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Reduced long-range dependence combining Poisson bursts with on–off sources

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Abstract. A workload model using the infinite source Poisson model for bursts is combined with the on–off model for within burst activity. Burst durations and on–off durations are assumed to have heavy-tailed distributions with infinite variance and finite mean. Since the number of bursts is random, one can consider limiting results based on "random centering" of a random sum for the total workload from all sources. Convergence results are shown to depend on the tail indices of both the on–off durations and the lifetimes distributions. Moreover, the results can be separated into cases depending on those tail indices. In one case where all distributions are heavy tailed it is shown that the limiting result is Brownian motion. In another case, convergence to fractional Brownian motion is shown, where the Hurst parameter depends on the heavy-tail indices of the distribution of the on, off and burst durations.

1 Introduction

Workload models for packet traffic have been described previously [Brichet et al. (1996, 2000); Levy and Taqqu (2000); Mandelbrot (1969); Taqqu and Levy (1986); Taqqu, Willinger and Sherman (1997); Willinger et al. (1997); Kurtz (1996)]. An important model has been the strictly alternating on–off model with heavy tailed on or off times [Taqqu, Willinger and Sherman (1997); Willinger et al. (1997)], in which each source creates work at constant rate for all time. Another important model has been the "$M/G/\infty$ queue" described by Cox (1984), also called the "infinite source Poisson" model [Mikosch et al. (2002)]. In this model, traffic arrives as independent bursts of heavy-tailed size or duration at the time points of a homogeneous Poisson process. There is no variability within each burst and bursts are considered to have the same constant rate. (See Mikosch et al. (2002) for a treatment of convergence results in both models.) The use of heavy tails is motivated by, for example, empirical evidence on the sizes of WWW objects [Crovella, Taqqu and Bestavros (1998)]. For these models, the cumulative work centered about its mean is shown to be a fractional Brownian motion (fBm) in appropriate limiting regimes, although other limiting regimes have been studied which show convergence to stable Lévy motion [e.g., Taqqu, Willinger and Sherman (1997); Mikosch et al. (2002); Kaj and Taqqu (2007)].

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A number of more recent models combine Poisson arrivals of bursts with assumptions on the burst volume, burst rate, or by introducing some dynamics within the burst. In Maulik, Resnick and Rootzén (2002) the infinite source Poisson model is used but with an independent random rate given to each burst. There is no variability within bursts. In Maulik and Resnick (2003) bursts are given a "transmission schedule" according to an H-self-similar process with nondecreasing, cadlag paths (necessarily $H \geq 1$), and a "volume" of data (work) whose distribution has infinite variance and finite mean. Under a "fast growth" condition they show convergence (of the finite-dimensional distributions) of the cumulative work, centered by its mean, to a Gaussian self-similar process that generally lacks stationary increments. So, their limit is generally not fractional Gaussian noise. In Çağlar (2004) "flows" arrive according to a Poisson process, have infinite variance Pareto-distributed holding times, but packets within a flow arrive according to a compound Poisson process, and packet sizes have finite variance. The increment of the cumulative workload is shown to be fractional Gaussian noise, with Hurst parameter H depending on holding time.

Cluster models have been proposed to model the packet arrival process. While not modeling the workload per se (absent a model for packet size), they also model variability within a "flow." In Hohn, Veitch and Abry (2003) and Faÿ et al. (2006) flows arrive according to a Poisson process. The number of packets in each flow and the times between packets are random and may have infinite variance. Long-range dependence arises when the number of packets has infinite variance and finite mean.

This paper studies the workload generated in a model where sources arrive at Poisson time points and have heavy-tailed "session" durations. During its session a source is stationary with independent on and off durations. The on times are independent and identically distributed (i.i.d.), as are the off times, and at least one of the two distributions is assumed to be heavy tailed with infinite variance but finite mean. Here "session" is a euphemism for a structure above that of the on–off behavior. Thus the model here can be considered as a hybrid of the infinite source Poisson model and the alternating on–off model, although the proof here owes more to the latter. The model studied here simplifies the one studied by Rolls (2003) without changing the main results, by assuming the number of sessions has achieved stationarity.

The approach used in this paper can establish a (traditional) workload limit result where the variability of the workload about its mean is fractional Brownian motion under certain assumptions. Moreover, the Hurst parameter is the same as that from a corresponding infinite source Poisson model (i.e., from the model if there was no variability within bursts). But the main point here is something else. The on–off behavior imposes additional variability on top of that from the sessions themselves. Under so-called "random centering" this additional variability is shown to have a corresponding Hurst parameter that depends on the indices of both the heavy-tailed session durations and the on–off durations. The formula for the Hurst parameter is a new expression, different from the standard limiting results from either the on–off or infinite source Poisson models. This might arise if session start and end times were announced and this information was incorporated into the centering. Two cases are considered, giving rise to either a fractional Brownian motion or a Brownian motion. It is noteworthy than under random centering, the limit can be Brownian motion even when both the session durations and the on or off durations are heavy tailed.

The rest of this paper is organized as follows. Section 2 provides background and introduces necessary notation. Section 3 describes the alternating on–off model with session lifetimes. Section 4 establishes convergence results for this model. Section 5 extends the convergence results to weak convergence. Section 6 contains the result from several simulations. Section 7 provides conclusions and discusses possible future work.

2 Background

For the on–off processes we use the notation of Taqqu, Willinger and Sherman (1997). Let $\{W(t), t \geq 0\}$ be a strictly alternating, stationary on–off process that takes the value 0 during an off period and 1 during an on period. Let $\{W_i(t), t \geq 0\}$, $i = 1, 2, \ldots$ be mutually independent copies of $\{W(t), t \geq 0\}$. Define the autocovariance and mean of $W(t)$ by

$$r(t) = \mathbb{E}[W^2(t)] - (\mathbb{E}[W(t)])^2 \quad \text{and} \quad \mu_W = \mathbb{E}[W(t)],$$

respectively.

The on and off times are nonnegative and independent of each other. Also, the lengths of the on periods are i.i.d. as are the off periods. Let $f_j(x)$, $F_j(x)$, and μ_j, be the density, distribution function, and mean for the duration of on periods ($j = 1$) or off periods ($j = 2$), respectively.

The on and off times are assumed to be heavy tailed with regularly varying tails: $\bar{F}_j(x) = 1 - F_j(x) \sim x^{-\alpha_j} L_j(x)$ as $x \to \infty$, $1 < \alpha_j < 2$, where $L_j > 0$ is a slowly varying function, $j = 1, 2$. (In fact, one can assume only one distribution is heavy tailed. The results would be unchanged.) Then

$$\mu_W = \frac{\mu_1}{\mu_1 + \mu_2} \quad \text{and} \quad \sigma_W^2 = \mathrm{Var}\big[W(t) - \mathbb{E}[W(t)]\big] = \frac{\mu_1 \mu_2}{(\mu_1 + \mu_2)^2}.$$

Finally, set $\alpha_{min} = \min\{\alpha_1, \alpha_2\}$ and to specify the indices, let $(min, max) = (1, 2)$ if $\alpha_1 < \alpha_2$ and $(min, max) = (2, 1)$ if $\alpha_2 < \alpha_1$. Under these assumptions Taqqu, Willinger and Sherman (1997) showed

$$\mathcal{L} \lim_{T \to \infty} \mathcal{L} \lim_{M \to \infty} \frac{(\int_0^{Tt} (\sum_{i=1}^{M} W_i(u))\, du - T M \mu_W t)}{T^H L^{1/2}(T) M^{1/2}} = \sigma_{lim} Z_H(t),$$

where $\mathcal{L}\lim$ means convergence of the finite-dimensional distributions, $Z_H(t)$ is standard fractional Brownian motion, $H = (3 - \alpha_{min})/2$ and

$$\sigma_{lim}^2 = \frac{2\mu_{max}^2}{\mu_W^3(\alpha_{min} - 1)(3 - \alpha_{min})(2 - \alpha_{min})}. \tag{2.1}$$

(See Taqqu, Willinger and Sherman (1997) for $\alpha_1 = \alpha_2$.)

Conditions so that $r(t)$ has a tail regularly varying at infinity are given by Brichet et al. (2000) and Heath, Resnick and Samorodnitsky (1998). For the results here it is assumed that

$$r(u) = \frac{\sigma_{lim}^2}{2}(3 - \alpha_{min})(2 - \alpha_{min})u^{1-\alpha_{min}} L_r(u) \tag{2.2}$$

for some slowly varying function $L_r(u)$.

Analogous convergence results to fractional Brownian motion have been obtained for the infinite source Poisson model [Kurtz (1996)]. In those results it is not the number of sources, M, but rather the arrival rate of a homogeneous Poisson process, say λ, that goes to infinity, followed by the time rescaling T.

3 The model description

In this section we define the alternating on–off model with heavy-tailed session lifetimes. For the on–off processes we use the same notation and assumptions as in Section 2. In particular, at least one of the on or off-period distributions is assumed heavy tailed so that $1 < \alpha_{min} < 2$. Also, let $\{G(t), t \geq 0\}$ be the mean-zero Gaussian process with autocovariance $r(t)$ in Taqqu, Willinger and Sherman (1997) arising as

$$\mathcal{L}\lim_{n\to\infty} \frac{\sum_{i=1}^n (W_i(t) - \mu_W)}{\sqrt{n}} = G(t).$$

For the sessions, we make assumptions on their arrivals and durations. Let $\{T_{\lambda_n,i}, -\infty < i < \infty\}$ be the arrival times of a rate λ_n Poisson process on $\mathbb{R}$, labeled so that $T_{\lambda_n,0} < 0 < T_{\lambda_n,1}$, where $\{\lambda_n, n = 1, 2, \ldots\}$ is a sequence of positive constants such that $\lambda_n \to \infty$ as $n \to \infty$. For the durations of the sessions, let $\{V_i\}$ be an i.i.d. sequence with continuous distribution H and finite mean μ_V. Here V_i will be the lifetime (i.e., holding time, session length) for source i. It is assumed that the distribution function $H(x)$ of the lifetimes is Lipschitz continuous (e.g., satisfied if H has a bounded derivative), and has a regularly varying tail so we may write

$$\bar{H}(x) = x^{-\alpha_{sess}} L_V(x), \qquad \alpha_{sess} > 0,$$

where $L_V(x)$ is slowly varying. Consideration of the integrated tail of $H(x)$, $\bar{H}_I(x) = \int_x^\infty \bar{H}(z)\,dz$ will be necessary. If $\alpha_{sess} > 1$, by Karamata's theorem

[Bingham, Goldie and Teugels (1987), p. 28]

$$\bar{H}_I(x) \sim \frac{x^{-\alpha_{sess}+1}}{\alpha_{sess}-1} L_V(x) \qquad \text{as } x \to \infty. \tag{3.1}$$

In the boundary case $\alpha_{sess} = 1$, with the additional assumption that $\int_1^\infty L_V(t)/t\,dt < \infty$ then

$$\frac{1}{L_V(x)} \int_x^\infty \frac{L_V(t)}{t}\,dt \to \infty \qquad \text{as } x \to \infty.$$

The processes $\{T_{\lambda_n,i}\}$, $\{V_i\}$ and $\{W_i(t)\}$ are all assumed independent. Together, the combination of the Poisson arrival process and the sequence of holding times defines the busy server process of an $M/G/\infty$ queueing system.

A key idea in this paper is the use of random sums. That is, sums whose upper limit of summation is random and obeys some convergence result of its own [Gnedenko and Korolev (1996)]. With random sums one must distinguish between nonrandom centering of the sum, and nonrandom centering of the summands (really a *random* centering of the sum). For the model presented here, convergence results for both kinds of centering will play a role and so both are discussed.

Let

$$A_n(t) = \int_0^t \sum_{i=-\infty}^{\infty} W_i(u) \mathbf{1}_{[T_{\lambda_n,i}, T_{\lambda_n,i}+V_i)}(u)\,du$$

and

$$B_n(t) = \int_0^t \sum_{i=-\infty}^{\infty} \mathbf{1}_{[T_{\lambda_n,i}, T_{\lambda_n,i}+V_i)}(u)\,du,$$

where $\mathbf{1}_A(x)$ is 1 on the set A and 0 otherwise. With this notation $A_n(t)$ is the total cumulative work in $[0,t)$ for the alternating on–off sources with lifetimes. Note that $B_n(t)$ is the total cumulative work in $[0,t)$ from the infinite source Poisson model whose "bursts" are exactly the "sessions" in $A_n(t)$ and sources are on for the complete burst.

The total cumulative work for the infinite source Poisson model with nonrandom centering is easy to express

$$B_n(Tt) - \mu_V \lambda_n Tt = \int_0^{Tt} \left[\sum_{i=-\infty}^{\infty} \mathbf{1}_{[T_{\lambda_n,i}, T_{\lambda_n,i}+V_i)}(u) - \mu_V \lambda_n \right] du. \tag{3.2}$$

One can show weak convergence of a rescaled version of this quantity to fBm with $H = (3-\alpha_{sess})/2$ under suitable assumptions. Similarly, for the alternating on–off model with lifetimes described here, with nonrandom centering, one can show weak convergence of a rescaled version of

$$A_n(Tt) - \mu_W \mu_V \lambda_n Tt$$

to fBm with the same Hurst parameter as for (3.2).

The focus of this paper is something different, namely

$$A_n(Tt) - \mu_W B_n(Tt) = \int_0^{Tt} \sum_{i=-\infty}^{\infty} X_{n,i}(u)\, du, \tag{3.3}$$

where

$$X_{n,i}(t) = [W_i(t) - \mu_W]\mathbf{1}_{[T_{\lambda_n,i}, T_{\lambda_n,i}+V_i)}(t). \tag{3.4}$$

This quantity captures the variability from the on–off dynamics on top of the variability (at a larger time scale) from the session arrivals and departures. The goal now will be to establish the convergence properties of this process.

4 Main results

Here we present our main results, which depend on the tail indices α_{min} and α_{sess}. The results can be separated into several cases by these values, as shown in Figure 1. Our first theorem is the limit result for case 3, while the second theorem is for case 4. In particular, note that although the tail indices of α_{min} and α_{sess} both correspond to heavy tails, the limiting result is Brownian motion for case 4. Cases 1 and 2 are left for future work.

Theorem 4.1. *For* $1 < \alpha_{min} < 2$ *and* $1 < 4 - \alpha_{min} - \alpha_{sess} < 2$

$$\mathcal{L} \lim_{T\to\infty} \mathcal{L} \lim_{n\to\infty} \frac{\int_0^{Tt} \sum_{i=-\infty}^{\infty} X_{n,i}(u)\, du}{\sqrt{\lambda_n L_V(T) L_r(T)} T^H} = \sigma Z_H(t), \tag{4.1}$$

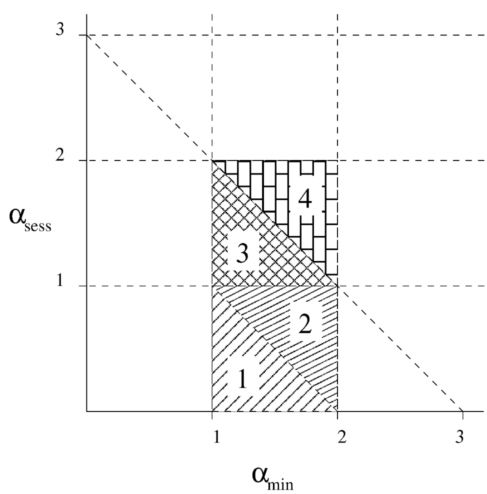

Figure 1 *Differing cases arising from the tail indices* α_{min} *and* α_{sess}. *Case* 3 *leads to fBm, while case* 4 *leads to Brownian motion. The long diagonal line corresponds to* $3 - \alpha_{min} - \alpha_{sess} = 0$.

where $H = (4 - \alpha_{min} - \alpha_{sess})/2$, σ^2_{lim} *is defined by* (2.1),

$$\sigma^2 = \frac{\sigma^2_{lim}(3-\alpha_{min})(2-\alpha_{min})}{(\alpha_{sess}-1)(3-\alpha_{min}-\alpha_{sess})(4-\alpha_{min}-\alpha_{sess})},$$

and $\{Z_H(t)\}$ *is standard fractional Brownian motion with Hurst parameter* H.

Theorem 4.2. *For* $1 < \alpha_{min} < 2$ *and* $-2 < 2 - \alpha_{min} - \alpha_{sess} < -1, \alpha_{sess} < 2$

$$\mathcal{L}\lim_{T\to\infty}\mathcal{L}\lim_{n\to\infty}\frac{\int_0^{Tt}\sum_{i=-\infty}^{\infty}X_{n,i}(u)\,du}{\sqrt{\lambda_n T}} = \sqrt{2c}Z_{0.5}(t), \tag{4.2}$$

where

$$c = \int_0^\infty r(x)\bar{H}_I(x)\,dx$$

and $\{Z_{0.5}(t)\}$ *is standard Brownian motion.*

The rest of this section will prove these results in three steps. Section 4.1 presents limit results for the incremental process (i.e, for $n \to \infty$). Section 4.2 describes the asymptotic behavior of the variance of the incremental process, which is useful for finding the limiting process as $T \to \infty$. Section 4.3 proves the main results.

4.1 Limit results for the incremental process

Let $\{Z(t)\}$ be a Gaussian process such that for $s < t$:

$$\begin{aligned} &\text{(i)} && E[Z(t)] = 0, \\ &\text{(ii)} && \mathrm{Var}[Z(t)] = \sigma^2_W\mu_V \quad \text{and} \\ &\text{(iii)} && \mathrm{Cov}[Z(s), Z(t)] = r(t-s)\int_{t-s}^\infty \bar{H}(z)\,dz, \end{aligned} \tag{4.3}$$

where $\bar{H}(z) = 1 - H(z)$. It will be shown that to obtain limiting results as $T \to \infty$, $\{Z(t)\}$ takes the place of the incremental process $\{G(t)\}$ in Taqqu, Willinger and Sherman (1997).

Theorem 4.3. *For the processes* $\{X_{n,i}(t)\}$ *of* (3.4) *and* $\{Z(t)\}$ *of* (4.3)

$$\mathcal{L}\lim_{n\to\infty}\frac{\sum_{i=-\infty}^{\infty}X_{n,i}(t)}{\sqrt{\lambda_n}} = Z(t).$$

Proof. For any $d \in \mathbb{Z}^+$, let $(c_1, \ldots, c_d)$ be arbitrary and let $0 < t_1 < t_2 < \cdots < t_d$ be a partition of the time axis. To simplify the exposition, let $I_k = [t_k, t_{k+1})$, $k = 1, 2, \ldots, d-1$, $I_0 = (-\infty, t_1)$, $I_d = [t_d, \infty)$ and let

$$Z_n = \frac{c_1\sum_{k=-\infty}^{\infty}X_{n,k}(t_1) + \cdots + c_d\sum_{k=-\infty}^{\infty}X_{n,k}(t_d)}{\sqrt{\lambda_n}}.$$

Classify an arrival as type (i, j), $i = 0, \dots, d$, $j = i, \dots, d$, if it arrives in I_i and ends in I_j. Let $N_{n,i,j}(t)$ be the number of type (i, j) arrivals by time t, $t > 0$. Conditional on an arrival at time y, the probability an arrival is of type (i, j), $P_{(i,j)}(y)$, is

$$P_{(i,j)}(y) = \begin{cases} 0, & y \notin I_i, \\ H(t_{i+1} - y), & y \in I_i, i = j \neq d, \\ H(t_{j+1} - y) - H(t_j - y), & y \in I_i, i < j, j \neq d, \\ \bar{H}(t_d - y), & y \in I_i, i < j, j = d, \\ 1, & y \in I_i, i = j = d. \end{cases}$$

By Proposition 5.3 of Ross (2000, p. 273), for fixed t, the random variables $N_{n,i,j}(t)$, $i = 0, \dots, d$, $j = i, \dots, d$ are mutually independent and Poisson distributed with known mean (and variance)

$$\mathbb{E}[N_{n,i,j}(t)] = \mathrm{Var}[N_{n,i,j}(t)] = \lambda_n \int_0^t P_{(i,j)}(y)\, dy = \lambda_n p_{(i,j)}(t), \qquad i = 0, \dots, d, j = i, \dots, d,$$

where $p_{(i,j)}(t) = \int_0^t P_{(i,j)}(y)\, dy$. (See Rolls (2003) for details.) For each process k of type (i, j) it contributes to Z_n during its holding period at $t_{i+1}, \dots, t_j$. Let $(X_{n,k}^{(i,j)}(t_{i+1}), \dots, X_{n,k}^{(i,j)}(t_j))$ be copies of $\Big(W(t_{i+1}) - \mu_W, \dots, W(t_j) - \mu_W\Big)$, mutually independent in i, j and k. Notice that for fixed i and j the sequence of random vectors is i.i.d. Then the contribution of the kth type (i, j) process is

$$c_{i+1} X_{n,k}^{(i,j)}(t_{i+1}) + \cdots + c_j X_{n,k}^{(i,j)}(t_j)$$

and there are $N_{n,i,j}(t_d)$ such processes. Thus we can write

$$Z_n \overset{D}{=} \sum_{i=0}^{d-1} \sum_{j=i+1}^{d} \left[\frac{\sum_{k=1}^{N_{n,i,j}(t_d)} (c_{i+1} X_{n,k}^{(i,j)}(t_{i+1}) + \cdots + c_j X_{n,k}^{(i,j)}(t_j))}{\sqrt{\lambda_n}} \right].$$

(Strictly speaking there is also a term from arrivals occurring exactly at t_k, $k = 1, 2, \dots, d$. Since these arrivals have probability zero that term can be safely ignored.)

By Lemmas 8.2.2 and 8.2.3 of Rolls (2003) we have

$$\frac{\sum_{k=1}^{N_{n,i,j}(t_d)} (c_{i+1} X_{n,k}^{(i,j)}(t_{i+1}) + \cdots + c_j X_{n,k}^{(i,j)}(t_j))}{\sqrt{\lambda_n}} \xrightarrow{D} Z^{(i,j)} \qquad \text{as } n \to \infty,$$

$i = 0, \dots, d-1$, $j = i+1, \dots, d$, where $Z^{(i,j)}$ are mutually independent random variables such that

$$Z^{(i,j)} = \sqrt{p_{(i,j)}(t_d)} \big(c_{i+1} G^{(i,j)}(t_{i+1}) + \cdots + c_j G^{(i,j)}(t_j)\big)$$
$$\sim \mathrm{N}\left(0, p_{(i,j)}(t_d) \left[\sigma_W^2 c_{i+1}^2 + \cdots + \sigma_W^2 c_j^2 + 2 \sum_{u=i+1}^{j} \sum_{v=u+1}^{j} c_u c_v r(|t_v - t_u|) \right] \right),$$

where $r(u)$ is the autocovariance of $\{G(t), t \geq 0\}$ and so

$$Z_n \xrightarrow{D} \sum_{i=0}^{d-1} \sum_{j=i+1}^{d} Z^{(i,j)} \stackrel{D}{=} \sum_{i=1}^{d} c_i Z(t_i) \qquad \text{as } n \to \infty.$$

Here $\{G^{(i,j)}(t), t \geq 0\}$ are independent copies of $\{G(t), t \geq 0\}$ in i and j. Since $(c_1, \ldots, c_d)$ and $(t_1, \ldots, t_d)$ are arbitrary, the result follows from the Cramér–Wold theorem. □

4.2 Variance and covariance of the integrated process

The goal of this section is to establish the asymptotic behavior of the variance $V(Tt) = \mathrm{Var}[\tilde{Y}_{Tt}]$ as $T \to \infty$ of the integrated process

$$\tilde{Y}_t = \int_0^t Z(u)\, du. \tag{4.4}$$

Since $\{Z(u)\}$ is a Gaussian process with mean zero, so is $\{\tilde{Y}_t\}$. Since $\{Z(u)\}$ is stationary, $\{\tilde{Y}_t, t \geq 0\}$ has stationary increments.

It is helpful to relate $V(t)$ exactly to the autocovariance $r(u)$. This is the content of the following lemma. Notice in particular that since $0 < \bar{H}(s) < 1$, I_1 is a finite constant for any fixed X since it does not depend on t.

Lemma 4.1. *For the variance $V(t)$, any constant $X \geq 0$, and $t \geq X$*

$$V(t) = I_1 + I_2 + I_3,$$

where $I_1 = 2\int_0^X \int_0^y r(x)\bar{H}_I(x)\, dx\, dy$, $I_2 = 2c_X(t - X)$, $c_X = \int_0^X r(x)\bar{H}_I(x)\, dx$, and $I_3 = 2\int_X^t \int_X^y r(x)\bar{H}_I(x)\, dx\, dy$.

Proof. Since $Z(u)$ is a mean zero process it can be shown than

$$V(t) = \mathbb{E}\left[\int_0^t Z(u)\, du \int_0^t Z(v)\, dv\right] = 2\int_0^t \int_0^y r(x)\bar{H}_I(x)\, dx\, dy.$$

Now separate the various regions of integration. □

The following theorem establishes the asymptotic results for the variance $V(t)$ as $t \to \infty$ which is needed to prove the main results.

Lemma 4.2. *Assume the autocovariance $r(u)$ satisfies the slowly varying function condition in (2.2). For the processes $\{Z(t), t \geq 0\}$ and $\{\tilde{Y}_t, t \geq 0\}$ defined in (4.3) and (4.4), respectively, and with*

$$\sigma^2 = \frac{\sigma_{lim}^2 (3 - \alpha_{min})(2 - \alpha_{min})}{(\alpha_{sess} - 1)(3 - \alpha_{min} - \alpha_{sess})(4 - \alpha_{min} - \alpha_{sess})},$$

Case 3 ($1 < \alpha_{min} < 2$, $-1 < 2 - \alpha_{min} - \alpha_{sess} < 1$, $\alpha_{sess} > 1$):

$$V(t) \sim \sigma^2 t^{4-\alpha_{min}-\alpha_{sess}} L_V(t) L_r(t) \qquad \text{as } t \to \infty.$$

Case 4 ($1 < \alpha_{min} < 2$, $-2 < 2 - \alpha_{min} - \alpha_{sess} < -1$, $\alpha_{sess} < 2$):

$$\lim_{T\to\infty} \frac{V(Tt)}{T L_V(T) L_r(T)} = \lim_{T\to\infty} \frac{2ct}{L_V(T) L_r(T)} \qquad \text{and}$$

$$V(t) \sim 2ct \qquad \text{as } t \to \infty \text{ where } c = \int_0^\infty r(x)\bar{H}_I(x)\,dx.$$

Boundary between cases 3 *and* 4 ($1 < \alpha_{min} < 2$, $2 - \alpha_{min} - \alpha_{sess} = -1$): *Assume* $c = \int_0^\infty r(x)\bar{H}_I(x)\,dx < \infty$. *Then*

$$\lim_{T\to\infty} \frac{V(Tt)}{T L_V(T) L_r(T)} = \infty \quad \text{and} \quad V(t) \sim 2ct \qquad \text{as } t \to \infty.$$

Proof. By Corollary 1.4.2 of the Characterization theorem [Bingham, Goldie and Teugels (1987), pp. 17–18] there exists $X \geq 0$ such that $L_V(x)$ and $L_V(x)L_r(x)$ are locally bounded and locally integrable on $[X, \infty)$. Since only the tail of $V(t)$ will be important, it is assumed $t > X$. In particular, this means the integrals I_1 and I_2 are nonzero.

Case 3: By assumption $1 < \alpha_{sess} < 2$ and $\alpha_{sess} < 3 - \alpha_{min}$. For I_3, let $J_1(y)$ be the inner integral

$$J_1(y) = \int_X^y r(x)\bar{H}_I(x)\,dx$$

so that $I_3 = 2\int_X^t J_1(y)\,dy$. By (3.1) and Karamata's theorem [Bingham, Goldie and Teugels (1987), p. 28], as $t \to \infty$

$$I_3 \sim \frac{\sigma_{lim}^2(3-\alpha_{min})(2-\alpha_{min})}{(\alpha_{sess}-1)(3-\alpha_{min}-\alpha_{sess})(4-\alpha_{min}-\alpha_{sess})} t^{4-\alpha_{min}-\alpha_{sess}} L_V(t) L_r(t).$$

Since $4 - \alpha_{min} - \alpha_{sess} > 1$, I_3 is regularly varying with higher order than I_1 and I_2 (which are constant and linear in t, respectively), the result follows.

Case 4: By Karamata's theorem [Bingham, Goldie and Teugels (1987), p. 28]

$$\begin{aligned} I_2 + I_3 &= 2\int_X^t \int_0^y r(x)\bar{H}_I(x)\,dx\,dy \\ &= 2c(t - X) - 2\int_X^t \int_y^\infty r(x)\bar{H}_I(x)\,dx\,dy, \end{aligned} \tag{4.5}$$

where $c = \int_0^\infty r(x)\bar{H}_I(x)\,dx < \infty$ since $2 - \alpha_{min} - \alpha_{sess} < -1$, as $y \to \infty$

$$\int_y^\infty r(x)\bar{H}_I(x)\,dx \sim \frac{-\sigma_{lim}^2(3-\alpha_{min})(2-\alpha_{min})}{2(\alpha_{sess}-1)(3-\alpha_{min}-\alpha_{sess})} y^{3-\alpha_{min}-\alpha_{sess}} L_V(y) L_r(y)$$

and so for the integral in (4.5)

$$\lim_{t\to\infty} \frac{\int_X^t \int_y^\infty r(x)\bar{H}_I(x)\,dx\,dy}{t^{4-\alpha_{min}-\alpha_{sess}} L_V(t) L_r(t)} = \frac{-\sigma_{lim}^2(3-\alpha_{min})(2-\alpha_{min})}{2(\alpha_{sess}-1)(3-\alpha_{min}-\alpha_{sess})(4-\alpha_{min}-\alpha_{sess})}.$$

Since $4-\alpha_{min}-\alpha_{sess} < 1$ the linear term of (4.5) is regularly varying with higher order, and $V(t) \sim 2ct$ as $t \to \infty$ while

$$\lim_{T\to\infty} \frac{V(Tt)}{TtL_V(T)L_r(T)} = \lim_{T\to\infty} \frac{2c}{L_V(T)L_r(T)}.$$

The second limit requires more information about $L_V(T)$ and $L_r(T)$. Both positive constants and the logarithm function are examples of slowly varying functions, and they would give quite different limits.

Boundary case: By assumption, $2-\alpha_{min}-\alpha_{sess} = -1$ and $c = \int_0^\infty r(x) \times \bar{H}_I(x)\,dx < \infty$. Since

$$r(x)\bar{H}_I(x) \sim \frac{\sigma_{lim}^2}{2} \frac{1}{\alpha_{sess}-1}(3-\alpha_{min})(2-\alpha_{min})x^{-1}L_V(x)L_r(x),$$

we know $L_V(x)L_r(x) \to 0$ as $x \to \infty$ since otherwise would make the tail too heavy for integrability.

Let $J_2(y) = \int_y^\infty r(x)\bar{H}_I(x)\,dx$ and $c_X = \int_0^X r(x)\bar{H}_I(x)\,dx$. Then

$$V(t) = I_1 + 2\int_X^t \bigl(c - J_2(y)\bigr)\,dy,$$

and so the asymptotic properties of $V(t)$ depend, in part, on those of $J_2(y)$ as $y \to \infty$. For $J_2(y)$, $\lim_{y\to\infty} J_2(y) = 0$, by [Bingham, Goldie and Teugels (1987), Proposition 1.5.9(b)] $J_2(y)$ is slowly varying with

$$\lim_{y\to\infty} \frac{J_2(y)}{L_V(y)L_r(y)} = \infty,$$

and by Karamata's theorem [Bingham, Goldie and Teugels (1987), p. 28] it follows that

$$\lim_{t\to\infty} \frac{\int_X^t J_2(y)\,dy}{tJ_2(t)} = 1, \qquad \lim_{t\to\infty} \frac{\int_X^t J_2(y)\,dy}{tL_V(t)L_r(t)} = \infty \quad \text{and}$$

$$\lim_{t\to\infty} \frac{\int_X^t J_2(y)\,dy}{t} = 0.$$

The results follow immediately. □

4.3 Proof of main results

Proof of Theorem 4.1. By Theorem 4.3,

$$\mathcal{L}\lim_{n\to\infty}\frac{\sum_{i=-\infty}^{\infty}X_{n,i}(t)}{\sqrt{\lambda_n}}=Z(t)$$

is Gaussian, mean zero and has covariance given by $\mathrm{Cov}[Z(s),Z(t)]=r(t-s)\int_{t-s}^{\infty}\bar{H}(z)\,dz$, $s<t$. Now, for any $t\geq 0$ and $T>0$ let $Y(t)$ be defined by

$$Y(t)=\frac{\tilde{Y}_t}{\sqrt{L_V(T)L_r(T)T^H}}.$$

Then for the characteristic function

$$\begin{aligned}\mathbb{E}[\exp(isY(Tt))] &= \exp\Big(-\frac{V(Tt)}{2L_V(T)L_r(T)T^{2H}}s^2\Big)\\ &\to \exp\Big(-\frac{\sigma_{lim}^2(2H-1+\alpha_{sess})}{2H(2H-1)}s^2\Big)\qquad \text{as } T\to\infty\end{aligned}$$

since

$$\lim_{T\to\infty}\frac{V(Tt)}{L_V(T)L_r(T)T^{2H}}=\frac{\sigma_{lim}^2(2H-1+\alpha_{sess})}{2H(2H-1)}t^{2H}$$

by Lemma 4.2. Thus, for fixed $t\geq 0$

$$Y(Tt)\xrightarrow{D}Y_1\qquad \text{as } T\to\infty,$$

where

$$Y_1\sim \mathrm{N}\Big(0,\frac{\sigma_{lim}^2(2H-1+\alpha_{sess})}{2H(2H-1)}t^{2H}\Big).$$

For the covariances, by Lemma 4.2 and stationary increments

$$\begin{aligned}&\mathrm{Cov}[Y(Ts),Y(Tt)]\\ &\quad=\frac{1}{2}\frac{(V(Ts)+V(Tt)-V(T(t-s)))}{L_V(T)L_r(T)T^{4-\alpha_{min}-\alpha_{sess}}}\\ &\quad\to \sigma^2\big(s^{4-\alpha_{min}-\alpha_{sess}}+t^{4-\alpha_{min}-\alpha_{sess}}-(t-s)^{4-\alpha_{min}-\alpha_{sess}}\big)\qquad \text{as } T\to\infty\\ &\quad=\sigma^2\big(s^{2H}+t^{2H}-(t-s)^{2H}\big).\end{aligned}$$

Thus, by the definition of fractional Brownian motion [Samorodnitsky and Taqqu (1994)] the left and right-hand sides of (4.1) have the same finite-dimensional distributions. □

Proof of Theorem 4.2. The proof is similar to that for Theorem 4.1 but shows, for

$$Y(t) = \frac{\tilde{Y}_t}{T^{1/2}}$$

and fixed $t \geq 0$

$$Y(Tt) \xrightarrow{D} Y_2 \qquad \text{as } T \to \infty \text{ where } Y_2 \sim \mathrm{N}(0, 2ct)$$

and for $s < t$

$$\mathrm{Cov}[Y(Ts), Y(Tt)] = \frac{1}{2}\frac{(V(Ts) + V(Tt) - V(T(t-s)))}{T} \to 2cs \qquad \text{as } T \to \infty$$

so the left and right-hand sides of (4.2) have the same finite-dimensional distributions. □

5 Weak convergence

In this section, weak convergence is established for both the first limit ($n \to \infty$) providing the incremental process, and the second limit ($T \to \infty$) providing the limiting process.

5.1 Weak convergence for the first limit

Theorem 5.1. *The convergence in Theorem 4.3 can be strengthened to weak convergence in the space $D[0,\infty)$ equipped with the J_1 topology, and the limiting Gaussian process is almost surely continuous.*

Proof. Convergence of the finite-dimensional distributions was established in Theorem 4.3 so it remains to prove tightness. For any $M > 0$, take $0 \leq t_1 \leq t_2 \leq t_3 \leq M$, $t_3 - t_1 < 1$ and define

$$U_n(t) = \frac{\sum_{i=1}^{N_n(t)} X_{n,i}(t)}{\sqrt{\lambda_n}} \quad \text{and}$$
$$\Delta = \big(U_n(t_2) - U_n(t_1)\big)^2 \big(U_n(t_3) - U_n(t_2)\big)^2. \tag{5.1}$$

To prove tightness in $D[0,\infty)$ it will be shown that there exist constants $K_1 > 0$ and $K_2 > 0$ (possibly depending on M) such that

$$E[\Delta] \leq K_1(t_3 - t_1)^2, \qquad 0 < t_3 - t_1 < 1 \quad \text{and} \tag{5.2}$$
$$E[U_n(t_2) - U_n(t_1)] \leq K_2(t_3 - t_1), \qquad 0 < t_3 - t_1 < 1. \tag{5.3}$$

Then the result will follow by a result due to Whitt (2002, pp. 226–227).

First we prove the bound of (5.2). Establishing the bound of (5.3) will require little additional work. Let $(X_{n,k}^{(i,j)}(t_{i+1}), \ldots, X_{n,k}^{(i,j)}(t_j))$ and $N_{n,i,j}(t)$ be defined as in the proof of Theorem 4.3 (so mutually independent in i, j, and k). For simplicity define $N_{n,i,j} = N_{n,i,j}(t_3)$. Define

$$U_{i,j,k} = \frac{1}{\sqrt{\lambda_n}} \sum_{u=1}^{N_{n,i,j}} X_{n,u}^{(i,j)}(t_k)$$

and

$$T_{i,j,k} = \frac{1}{\sqrt{\lambda_n}} \sum_{u=1}^{N_{n,i,j}} \left(X_{n,u}^{(i,j)}(t_k) - X_{n,u}^{(i,j)}(t_{k-1})\right) = U_{i,j,k} - U_{i,k,(k-1)}.$$

Since $E[X_{n,u}^{(i,j)}(t)] = 0$, by conditioning arguments it follows that $E[U_{i,j,k}] = E[T_{i,j,k}] = 0$. In what follows below both $U_{i,j,k}$ and $T_{i,j,k}$ will contribute to $E[\Delta]$ through the nonzero higher moments of the summands. In addition, $T_{i,j,k}$ will contribute through the temporal correlation within each summand.

It can be shown that

$$U_n(t_2) - U_n(t_1) = T_{0,2,2} + T_{0,3,2} + U_{1,2,2} + U_{1,3,2} - U_{0,1,1} \qquad \text{and}$$
$$U_n(t_3) - U_n(t_2) = T_{0,3,3} - T_{1,3,3} + U_{2,3,3} - U_{0,2,2} - U_{1,2,2}.$$

Then the expansion of Δ in (5.1) has 25^2 terms before simplifications. Those terms whose factors differ in either the first or second indices (i or j) are the product of independent terms. Since an individual term has expected value zero, $E[\Delta]$ gets contributions from only 31 nonzero terms. That is,

$$\begin{aligned} E[\Delta] = E[&(U_{1,2,2}^2 + U_{1,3,2}^2 + T_{0,3,2}^2 + U_{0,1,1}^2 + T_{0,2,2}^2) \\ &\times (U_{1,2,2}^2 + U_{0,2,2}^2 + U_{2,3,3}^2 + T_{1,3,3}^2 + T_{0,3,3}^2)] \\ &+ 4E[-T_{0,2,2}U_{1,3,2}T_{1,3,3}U_{0,2,2} - T_{0,3,2}U_{1,2,2}T_{0,3,3}U_{1,2,2} \\ &\quad - U_{1,2,2}U_{1,3,2}T_{1,3,3}U_{1,2,2} + T_{0,2,2}U_{1,2,2}U_{0,2,2}U_{1,2,2} \\ &\quad - T_{0,2,2}T_{0,3,2}T_{0,3,3}U_{0,2,2} + T_{0,3,2}U_{1,3,2}T_{0,3,3}T_{1,3,3}]. \end{aligned} \tag{5.4}$$

The idea now will be to bound both $\sigma_W^2 - r(u)$ and $p_{i,j}(t_3)$ (recall $E[N_{n,i,j}] = \lambda_n p_{i,j}(t_3)$) suitably.

From now K will be an arbitrary constant whose exact value will vary from one calculation to the next. By the assumed Lipschitz continuity of $H(x)$ we have

$$\begin{aligned} p_{1,2}(t_3) &= \int_{t_1}^{t_2} H(t_3 - s) - H(t_2 - s)\, ds \\ &\le \int_{t_1}^{t_2} K(t_3 - t_2)\, ds \le K(t_3 - t_1)^2. \end{aligned}$$

Since $0 \leq H(x) \leq 1$, $p_{0,1}(t_3)$, $p_{0,2}(t_3)$, $p_{1,3}(t_3)$, $p_{2,3}(t_3)$ are bounded above by $(t_3 - t_1)$. Thus we have

$$E[U_{1,2,2}^2] = \frac{1}{\lambda_n} E\left[\sum_{u=1}^{N_{n,1,2}} X_{n,u}^{(1,2)}(t_2) \sum_{v=1}^{N_{n,1,2}} X_{n,v}^{(1,2)}(t_2)\right]$$
$$= \frac{1}{\lambda_n} E[N_{n,1,2}\sigma_W^2]$$
$$\leq K(t_3 - t_1)^2$$

and for $(i, j) \neq (1, 2)$

$$E[U_{i,j,k}^2] \leq K p_{i,j}(t_3) \leq K(t_3 - t_1). \tag{5.5}$$

Since $E[N_{n,i,j}^2 - N_{n,i,j}] = \lambda_n^2 p_{i,j}^2$ it follows that

$$E[U_{1,2,k}^4] = \frac{1}{\lambda^2} E\big[N_{n,1,2} E[X_{n,i}^4(t_k)] + 3(N_{n,1,2}^2 - N_{n,1,2})\sigma_W^4\big] \leq K(t_3 - t_1)^2.$$

To understand contributions from $T_{i,j,k}$ notice that we can write

$$\sigma_W^2 - r(t) = r(0) - r(t) = \frac{\mu_1}{\mu_1 + \mu_2}\big(1 - \pi_{11}(t)\big), \tag{5.6}$$

where $\pi_{11}(t) = P(W(t) = 1 | W(0) = 1)$, which is continuous in t since the on and off distributions are assumed continuous. Now by Taqqu, Willinger and Sherman (1997), equation (13),

$$1 - \pi_{11}(t) = \frac{1}{\mu_1} \int_0^t \bar{F}_1(u)\,du + \int_0^t \bar{F}_1(t-u) h_{12}(u)\,du,$$

where H_{12} is the renewal function for the interrenewal distribution $F_1 * F_2$ (i.e., $H_{12} = \sum_{k=1}^{\infty} (F_1 * F_2)^k$) with the density $h_{12}(u)$. Using the Fundamental Theorem of Calculus for the first term, and Laplace transform arguments for the second term,

$$\frac{d}{dt}\big(1 - \pi_{11}(t)\big) = \frac{1}{\mu_1}\bar{F}_1(t) - \int_0^t \bar{f}_1(t-u) h_{12}(u)\,du.$$

On the right side, the first term is bounded between 0 and $1/\mu_1$. The second term is also bounded [Bhat (1972), p. 167]. Therefore, since $(1 - \pi_{11}(t))$ is continuous with a bounded density, it is Lipschitz continuous. Using (5.6) it follows that

$$|\sigma_W^2 - r(u)| \leq Ku.$$

It can also be shown that

$$E[U_{i,j,k} U_{i,j,(k-1)}] = \frac{1}{\lambda_n} E\left[\sum_{u=1}^{N_{n,i,j}} X_{n,u}^{(i,j)}(t_k) \sum_{v=1}^{N_{n,i,j}} X_{n,v}^{(i,j)}\big(t_{(k-1)}\big)\right]$$
$$= p_{i,j}(t_3) r\big(t_k - t_{(k-1)}\big)$$

and by expanding the square

$$\begin{aligned} E[T^2_{i,j,k}] &= E\big[\big(U_{i,j,k} - U_{i,j,(k-1)}\big)^2\big] \\ &\le 2p_{i,j}(t_3)\big|\sigma^2_W - r\big(t_k - t_{(k-1)}\big)\big| \\ &\le K(t_3 - t_1)^2. \end{aligned} \tag{5.7}$$

Notice that since $|\sigma^2_W - r(u)| \le 2\sigma^2_W$ the looser bound $E[T^2_{i,j,k}] \le Kp_{i,j}(t_3)$ is also available.

Exploiting the idea that since $\{W(t)\}$ is a 0–1 process we know for the joint moments $E[W^r(t_i)W^s(t_j)] = E[W(t_i)W(t_j)]$ for $r, s \in \mathbb{Z}^+$ it can also be shown that

$$E[U_{1,3,2}U^3_{1,3,3}] = \frac{p_{1,3}(t_3)}{\lambda_n}(1 - 3\mu + 3\mu^2)r(t_3 - t_2) + 3p^2_{1,3}(t_3)\sigma^2_W r(t_3 - t_2)$$

and

$$\begin{aligned} E[U^2_{1,3,2}U^2_{1,3,3}] = {}& \frac{p_{1,3}(t_3)}{\lambda_n}(1 - 2\mu^2)r(t_3 - t_2) + \frac{p_{1,3}(t_3)}{\lambda_n}\sigma^4_w + p^2_{1,3}(t_3)\sigma^4_w \\ &+ 2p^2_{1,3}(t_3)[r(t_3 - t_2)]^2. \end{aligned}$$

Fourth moments of $T_{i,j,k}$ can now be found. For example, we have that

$$\begin{aligned} E[T^4_{1,3,3}] &= E[U^4_{1,3,3} - 4U^3_{1,3,3}U_{1,3,2} + 6U^2_{1,3,3}U^2_{1,3,2} - 4U_{1,3,3}U_{1,3,2} + U^4_{1,3,2}] \\ &= \frac{2p_{1,3}(t_3)}{\lambda_n}\big(\sigma^2_W - r(t_3 - t_2)\big) + 12p_{1,3}(t_3)\sigma^2_W\big(\sigma^2_W - r(t_3 - t_2)\big) \\ &\quad + 12p^2_{1,3}\big(\sigma^4_W - r^2(t_3 - t_2)\big) \\ &\le \frac{2p_{1,3}(t_3)}{\min_n\{\lambda_n\}}K(t_3 - t_2) \\ &\quad + 12[p_{1,3}(t_3)\sigma^2_W K(t_3 - t_2) + p^2_{1,3}(t_3)(2\sigma^2_W)K(t_3 - t_2)] \\ &\le K(t_3 - t_1)^2. \end{aligned}$$

The inequalities above provide a means to bound the first 25 terms in the expansion of (5.4). For the remaining six terms we use the Cauchy–Schwarz inequality and independence. For example, it can be shown that

$$\begin{aligned} E[T_{0,2,2}U_{1,3,2}T_{1,3,3}U_{0,2,2}] &\le (E[T^2_{0,2,2}U^2_{1,3,2}])^{1/2}(E[T^2_{1,2,3}U^2_{0,2,2}])^{1/2} \\ &= (E[T^2_{0,2,2}]E[U^2_{1,3,2}]E[T^2_{1,3,3}]E[U^2_{0,2,2}])^{1/2} \\ &\le K(t_3 - t_1)^2. \end{aligned}$$

Now every term in the expansion of (5.4) is bounded by $K(t_3 - t_1)^2$ for some constant $K > 0$ and (5.2) is shown.

To establish (5.3) note that for $t_3 - t_1 < 1$

$$\begin{aligned} E\big[(U_n(t_3) - U_n(t_1))^2\big] &= E[(T_{0,3,3} + U_{1,3,3} + U_{2,3,3} - U_{0,1,1} - U_{0,2,1})^2] \\ &\le K\,E[(T_{0,3,3}^2 + U_{1,3,3}^2 + U_{2,3,3}^2 + U_{0,1,1}^2 + U_{0,2,1}^2)] \\ &\le K[(t_3 - t_1)^2 + (t_3 - t_1)] \\ &\le K_2(t_3 - t_1), \end{aligned}$$

where the second line follows from independence for distinct indices and the third line follows from (5.5) and (5.7). Thus, weak convergence in $D[0,\infty)$ and limiting almost surely continuous paths are established [Whitt (2002), pp. 126–127]. □

5.2 Weak convergence for the second limit

Theorem 5.2. *The convergence (as $T \to \infty$) in Theorems 4.1 and 4.2 can be strengthened to weak convergence in $C[0,\infty)$ with the uniform topology.*

Proof. Recall the integrated process

$$\tilde{Y}_t = \int_0^t Z(u)\,du$$

defined in Section 4.2 which has stationary increments. Since convergence of the finite-dimensional distributions is established above (Theorems 4.1 and 4.2) it remains to prove tightness, which we do using a moment condition [Billingsley (1968), p. 95].

Case 3: It will be shown that for $0 < u < 1$ and T sufficiently large

$$E\left[\frac{(\tilde{Y}_{T(s+u)} - \tilde{Y}_{Ts})^2}{T^{2H} L_V(t) L_r(t)}\right] = \frac{V(Tu)}{T^{2H} L_V(t) L_r(t)} \le K u^{1+\delta}$$

for some constant $K > 0$ and some constant $\delta > 0$.

Since $V(t) \sim \sigma^2 t^{2H} L_V(t) L_r(t)$ we have

$$\frac{V(Tu)}{T^{2H} L_V(t) L_r(t)} \le \frac{K_1 (Tu)^{2H} L_V(Tu) L_r(Tu)}{T^{2H} L_V(T) L_r(T)}.$$

Proceeding as in [Mikosch et al. (2002)] (or using the Potter bounds [Bingham, Goldie and Teugels (1987), p. 25]), there exists t_0 such that for all $u \le 1$ and $Tu \ge t_0$

$$\frac{L_V(Tu) L_r(Tu)}{L_V(T) L_r(T)} \le K_2 u^{-\delta}$$

with small $\delta > 0$ chosen so that $2H - 2\delta > 1$. Then

$$\frac{V(Tu)}{T^{2H} L_V(T) L_r(T)} \le K_1 K_2 u^{2H-\delta}.$$

For $0 < Tu < t_0$, since $V(t) \sim \sigma^2 t^{2H} L_V(t) L_r(t)$ for T sufficiently large we have

$$\frac{V(Tu)}{T^{2H} L_V(T) L_r(T)} \leq \frac{2(Tu)^2}{T^{2H} L_V(T) L_r(T)} \leq \frac{2(Tu)^{1+\delta}}{T^{2H} L_V(T) L_r(T)} t_0^{1-\delta} u^{1+\delta}$$
$$\leq K_3 u^{1+\delta}.$$

Since $2H - \delta < 1 + \delta$ we have

$$\frac{V(Tu)}{T^{2H} L_V(T) L_r(T)} \leq K u^{1+\delta}$$

for T large enough.

Boundary case and case 4: Since $\{\tilde{Y}_t, t \geq 0\}$ has stationary increments it is enough to show for that for $u > 0$ and some constant K

$$E\left[\left(\frac{\tilde{Y}_{Tu}}{T^{1/2}}\right)^4\right] \leq K u^2.$$

Now, we know $\tilde{Y}_t \sim N(0, V(t))$ and so $E[\tilde{Y}_t^4] = 3V^2(t)$, while

$$V(t) = 2ct - 2 \int_0^t \int_y^\infty r(x) \bar{H}_I \, dx \, dy \leq 2ct.$$

Thus for all $T > 0$ and $u > 0$ and some constant K

$$E\left[\left(\frac{\tilde{Y}_{Tu}}{T^{1/2}}\right)^4\right] = 3 \frac{V^2(Tu)}{T^2} \leq K u^2,$$

tightness follows, and weak convergence is established [Billingsley (1968), p. 95]. □

6 Simulation results

In this section the limit results obtained above are demonstrated through simulation. The starting point for each simulation is the "session" which arrives according to a Poisson process with intensity λ. In these simulations each session has a Pareto distributed lifetime with mean μ_{sess} and characteristic exponent α_{sess}, $1 < \alpha_{sess} < 2$. To simplify the implementation, session lengths were truncated to integers. This step produced a list of start and end times for the "sessions." In a second step, these start and end times were converted to a timeseries $\{B(k), k = 1, \ldots, 25 \times 10^6\}$, the number of sessions alive at time k. (For these simulations, the first 50×10^6 observations were discarded to achieve stationarity and the next 25×10^6 observations were retained.) This corresponds to the number of busy servers in the infinite source Poisson model (i.e., the busy server process of an $M/G/\infty$ queue).

On–off activity within the sessions was simulated in a third step. Starting with the same session start and end times, session activity within each session was simulated as a stationary on–off process with parameters α_1, μ_1, α_2 and μ_2. This produced a dataset $\{A(k), k = 1, \dots, 25 \times 10^6\}$ which represents the number of active (i.e., alive and on) sources for each k. Finally, a dataset $\{C(k), k = 1, \dots, 25 \times 10^6\}$ was produced where

$$C(k) = \frac{\mu_1}{\mu_1 + \mu_2} B(k) - A(k)$$

and corresponds to the increments of (3.3) above.

As shown in Table 1, several series, each corresponding to different parameter values, were simulated. For the datasets in series 1 through series 3, five simulations were performed in each series. For series 4, 10 simulations were performed. For all simulations, the parameters for the on–off activity were unchanged, with $\alpha_1 = \alpha_2 = 1.4$ and $\mu_1 = \mu_2 = 100$. The parameters for the sessions, namely α_{sess}, μ_{sess}, and λ are shown in Table 1. The values of α_1, α_2 and α_{sess} were chosen mainly for illustrative purposes. Hurst parameters can be calculated for the infinite source Poisson model ($H_{ISP} = (3 - \alpha_{sess})/2$), the stationary on–off model [Taqqu, Willinger and Sherman (1997)] ($H_{on/off} = (3 - \min(\alpha_1, \alpha_2))/2$), and the Hurst parameter $H_{burst/on/off}$ described above in Theorems 4.1 or 4.2. The theoretical values are all shown in the table.

Hurst parameter estimates for the simulated data were made using the "logscale diagram" [Abry and Veitch (1999)]. This technique divides the data into blocks of size 2^j and calculates the variance of the nth order Daubechies wavelet coefficients for each value of j. Values of j are sometimes called "octaves." Collectively, the points show how the data scales with increasing block length. H is estimated using weighted linear regression to find the slope of the graph over an interval of octaves chosen by the user. Besides being quick, this technique provides approximate 95% confidence intervals.

Figure 2 shows the results from series 1–4. For each run, the central dot is the value of the estimated Hurst parameter, while the high and low bars show the ends of the approximate 95% confidence inteval. For series 1 (top left), the estimate is peformed over the interval $j = 15, \dots, 21$. Since the mean session length is 12,000, and $\log_2 12{,}000 \sim 13.6$, $j = 15$ provides a starting octave beyond any artifacts associated with the mean session length. For series 2 (top right) and series 3

Table 1 *Parameter values for the simulations*

	α_{sess}	μ_{sess}	λ	μ_1, μ_2	α_1, α_2	$H_{burst/on/off}$	H_{ISP}	$H_{on/off}$
Series 1	1.2	12000	1	100	1.4	0.7	0.9	0.8
Series 2	1.2	120	5	100	1.4	0.7	0.9	0.8
Series 3	1.2	1200	1	100	1.4	0.7	0.9	0.8
Series 4	1.8	1200	1	100	1.4	0.5	0.6	0.8

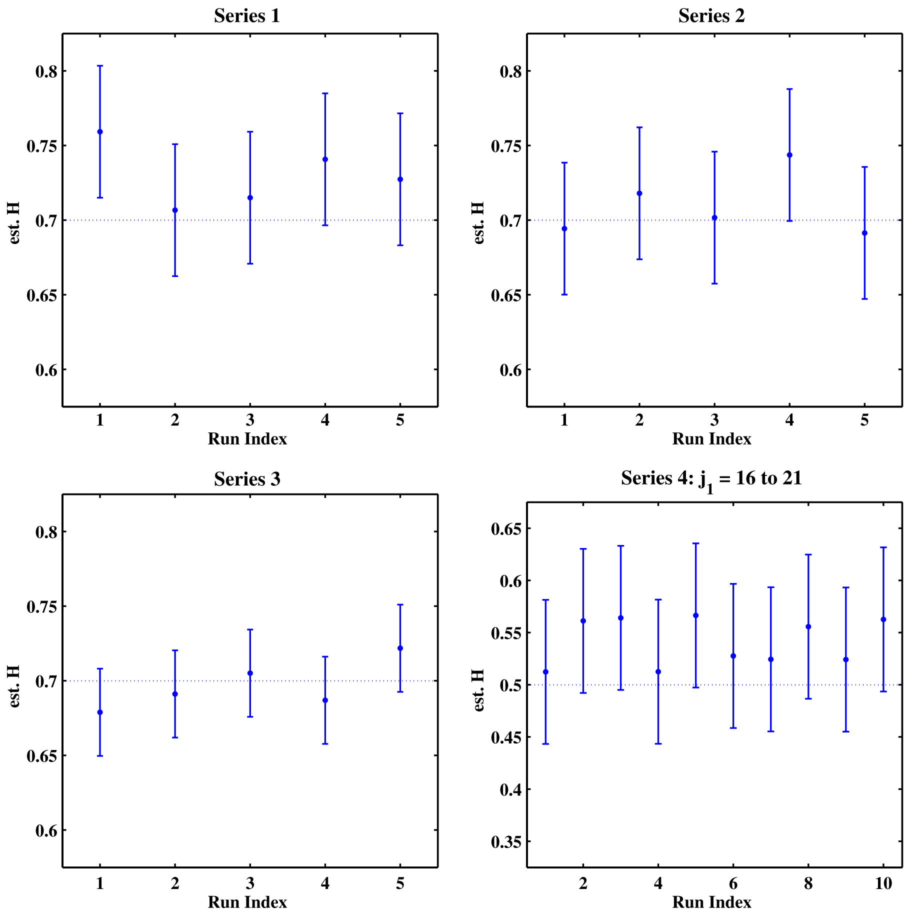

Figure 2 *Estimated Hurst parameters with approximate 95% confidence intervals for the simulated data.*

(bottom left), with shorter mean session lengths, $j = 14, \ldots, 21$ was used for the regression. In all cases the estimated Hurst parameter is within 0.06 of the value described in Theorem 4.1, and in all but one case the theoretical value is within the confidence interval.

Figure 2 (bottom right) shows the Hurst parameter estimates for the series 4 data using octaves 16 to 21. The horizontal line corresponds to a value of 0.5, which is the Hurst parameter predicted by Theorem 4.2. The estimated values are somewhat higher than that predicted by theory, although the confidence intervals do cover the theoretical value. The logscale diagram in Figure 3 shows the difficulties in estimating the Hurst parameters with this data. One hopes to find an interval of linear increase on which to obtain the Hurst parameter estimate, but where does such an interval start? A horizontal line would correspond to a Hurst parameter

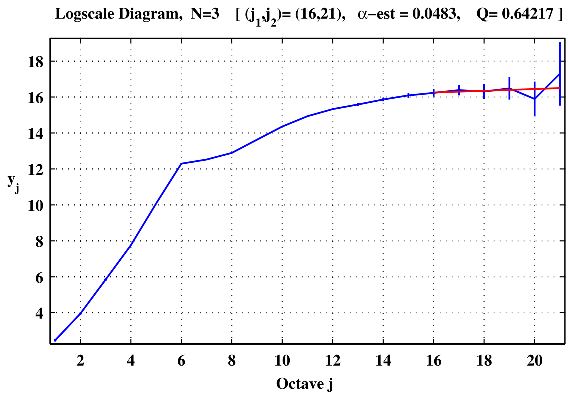

Figure 3 *Logscale diagram for run 8 of series 4 (right). An ultimately horizontal line corresponds to* $H = 0.5$*, and is predicted by Theorem 4.2.*

of 0.5. A reasonable start is octave 16 for which the estimate would be 0.524. But, it would be equally reasonable to start at a larger octave, for which the estimate would be even closer to 0.5. All the data for series 4 is similar, and so estimates using octaves 16 to 21 are shown here. Moreover, since the trend in each logscale diagram is to flatness, the point estimates are likely to be consistently biased too large. Even so, the results from these simulations appear to agree with the value predicted in Theorem 4.2. In all cases the confidence interval covers the predicted value.

7 Conclusions

In this paper, the infinite source Poisson model for bursts has been combined with the on–off model for in-burst activity. Burst and on–off durations are assumed heavy tailed with infinite variance and finite mean. By using convergence results for random centering of random sums, weak convergence of the workload to fractional Brownian motion is shown. The degree of long-range dependence is shown to depend on the tail indices of both the on–off durations and the lifetimes distributions. Moreover, the results can be separated into cases depending on those tail indices. In one case where all distributions are heavy tailed it is shown that the limiting result is Brownian motion.

The method of proof here (and in Rolls (2003)) uses iterated limits for first the arrival rate, and then the time scale, as in Taqqu, Willinger and Sherman (1997) and Brichet et al. (2000). As in those papers, one could study convergence with the limits taken in reverse order. This is left for future work. Moreover, the iterated limits are not entirely satisfactory, and a single limit as in Mikosch et al. (2002) and Gaigalas and Kaj (2003) would be preferable. Unfortunately, the change in details within the proof would make it essentially a new proof, and is left for future work.

Random centering in traffic models appears to be a new idea. The application is not necessarily targeted at current TCP/IP data networks but simply any network where the start and end of some kind of "sessions" are signaled. It would be interesting to see if in a network element this can be practically used to advantage, possibly leveraging the "nicer" queueing behavior of Brownian motion workloads.

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