Brazilian Journal of Probability and Statistics
2011, Vol. 25, No. 1, 1–13
DOI: 10.1214/09-BJPS107

On generalized multivariate analysis of variance

José A. Díaz-García

Universidad Autónoma Agraria Antonio Narro

Abstract. This work studies the behavior of certain test criteria in multivariate analysis of variance, under the existence of multiplicity in the sample eigenvalues of the matrix $\mathbf{S}_{\mathbf{E}}^{-1}\mathbf{S}_{\mathbf{H}}$; where $\mathbf{S}_{\mathbf{H}}$ is the matrix of sum of squares and sum of products due to the hypothesis and $\mathbf{S}_{\mathbf{E}}$ is the matrix of sum of squares and sum of products due to the error.

1 Introduction

Consider the multivariate linear model

$$\mathbf{Y} = \mathbf{X}\mathbb{B} + \mathbf{E},$$

where $\mathbf{Y}$ and $\mathbf{E}$ are $n \times m$ random matrices, $\mathbf{X}$ is a known $n \times p$ matrix, and $\mathbb{B}$ is a unknown $p \times m$ of parameters termed regression coefficients. We shall assume that $\mathbf{X}$ has rank $r \le p$, that $n \ge m + r$, and that the rows of the error matrix $\mathbf{E}$ are independent and identically distributed as multivariate normal with mean vector zero and covariance matrix Σ, denoted as $E_{(i)} \sim \mathcal{N}_m(\mathbf{0}, \Sigma)$ where $\mathbf{E}' = (E_{(1)}, \dots, E_{(n)})$. Using matrix variate notation, $\mathbf{E} \sim \mathcal{N}_{n\times m}(\mathbf{0}, \mathbf{I}_n \otimes \Sigma)$ then that $\mathbf{Y} \sim \mathcal{N}_{n\times m}(\mathbf{X}\mathbb{B}, \mathbf{I}_n \otimes \Sigma)$. Given $\mathbf{M}$ a $q \times n$ matrix of known constants, we known that for $\mathbf{M}\mathbb{B}$ estimable, the maximum likelihood or the least square estimate of $\mathbf{M}\mathbb{B}$ is given by

$$\widehat{\mathbf{M}\mathbb{B}} \equiv \mathbf{M}\widehat{\mathbb{B}} = \mathbf{M}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{Y} = \mathbf{M}\mathbf{X}^{+}\mathbf{Y},$$

where $\mathbf{A}^{-}$ is any generlized inverse of $\mathbf{A}$ (this is, $\mathbf{A} = \mathbf{A}\mathbf{A}^{-}\mathbf{A}$) and $\mathbf{X}^{+}$ is the Moore–Penrose generalised inverse of $\mathbf{X}$.

The $m \times m$ covariance matrix Σ can be unbiasedly estimated by

$$\widehat{\Sigma} = \mathbf{S}_{\mathbf{E}}/(n - r),$$

where $\mathbf{S}_{\mathbf{E}} = (\mathbf{Y} - \mathbf{X}\widehat{\mathbf{B}})'(\mathbf{Y} - \mathbf{X}\widehat{\mathbf{B}})$ is termed matrix of sum of squares and sum of products due to the error. We consider the problem of testing the general linear hypothesis

$$H_0 : \mathbf{C}\mathbb{B} = \mathbf{0} \quad \text{vs} \quad H_a : \mathbf{C}\mathbb{B} \ne \mathbf{0},$$

Key words and phrases. Multiplicity, MANOVA, Wilks' criteria, Lawley–Hotelling criterion, Pillai's criterion, Roy criteria.

Received January 2009; accepted June 2009.

where $\mathbf{C}$ a $q \times n$ matrix of rank $q \leq n$ of known constants. The matrix of sum of squares and sum of products due to the hypothesis is given by

$$\mathbf{S_H} = (\mathbf{C}\widehat{\mathbb{B}})'(\mathbf{C}(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}')^{-1}(\mathbf{C}\widehat{\mathbb{B}}).$$

Let $\delta_1, \ldots, \delta_m$ and $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of the matrices $\mathbf{S_H}\mathbf{S_E}^{-1}$ and $\mathbf{S_H}(\mathbf{S_H} + \mathbf{S_E})^{-1}$, respectively; where under the null hypothesis $H_0 : \mathbf{C}\mathbb{B} = \mathbf{0}$, $\mathbf{S_H} : m \times m$ is Wishart distributed with ν_H degrees of freedom, $\mathbf{S_H} \sim \mathcal{W}_m(\nu_H, I_m)$ and $\mathbf{S_E} \sim \mathcal{W}_m(\nu_E, I_m)$. Specifically, ν_H and ν_E denote the degrees of freedom of the hypothesis and error, respectively. Various authors have proposed a number of different criteria for testing the multivariate general linear hypothesis; see Kress (1983), Anderson (1984) and Díaz-García and Caro-Lopera (2008). Then all of the test statistics may be represented as functions of the $s = \min(m, \nu_H)$ nonzero eigenvalues $\lambda' s$ and/or $\delta' s$, observing that $\lambda_i = \delta_i/(1 + \delta_i)$ and $\delta_i = \lambda_i/(1 - \lambda_i)$, $i = 1, \ldots, s$. Moreover, from Kress (1983) we known that:

(1) The likelihood ratio criterion Λ of Wilks,

$$\Lambda = \frac{|\mathbf{S_E}|}{|\mathbf{S_H} + \mathbf{S_E}|} = \prod_{i=1}^{s}(1 - \lambda_i) = \prod_{i=1}^{s} \frac{1}{(1 + \delta_i)}.$$

(2) The trace criterion of Hotelling and Lawley,

$$V = \operatorname{tr} \mathbf{S_H}\mathbf{S_E}^{-1} = \sum_{i=1}^{s} \frac{\lambda_i}{(1 - \lambda_i)} = \sum_{i=1}^{s} \delta_i.$$

(3) The maximal root criterion of Roy,

$$\delta_{\max} = \delta_{\max}(\mathbf{S_H}\mathbf{S_E}^{-1}) = \frac{\lambda_{\max}}{(1 - \lambda_{\max})}.$$

(4) The maximal root criterion of Pillai and Roy (version due to Forster and Rees),

$$\lambda_{\max} = \lambda_{\max}(\mathbf{S_H}(\mathbf{S_H} + \mathbf{S_E})^{-1}) = \frac{\delta_{\max}}{(1 + \delta_{\max})}.$$

(5) The trace criterion of Hotelling–Lawley–Pillai–Nanda–Bartlett,

$$V^{(s)} = \operatorname{tr} \mathbf{S_H}(\mathbf{S_H} + \mathbf{S_E})^{-1} = \sum_{i=1}^{s} \lambda_i = \sum_{i=1}^{s} \frac{\delta_i}{(1 + \delta_i)}.$$

(6) Third criterion of Wilks (S-criterion of Olson),

$$S = |\mathbf{S_H}\mathbf{S_E}^{-1}| = \prod_{i=1}^{s} \frac{\lambda_i}{(1 - \lambda_i)} = \prod_{i=1}^{s} \delta_i.$$

The decision rule for all the criteria is:

$$\text{reject } H_0 \text{ if the statistic} \geq \text{critical value.}$$

However, for Wilks's Λ criterium, the decision rule is [this class of test are known in statistical literature as *inverse test*; see Rencher (1995), page 162]:

$$\text{reject } H_0 \text{ if the statistic} \leq \text{critical value.}$$

Our interest is to study the distributions of these criteria under null hypothesis when the multiplicity in the eigenvalues $\lambda' s$ and $\delta' s$ is considered.

Let $\mathbf{A}$ be a $m \times m$ symmetric matrix with spectral decomposition

$$\mathbf{A} = \mathbf{HLH}', \tag{1.1}$$

where $\mathbf{H}$ is a $m \times m$ orthogonal matrix and $\mathbf{L}$ is a diagonal matrix, such that $\mathbf{L} = \operatorname{diag}(l_1, \dots, l_m)$. The representation (1.1) is unique if the eigenvalues $l_1, \dots, l_m$ are distinct and the sign of the first element in each column is nonnegative, Muirhead (1982), page 588.

For $\mathbf{A}$ a positive definite matrix ($\mathbf{A} > \mathbf{0}$), that is, for $l_1 > \cdots > l_m > 0$, the Jacobian of the transformation (1.1) has been computed by different authors, James (1954), Muirhead (1982), pages 104–105 and Anderson (1984), Section 13.2.2, among many others. Similarly, when $\mathbf{A}$ is a positive semidefinite matrix ($\mathbf{A} \geq \mathbf{0}$), that is, when $l_1 > \cdots > l_r > 0$ and $l_{r+1} = \cdots = l_m = 0$, $r < m$, the respective Jacobian was computed by Uhlig (1994); see also DíazGarcía, Gutiérrez Jáimez and Mardia (1997).

Note that under the spectral decomposition, the Lebesgue measure defined on the homogeneous space of $m \times m$ positive definite symmetric matrices $\mathcal{S}_m^+$ [and implicitly the Jacobian of the transformation (1.1)] is given by

$$(d\mathbf{A}) = 2^{-m} \prod_{i<j}^{m} (l_i - l_j)(\mathbf{H}' \, d\mathbf{H}) \wedge (d\mathbf{L}), \tag{1.2}$$

see Muirhead (1982), pages 104–105, where

$$(\mathbf{H}' \, d\mathbf{H}) = \bigwedge_{i<j}^{m} h_j' \, dh_i, \qquad (d\mathbf{L}) = \bigwedge_{i=1}^{m} dl_i$$

and $(d\mathbf{B})$ denotes the exterior product of the distinct elements of the matrix differentials (db_{ij}) and in particular $(\mathbf{H}' \, d\mathbf{H})$ denotes the Haar measure; see James (1954) and Muirhead (1982), Chapter 2.

By applying the definition of exterior product, it is easy to see that under multiplicity in the eigenvalues of the matrix $\mathbf{A}$, that is, $l_i = l_j$ at least for a $i \neq j$, we obtain that $(d\mathbf{L}) = 0$, moreover, in (1.2)

$$\prod_{i<j}^{m} (l_i - l_j) = 0,$$

then $(d\mathbf{A}) = 0$. This happens because the multiplicity of the eigenvalues of $\mathbf{A}$ forces it to live in a $(ml - l(l-1)/2)$-dimensional manifold of rank l (where l is the number of distinct eigenvalues of $\mathbf{A}$) on the homogeneous space of $m \times m$ $\mathcal{S}_{m,l}^{+} \subset \mathcal{S}_m^{+}$.

Observe that $\mathcal{S}_m^{+}$ is a subset of the $m(m+1)/2$-dimensional $\mathcal{S}_m$ Euclidian space of $m \times m$ symmetric matrices, and, in fact, it forms an open cone described by the following system of inequalities; see Muirhead (1982), page 61 and page 77, Problem 2.6:

$$\mathbf{A} > 0 \quad \Leftrightarrow \quad a_{11} > 0, \qquad \det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} > 0, \ldots, \det(\mathbf{A}) > 0. \tag{1.3}$$

In particular, let $m = 2$, after factorizing the Lebesgue measure in $\mathcal{S}_m$ by the spectral decomposition, then the inequalities (1.3) are as follows:

$$\mathbf{A} > 0 \quad \Leftrightarrow \quad l_1 > 0, \qquad l_2 > 0, \qquad l_1 l_2 > 0. \tag{1.4}$$

But if $l_1 = l_2 = \varrho$, (1.4) reduces to

$$\mathbf{A} > 0 \quad \Leftrightarrow \quad \varrho > 0, \qquad \varrho^2 > 0. \tag{1.5}$$

Which defines a curve (a parabola) in the space, over the line $l_1 = l_2 (= \varrho)$ in the subspace of points (l_1, l_2). Formally, we say that $\mathbf{A}$ has a density respect to the Hausdorff measure [Billingsley (1986)].

When $\mathbf{A} \in \mathcal{S}_m^{+}$, the eigenvalue distributions have been studied by several authors, Srivastava and Khatri (1979), Muirhead (1982), Anderson (1984), among many others. If $\mathbf{A} \in \mathcal{S}_m^{+}(q)$, that is, $\mathbf{A}$ is a positive semidefinite matrix with q distinct positive eigenvalues, the eigenvalue distributions have been founded by Díaz-García and Gutiérrez Jáimez (1997), DíazGarcía, Gutiérrez Jáimez and Mardia (1997), Srivastava (2003), Díaz-García and Gutiérrez Jáimez (2006) and Díaz-García (2007a).

In general, we can consider multiplicity in the eigenvalues of any symmetric matrix, but in some applied cases [multivariate analysis of variance (MANOVA) problems] the eigenvalues are always assumed distinct, for instance, Okamoto (1973) studies the matrix $\mathbf{S_E}$ assuming that N (the sample size) $\geq m$ (the dimension) and the sample is independent, that is, the population has an absolutely continuous distribution. However, recall that if $\mathbf{S_E}^{-1/2}\mathbf{S_H}\mathbf{S_E}^{-1/2} \geq \mathbf{0}$ of rank $r \leq m$, then $\mathbf{S_E}^{-1/2}\mathbf{S_H}\mathbf{S_E}^{-1/2}$ has an eigenvalue $\lambda = 0$ with multiplicity $m - r$. In the present work, we will not assume such conditions and then we will study the test criteria for a general multivariate linear model.

The main aim of this study is to highlight the fact that on some occasions there may occur multiplicity among the eigenvalues of the matrices $\mathbf{S_H}\mathbf{S_E}^{-1}$ and $\mathbf{S_H}(\mathbf{S_H} + \mathbf{S_E})^{-1}$. In addition, we show that the distributions, and therefore the critical values published in the classical tables [see Kress (1983)] are not generally the most appropriate for taking a decision with respect to the null hypothesis in a multivariate general linear model when there exists multiplicity among the eigenvalues of the matrices $\mathbf{S_H S_E^{-1}}$ and $\mathbf{S_H}(\mathbf{S_H} + \mathbf{S_E})^{-1}$. As examples, we examine the cases $m = 2$ and $m = 3$, in which it is apparent that it would be very laborious to attempt to prepare tables for a general m addressing all the possible combinations occurring in the multiplicity of the $s = \min(m, \nu_H)$ eigenvalues of the matrices $\mathbf{S_H S_E^{-1}}$ and $\mathbf{S_H}(\mathbf{S_H} + \mathbf{S_E})^{-1}$; see Section 3. Thus, in our conclusions, we propose a (heuristic) practical solution to this problem. We wish to make it clear that the most important properties of these modified tests are obtained immediately. These properties are their exact distribution and the identification of the critical values; the solution then becomes a simple modification of the parameters of the distributions of the standard tests and their corresponding tabulated values; see Díaz-García (2007b).

2 Preliminary note

Statistics books include often the following assertions:

(1) let $\mathbf{X}$ be an m-dimensional normal distributed random vector with parameters $E(\mathbf{X}) = \mu$ and $\text{Cov}(\mathbf{X}) = \Sigma$, so, if $\Sigma \geq \mathbf{0}$, then $\mathbf{X}$ has not a density; and,
(2) consider a follow nultivariate sample $X_1, \dots, X_n$ and let $\mathbf{S}: m \times m$ be the corresponding sample covariance matrix, then with probability 1, all the eigenvalues of $\mathbf{S}$ are distinct; among many other examples.

The corresponding appropriate conclusions can be given as follows:

(1) indeed $\mathbf{X}$ has not a density respect to the Lebesgue measure in $\Re^m$, but, definitely, $\mathbf{X}$ has a density in a subspace with dimension equal to the rank of Σ; explicitly, $\mathbf{X}$ has a density respect to the Hausdorff measure [see DíazGarcía, Gutiérrez Jáimez and Mardia (1997)]; and,
(2) in this case, no measure and/or density can be specified in the computation of the probability, but the situation becomes clear if we explain it as follows:

$$P(\rho_1 > \cdots > \rho_m > 0) = \int_{\mathcal{B}} dF(\rho_1, \dots \rho_m),$$

where $\mathcal{B}$ is the corresponding space, $\rho_1, \dots, \rho_m$, $\rho_1 > \cdots > \rho_m > 0$ are the eigenvalues of $\mathbf{S}$ and $dF(\rho_1, \dots, \rho_m)$ denotes the joint density of the eigenvalues $\rho_1, \dots, \rho_m$.

Now, consider the following example: assume that we have a density function $f(\mathbf{X})$ with respect to the Lebesgue measure $(d\mathbf{X})$ in $\Re^n$ and let $\mathcal{N}(\mathbf{X})$ be a surface in a subspace with dimension $r < n$ in $\Re^n$. Certainly, we can perform any operation of kind

$$\int_{\mathcal{A}} f(\mathbf{X})(d\mathbf{X}),$$

for some $\mathcal{A} \subset \Re^n$. However

$$\int_{\mathcal{C}} \mathcal{N}(\mathbf{X})(d\mathbf{X}) = 0,$$

for every $\mathcal{C} \subset \Re^n$. Of course, this is not a sufficient reason for refusing to do computations involving the surface $\mathcal{N}(\mathbf{X})$. We just need to define an adequate measure for this surface and we can perform computations of the following type

$$\int_{\mathcal{C}} d\mathcal{N}(\mathbf{X}) = \int_{\mathcal{C}} \mathcal{N}(\mathbf{X})\, d\nu,$$

where $d\nu$ is an appropriate measure.

This example suggests an unified treatment of the ideas involved in the cases (1) and (2) and their context: first, define the appropriate measures and their corresponding density functions and second, perform the analysis based on $\boldsymbol{\Sigma} \geq \mathbf{0}$, item (1), and recalling that not all the eigenvalues ρ_j are distinct, item (2).

3 Multiplicity in MANOVA

Suppose that the eigenvalues $\lambda's$ and $\delta's$ have multiplicity, then, in particular we get: $\lambda_1, \dots, \lambda_l, \lambda_{l+1}, \dots, \lambda_m$, such that $1 > \lambda_1 > \cdots > \lambda_l > 0$ and $1 \geq \lambda_{l+1} \geq \cdots \geq \lambda_l \geq 0$, this is, $l \leq s \leq m$ denotes the number of nonnull distinct eigenvalues of the matrix $\mathbf{U} = (\mathbf{S_H} + \mathbf{S_E})^{-1/2}\mathbf{S_H}(\mathbf{S_H} + \mathbf{S_E})^{-1/2}$. Consider the spectral decomposition of $\mathbf{U}$, such that

$$\mathbf{U} = \mathbf{HLH}' = (\mathbf{H}_1\mathbf{H}_2)\begin{pmatrix} \mathbf{L}_1 & 0 \\ 0 & \mathbf{L}_2 \end{pmatrix}\begin{pmatrix} \mathbf{H}_1' \\ \mathbf{H}_2' \end{pmatrix} = \mathbf{H}_1\mathbf{L}_1\mathbf{H}_1' + \mathbf{H}_2\mathbf{L}_2\mathbf{H}_2' = \mathbf{U}_1 + \mathbf{U}_2.$$

We want to find the distribution of $\mathbf{U}_1$ and the distribution of $\mathbf{L}_1$, where $\mathbf{L}_1 = \operatorname{diag}(\lambda_1, \dots, \lambda_l)$, $\mathbf{H}_1 \in \mathcal{V}_{l,m} = \{\mathbf{H}_1 \in \Re^{m\times l} | \mathbf{H}_1'\mathbf{H}_1 = \mathbf{I}_l\}$ (the Stiefel manifold). Also observe that $\mathbf{U}_1 \in \mathcal{S}_{m,l}^{+}$, so if $l = \nu_H \leq m$, then by Uhlig (1994), Theorem 2,

$$(d\mathbf{U}_1) = 2^{-l}\prod_{i=1}^{l} l_i^{m-l} \prod_{i<j}^{l}(l_i - l_j)(\mathbf{H}_1'\, d\mathbf{H}_1) \wedge (d\mathbf{L}_1),$$

where

$$(\mathbf{H}_1'\, d\mathbf{H}_1) = \bigwedge_{i=1}^{m}\bigwedge_{j=i+1}^{l} h_j'\, dh_i, \qquad (d\mathbf{L}_1) = \bigwedge_{i=1}^{l} dl_i;$$

Díaz-García and González-Farías (2005a) give alternative expressions of $(d\mathbf{U}_1)$ in terms of other factorizations. Under this context from Díaz-García and Gutiérrez Jáimez (1997), Theorem 4, we have:

Theorem 1.

(1) *The distribution of the nonnull distinct eigenvalues of* $\mathbf{U}$ (*the eigenvalues of* $\mathbf{U}_1$) *is*

$$f(\lambda_1,\dots,\lambda_l)=\frac{\pi^{l^2/2}\Gamma_m[(l+\nu_E)/2]}{\Gamma_m[\nu_E/2]\Gamma_l[l/2]\Gamma_l[m/2]}\times\prod_{i=1}^{l}\lambda_i^{(m-l-1)/2}\prod_{i=1}^{l}(1-\lambda_i)^{(\nu_E-m-1)/2}\prod_{i<j}^{l}(\lambda_i-\lambda_j).$$

(2) *If* $\mathbf{F}=\mathbf{S}_\mathbf{E}^{-1/2}\mathbf{S}_\mathbf{H}\mathbf{S}_\mathbf{E}^{-1/2}$, *the distribution of the nonnull distinct eigenvalues of* $\mathbf{F}$ *is given by*

$$f(\delta_1,\dots,\delta_l)=\frac{\pi^{l^2/2}\Gamma_m[(l+\nu_E)/2]}{\Gamma_m[\nu_E/2]\Gamma_l[l/2]\Gamma_l[m/2]}\times\prod_{i=1}^{l}\delta_i^{(m-l-1)/2}\prod_{i=1}^{l}(1+\delta_i)^{-(l+\nu_E)/2}\prod_{i<j}^{l}(\delta_i-\delta_j).$$

Proof. See Díaz-García and Gutiérrez Jáimez (1997), Theorem 4.

Case $m=2$. Consider the case $m=2$ such that the eigenvalues of the matrices $\mathbf{U}$ and $\mathbf{F}$ have multiplicity, namely, $\lambda_1=\lambda_2=\lambda$ and $\delta_1=\delta_2=\delta$, then from Theorem 1,

$$f_\lambda(\lambda)=\frac{\Gamma[(\nu_E+1)/2]}{\Gamma[(\nu_E-1)/2]}(1-\lambda)^{(\nu_E-3)/2},\qquad 0<\lambda<1,\tag{3.1}$$

and

$$f_\delta(\delta)=\frac{\Gamma[(\nu_E+1)/2]}{\Gamma[(\nu_E-1)/2]}(1+\delta)^{-(\nu_E+1)/2},\qquad 0<\delta.\tag{3.2}$$

For the present particular case ($m=2$, $\nu_H=l=1$), the test statistics are given by $\Lambda=(1-\lambda)^2$, $V=2\delta$, $\delta_{\max}=\delta$, $\lambda_{\max}=\lambda$, $V^{(s)}=2\lambda$ and $S=\delta^2$, and the associated density functions are, respectively:

(1) $$f_\Lambda(\Lambda)=\frac{\Gamma[(\nu_E+1)/2]}{2\Gamma[(\nu_E-1)/2]}\Lambda^{(\nu_E-5)/4},\qquad 0<\Lambda<1,$$

(2) $$f_V(V)=\frac{\Gamma[(\nu_E+1)/2]}{2\Gamma[(\nu_E-1)/2]}(1+V/2)^{-(\nu_E+1)/2},\qquad 0<V,$$

(3) $$f_{\delta_{\max}}(\delta_{\max})=\frac{\Gamma[(\nu_E+1)/2]}{\Gamma[(\nu_E-1)/2]}(1+\delta_{\max})^{-(\nu_E+1)/2},\qquad 0<\delta_{\max},$$

(4) $$f_{\lambda_{\max}}(\lambda_{\max})=\frac{\Gamma[(\nu_E+1)/2]}{\Gamma[(\nu_E-1)/2]}(1-\lambda_{\max})^{(\nu_E-3)/2},\qquad 0<\lambda_{\max}<1,$$

(5) $$f_{V^{(s)}}(V^{(s)})=\frac{\Gamma[(\nu_E+1)/2]}{2\Gamma[(\nu_E-1)/2]}(1-V^{(s)}/2)^{(\nu_E-3)/2},\qquad 0<V^{(s)}<2,$$

$$(6)\quad f_S(S) = \frac{\Gamma[(\nu_E+1)/2]}{2\Gamma[(\nu_E-1)/2]\sqrt{S}}(1+\sqrt{S})^{-(\nu_E+1)/2}, \qquad 0 < S.$$

The following six tables resume results on the six mentioned criteria: the first two columns show the critical values of the corresponding criterion for $\alpha = 0.05$ [or $(1-\alpha) = 0.95$] and $\alpha = 0.01$ [or $(1-\alpha) = 0.99$], when we *do not* consider multiplicity in the eigenvalues; in contrast, the third and fourth columns present the critical values for $\alpha = 0.05$ and $\alpha = 0.01$, when we *do* consider multiplicity in the eigenvalues; and finally, the fifth and sixth columns show the p-values for which the null hypothesis could be rejected or accepted if the decision is taken in function of the critical values $\alpha = 0.05$ and $\alpha = 0.01$ computed without multiplicity of the eigenvalues, that is, we use the criteria distributions involving multiplicity for computing the p-values associated to the critical values without multiplicity.

For example, with the criterion of Table 1, we conclude that a rejected (nonmultiplicity) null hypothesis with a significance level of $\alpha = 0.05$, really reaches an $\alpha \geq 0.2$ when we consider multiplicity in the eigenvalues. Similarly, for a rejected (nonmultiplicity) null hypothesis with $\alpha = 0.01$, we really obtain $\alpha \geq 0.1$ if we consider multiplicity in the eigenvalues. Analogous conclusions can be provided from Tables 2–6 for the remaining criteria.

Case $m = 3$. Now, consider $m = 3$, $\nu_H = 2$, namely, the matrices $\mathbf{U}$ and $\mathbf{F}$ have rank 2. Also, assume that $l = 1$, that is, the nonnull eigenvalues of $\mathbf{U}$ ($\mathbf{F}$) are equal, $\lambda_1 = \lambda_2 = \lambda$ and $\delta_1 = \delta_2 = \delta$. In particular, we will study in this section the behavior of the criterion Λ of Wilks. Then, by Theorem 1, we obtain:

$$(1)\quad f_\lambda(\lambda) = \frac{2\Gamma[(\nu_E+1)/2]}{\sqrt{\pi}\,\Gamma[(\nu_E-2)/2]}\lambda^{1/2}(1-\lambda)^{(\nu_E-4)/2},$$

Table 1 *Table of comparisons for third criterion of Wilks (S-criterion of Olson)*

	Critical value* (nonmultiplicity)		Critical value (multiplicity)		$(1-p)$-value	
ν_E	0.95	0.99	0.95	0.99	0.95	0.99
2	361.00	9801.00	159201.0	1E08	0.804	0.900
5	1.2426	13.26111	12.0557	81.0	0.776	0.953
10	0.1559	0.4463	0.8947	3.17752	0.776	0.899
20	0.0291	0.0752	0.1374	0.38909	0.776	0.899
30	0.0118	0.0296	0.0526	0.13974	0.775	0.899
40	0.0063	0.0157	0.02757	0.07095	0.775	0.899
60	0.0027	0.0065	0.01142	0.02854	0.775	0.898
80	0.0014	0.0036	0.0062	0.01529	0.765	0.899
100	0.0009	0.0022	0.0039	0.0095	0.768	0.896
440	0.00005	0.00011	0.00018	0.00044	0.787	0.898
1000	0.00001	0.00002	0.000036	0.00008	0.793	0.898

*From Díaz-García and Caro-Lopera (2008).

Table 2 *Comparisons for the criterion Λ of Wilks*

	Critical value* (nonmultiplicity)		Critical value (multiplicity)		p-value	
ν_E	0.05	0.01	0.05	0.01	0.05	0.01
2	6.41E-4	2.5E-5	6.25e-6	1.00E-8	0.150	0.070
5	0.117368	0.049316	0.05000	0.01000	0.117	0.049
10	0.367038	0.245660	0.264098	0.129155	0.105	0.042
20	0.614483	0.505819	0.522230	0.379269	0.099	0.039
30	0.724899	0.637459	0.661527	0.529832	0.097	0.038
40	0.786433	0.714476	0.735463	0.623551	0.096	0.037
60	0.852599	0.799984	0.816196	0.731824	0.095	0.037
80	0.887496	0.846188	0.859261	0.792016	0.094	0.036
100	0.909051	0.875081	0.885999	0.880218	0.094	0.036
440	0.978644	0.970243	0.973073	0.958908	0.094	0.036
1000	0.990552	0.986804	0.988077	0.981730	0.093	0.036

*From Table 1 in Kress (1983).

Table 3 *Comparisons for the maximal root criterion of Roy*

	Critical value* (nonmultiplicity)		Critical value (multiplicity)		$(1-p)$-value	
ν_E	0.95	0.99	0.95	0.99	0.95	0.99
12	12.23	20.36	3.51	6.18	0.99	0.999
20	10.78	16.90	3.30	5.05	0.99	0.999
25	10.24	15.64	3.23	5.31	0.99	0.999
30	9.95	14.98	3.19	5.18	0.99	0.999
40	9.59	14.20	3.14	5.03	0.99	0.999
50	9.39	13.77	3.11	4.94	0.99	0.999
60	9.27	13.50	3.09	4.88	0.99	0.999
80	9.12	13.17	3.07	4.81	0.99	0.999
100	9.04	13.01	3.05	4.77	0.99	0.999
300	8.79	12.49	3.01	4.66	0.99	0.999
1000	8.71	12.32	3.00	4.62	0.99	0.999

*From Table 3 in Kress (1983).

$$(2)\quad f_\delta(\delta) = \frac{2\Gamma[(\nu_E+1)/2]}{\sqrt{\pi}\Gamma[(\nu_E-2)/2]}\delta^{1/2}(1+\delta)^{-(\nu_E+1)/2}.$$

Similar results can be derived for the joint distribution of λ_1, λ_2 and δ_1, δ_2 by using Theorem 1. However, these are not necessary if use the statements by Díaz-García and Gutiérrez Jáimez (1997) for the coincidence of the nonnull eigenvalue distribution, via singular distributions [Díaz-García and Gutiérrez Jáimez (1997)], and

Table 4 *Comparisons for maximum root criterion of Pillai and Roy (version of Foster and Rees)*

	Critical value* (nonmultiplicity)		Critical value (multiplicity)		$(1-p)$-value	
ν_E	0.95	0.99	0.95	0.99	0.95	0.99
5	0.8577	0.9377	0.7763	0.9000	0.9797	0.9961
15	0.4475	0.5687	0.3481	0.4820	0.9843	0.9972
21	0.3427	0.4479	0.2588	0.3690	0.9849	0.9973
25	0.2960	0.3915	0.2209	0.3187	0.9851	0.9974
31	0.2457	0.3290	0.1810	0.2643	0.9854	0.9974
35	0.2206	0.2972	0.1615	0.2373	0.9855	0.9975
41	0.1912	0.2594	0.1391	0.2056	0.9856	0.9975
61	0.1324	0.1821	0.0950	0.1423	0.9858	0.9976
81	0.1013	0.1402	0.0721	0.1087	0.9860	0.9976
101	0.0820	0.1140	0.0581	0.0879	0.9861	0.9976
161	0.0521	0.0730	0.0367	0.0559	0.9861	0.9977

*From Table 5 in Kress (1983) and Anderson (1984), Table 4.

Table 5 *Comparisons for trace criterion of Hotelling and Lawley*

	Critical value* (nonmultiplicity)		Critical value (multiplicity)		$(1-p)$-value	
ν_E	0.95	0.99	0.95	0.99	0.95	0.99
2	985.9	24670	798	19998	0.955	0.991
5	6.2550	15.318	6.9443	18.0000	0.941	0.986
10	1.5818	2.7402	1.8919	3.5651	0.927	0.979
20	0.6019	0.9236	0.7414	1.2475	0.918	0.973
30	0.3693	0.5479	0.4589	0.7475	0.914	0.970
40	0.2661	0.3886	0.3321	0.5327	0.912	0.968
60	0.1706	0.2454	0.2137	0.3379	0.910	0.967
80	0.1255	0.1792	0.1576	0.2473	0.909	0.966
100	0.0993	0.1412	0.1248	0.1499	0.909	0.965
200	0.0485	0.0684	0.0611	0.0947	0.908	0.965

*From Table 6 in Kress (1983) and Anderson (1984), Table 2.

the respective nonsingular distribution; see, for example, [Muirhead (1982), Section 10.4, Case 2, pages 451–455]. Then the critical values of the cited criteria can be computed from the existing tables ($\nu_H < m$) by making the parameter transformation $(m, \nu_H, \nu_E) \rightarrow (\nu_H, m, \nu_E + \nu_H - m)$, see Muirhead (1982), page 455. Observe that for the criterion Λ of Wilks we do not need to perform that transformation, because the critical values coincide under both parameter definitions; see Anderson (1984), Theorem 8.4.2, page 302.

Table 6 *Comparisons for trace criterion of Hotelling–Lawley–Pillai–Nanda and Bartlett*

	Critical value* (nonmultiplicity)		Critical value (multiplicity)		$(1-p)$-value	
ν_E	0.95	0.99	0.95	0.99	0.95	0.99
13	0.5666	0.7212	0.7860	1.0710	0.864	0.931
15	0.5070	0.6516	0.6963	0.9641	0.870	0.936
23	0.3562	0.4694	0.4768	0.6841	0.884	0.947
33	0.2593	0.3474	0.3415	0.5002	0.891	0.952
43	0.2038	0.2756	0.2659	0.3938	0.895	0.955
63	0.1426	0.1948	0.1842	0.2761	0.899	0.985
83	0.1096	0.1507	0.1409	0.2125	0.900	0.964
123	0.0750	0.1036	0.0958	0.1454	0.902	0.964
243	0.0384	0.0535	0.0489	0.0747	0.904	0.962

*From Table 7 in Kress (1983) and Anderson (1984), Table 3.

Table 7 *Comparisons for the criterion* Λ *of Wilks*

	Critical value* (nonmultiplicity)		Critical value (multiplicity)		p-value	
ν_E	0.05	0.01	0.05	0.01	0.05	0.01
2	0.000000	0.000000	0.000000	0.000000	0.000	0.000
5	0.243139	0.011210	0.009468	0.001078	0.131	0.056
10	0.514622	0.150746	0.156870	0.067583	0.113	0.046
20	0.647501	0.411734	0.429062	0.292612	0.105	0.042
30	0.723938	0.559656	0.577664	0.450770	0.102	0.040
40	0.807778	0.649620	0.666239	0.554541	0.101	0.040
60	0.852653	0.751990	0.765511	0.678456	0.100	0.039
80	0.880557	0.808282	0.819453	0.748957	0.099	0.039
100	0.971785	0.843804	0.853266	0.794241	0.099	0.039
440	0.978644	0.962428	0.964985	0.949572	0.098	0.038
1000	0.987475	0.983312	0.984469	0.977532	0.0938	0.038

*From Table 1 in Kress (1983).

Next we tabulate a comparison between the nonmultiplicity and multiplicity critical values, and we also provide the p-values for a sort of ν_E.

From Table 7 we see that for a rejected (nonmultiplicity) null hypothesis with a significance level of 0.05 (0.01), we need a significance level of $\alpha \geq 0.09$ ($\alpha \geq 0.03$) for rejecting the same hypothesis if we consider multiplicity in the eigenvalues. □

4 Conclusions

We highlight the variation of the criterion distributions, for testing hypothesis in a general linear model, when multiplicity of the eigenvalues is considered. The change is high in the sense that for rejecting a null hypothesis, in general, the significance level α increases. A practical way for handling the inclusion of multiplicity proposes the following modifications of the usual test statistics:

- *Consider only the nonnull distinct eigenvalues* in the computation of the different test statistics, namely, take l instead of ν_H; and compare those values with the tabulated critical values, but make the parameter transformation

$$(m, \nu_H, \nu_E) \to (m, l, \nu_E), \qquad 1 \le l \le \nu_H \le m,$$

  where l is the number of nonnull distinct eigenvalues.

Finally, note that the present work considers only the case when $\nu_H \le m$, otherwise the procedure for finding the distribution of the nonnull distinct eigenvalues of the matrices **U** and **F** remains as an open problem.

Acknowledgments

The author wish to express his gratitude to the editor and the referees for their helpful comments and suggestions. This research work was partially supported by CONACYT-Mexico, research Grant 81512 and by IDI-Spain, Grants FQM2006-2271 and MTM2008-05785. This paper was partially written during J. A. Díaz-García's stay as a visiting professor at the Department of Statistics and O. R. of the University of Granada, Spain. Also, thanks to Francisco José Caro-Lopera (CIMAT-México) for his careful reading and excellent comments.

References

Anderson, T. W. (1984). *An Introduction to Multivariate Statistical Analysis*, 2nd ed. Wiley, New York. MR0771294

Billingsley, P. (1986). *Probability and Measure*, 2nd ed. Wiley, New York. MR0830424

Díaz-García, J. A. (2007a). A note about measures and Jacobians of singular random matrices. *Journal of Multivariate Analysis* **98** 960–969.

Díaz-García, J. A. (2007b). Multivariate analysis of variance under multiplicity. Technical Report, No. I-07-13 (PE/CIMAT). Available at http://www.cimat.mx/biblioteca/RepTec.

Díaz-García, J. A., Gutiérrez Jáimez, R. and Mardia, K. V. (1997). Wishart and pseudo-Wishart distributions and some applications to shape theory. *Journal of Multivariate Analysis* **63** 73–87. MR1491567

Díaz-García, J. A. and Gutiérrez Jáimez, R. (1997). Proof of the conjectures of H. Uhlig on the singular multivariate beta and the Jacobian of a certain matrix transformation. *The Annals of Statistics* **25** 2018–2023. MR1474079

Díaz-García, J. A. and Gutiérrez Jáimez, R. (2006). Wishart and pseudo-Wishart distributions under elliptical laws and related distributions in the shape theory context. *Journal of Statistics Planning and Inference* **136** 4176–4193. MR2323409

Díaz-García, J. A. and Gutiérrez Jáimez, R. (2006). A note on certain singular transformations: Singular beta distribution. *Far East Journal of Theoretical Statistics* **20** 27–36. MR2279462

Díaz-García, J. A. and González-Farías, G. (2005a). Singular random matrix decompositions: Jacobians. *Journal of Multivariate Analysis* **93** 196–212.

Díaz-García, J. A. and Caro-Lopera, F. J. (2008). About test criteria in multivariate analysis. *Brazilian Journal of Probability and Statistics* **22** 1–25.

Kress, H. (1983). *Statistical Tables for Multivariate Analysis*. Springer, New York. MR0716498

James, A. T. (1954). Normal multivariate analysis and the orthogonal group. *Annals of Mathematical Statistics* **25** 40–75. MR0060779

Muirhead, R. J. (1982). *Aspects of Multivariated Statistical Theory*. Wiley, New York. MR0652932

Okamoto, M. (1973). Distinctness of the eigenvalues of a quadratic form in a multivariate sample. *The Annals of Statistics* **1** 763–765. MR0331643

Rencher, A. C. (1995). *Methods of Multivariate Analysis*. Wiley, New York. MR1313913

Srivastava, S. M. and Khatri, C. G. (1979). *An Introduction to Multivariate Statistics*. North Holland, New York. MR0544670

Srivastava, M. S. (2003). Singular Wishart and multivariate beta distributions. *The Annals of Statistics* **31** 1537–1560. MR2012825

Uhlig, H. (1994). On singular Wishart and singular multivartiate beta distributions. *The Annals of Statistics* **22** 395–405. MR1272090

Department of Statistics and Computation
Universidad Autónoma Agraria Antonio Narro
25315 Buenavista, Saltillo
Coahuila
México
E-mail: jadiaz@uaaan.mx