_\mathrm{i} - \mathrm{r_{M_i}})]\}^{1/2}, \quad (5.1)$$

where r_{M_i} is the martingale residual. Ortega, Paula and Bolfarine (2008) and Carrasco, Ortega and Paula (2008) investigated the empirical distributions of r_{M_i} and r_{D_i} for the generalized log-gamma and LMW regression models varying the sample sizes and censoring proportions, respectively.

5.1 Simulation studies

We investigate the form of the empirical distribution of the deviance component residual r_{D_i} for different values of n and censoring percentages through some simulation studies. Plots of the ordered residuals obtained from the simulations against the expected quantiles of the standard normal distribution are displayed in Figures 2 and 3. We fixed $n = 30, 50$ and 100 and the lifetimes $t_1, \ldots, t_n$ were generated from the GMW distribution (2.1) considering $\gamma = 1.4$, $\lambda = 0.1$ and $\varphi = 0.5$ (with $\varphi < 1$) and $\gamma = 1.4$, $\lambda = 0.1$ and $\varphi = 1.8$ (with $\varphi > 1$), taking again the reparametrization $\gamma = 1/\sigma$ and $\alpha = \exp(-\mu/\sigma)$. Further, we assume $\mu_i = \beta_0 + \beta_1 x_i$, where x_i was generated from a uniform distribution on the interval (0, 1), and $\beta_0 = 0.5$ and $\beta_1 = 1.0$. The censoring times $c_1, \ldots, c_n$ were generated from a uniform distribution $(0, \theta)$, where θ was adjusted until the censoring percentages 10%, 30% or 50% are reached. The lifetimes considered in each fit were calculated as $\min\{\mathrm{c_i}, \mathrm{t_i}\}$. For each combination of n, σ, λ, φ and censoring percentages, 1000 samples were generated. For each generated dataset, we fitted the LGMW regression model (3.1), where $\mu_i = \beta_0 + \beta_1 x_i$ and calculated the residuals r_{D_i}. Thus, the ordered residuals were plotted against the expected quantiles of the standard normal distribution.

Figures 2 and 3 lead to some conclusions. The main conclusion from the generated plots is that the empirical distributions of the residual r_{D_i} present a good agreement with the standard normal distribution. When the censoring percentage decreases or the sample size increases, the empirical distribution of the residuals r_{D_i} performs better agreement with the standard normal distribution, as expected in both situations. Thus, we can use normal probability plots for the residuals r_{D_i} with

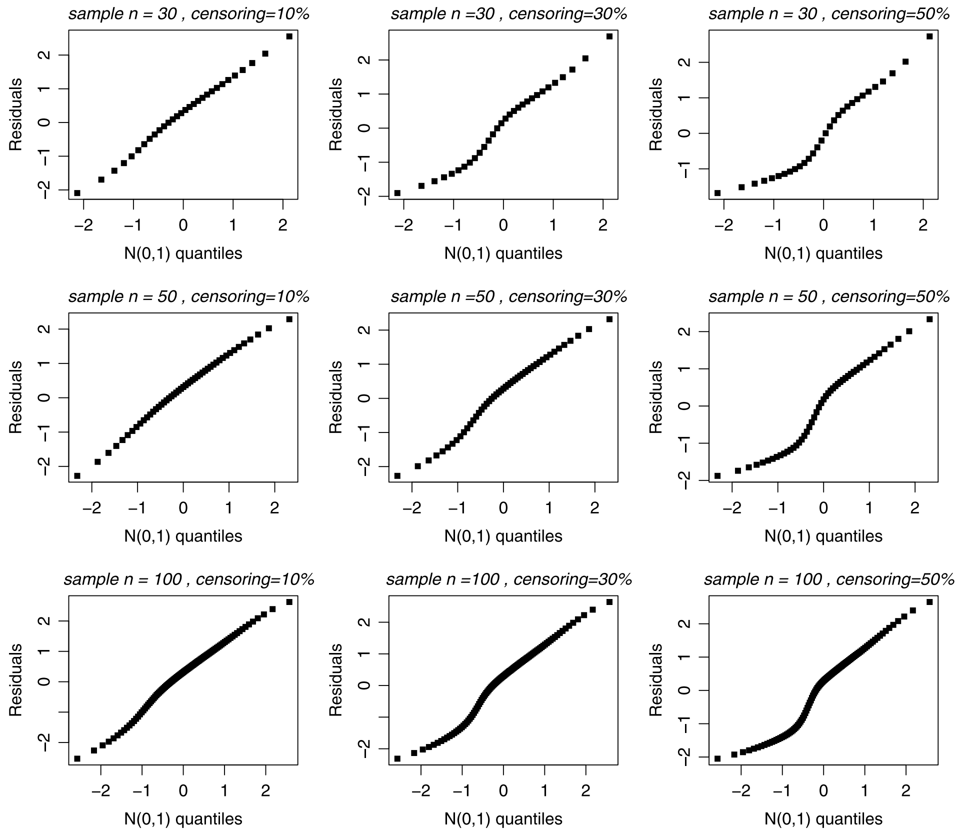

Figure 2 *Normal probability plots for the residuals r_{D_i}. Sample sizes $n = 30$, $n = 50$ and $n = 100$, percentages of censoring = 10%, 30% and 50%, parameter values $\gamma = 1.4$, $\lambda = 0.1$ and $\varphi = 0.5$.*

simulated envelopes for both models, as suggested by Atkinson (1985), obtained as follows: (i) fit the model and generate a sample of n independent observations using the fitted model as if it were the true model; (ii) fit the model to the generated sample using the dataset $(\delta_i, \mathbf{x}_i)$ and compute the values of the residuals; (iii) repeat steps (i) and (ii) m times; (iv) obtain ordered values of the residuals, $r^*_{(i)v}$, $i = 1, \ldots, n$ and $v = 1, \ldots, m$; (v) consider n sets of the m ordered statistics and for each set compute the mean, minimum and maximum values; (vi) plot these values and the ordered residuals of the original sample against the normal scores. The minimum and maximum values of the m ordered statistics yield the envelope. The observations corresponding to residuals outside the limits provided by the simulated envelope require further investigation. Additionally, if a considerable proportion of points falls outside the envelope, then we have evidence against the adequacy of the fitted model. Plots of such residuals against the fitted values can also be useful.

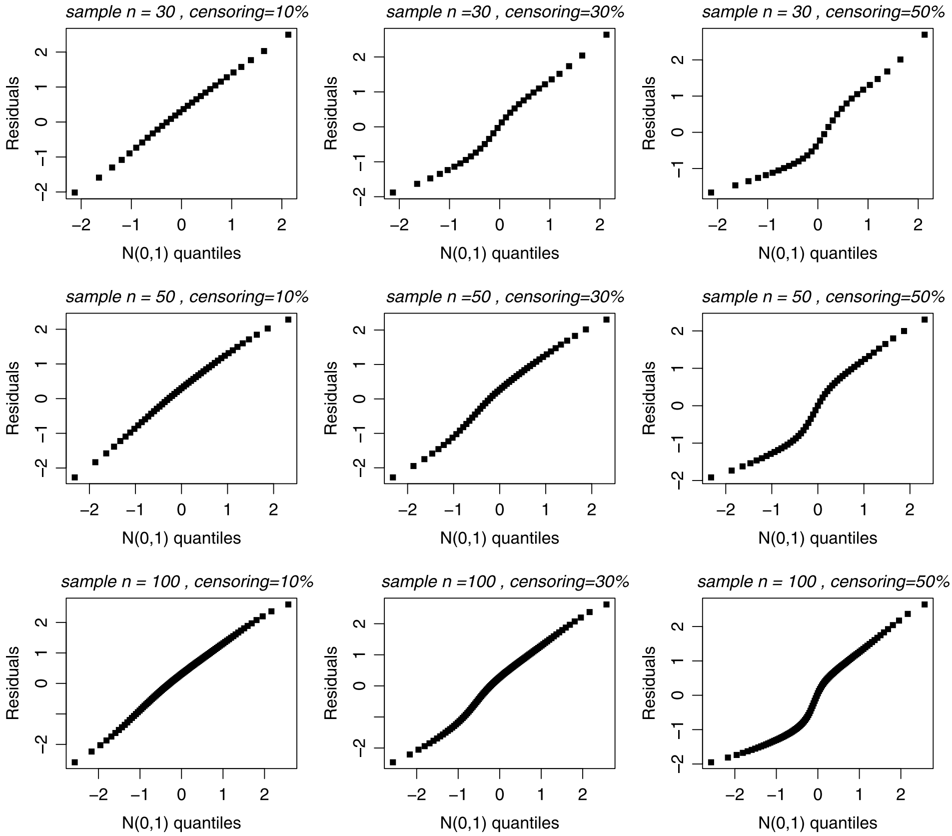

Figure 3 *Normal probability plots for the residuals* r_{D_i}. *Sample sizes* $n = 30$, $n = 50$ *and* $n = 100$, *percentages of censoring* = 10%, 30% *and* 50%, *parameter values* $\gamma = 1.4$, $\lambda = 0.1$ *and* $\varphi = 1.8$.

6 Application

Survival times for the Golden shiner data, Notemigonus crysoleucas, were obtained from field experiments conducted in Lake Saint Pierre, Quebec, in 2005 [Laplante-Albert (2008)]. Each individual fish was attached by means of a monofilament chord to a chronographic tethering device that allowed the fish to swim in midwater. A timer in the device was set off when the tethered fish was captured by a predator. The device was retrieved approximately 24 hours after the onset of the experiment and survival time was then obtained from the difference: time elapsed between onset of the experiment and retrieval time elapsed in device timer since predation event. The variables involved in the study are: y_i—observed survival time (in hours); $cens_i$—censoring indicator (0 = censoring, 1 = lifetime observed); x_{i1}—north or south bank of the lake (0 = north, 1 = south); x_{i2}—distance over the longitudinal axis of the lake (in km); x_{i3}—size of the fish (in

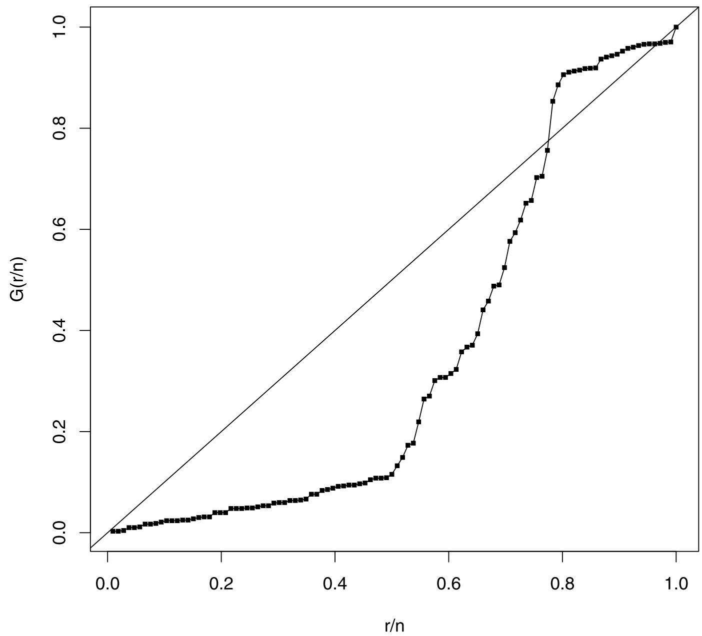

Figure 4 *TTT plot for the Golden shiner data.*

cm); x_{i4}—depth of the place (in cm); x_{i5}—abundance index of macro-thin plants (in percentage) and x_{i6}—transparency of the water (in cm).

In many applications there is qualitative information about the hazard shape which can support a specified model. In this context, a device called the total time on test (TTT) plot [Aarset (1987)] is very useful. The TTT plot is obtained by plotting $\mathsf{G}(r/n) = [(\sum_{i=1}^{r} T_{i:n}) + (n-r)T_{r:n}]/(\sum_{i=1}^{n} T_{i:n})$ for $r = 1, \dots, n$ against r/n, where $T_{i:n}$ are the order statistics of the sample ($i = 1, \dots, n$). The TTT plot for Golden shiner data given in Figure 4 has first a convex shape and then a concave shape, thus indicating a bathtub shaped failure rate function.

The Golden shiner data have been analyzed by Carrasco, Ortega and Paula (2008) using the LMW regression model. We now reanalyzed these data using the LGMW regression model. First, we consider the equation

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \beta_5 x_{i5} + \beta_6 x_{i6} + \sigma z_i, \quad i = 1, \dots, 106, \tag{6.1}$$

where the random variable y_i has the LGMW distribution. The MLEs (approximate standard errors and p-values in parentheses) are: $\hat{\lambda} = 0.001$ (0.003), $\hat{\varphi} = 12.855$ (20.066), $\hat{\sigma} = 5.086$ (2.776), $\hat{\beta}_0 = -1.894$ (5.904) (0.748), $\hat{\beta}_1 = 2.197$ (0.536) (<0.001), $\hat{\beta}_2 = 0.097$ (0.037) (0.008), $\hat{\beta}_3 = -0.125$ (0.032) (<0.001), $\hat{\beta}_4 = 0.035$ (0.009) (<0.001), $\hat{\beta}_5 = 0.022$ (0.017) (0.202) and $\hat{\beta}_6 = 0.222$ (0.204) (0.278).

Table 1 *Statistics AIC, BIC and CAIC for comparing the LGMW and LMW models*

Model	AIC	BIC	CAIC
LGMW	422.3	424.6	448.9
LWM	427.2	429.0	451.1

Further, we calculate the maximum unrestricted and restricted log-likelihoods and the LR statistics for testing some submodels. An analysis under the LGMW regression model provides a check on the appropriateness of the LW, LEW and LMW submodels and indicates the extent for which inferences depend upon the model. For example, the LR statistic for testing the hypotheses $H_0 : \varphi = 1$ versus $H_1 : H_0$ is not true, that is, to compare the LMW and LGMW regression models, is $w = 2\{-201.142 - (-204.577)\} = 6.87$ (p-value <0.05) which yields favorable indications toward to the LGMW regression model. A summary of the values of the Akaike Information Criterion (AIC), the Bayesian Information Criterion (BIC) and the Consistent Akaike Information Criterion (CAIC) to compare the LGMW and LMW regression models is given in Table 1. The LGMW regression model outperforms the LMW model irrespective of the criteria and can be used effectively in the analysis of these data. The explanatory variables x_1, x_2, x_3 and x_4 are marginally significant for the LGMW model at the significance level of 5%. We use Ox to compute case-deletion measures $GD_i(\boldsymbol{\theta})$ and $LD_i(\boldsymbol{\theta})$ defined in Section 4.1. The results of such influence measure index plots are displayed in Figure 5. These plots show that the cases ♯5, ♯34 and ♯101 are possible influential observations. We apply the local influence theory developed in Section 4.2, where

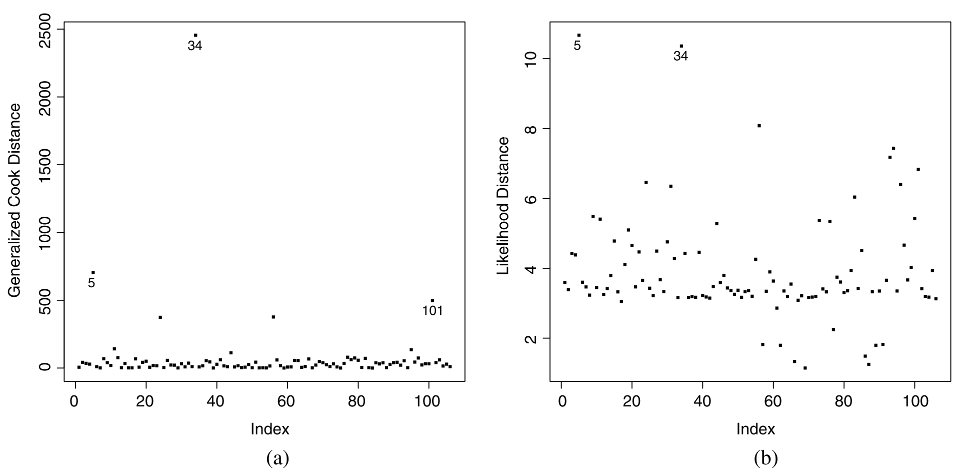

Figure 5 (a) *Index plot of $GD_i(\boldsymbol{\theta})$ for $\boldsymbol{\theta}$ on the Golden shiner data.* (b) *Index plot of $LD_i(\boldsymbol{\theta})$ for $\boldsymbol{\theta}$ on the Golden shiner data.*

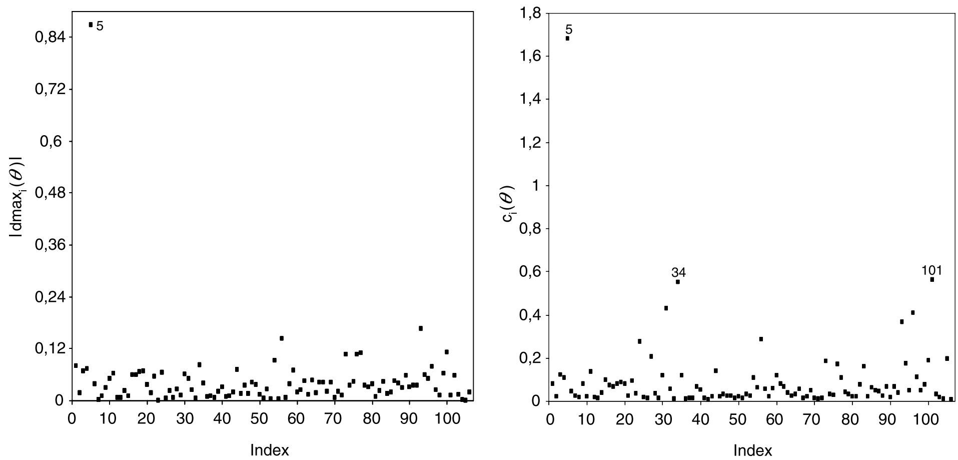

Figure 6 (a) *Index plot of* $|\mathbf{d}_{\max}|$ *for* $\boldsymbol{\theta}$ *on the Golden shiner data (case-weight perturbation).* (b) *Total local influence on estimates* $\boldsymbol{\theta}$ *in the Golden shiner data (case-weight perturbation).*

case-weight perturbation is used, and obtain the value of the maximum curvature $C_{\mathbf{d}_{\max}} = 2.136$. Figure 6(a) plots the eigenvector corresponding to $|\mathbf{d}_{\max}|$, whereas Figure 6(b) plots the total influence C_i versus the index, where we verify that the observations ♯5, ♯34 and ♯101 are again very distinguished related to the others.

The influence of perturbations on the observed survival times is now analyzed (response variable perturbation). The value of the maximum curvature is $C_{\mathbf{d}_{\max}} = 10.845$. Figure 7a plots $|\mathbf{d}_{\max}|$ versus the observation index and shows that the observation ♯5 is far way from the others. Figure 7b plots the total local influence (C_i), where the observation ♯5 again stand out. The index plot of $|\mathbf{d}_{\max}|$ as well as the total local influence C_i for the explanatory variable perturbations (x_2, x_3, x_4, x_5, x_6 and x_7), omitted here, also confirm the influence of the observations ♯5, ♯34 and ♯101. We perform the residual analysis by plotting in Figure 8a the deviance component residual r_{D_i} (see Section 5) against the index of observations. Figure 8b gives the normal probability plot with generated envelope. Figure 8a shows some large residuals (observations ♯5, ♯34 and ♯101), although Figure 8b supports the hypothesis that the LGMW model is very suitable for these data, since there are no observations falling outside the envelope.

6.1 Impact of the detected influential observations

We conclude that the diagnostic analysis (global influence and local influence) detected as potentially influential observations, the following three cases: ♯5, ♯34 and ♯101. The observations ♯5 and ♯101 are censored. The lifetime ♯5 is the highest in the sample, whereas ♯101 is the smallest for the uncensored observations. On the other hand, the observation ♯34 refers to the fish with smallest survival time. In order to reveal the impact of these three observations on the parameter estimates, we

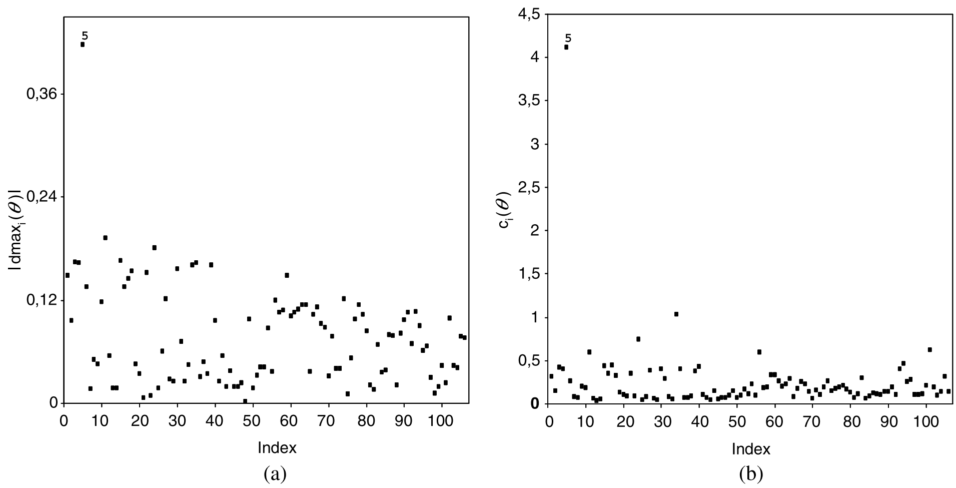

Figure 7 (a) *Index plot of* $|\mathbf{d}_{max}|$ *for* $\boldsymbol{\theta}$ *on the Golden shiner data (response perturbation).* (b) *Total local influence for* $\boldsymbol{\theta}$ *on the Golden shiner data (response perturbation).*

refitted the model under some situations. First, we individually eliminated each one of these three observations. Next, we removed from the set "A" (original dataset) the totality of potentially influential observations.

Table 2 gives the relative change (in percentage) of each estimate defined by $RC_{\boldsymbol{\theta}_j} = [(\hat{\boldsymbol{\theta}}_j - \hat{\boldsymbol{\theta}}_j(I))/\hat{\boldsymbol{\theta}}_j] \times 100$, and the corresponding p-value, where $\hat{\boldsymbol{\theta}}_j(I)$ is the MLE of $\boldsymbol{\theta}_j$ after the "set I" of observations being removed. Table 2 pro-

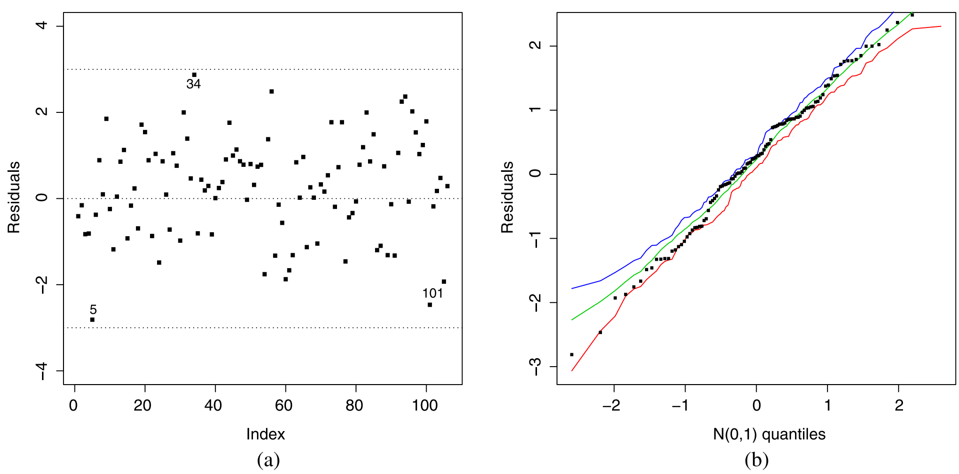

Figure 8 (a) *Index plot of the deviance component residual for the Golden shiner data.* (b) *Normal probability plot for the deviance component residual from the fitted LGMW regression model to the Golden shiner data.*

Table 2 *Relative changes [-RC- in %], estimates and their p-values (in parentheses) for the corresponding set*

Dropping	$\hat{\lambda}$	$\hat{\varphi}$	$\hat{\sigma}$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	$\hat{\beta}_5$	$\hat{\beta}_6$
	–	–	–	–	–	–	–	–	–	–
None	0.001	12.86	5.09	−1.89	2.20	0.10	−0.13	0.04	0.02	0.22
	(0.78)	(0.52)	(0.07)	(0.75)	(0.00)	(0.01)	(0.00)	(0.00)	(0.20)	(0.28)
	[217]	[56]	[27]	[−140]	[5]	[9]	[−5]	[15]	[89]	[22]
Set I_1	0.00	5.61	3.73	0.76	2.31	0.11	−0.13	0.04	0.00	0.27
	(0.52)	(0.37)	(0.03)	(0.83)	(0.00)	(0.00)	(0.00)	(0.00)	(0.88)	(0.18)
	[5]	[181]	[30]	[−162]	[0]	[6]	[−6]	[4]	[19]	[13]
Set I_2	0.00	36.09	6.64	−4.95	2.19	0.10	−0.13	0.03	0.02	0.19
	(0.70)	(0.69)	(0.14)	(0.63)	(0.00)	(0.00)	(0.00)	(0.00)	(0.32)	(0.33)
	[54]	[52]	[26]	[−150]	[8]	[14]	[−8]	[0]	[29]	[33]
Set I_3	0.00	6.13	3.78	0.95	2.37	0.11	−0.14	0.03	0.03	0.15
	(0.92)	(0.42)	(0.05)	(0.81)	(0.00)	(0.00)	(0.00)	(0.00)	(0.09)	(0.47)
	[177]	[25]	[13]	[−90]	[5]	[14]	[−11]	[11]	[95]	[11]
Set I_4	0.00	9.60	4.41	−0.19	2.31	0.11	−0.14	0.04	0.00	0.25
	(0.51)	(0.49)	(0.06)	(0.97)	(0.00)	(0.00)	(0.00)	(0.00)	(0.95)	(0.21)
	[266]	[72]	[41]	[−223]	[14]	[25]	[−14]	[16]	[62]	[18]
Set I_5	0.00	3.60	3.02	2.33	2.50	0.12	−0.14	0.04	0.01	0.18
	(0.54)	(0.29)	(0.02)	(0.39)	(0.00)	(0.00)	(0.00)	(0.00)	(0.60)	(0.37)
	[34]	[5]	[8]	[−72]	[7]	[18]	[−13]	[3]	[11]	[38]
Set I_6	0.00	12.15	4.68	−0.53	2.35	0.12	−0.14	0.03	0.02	0.14
	(0.86)	(0.56)	(0.09)	(0.93)	(0.00)	(0.00)	(0.00)	(0.00)	(0.15)	(0.48)
	[224]	[73]	[6]	[−196]	[13]	[28]	[−18]	[12]	[69]	[22]
Set I_7	0.00	3.46	5.39	1.82	2.49	0.12	−0.15	0.04	0.01	0.17
	(0.54)	(0.04)	(0.38)	(0.58)	(0.00)	(0.00)	(0.00)	(0.00)	(0.67)	(0.37)

vides the following sets: $I_1 = \{\sharp 5\}$, $I_2 = \{\sharp 34\}$, $I_3 = \{\sharp 101\}$, $I_4 = \{\sharp 5, \sharp 34\}$, $I_5 = \{\sharp 5, \sharp 101\}$, $I_6 = \{\sharp 34, \sharp 101\}$ and $I_7 = \{\sharp 5, \sharp 34, \sharp 101\}$.

The figures in Table 2 show that the estimates for the LGMW regression model are not highly sensitive under deletion of the outstanding observations. Few variations are only observed for the estimates of the parameters λ and β_0, but inferential changes are not observed. In general, the significance of the estimates does not change (at the 5% level) after removing the set I. Hence, we do not have inferential changes after removing the observations handed out in the diagnostic plots. The LR statistic for testing the null hypothesis $H_0 : (\beta_5, \beta_6)^T = (0, 0)^T$ versus $H_1 : H_0$ *is not true*, that is, to verify the joint contribution effects of the explanatory variables x_5 and x_6, is $w = 1.4$ (p-value $= 0.497$), and then we conclude that the parameters β_5 and β_6 are not jointly significant for the model. Based on this analysis, we conclude that the LGMW regression model is more appropriate for fitting these data leading to the final equation

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \sigma z_i, \qquad i = 1, \ldots, 106, \quad (6.2)$$

where the estimates (approximate standard errors and p-values in parentheses) of the parameters are: $\hat{\lambda} = 0.001$ (0.003), $\hat{\varphi} = 35.910$ (73.936), $\hat{\sigma} = 7.043$ (4.039), $\hat{\beta}_0 = -6.318$ (9.322) (0.497), $\hat{\beta}_1 = 2.356$ (0.543) (<0.001), $\hat{\beta}_2 = 0.072$ (0.034) (0.037), $\hat{\beta}_3 = -0.117$ (0.033) (0.0004) and $\hat{\beta}_4 = 0.034$ (0.009) (0.0002).

Finally, the expected survival time should decrease (approximately) 11% ($[1 - e^{-0.117}] \times 100\%$) when the size of the fish measurement increases one unity, all the others variables being fixed.

7 Concluding remarks

We introduce the so-called log-generalized modified Weibull (LGMW) distribution whose hazard rate function accommodates four types of shape forms, namely increasing, decreasing, bathtub and unimodal. We derive an expansion for its moments. Based on this new distribution, we propose a LGMW regression model very suitable for modeling censored and uncensored lifetime data. The new regression model permits testing the goodness of fit of some known regression models as special submodels. Hence, the proposed regression model serves as a good alternative for lifetime data analysis. Further, the new regression model is much more flexible than the exponentiated Weibull, modified Weibull and generalized Rayleigh submodels. We use the matrix programming language Ox (MaxBFGS function) to obtain the maximum likelihood estimates and perform asymptotic tests for the parameters based on the asymptotic distribution of these estimates. We examine a simulation study. We discuss influence diagnostics and model checking analysis in the LGMW regression models fitted to censored data. We also discuss the sensitivity of the maximum likelihood estimates from the fitted model via deviance component residuals and sensitivity analysis. We demonstrate in one application to real data that the LGMW model can produce better fit than its submodels.

Appendix A: Matrix of second derivatives $-\ddot{\mathbf{L}}(\boldsymbol{\theta})$

Here we give the necessary formulas to obtain the second-order partial derivatives of the log-likelihood function. After some algebraic manipulations, we obtain

$$\begin{aligned}
\mathbf{L}_{\lambda\lambda} = {} & -[(\dot{u}_i)_\lambda]^2\Big[\sum_{i\in F}(\sigma^{-1}+u_i)^{-2}+\sum_{i\in F}v_i\Big] \\
& + \sum_{i\in F}(\varphi-1)\{[(\ddot{h}_i)_{\lambda\lambda}]h_i^{-1}-[(\dot{h}_i)_\lambda]^2h_i^{-2}\} \\
& - \sum_{i\in C}\varphi\left(\frac{h_i^{\varphi-1}}{1-h_i^{\varphi}}\right)\{(1-h_i^{\varphi})^{-1}[(\dot{h}_i)_\lambda]^2 \\
& \qquad \times [(\varphi-1)h_i^{-1}(1-h_i^{\varphi})+\varphi h_i^{\varphi-1}]+[(\ddot{h}_i)_{\lambda\lambda}]\},
\end{aligned}$$

$$\mathbf{L}_{\lambda\varphi} = \sum_{i\in F} h_i^{-1}[(\dot{h}_i)_\lambda] - \sum_{i\in C}[(\dot{h}_i)_\lambda]\left(\frac{h_i^{\varphi-1}}{1-h_i^{\varphi}}\right)[1+\varphi\log(h_i)(1-h_i^{\varphi})^{-1}],$$

$$\mathbf{L}_{\lambda\sigma} = \sum_{i\in F}\sigma^{-2}(\sigma^{-1}+u_i)^{-2}\exp(y_i) + \sum_{i\in F}\sigma^{-1}z_i\exp(y_i)v_i$$
$$+\sum_{i\in F}(\varphi-1)h_i^{-2}\{[(\ddot{h}_i)_{\lambda\sigma}]h_i - [(\dot{h}_i)_\lambda][(\dot{h}_i)_\sigma]\}$$
$$-\sum_{i\in C}\varphi\left(\frac{h_i^{\varphi-1}}{1-h_i^{\varphi}}\right)$$
$$\times\{[(\dot{h}_i)_\sigma][(\dot{h}_i)_\lambda](1-h_i^{\varphi})^{-1}(\varphi-1)h_i^{-1}(1-h_i^{\varphi}) - [(\ddot{h}_i)_{\lambda\sigma}]\},$$

$$\mathbf{L}_{\lambda\beta_j} = \sum_{i\in F}\sigma^{-1}x_{ij}\exp(y_i)v_i + \sum_{i\in F}(\varphi-1)h_i^{-2}\{[(\ddot{h}_i)_{\lambda\beta_j}]h_i - [(\dot{h}_i)_\lambda][(\dot{h}_i)_{\beta_j}]\}$$
$$-\sum_{i\in C}\varphi\left(\frac{h_i^{\varphi-1}}{1-h_i^{\varphi}}\right)$$
$$\times\{[(\dot{h}_i)_{\beta_j}][(\dot{h}_i)_\lambda](1+h_i^{\varphi})^{-2}[(\varphi-1)h_i^{-1}(1-h_i^{\varphi})+\varphi h_i^{\varphi-1}]\}$$
$$-\sum_{i\in C}\varphi\left(\frac{h_i^{\varphi-1}}{1-h_i^{\varphi}}\right)[(\ddot{h}_i)_{\lambda\beta_j}],$$

$$\mathbf{L}_{\varphi\varphi} = -r\varphi^{-2} - \sum_{i\in C}h_i^{\varphi}[\log(h_i)]^2(1-h_i^{\varphi})^{-2},$$

$$\mathbf{L}_{\varphi\sigma} = \sum_{i\in F}h_i^{-1}[(\dot{h}_i)_\sigma]$$
$$-\sum_{i\in C}[(\dot{h}_i)_\sigma]\left(\frac{h_i^{\varphi-1}}{1-h_i^{\varphi}}\right)$$
$$\times\{(1-h_i^{\varphi})^{-1}\log(h_i)[(\varphi-1)h_i^{-1}(1-h_i^{\varphi})+\varphi h_i^{\varphi-1}]+1\},$$

$$\mathbf{L}_{\varphi\beta_j} = \sum_{i\in F}h_i^{-1}[(\dot{h}_i)_{\beta_j}] - \sum_{i\in C}[(\dot{h}_i)_{\beta_j}]\left(\frac{h_i^{\varphi-1}}{1-h_i^{\varphi}}\right)$$
$$\times\{\log(h_i)[(\varphi-1)h_i^{-1}+\varphi h_i^{\varphi-1}(1-h_i^{\varphi})^{-1}]+1\},$$

$$\mathbf{L}_{\sigma\sigma} = \sum_{i\in F}\sigma^{-3}(\sigma^{-1}+u_i)^{-1}[2+\sigma^{-1}(\sigma^{-1}+u_i)^{-1}]$$
$$+\sum_{i\in F}\sigma^{-2}z_i[2(1-v_i)+z_iv_i]$$
$$+\sum_{i\in F}(\varphi-1)h_i^{-1}\{-[(\dot{h}_i)_\sigma]^2h_i^{-1}+[(\ddot{h}_i)_{\sigma\sigma}]\}$$

$$+\sum_{i\in C}\varphi h_i^{-1}[(\dot{h}_i)_\sigma]^2\{(1-h_i^{\varphi})^{-2}[(\varphi-1)h_i^{-1}(1-h_i^{\varphi})+\varphi h_i^{\varphi-1}]\}$$

$$+\sum_{i\in C}\varphi\left(\frac{h_i^{\varphi-1}}{1-h_i^{\varphi}}\right)[(\ddot{h}_i)_{\sigma\sigma}],$$

$$\mathbf{L}_{\sigma\beta_j}=-\sum_{i\in F}\sigma^{-2}x_{ij}[(1+z_i)v_i-1]$$

$$+\sum_{i\in F}(\varphi-1)h_i^{-2}\{[(\ddot{h}_i)_{\beta_j\sigma}]h_i-[(\dot{h}_i)_{\beta_j}][(\dot{h}_i)_\sigma]\}$$

$$-\sum_{i\in C}h_i^{\varphi-1}[(\dot{h}_i)_{\beta_j}][(\dot{h}_i)_\sigma](1-h_i^{\varphi})^{-2}[(\varphi-1)h_i^{-1}(1-h_i^{\varphi})+\varphi h_i^{\varphi-1}]$$

$$-\sum_{i\in C}\left(\frac{h_i^{\varphi-1}}{1-h_i^{\varphi}}\right)[(\ddot{h}_i)_{\beta_j\sigma}]$$

and

$$\mathbf{L}_{\beta_j\beta_s}=-\sum_{i\in F}\sigma^{-1}x_{ij}x_{is}v_i+\sum_{i\in F}(\varphi-1)h_i^{-2}\{[(\ddot{h}_i)_{\beta_j\beta_s}]h_i-[(\dot{h}_i)_{\beta_j}][(\dot{h}_i)_{\beta_s}]\}$$

$$-\sum_{i\in C}\varphi\left(\frac{h_i^{\varphi-1}}{1-h_i^{\varphi}}\right)\{[(\dot{h}_i)_{\beta_j}][(\dot{h}_i)_{\beta_s}][(\varphi-1)h_i^{-1}+\varphi h_i^{\varphi-1}(1-h_i^{\varphi})^{-1}]$$

$$+[(\ddot{h}_i)_{\beta_j\beta_s}]\},$$

where $z_i=(y_i-\mathbf{x}_i^T\boldsymbol{\beta})/\sigma$, $g_i=\exp(z_i+y_i)$, $v_i=\exp(z_i+u_i)$, $u_i=\lambda\exp(y_i)$, $h_i=1-\exp(-v_i)$, $(\dot{z}_i)_\sigma=-\sigma^{-1}z_i$, $(\dot{z}_i)_{\beta_j}=-\sigma^{-1}x_{ij}$, $(\dot{z}_i)_{\beta_s}=-\sigma^{-1}x_{is}$, $(\ddot{z}_i)_{\sigma\sigma}=-\sigma^{-2}\{[(\dot{z}_i)_\sigma]\sigma-z_i\}$, $(\ddot{z}_i)_{\sigma\beta_j}=\sigma^{-2}x_{ij}$, $(\dot{u}_i)_\lambda=\exp(y_i)$, $(\ddot{u}_i)_{\lambda\lambda}=0$, $(\dot{h}_i)_\lambda=\exp(y_i)g_i\exp(-v_i)$, $(\ddot{h}_i)_{\lambda\lambda}=\exp(2y_i)v_i\exp(-v_i)(1-v_i)$, $(\dot{h}_i)_\sigma=[(\dot{z}_i)_\sigma]g_i\exp(-v_i)$, $(\ddot{h}_i)_{\sigma\sigma}=v_i\exp(-v_i)\{[(\dot{z}_i)_\sigma]^2(1-v_i)+[(\ddot{z}_i)_{\sigma\sigma}]\}$, $(\dot{h}_i)_{\beta_j}=-\sigma^{-1}x_{ij}g_i\exp(-v_i)$, $(\dot{h}_i)_{\beta_s}=-\sigma^{-1}x_{is}g_i\exp(-v_i)$, $(\ddot{h}_i)_{\beta_j\beta_s}=\sigma^{-2}x_{ij}x_{is}v_i\times\exp(-v_i)(1-v_i)$, $(\ddot{h}_i)_{\lambda\sigma}=-\sigma^{-1}z_i\exp(y_i)v_i\exp(-v_i)(1-v_i)$, $(\ddot{h}_i)_{\lambda\beta_j}=-\sigma^{-1}x_{ij}\exp(y_i)v_i\exp(-v_i)(1-v_i)$ and $(\ddot{h}_i)_{\sigma\beta_j}=\sigma^{-2}x_{ij}v_i\exp(-v_i)[1+z_i(1-v_i)]$.

Appendix B: Case-weight perturbation scheme

The elements of the matrix $\boldsymbol{\Delta}=(\boldsymbol{\Delta}_\lambda^T,\boldsymbol{\Delta}_\varphi^T,\boldsymbol{\Delta}_\sigma^T,\boldsymbol{\Delta}_{\boldsymbol{\beta}}^T)^T$ for the case-weight perturbation scheme are expressed as

$$\Delta_{\lambda i}=\begin{cases}\exp(y_i)[(\hat{\sigma}^{-1}+\hat{u}_i)^{-1}+1-\hat{v}_i+(\hat{\varphi}-1)\hat{h}_i^{-1}], & \text{if } i\in F,\\ -\hat{\varphi}\hat{h}_i^{\hat{\varphi}-1}(1-\hat{h}_i^{\hat{\varphi}})^{-1}[(\dot{h}_i)_\lambda], & \text{if } i\in C.\end{cases}$$

$$\Delta_{\varphi i} = \begin{cases} \hat{\varphi}^{-1} + \log(\hat{h}_i), & \text{if } i \in F \\ -(1-\hat{h}_i^{\hat{\varphi}})^{-1}\hat{h}_i^{\hat{\varphi}}\log(\hat{h}_i), & \text{if } i \in C. \end{cases}$$

$$\Delta_{\sigma i} = \begin{cases} \hat{\sigma}^{-2}(\hat{\sigma}^{-1}+\hat{u}_i)^{-1} - \hat{\sigma}^{-1}\hat{z}_i(1-\hat{v}_i) + (\hat{\varphi}-1)\hat{h}_i^{-1}[(\dot{h}_i)_\sigma], & \text{if } i \in F, \\ -\hat{\varphi}(1-\hat{h}_i^{\hat{\varphi}})^{-1}\hat{h}_i^{\hat{\varphi}-1}[(\dot{h}_i)_\sigma], & \text{if } i \in C. \end{cases}$$

$$\Delta_{\boldsymbol{\beta} ji} = \begin{cases} -\hat{\sigma}^{-1}x_{ij}(1-\hat{v}_i) + (\hat{\varphi}-1)\hat{h}_i^{-1}[(\dot{h}_i)_{\beta_j}], & \text{if } i \in F, \\ -\hat{\varphi}(1-\hat{h}_i^{\hat{\varphi}})^{-1}\hat{h}_i^{\hat{\varphi}-1}[(\dot{h}_i)_{\beta_j}], & \text{if } i \in C, \end{cases}$$

where $\hat{z}_i = (y_i - \mathbf{x}_i^T\hat{\boldsymbol{\beta}})/\hat{\sigma}$, $\hat{u}_i = \hat{\lambda}\exp(y_i)$, $\hat{h}_i = 1-\exp(-\hat{v}_i)$, $\hat{g}_i = \exp(\hat{z}_i + y_i)$, $\hat{v}_i = \exp(\hat{z}_i + \hat{u}_i)$, $(\dot{h}_i)_\lambda = \exp(y_i)\hat{g}_i\exp(-\hat{v}_i)$, $(\dot{h}_i)_\sigma = -\hat{\sigma}\hat{z}_i\hat{g}_i\exp(-\hat{v}_i)$ and $(\dot{h}_i)_{\beta_j} = -\hat{\sigma}^{-1}x_{ij}\hat{g}_i\exp(-\hat{v}_i)$.

Appendix C: Response perturbation scheme

The elements of the matrix $\boldsymbol{\Delta} = (\boldsymbol{\Delta}_\lambda^T, \boldsymbol{\Delta}_\varphi^T, \boldsymbol{\Delta}_\sigma^T, \boldsymbol{\Delta}_{\boldsymbol{\beta}}^T)^T$ for the response variable perturbation scheme are expressed as

$$\Delta_{\lambda i} = \begin{cases} [(\ddot{u}_i^*)_{\omega_i\lambda}][\hat{\sigma}^{-1} + \hat{u}_i + 1 - \hat{v}_i - (\hat{\varphi}-1)\hat{h}_i] \\ \quad - \hat{\varphi}[(\dot{u}_i^*)_{\omega_i}][(\dot{u}_i^*)_\lambda] \\ -\hat{v}_i[(\dot{u}_i^*)_\lambda]\{[(\dot{z}_i^*)_{\omega_i}][(\dot{u}_i^*)_{\omega_i}]\}, & \text{if } i \in F, \\ -\hat{\varphi}\hat{h}_i^{\hat{\varphi}-1}\{[(\dot{h}_i^*)_{\omega_i}][(\dot{h}_i^*)_\lambda][(\hat{\varphi}-1)\hat{h}_i^{-1}(1-\hat{h}_i^{\hat{\varphi}}) + \hat{\varphi}\hat{h}_i^{\hat{\varphi}-1}] \\ \qquad + (1-\hat{h}_i^{\hat{\varphi}})[(\ddot{h}_i^*)_{\omega_i\lambda}]\}, & \text{if } i \in C. \end{cases}$$

$$\Delta_{\varphi i} = \begin{cases} \hat{h}_i^{-1}[(\dot{h}_i^*)_{\omega_i}], & \text{if } i \in F, \\ -\hat{h}_i^{\hat{\varphi}-1}\log(\hat{h}_i)[(\dot{h}_i^i)_{\omega_i}]\{(\hat{\varphi}-1)\hat{h}_i^{-1}(1-\hat{h}_i^{\hat{\varphi}}) + \hat{\varphi}\hat{h}_i^{\hat{\varphi}-1}\} \\ \quad - \hat{h}_i^{\hat{\varphi}-1}(1-\hat{h}_i^{\hat{\varphi}})^{-1}[(\dot{h}_i^*)_{\omega_i}], & \text{if } i \in C. \end{cases}$$

$$\Delta_{\sigma i} = \begin{cases} \hat{\sigma}^{-2}(\hat{\sigma}^{-1}+\hat{u}_i)^{-2}[(\dot{u}_i^*)_{\omega_i}] + (1-\hat{v}_i)[(\ddot{z}_i^*)_{\omega_i\sigma}] \\ \quad -\hat{v}_i[(\dot{z}_i^*)_\sigma]\{[(\dot{z}_i^*)_{\omega_i}] + [(\dot{u}_i^*)_{\omega_i}]\} \\ \quad +(\hat{\varphi}-1)\hat{h}_i^{-2}\{[(\ddot{h}_i^*)_{\omega_i\sigma}]\hat{h}_i - [(\dot{h}_i^*)_{\omega_i}][(\dot{h}_i^*)_\sigma]\}, & \text{if } i \in F, \\ -\hat{\varphi}\hat{h}_i^{\hat{\varphi}-1}\{[(\dot{h}_i^*)_{\omega_i}][(\dot{h}_i^*)_\sigma][(\hat{\varphi}-1)\hat{h}_i^{-1}(1-\hat{h}_i^{\hat{\varphi}}) + \hat{\varphi}h_i^{\hat{\varphi}-1}] \\ \quad + (1-\hat{h}_i^{\hat{\varphi}})^{-1}[(\ddot{h}_i^*)_{\omega_i\sigma}]\}, & \text{if } i \in C. \end{cases}$$

$$\Delta_{\boldsymbol{\beta} ji} = \begin{cases} -\hat{v}_i[(\dot{z}_i^*)_{\beta_j}]\{[(\dot{z}_i^*)_{\omega_i}] + [(\dot{z}_i^*)_{\omega_i}]\} \\ \quad + (\hat{\varphi}-1)\hat{h}_i^{-2}\{[(\ddot{h}_i^*)_{\omega_i\beta_j}]\hat{h}_i - [(\dot{h}_i^*)_{\omega_i}][(\dot{h}_i^*)_{\beta_j}]\}, & \text{if } i \in F, \\ -\hat{\varphi}h_i^{\hat{\varphi}-1}\{[(\dot{h}_i^*)_{\omega_i}][(\dot{h}_i^*)_{\beta_j}][(\hat{\varphi}-1)\hat{h}_i^{-1}(1-\hat{h}_i^{\hat{\varphi}}) + \hat{\varphi}h_i^{\hat{\varphi}-1}] \\ \qquad + (1-h_i^{\hat{\varphi}})^{-1}[(\ddot{h}_i^*)_{\omega_i\beta_j}]\}, & \text{if } i \in C, \end{cases}$$

where $\hat{z}_i = (y_i - \mathbf{x}_i^T\hat{\boldsymbol{\beta}})/\hat{\sigma}$, $\hat{u}_i = \hat{\lambda}\exp(y_i)$, $\hat{h}_i = 1-\exp(-\hat{v}_i)$, $\hat{g}_i = \exp(\hat{z}_i + y_i)$, $\hat{v}_i = \exp(\hat{z}_i + \hat{u}_i)$, $(\dot{z}_i^*)_\sigma = -\hat{\sigma}^{-1}\hat{z}_i$, $(\dot{z}_i^*)_{\beta_j} = -\hat{\sigma}^{-1}x_{ij}$, $(\dot{z}_i^*)_{\omega_i} = \hat{\sigma}^{-1}S_x$,

$(\dot{u}_i^*)_\lambda = \exp(y_i)$, $(\dot{u}_i^*)_{\omega_i} = \hat{\lambda} S_x \exp(y_i)$, $(\dot{h}_i^*)_\lambda = \exp(y_i)\hat{g}_i \exp(-\hat{v}_i)$, $(\dot{h}_i^*)_\sigma = -\hat{\sigma}^{-1}\hat{z}_i \hat{g}_i \exp(-\hat{v}_i)$, $(\dot{h}_i^*)_{\beta_j} = -\hat{\sigma}^{-1} x_{ij} \hat{g}_i \exp(-\hat{v}_i)$, $(\dot{h}_i^*)_{\omega_i} = S_x \hat{g}_i \exp\{-\hat{v}_i \times [\hat{\sigma}^{-1} + \hat{\lambda}\exp(y_i)]\}$, $(\ddot{z}_i^*)_{\omega_i\sigma} = -S_x\hat{\sigma}^{-2}$, $(\ddot{z}_i^*)_{\omega_i\beta_j} = 0$, $(\ddot{u}_i^*)_{\omega_i\lambda} = S_x \exp(y_i)$, $(\ddot{h}_i^*)_{\omega_i\lambda} = \hat{g}_i \exp(-\hat{v}_i)\{(1 - \hat{v}_i)([(\dot{z}_i^*)_{\omega_i}] + [(\dot{u}_i^*)_{\omega_i}]) + S_x\}$, $(\ddot{h}_i^*)_{\omega_i\sigma} = \hat{g}_i \times \exp(-\hat{v}_i)\{[(\dot{z}_i^*)_\sigma]([(\dot{z}_i^*)_{\omega_i}] + [(\dot{u}_i^*)_{\omega_i}])(1 - \hat{v}_i) + [(\ddot{z}_i^*)_{\omega_i\sigma}]\}$ and $(\ddot{h}_i^*)_{\omega_i\beta_j} = \hat{g}_i \times \exp(-\hat{v}_i)\{[(\dot{z}_i^*)_{\beta_j}]([(\dot{z}_i^*)_{\omega_i}] + [(\dot{u}_i^*)_{\omega_i}])(1 - \hat{v}_i) + [(\ddot{z}_i^*)_{\omega_i\beta_j}]\}$.

Appendix D: Explanatory variable perturbation scheme

The elements of the matrix $\boldsymbol{\Delta} = (\boldsymbol{\Delta}_\lambda^T, \boldsymbol{\Delta}_\varphi^T, \boldsymbol{\Delta}_\sigma^T, \boldsymbol{\Delta}_{\boldsymbol{\beta}}^T)^T$ are expressed as

$$\Delta_{\lambda i} = \begin{cases} -\hat{v}_i[(\dot{u}_i^*)_\lambda][(\dot{z}_i^*)_{\omega_i}] + (\hat{\varphi}-1)\hat{h}_i^{-2} \\ \quad \times \{[(\ddot{h}_i^*)_{\omega_i\lambda}]\hat{h}_i - [(\dot{h}_i^*)_{\omega_i}][(\dot{h}_i^*)_\lambda]\}, & \text{if } i \in F, \\ -\hat{\varphi} h_i^{\hat{\varphi}-1}\{[(\dot{h}_i^*)_{\omega_i}][(\dot{h}_i^*)_\lambda][(\hat{\varphi}-1)\hat{h}_i^{-1}(1-\hat{h}_i^{\hat{\varphi}}) + \hat{\varphi}\hat{h}_i^{\hat{\varphi}-1}] \\ \qquad + (1-\hat{h}_i^{\hat{\varphi}})^{-1}[(\ddot{h}_i^*)_{\omega_i\lambda}]\}, & \text{if } i \in C. \end{cases}$$

$$\Delta_{\varphi i} = \begin{cases} \hat{h}_i^{-1}[(\dot{h}_i^*)_{\omega_i}], & \text{if } i \in F, \\ \hat{h}_i^{\hat{\varphi}-1}\log(\hat{h}_i)[(\dot{h}_i^*)_{\omega_i}]\{(\hat{\varphi}-1)\hat{h}_i^{-1}(1-\hat{h}_i^{\hat{\varphi}}) + \hat{\varphi}\hat{h}_i^{\hat{\varphi}-1}\} \\ \quad + \hat{h}_i^{\hat{\varphi}-1}(1-\hat{h}_i^{\hat{\varphi}})^{-1}[(\dot{h}_i^*)_{\omega_i}], & \text{if } i \in C. \end{cases}$$

$$\Delta_{\sigma i} = \begin{cases} [(\ddot{z}_i^*)_{\omega_i\sigma}](1-\hat{v}_i) - \hat{v}_i[(\dot{z}_i^*)_{\omega_i}][(\dot{z}_i^*)_\sigma] \\ \quad + (\hat{\varphi}-1)\hat{h}_i^{-2}\{[(\ddot{h}_i^*)_{\omega_i\sigma}]\hat{h}_i - [(\dot{h}_i^*)_{\omega_i}][(\dot{h}_i^*)_\sigma]\}, & \text{if } i \in F, \\ -\hat{\varphi}\hat{h}_i^{\hat{\varphi}-1}\{[(\dot{h}_i^*)_{\omega_i}][(\dot{h}_i^*)_\sigma][(\hat{\varphi}-1)\hat{h}_i^{-1}(1-\hat{h}_i^{\hat{\varphi}}) + \hat{\varphi}\hat{h}_i^{\hat{\varphi}-1}] \\ \qquad + (1-\hat{h}_i^{\hat{\varphi}})^{-1}[(\ddot{h}_i^*)_{\omega_i\sigma}]\}, & \text{if } i \in C. \end{cases}$$

For $j \neq q$, the elements take the forms

$$\Delta_{\boldsymbol{\beta} ji} = \begin{cases} [(\ddot{z}_i^*)_{\omega_i\beta_j}](1-\hat{v}_i) - \hat{v}_i[(\dot{z}_i^*)_{\omega_i}][(\dot{z}_i^*)_{\beta_j}] \\ \quad + (\hat{\varphi}-1)\hat{h}_i^{-2}\{[(\ddot{h}_i^*)_{\omega_i\beta_j}]\hat{h}_i - [(\dot{h}_i^*)_{\omega_i}][(\dot{h}_i^*)_{\beta_j}]\}, & \text{if } i \in F, \\ -\hat{\varphi}\hat{h}_i^{\hat{\varphi}-1}\{[(\dot{h}_i^*)_{\omega_i}][(\dot{h}_i^*)_{\beta_j}][(\hat{\varphi}-1)\hat{h}_i^{-1}(1-\hat{h}_i^{\hat{\varphi}}) + \hat{\varphi}\hat{h}_i^{\hat{\varphi}-1}] \\ \qquad + (1-\hat{h}_i^{\hat{\varphi}})^{-1}[(\ddot{h}_i^*)_{\omega_i\beta_j}]\}, & \text{if } i \in C \end{cases}$$

and for $j = q$, the elements take the forms

$$\Delta_{\boldsymbol{\beta} qi} = \begin{cases} [(\ddot{z}_i^*)_{\omega_i\beta_q}](1-\hat{v}_i) - \hat{v}_i[(\dot{z}_i^*)_{\omega_i}][(\dot{z}_i^*)_{\beta_q}] \\ \quad +(\hat{\varphi}-1)\hat{h}_i^{-2}\{[(\ddot{h}_i^*)_{\omega_i\beta_q}]\hat{h}_i - [(\dot{h}_i^*)_{\omega_i}][(\dot{h}_i^*)_{\beta_q}]\}, & \text{if } i \in F, \\ -\hat{\varphi}\hat{h}_i^{\hat{\varphi}-1}\{[(\dot{h}_i^*)_{\omega_i}][(\dot{h}_i^*)_{\beta_q}][(\hat{\varphi}-1)\hat{h}_i^{-1}(1-\hat{h}_i^{\hat{\varphi}}) + \hat{\varphi}\hat{h}_i^{\hat{\varphi}-1}] \\ \qquad + (1-\hat{h}_i^{\hat{\varphi}})^{-1}[(\ddot{h}_i^*)_{\omega_i\beta_q}]\}, & \text{if } i \in C, \end{cases}$$

where $\hat{z}_i = (y_i - \mathbf{x}_i^T\hat{\boldsymbol{\beta}})/\hat{\sigma}$, $\hat{u}_i = \hat{\lambda}\exp(y_i)$, $\hat{h}_i = 1 - \exp(-\hat{v}_i)$, $\hat{g}_i = \exp(\hat{z}_i + y_i)$, $\hat{v}_i = \exp(\hat{z}_i + \hat{u}_i)$, $(\dot{z}_i^*)_\sigma = -\hat{\sigma}^{-1}\hat{z}_i$, $(\dot{z}_i^*)_{\beta_j} = -\sigma^{-1}x_{ij}, \forall(j \neq q)$, $(\dot{z}_i^*)_{\beta_q} =$

$-\hat{\sigma}^{-1}x_{it}, \forall(j = q)$, $(\dot{z}_i^*)_{\omega_i} = -\hat{\sigma}^{-1}S_q\beta_q$, $(\dot{u}_i^*)_\lambda = \exp(y_i)$, $(\dot{u}_i^*)_{\omega_i} = 0$, $(\dot{h}_i^*)_\lambda = \exp(y_i)\hat{g}_i\exp(-\hat{v}_i)$, $(\dot{h}_i^*)_\sigma = -\hat{\sigma}^{-1}\hat{z}_i\hat{g}_i\exp(-\hat{v}_i)$, $(\dot{h}_i^*)_{\beta_j} = -\hat{\sigma}^{-1}x_{ij}\hat{g}_i\exp(-\hat{v}_i)$, $\forall(j \neq q)$, $(\dot{h}_i^*)_{\beta_q} = -\hat{\sigma}^{-1}x_{it}\hat{g}_i\exp(-\hat{v}_i), \forall(j = q)$, $(\dot{h}_i^*)_{\omega_i} = -\hat{\sigma}^{-1}S_q\hat{\beta}_q\hat{g}_i \times \exp(-\hat{v}_i)$, $(\ddot{z}_i^*)_{\omega_i\sigma} = \hat{\sigma}^{-2}S_q\hat{\beta}_q$, $(\ddot{z}_i^*)_{\omega_i\beta_j} = 0, \forall(j \neq q)$, $(\ddot{z}_i^*)_{\omega_i\beta_q} = -\hat{\sigma}^{-1}S_q$, $\forall(j = q)$, $(\ddot{u}_i^*)_{\omega_i\lambda} = 0$, $(\ddot{h}_i^*)_{\omega_i\lambda} = -\hat{\sigma}^{-1}S_q\hat{\beta}_q\exp(y_i)\hat{g}_i\exp(-\hat{v}_i)(1 - \hat{v}_i)$, $(\ddot{h}_i^*)_{\omega_i\sigma} = \hat{g}_i\exp(-\hat{v}_i)\{[(\dot{z}_i^*)_\sigma]([(\dot{z}_i^*)_{\omega_i}] + [(\dot{u}_i^*)_{\omega_i}])(1 - \hat{v}_i) + [(\ddot{z}_i^*)_{\omega_i\sigma}]\}$, $(\ddot{h}_i^*)_{\omega_i\beta_j} = \hat{g}_i\exp(-\hat{v}_i)[(\dot{z}_i^*)_{\beta_q}] \times [(\dot{h}_i^*)_{\omega_i}](1 - \hat{v}_i), \forall(j \neq q)$, $(\ddot{h}_i^*)_{\omega_i\beta_q} = \hat{g}_i \times \exp(-\hat{v}_i)\{[(\dot{z}_i^*)_{\beta_q}][(\dot{h}_i^*)_{\omega_i}](1 - \hat{v}_i) + [(\ddot{h}_i^*)_{\omega_i\beta_q}]\}, \forall(j = q)$, $i = 1, \dots, n$ and $j = 1, \dots, p$.

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Edwin M. M. Ortega
Departamento de Ciências Exatas
ESALQ—USP

Gauss M. Cordeiro
Departamento de Estatística e Informática
DEINFO
Universidade Federal Rural de Pernambuco (UFRPE)
Recife, Pernambuco
Brazil
E-mail: gausscordeiro@uol.com.br

Jalmar M. F. Carrasco
Departamento de Estatística
IME—USP