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Does reference prior alleviate the incidental parameter problem?

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Abstract. In this note we deal with a panel data model with fixed effects. We show that a Bayesian procedure based on the reference prior suffers from the incidental parameter problem, as also happens with the Jeffrey's prior [*Econom. Lett.* **82** (2004) 135–138]. Using an alternative prior distribution we present a solution to the problem.

1 Introduction

Some models for panel data include incidental parameters associated to fixed effects, leading to a lack of robustness in the sense discussed in Hahn (2004) and references therein. In fact, Hahn (2004) concluded that in a Bayesian setup under Jeffrey's prior, the resulting estimator for the parameter of interest suffers from the same problem of the maximum likelihood estimator (MLE). We tackle the incidental parameter problem, but instead of the Jeffrey's prior, we adopt another kind of noninformative priors; namely, the so-called reference prior (briefly described below) and a prior motivated by the work of Li and Leon-Gonzalez (2009). For the sake of space, the algebraic details are omitted.

Reference analysis, introduced by Bernardo (1979) and further developed by Berger and Bernardo (1989, 1992a, 1992b, 1992c) and Bernardo (2005) seems to be the only available method to achieve posterior distributions which produces objective Bayesian inference, meaning that inferential statements depend only on the postulated model and the available data. Moreover, there is the requirement that the prior distribution is minimally informative in a precise information-theoretic sense. Here, the driving idea is to maximize the expected Kullback–Leibler divergence of the posterior distribution with respect to the prior. Starting from a reference prior, the reference posterior is a consequence of a formal application of the Bayes theorem. Reference analysis provides posterior distributions with some nice properties, such as generality and invariance.

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2 Model and estimation

As in Hahn (2004), we consider a simple normal ($\mathcal{N}$) model defined as

$$x_{it} \sim \mathcal{N}(\alpha_i, \theta) \text{ independent}, \qquad i = 1, \dots, N, t = 1, \dots, T, \tag{2.1}$$

where α_i is a fixed effect and θ is the common variance (here, the parameter of interest). Hahn (2004) showed that for this model the Jeffrey's prior is proportional to $\theta^{-(N+2)/2}$. Letting $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N)$, $\mathbf{x} = (x_{11}, \dots, x_{NT})$, and $\overline{x}_i = \sum_{t=1}^{T} x_{it}/T$, and taking into account that $\sum_{i=1}^{N} \sum_{t=1}^{T} (x_{it} - \overline{x}_i)(\alpha_i - \overline{x}_i) = 0$, it can be shown that the posterior distribution $p(\theta, \boldsymbol{\alpha} \mid \mathbf{x})$ is proportional to

$$\theta^{-(N(T+1)+2)/2} \exp\left\{-\frac{1}{2\theta} \sum_{i=1}^{N} \left[\sum_{t=1}^{T} (x_{it} - \overline{x}_i)^2 + T(\alpha_i - \overline{x}_i)^2\right]\right\},$$

that can be written as

$$p(\theta, \boldsymbol{\alpha} \mid \mathbf{x}) \propto \theta^{-(NT+2)/2} \exp\left(-\frac{S_{xx}}{2\theta}\right) \prod_{i=1}^{N} \frac{1}{(\theta/T)^{1/2}} \exp\left\{-\frac{1}{2\theta/T}(\alpha_i - \overline{x}_i)^2\right\},$$

where $S_{xx} = \sum_{i=1}^{N} \sum_{t=1}^{T} (x_{it} - \overline{x}_i)^2$. By integrating out $\boldsymbol{\alpha}$ we arrive at

$$p(\theta \mid \mathbf{x}) \propto \theta^{-(NT+2)/2} \exp\left(-\frac{S_{xx}}{2\theta}\right),$$

so that the posterior distribution for θ can be promptly recognized as an inverted gamma distribution with parameters $\beta_1 = NT/2$ and $\beta_2 = S_{xx}/2$. Hence, the posterior mode for θ comes out to be $\beta_2/(\beta_1 + 1)$, that is, an estimator for θ is given by

$$\hat{\theta}_J = \frac{S_{xx}}{NT + 2}.$$

Holding T fixed and taking $N \to \infty$ we get $\operatorname{plim}_{N\to\infty}(\hat{\theta}_J - \theta) = \theta/T = O(1/T)$, as it happens with the MLE of θ. We argue that in Hahn (2004) it should be $\overline{x}_i$ in place of $\overline{x}$ (with $\overline{x} = \sum_{i=1}^{N} \overline{x}_i/N$), but we emphasize that this amendment does not change the conclusions in Hahn (2004).

According to Bernardo (2005), in this problem the reference prior for the parameters is proportional to θ^{-1}. So, by direct manipulations (as above) or recurring to Example 18 in Bernardo (2005), we have that the posterior distribution for θ is an inverted gamma with parameters $\beta_1 = (N - 1)T/2$ and $\beta_2 = S_{xx}/2$, whose mode furnishes another Bayesian estimator for θ given by

$$\hat{\theta}_R = \frac{S_{xx}}{(N - 1)T + 2}.$$

With T fixed we obtain $\operatorname{plim}_{N\to\infty}(\hat{\theta}_R - \theta) = \theta/T = O(1/T)$ as before. This is not surprising, for Liseo (2005) stressed that reference priors are well suited to the incidental parameter problem only for a few examples. Furthermore, Li and Leon-Gonzalez (2009) pointed out that these priors are not intrinsically designed to solve this problem.

Following the approach in Li and Leon-Gonzalez (2009), now we turn our attention to another way to solve the problem. We denote the likelihood function corresponding to (2.1) by $p(\mathbf{x} \mid \theta, \boldsymbol{\alpha})$. Our choice of prior distribution for $(\theta, \boldsymbol{\alpha})$ is such that $p(\boldsymbol{\alpha} \mid \theta) \propto 1$ and $p(\theta) \propto 1$. Since

$$\int_{\mathbb{R}^N} p(\boldsymbol{\alpha} \mid \theta) p(\mathbf{x} \mid \theta, \boldsymbol{\alpha})\, d\boldsymbol{\alpha} < \infty \quad \text{and} \quad \mathbb{E}_{\mathbf{x}}\left[\frac{\partial^2 \log p(\mathbf{x} \mid \theta, \boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}\, \partial \theta}\right] = \mathbf{0},$$

the correction proposed by Li and Leon-Gonzalez (2009) ensures that by integrating out $\boldsymbol{\alpha}$ in $p(\theta, \boldsymbol{\alpha} \mid \mathbf{x})$, the mode of the resulting posterior distribution $p(\theta \mid \mathbf{x})$ is a consistent estimator for θ. Proceeding analogously as above we get

$$p(\theta \mid \mathbf{x}) \propto \theta^{-(N(T-1))/2} \exp\left(-\frac{S_{xx}}{2\theta}\right).$$

Therefore, the posterior distribution is an inverted gamma with parameters $\beta_1 = N(T-1)/2 - 1$, $\beta_2 = S_{xx}/2$, and mode

$$\hat{\theta}_C = \frac{S_{xx}}{N(T-1)}.$$

It follows that $\operatorname{plim}_{N\to\infty}(\hat{\theta}_C - \theta) = 0$, as desired.

3 Conclusion

In this short note we contributed to the question raised by Hahn (2004). We have seen that in the incidental parameter problem, a Bayesian estimator stemming from the reference prior has the same drawback of the MLE and an estimator based on the Jeffrey's prior as well. However, resorting to a technique recently presented by Li and Leon-Gonzalez (2009), we found a Bayesian estimator asymptotically unbiased for the parameter of interest.

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