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A note on r-processes

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Abstract. R-processes are a new type of discrete stationary stochastic process which we have recently shown to be finitarily isomorphic to Bernoulli schemes. Here, we present a simple example of an r-process and compute its entropy. Then, we prove that r-processes are weak Bernoulli.

1 Introduction

R-processes are a new type of discrete stationary stochastic process. Before we can properly introduce these new objects and motivate their study, we need a few definitions. We define a process as follows.

Definition 1. A process, X, is a quadruple $(X, \mathcal{U}, \mu, T)$ where X is the set of doubly infinite sequences of some alphabet A, $\mathcal{U}$ is the σ-algebra generated by the coordinates, μ is a probability measure on $(X, \mathcal{U})$ and T is the left shift by one.

We will always suppose our processes to be irreducible and to have at most countably many states. In this paper, X always refers to the above-defined quadruple with state space A.

Definition 2. We say $a \in A$ is a renewal state of X if the σ-algebras $\mathcal{U}(X_{n+1}, X_{n+2}, \ldots)$ and $\mathcal{U}(\ldots, X_{n-2}, X_{n-1})$ are conditionally independent given the event $[X_n = a]$. If there exists such an a, we say X is a renewal process.

Definition 3. Let $a \in A$ be a renewal state in X. We say $a \in A$ has n-Bernoulli distribution if for some nonnegative integer n, $P[X_{n'} = a | X_0 = a] = P[X_{n'} = a]$ for all $n' > n$.

We will say a state has n-Bernoulli distribution when we mean there exists such a finite n. If the precise n is of interest, we will make note, but this is, in general, not the case.

Definition 4. An r-process, X, is a renewal process such that a renewal state in X has n-Bernoulli distribution.

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We can now define a Markov process to be a process in which every state is a renewal state. We can define a Bernoulli scheme to be a Markov process in which every state has 0-Bernoulli distribution.

Common examples of r-processes include Bernoulli schemes and m-dependent Markov chains. For more information on the interplay of the Markov property and m-dependence, consult Matus (1998).

In Shea (2009) we extend the methods in Keane and Smorodinsky (1979a, 1979b) to show that entropy is a complete finitary isomorphism invariant for r-processes. We define finitary isomorphism and finitary factor as follows.

Definition 5. Let $(X, \mathcal{U}, \mu, T)$ and $(Y, \mathcal{V}, \nu, S)$ be two processes. An isomorphism, ϕ from $(X, \mathcal{U}, \mu, T)$ to $(Y, \mathcal{V}, \nu, S)$ is a bimeasurable equivariant map from a subset of X of measure one to a subset of Y of full measure which takes μ to ν. The isomorphism, ϕ is finitary if for almost every $x \in X$ there exist integers $m \leq n$ such that the zero coordinates of $\phi(x)$ and $\phi(x')$ agree for almost all $x' \in X$ with $x[m, n] = x'[m, n]$, and similarly for ϕ^{-1}. If we drop the requirement that ϕ be invertible, we say ϕ is a finitary factor map.

Our result is not the first occurrence of r-processes in the finitary isomorphism literature. In the Keane and Smorodinsky model, the first step in proving a finitary isomorphism exists between two processes is to find markers. Informally, this requires finding a renewal state in one process which has the same distribution as a renewal state in the other process. In this instance, the distribution of a state is a process defined as follows.

Definition 6. Let $a \in A$. The distribution of the state a is defined as the process $\hat{X}$ obtained by setting

$$\hat{X}_n = \begin{cases} 0, & \text{if } X_n \neq a, \\ 1, & \text{if } X_n = a. \end{cases}$$

We will also need the definition of k-stringing.

Definition 7. Let k be a positive integer. The process $X^{(k)}$ called the k-stringing of X is defined as follows. The state space of $X^{(k)}$ is all allowable sequences of length k in X, and $X_n^{(k)} = (X_n, X_{n+1}, \ldots, X_{n+k-1})$ $(n \in Z)$.

We formally define the marker methods of Keane and Smorodinsky with a notion of d-equivalence.

Definition 8. Two processes X and Y are 0-equivalent if X and Y have the same entropy, and if for some positive integers k and j, there exists a renewal state in $X^{(k)}$ with the same distribution as a renewal state in $Y^{(j)}$.

Definition 9. Let d be a positive integer. Two processes X and Z are d-equivalent if there exist d processes $Y_1, Y_2, \dots, Y_d$ such that X is 0-equivalent to Y_1, Z is 0-equivalent to Y_d, and Y_i is 0-equivalent to Y_{i+1} for $1 \leq i \leq d-1$.

In Keane and Smorodinsky (1979a, 1979b) and Shea (2009), showing two processes are finitarily isomorphic relies on first showing that the two processes are d-equivalent. The following lemma furthers our point that r-processes play an unavoidable role in the finitary theory.

Lemma 10. *If a process X is 0-equivalent to a Bernoulli scheme Z then there exists a positive integer j such that $X^{(j)}$ is an r-process.*

Proof. Suppose X is 0-equivalent to a Bernoulli scheme Z. Since Z is an independent process, every state in $Z^{(k')}$ for any ($k' \geq 1$) is a renewal state in $Z^{(k')}$. Since the preimage of any state in $Z^{(k')}$ is a block of length k' in Z, every state in $Z^{(k')}$ has k'-Bernoulli distribution. In order for X to be 0-equivalent to Z, there must exist positive integers j and k such that a renewal state in $X^{(j)}$ has the same distribution as a renewal state in $Z^{(k)}$. Therefore, $X^{(j)}$ must have a renewal state with k-Bernoulli distribution and is an r-process. □

In order to prove the existence of a finitary isomorphism between any discrete stationary stochastic process X and a Bernoulli scheme Z in the traditional methods of Keane and Smorodinsky, the above lemma implies that one must show that the given process X is d-equivalent to some r-process Y. The only result, we know of, that takes a process which is not an r-process and shows it is 0-equivalent to an r-process is the result mentioned below. This result is due to the work in Keane and Smorodinsky (1979b) and Akcoglu, del Junco and Rahe (1979).

Lemma 11. *If X is a mixing, irreducible, finite state Markov process, then X is 0-equivalent to an r-process.*

It is not just in the theory of finitary isomorphisms, that r-processes have played an important role. In Burton, Goulet and Meester (1993), Burton, Goulet and Meester find an r-process which is 1-dependent, and not a finite factor of an independent process. This was the first example of a process which is m-dependent for some positive integer m, but not a finite factor of a Bernoulli scheme.

We should remark that r-processes are new to Shea (2009) and this paper, and thus the term r-process does not occur in any of the above mentioned articles.

To build familiarity with this new type of discrete stationary process, in the next section, we define a simple r-process as a finite factor of an independent process. We then compute the entropy of this r-process. In the last section, we show that r-processes are weak Bernoulli.

2 A familiar example

Consider the following example of an r-process (as a finite factor of an independent process). We will also calculate this example's entropy. Our methods are not original. Entropy calculations of this nature can be traced back to Blackwell (1957) and Marcus, Petersen and Williams (1984).

Let $X = \{0, 1\}^{\mathbf{Z}}$, $\mu = (1/2, 1/2)^{\mathbf{Z}}$, and S be the left shift on X. Let $Y \subset X$, T the left shift on Y and define a map $\phi : X \to Y$ where

$$\phi(\ldots, x_{-1}, x_0, x_1, \ldots) = (\ldots, y_{-1}, y_0, y_1, \ldots)$$

with

$$y_i = x_i x_{i+1}$$

where we literally mean y_i equals x_i multiplied by x_{i+1}. For example, if we take

$$x = \ldots 01101001110010 \ldots$$

then

$$y = \ldots 01000001100000 \ldots .$$

If we define $\nu = \phi(\mu)$ $[\nu(B) = \mu(\phi^{-1}B)$ for B a subset of $Y]$, then (Y, ν, T) is measure-preserving ergodic. So naturally, we can ask, what is $h(Y, \nu, T)$. We claim that

$$h(Y, \nu, T) = \sum_{k=0}^{\infty} b_k \big(-a_k \log(a_k) - (1 - a_k) \log(1 - a_k)\big),$$

where

$$b_k = \frac{Fib(k+1)}{2^{k+2}},$$

$$a_k = \frac{Fib(k+2)}{2Fib(k+1)},$$

and

$$Fib(k) = \frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^k - \Big(\frac{1-\sqrt{5}}{2}\Big)^k\Big]$$

is the kth element of the Fibonacci sequence.

To see why this claim is true, notice that Y is an r-process with renewal state 1. Furthermore, the state 1 is such that the returns to that state form a countable state Bernoulli scheme. Thus we can compute the entropy from the occurrence of 1 in Y. For a more complete description of the behavior of a process such as this, see Marcus, Petersen and Williams (1984). a_k is then the probability the next element is a 0 given we have seen the previous $k+1$ coordinates equal 10^k (a one followed by k zeroes), and b_k is the probability of 10^k occurring in Y. Simple calculations then yield that

$$(b_k) = 1/4, 1/8, 1/8, 3/32, 5/64, 8/128, \ldots$$

and

$$(a_k) = 1/2, 1, 3/4, 5/6, 8/10, 13/16, \ldots.$$

We can generalize the above claim to class of processes defined as follows. We now let $\mu = (p, 1-p)^{\mathbf{Z}}$ for some $p \in \mathbf{R}$, and let everything else be as defined above.

Then again

$$h(Y, \nu, T) = \sum_{k=0}^{\infty} b_k\big(-a_k \log(a_k) - (1-a_k)\log(1-a_k)\big),$$

only this time, we define (b_k) recursively and (a_k) in terms of (b_k). Here,

$$b_0 = (1-p)^2, \qquad b_1 = p(1-p)^2, \qquad b_k = p\big((1-p)b_{k-2} + b_{k-1}\big)$$

and

$$a_k = \frac{b_{k+1}}{b_k}.$$

We can easily see that our results for $p = 1/2$ agree with this more general construction.

3 R-processes are weak Bernoulli

In Shea (2009), we show that r-processes are finitarily isomorphic to Bernoulli schemes. This result leads us to believe that r-processes might be weak Bernoulli. We will define weak Bernoulli (also called absolutely regular) as it is defined in Borovkova, Burton and Dehling (2001). There are equivalent definitions. One such example is an elegant coupling definition which is proven to be equivalent in Burton and Steif (1997).

Definition 12. A process $(X_n)_{n\in\mathbf{Z}}$ is weak Bernoulli if $\beta_k \to 0$ where

$$\beta_k = 2\sup_n\Big\{\sup_{A\in A_{n+k}^{\infty}}\big(P(A|A_1^n) - P(A)\big)\Big\}$$
$$= \sup_n\left\{\sup\sum_{i=1}^{I}\sum_{j=1}^{J}|P(A_i\cap B_j) - P(A_i)P(B_j)|\right\},$$

where the last supremum is taken over all finite A_1^n-measurable partitions $(A_1, \ldots, A_I)$, and all finite A_{n+k}^{∞}-measurable partitions $(B_1, \ldots, B_J)$.

In order to prove that r-processes are weak Bernoulli, we will need first the following result by S. Ito, H. Murata and H. Totoki, from Ito, Murata and Totoki (1971/1972).

Lemma 13. *Countable state mixing Markov shifts are weak Bernoulli.*

We will also use the following lemma, which is a combination the work in del Junco and Rahe (1979) and the results of Bowen (1975).

Lemma 14. *Let X and Y be processes, and Φ a finitary factor map from X to Y with finite expected coding time. If X is weak Bernoulli, then Y is weak Bernoulli.*

In del Junco and Rahe (1979), del Junco and Rahe show that there exists a weak Bernoulli process which is not a finitary factor of a Bernoulli scheme.

We are now ready to prove the following.

Theorem 15. *R-processes are weak Bernoulli.*

Proof. Let $X = (X, \mathcal{U}, \mu, T)$ be an r-process. We will begin by showing that there exists a finitary isomorphism with finite expected coding time from a mixing countable state Markov shift to X.

To construct the correct Markov shift, we will define the isomorphism first from our r-process X. Suppose X has alphabet A. Let $a \in A$ such that a is a renewal state with n-Bernoulli distribution. Since X is an r-process, we know that at least one such a exists. Let $\phi : X \to Y$ so that if $X_i = a$, $\phi(X_i) = Y_i = a$ and if $X_i \neq a$, $\phi(X_i) = Y_i = (a, \alpha_1, \alpha_2, \dots, \alpha_j)$ when $X_{i-j} = a$, $X_{i-j+1} = \alpha_1, \dots, X_i = \alpha_j$ and $\alpha_l \neq a$ for $1 \leq l \leq j$. With full probability, $X_{i-j} = a$ for some positive integer j. Therefore, ϕ is well defined on a subset of X which has full measure.

Since X is a countable state process, $Y = (Y, \mathcal{V}, \nu, S)$ is a countable state process.

Let b be a state in the alphabet of Y. To show that Y is Markov, we need to show that the σ-algebras $\mathcal{V}(Y_{n+1}, Y_{n+2}, \dots)$ and $\mathcal{V}(\dots, Y_{n-2}, Y_{n-1})$ are conditionally independent given the event $[Y_n = b]$. Recall that Y_n is simply the history of the chain X before time n back to the last occurrence of the renewal point a. By how ϕ is defined, $\mathcal{V}(\dots, Y_{n-2}, Y_{n-1})$ is completely determined by $\mathcal{U}(\dots, X_{n-2}, X_{n-1})$. Since a is a renewal state in X, $\mathcal{V}(Y_{n+1}, Y_{n+2}, \dots)$ is completely determined by $\mathcal{U}(X_{n+1-j}, X_{n+2-j}, \dots)$ where $X_{n+1-j} = a$ and $X_{n+1-i} \neq a$ for $0 \leq i < j$. If $X_{n+1-j} = a$ and $X_{n+1-i} \neq a$ for $0 \leq i < j$, then $b = (X_{n+1-j}, X_{n+2-j}, \dots, X_n)$. So $\mathcal{V}(Y_{n+1}, Y_{n+2}, \dots)$ is completely determined by the event $Y_0 = b$ and $\mathcal{U}(X_{n+1}, X_{n+2}, \dots)$. Therefore, the σ-algebras $\mathcal{V}(Y_{n+1}, Y_{n+2}, \dots)$ and $\mathcal{V}(\dots, Y_{n-2}, Y_{n-1})$ are conditionally independent given the event $[Y_n = b]$, and Y is Markov.

All that is left to show is that ϕ is invertible and that ϕ^{-1} is a finitary coding with finite expected coding time. If $Y_n = b$, where $b = (a_1, a_2, \ldots, a_m)$, $a_1 = a$, and $a_i \in A$ for $m \geq 0$ and $1 \leq i \leq m$, then $\phi^{-1}(Y_n) = X_n = a_m$. The inverse is a 1-block code and trivially has finite expected coding time. For a complete description of finite expected code length (FECT), see Parry (1979).

So, we have shown that any r-process can be realized as a 1-block isomorphism from a countable state Markov shift. Since we assumed our r-process X to be irreducible, and r-processes are by definition aperiodic, Y is mixing. Now by Lemmas 13 and 14, r-processes are weak Bernoulli. □

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