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Nonparametric density estimation for functional data by delta sequences

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Abstract. We consider the problem of estimation of density function by the method of delta sequences for functional data with values in an infinite dimensional separable Banach space.

1 Introduction

Methods of nonparametric estimation of density function and regression function are widely discussed in the literature starting from Prakasa Rao (1983, 1999a), Silverman (1986) and more recently in Efromovich (1999). Among the most interesting of recently developed statistical methods are those for analyzing data in the form of curves presently known as functional data. Nonparametric statistical models have been developed recently for such data. Functional data are present in many fields of application such as medicine, environmetrics, chemometrics, econometrics, etc. Analysis of such functional data is of importance in problems of classification, discrimination, regression, prediction and longitudinal studies. For an introduction to this area, see Ramsay and Silverman (2002, 2005). Gasser, Hall and Presnell (1998) consider density and mode estimation for data taking values in a normed vector space. Nonparametric regression estimation for functional data has been studied in Masry (2005), Rachdi and Vieu (2007) and Ferraty and Vieu (2006).

Our aim in this paper is to study density estimation for random elements taking values in an infinite dimensional separable Banach space such as the space of continuous functions on the interval [0, 1] endowed with the supremum norm. Examples of functional data where such spaces arise are stochastic processes with continuous sample paths on a finite interval associated with the supremum norm or stochastic processes whose sample paths are square integrable on the real line. Dabo-Niang (2004) and Dabo-Niang, Ferraty and Vieu (2006) developed a naive kernel estimator and a general kernel estimator for the estimation of a probability density function. Since there is no analog of the Lebesgue measure on a Banach space, the density function of a random element, if it exists, is related to the dominating measure with respect to which the density function or the Radon–Nikodym

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derivative is computed. Problems involving the density estimation of a random element taking values in a metric space were earlier studied by Geffroy (1974). Wertz (1972) and Craswell (1965) investigated the properties of kernel type density estimators for random elements taking values in locally compact topological groups [cf. Prakasa Rao (1983), page 226]. We study density estimation through the method of delta sequences generalizing the method of kernel density estimation in Dabo-Niang, Ferraty and Vieu (2006). It is known that the method of delta sequences unifies the kernel method of density estimation, histogram method and some other methods such as the method of orthogonal series for suitable choices of orthonormal bases in the one-dimensional and finite-dimensional cases. For a discussion of the method of delta sequences in the finite-dimensional cases, see Prakasa Rao (1983), pages 136–143 and pages 218–224, Walter and Blum (1979) and Susarla and Walter (1981). Density estimation for Markov processes using delta sequences is studied in Prakasa Rao (1978, 1979a) and sequential nonparametric estimation of density in the univariate case via delta sequences is investigated in Prakasa Rao (1979b). A different method of density estimation, for functional data by wavelets, was discussed recently in Prakasa Rao (2009).

Let $\{\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_t, t \geq 0\}$ be a nondecreasing family of sub-σ-algebras of $\mathcal{F}$. Let $\{W_t, t \geq 0\}$ be a standard Wiener process defined on $\{\Omega, \mathcal{F}, P)$ such that W_t is $\mathcal{F}_t$-measurable. Let $C[0, T]$ be the space of real-valued continuous functions defined on the interval $[0, T]$ associated with the supremum norm topology. It is known that the standard Wiener process induces a probability measure μ_W on the space $C[0, T]$ associated with Borel σ-algebra generated by the supremum norm topology. Consider a diffusion process $\{X(t), 0 \leq t \leq T\}$ governed by the stochastic differential equation

$$dX(t) = a(t, X(t))\,dt + b(t, X(t))\,dW(t), \qquad X(0) = x_0,\ 0 \leq t \leq T.$$

Under some conditions on the functions $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$, it can be shown that the probability measure μ_X induced by the process X on the space $C[0, T]$ is absolutely continuous with respect to the probability measure μ_W and one can compute the Radon–Nikodym derivative of μ_X with respect to μ_W by using the Girsanov's theorem. This can be considered as the probability density of the process X on the space $C[0, T]$. More details on such a frame work and other examples are given in Prakasa Rao (1999b). One of the motivations for analysis of functional data in our view is inference for stochastic processes [cf. Prakasa Rao (1999b, 1999c)]. We are assuming here that the complete path of the process is observable for inferential purposes. However, if the process can be observed only at discrete times either on a fine grid or when the data is sparse, other methods have to be developed as in the case of parametric inference for a discrete data, for instance, for the diffusion processes [cf. Prakasa Rao (1999b)]. Note that an important motivation for development of statistical methods for analysis of a functional data is that the parametric methods behave badly when the dimension is large or infinite due to "curse of dimensionality" and the infinite dimensional data can be viewed as a theoretical approximation of large dimensional data.

2 Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space and E be an infinite dimensional separable Banach space and $\mathcal{B}$ be the σ-algebra of Borel subsets of E. Suppose X is a random element defined on $(\Omega, \mathcal{F}, P)$ taking values in $(E, \mathcal{B})$ and that it has a density f with respect to a σ-finite measure μ on $(E, \mathcal{B})$ such that $0 < \mu(A) < \infty$ for every open ball $A \subset E$. Note that

$$P(X \in A) = \int_A f(x)\mu(dx), \qquad A \in \mathcal{B}.$$

Let $X_1, \ldots, X_n$ be independent and identically distributed random elements as X. Let $\|\cdot\|$ denote the norm on the Banach space E. Suppose C is a compact subset of E with the property that for any $r_n > 0$ there exist $t_k \in E, 1 \le k \le d_n$, where

$$C \subset \bigcup_{k=1}^{d_n} B(t_k, r_n)$$

and there exists $\alpha_n > 0$ such that $d_n r_n^{\alpha_n}$ is a constant $c > 0$. This condition gives a geometric link between the number d_n of open spheres and the radius of the r_n of the open spheres covering the compact set C [cf. Ferraty and Vieu (2008)]. Here $B(t_k, r_n)$ denotes the open sphere with center t_k and radius r_n.

(G1) Assume that, for every $\varepsilon > 0$, there exists $\gamma > 0$ such that

$$|f(y) - f(x)| \le \varepsilon \qquad \text{if } \|y - x\| \le \gamma, x \in C, y \in E.$$

Note that this condition is stronger than the uniform continuity of the function f on C as it refers to $x \in C$ and $y \in E$. It follows, in particular, that there exists a postive constant M such that

$$\sup_{x \in C} f(x) \le M < \infty.$$

Definition. A sequence of nonnegative functions $\{\delta_m(x, y), m \ge 1\}$ defined on $E \times E$ is said to be a *delta sequence* with respect to the measure μ if the following conditions hold:

(G2) for every $\gamma, 0 < \gamma \le \infty$,

$$\lim_{m \to \infty} \sup_{x \in C} \left| \int_{[y:\|y-x\| \le \gamma]} \delta_m(x, y)\mu(dy) - 1 \right| = 0;$$

(G3) there exists a constant $c_0 > 0$ such that

$$\sup_{x \in C, y \in E} \delta_m(x, y) \le c_0 s_m < \infty,$$

where $0 < s_m \to \infty$ as $m \to \infty$ and $\lim_{m \to \infty} \frac{m}{s_m \log m} = \infty$;

(G4) there exist $c > 0$, $\beta_1 > 0$ and $\beta_2 > 0$ such that

$$|\delta_m(x_1, y) - \delta_m(x_2, y)| \le c s_m^{\beta_2} \|x_1 - x_2\|^{\beta_1}$$

for all $x_1, x_2, y \in E$; and

(G5) for any $\gamma > 0$,

$$\lim_{m\to\infty} \sup_{(x,y)\in C\times[y:\|y-x\|>\gamma]} \delta_m(x, y)\|y - x\| = 0.$$

Further suppose that

(G6) $d_n = n^\alpha, \alpha > 0$, and

$$r_n^{\beta_1} s_m^{\beta_2} < ([s_m \log m]/m)^{1/2}$$

for large m and n.

Let

$$f_n(x) = \frac{1}{n}\sum_{i=1}^{n} \delta_m(x, X_i).$$

The choice of the sequence m might depend on n such that $m \to \infty$ as $n \to \infty$.

We now prove the following result leading to uniform strong consistency of the estimator $f_n(x)$ over the set C as an estimator of $f(x)$.

Theorem 1. *Suppose that $m \to \infty$ and there exists $0 < p < 1$ such that $n^p \le m \le n$ for n large. Under the conditions* (G1)–(G6),

$$\lim_{n\to\infty} \sup_{x\in C} |f_n(x) - f(x)| = 0 \qquad a.s.$$

Proof. Let $\gamma > 0$ and $x \in C$. Define

$$I_1(x) = \int_{[y:\|y-x\|\le\gamma]} \delta_m(x, y)\big(f(y) - f(x)\big)\mu(dy) \tag{2.1}$$

and

$$I_2(x) = \int_{[y:\|y-x\|>\gamma]} \delta_m(x, y)\big(f(y) - f(x)\big)\mu(dy). \tag{2.2}$$

Observe that

$$E[f_n(x)] - f(x) = \int_E \delta_m(x, y) f(y)\mu(dy) - f(x) \tag{2.3}$$

and hence

$$\begin{aligned} &E[f_n(x)] - f(x) - I_1(x) - I_2(x) \\ &\quad = \int_E \delta_m(x, y) f(y)\mu(dy) - f(x) - \int_E \delta_m(x, y)\big(f(y) - f(x)\big)\mu(dy) \qquad (2.4) \\ &\quad = f(x)\left[\int_E \delta_m(x, y)\mu(dy) - 1\right]. \end{aligned}$$

Observe that

$$\lim_{m\to\infty}\sup_{x\in C}\left|\int_E \delta_m(x,y)\mu(dy)-1\right|=0$$

by (G2). Hence

$$\lim_{n\to\infty}\sup_{x\in C}|E[f_n(x)]-f(x)-I_1(x)-I_2(x)|=0 \tag{2.5}$$

by the conditions (G1) and (G2). Furthermore, for every $x\in C$,

$$\begin{aligned}
|I_2(x)| &\le \int_{[y:\|y-x\|>\gamma]}\delta_m(x,y)f(y)\mu(dy)\\
&\quad + f(x)\int_{[y:\|y-x\|>\gamma]}\delta_m(x,y)\mu(dy)\\
&= \int_{[y:\|y-x\|>\gamma]}\delta_m(x,y)\frac{\|y-x\|}{\|y-x\|}f(y)\mu(dy)\\
&\quad + f(x)\int_{[y:\|y-x\|>\gamma]}\delta_m(x,y)\mu(dy)\\
&\le \frac{1}{\gamma}\sup_{(x,y)\in C\times[y:\|y-x\|>\gamma]}[\delta_m(x,y)\|y-x\|]\int_{[y:\|y-x\|>\gamma]}f(y)\mu(dy)\\
&\quad + f(x)\int_{[y:\|y-x\|>\gamma]}\delta_m(x,y)\mu(dy)\\
&\le \frac{1}{\gamma}\sup_{(x,y)\in C\times[y:\|y-x\|>\gamma]}[\delta_m(x,y)\|y-x\|]\\
&\quad + M\sup_{x\in C}\int_{[y:\|y-x\|>\gamma]}\delta_m(x,y)\mu(dy)
\end{aligned} \tag{2.6}$$

which implies that

$$\begin{aligned}
&\sup_{x\in C}|I_2(x)|\\
&\quad\le \frac{1}{\gamma}\sup_{(x,y)\in C\times[y:\|y-x\|>\gamma]}[\delta_m(x,y)\|y-x\|]\\
&\qquad + M\sup_{x\in C}\int_{[y:\|y-x\|>\gamma]}\delta_m(x,y)\mu(dy).
\end{aligned}$$

Assumptions (G2) and (G5) imply that the two terms on the right-hand side of the above inequality tend to zero as $m\to\infty$.

Note that, for every $\varepsilon>0$, there exists $\gamma>0$ such that

$$|f(y)-f(x)|\le\varepsilon \qquad \text{if } \|y-x\|\le\gamma,\ x\in C,\ y\in E$$

by condition (G1). Then there exists $\gamma > 0$ such that

$$|I_1(x)| \le \varepsilon \int_{[y:\|y-x\|\le\gamma]} \delta_m(x, y)\mu(dy). \tag{2.7}$$

Hence

$$\sup_{x\in C} |I_1(x)| \le \varepsilon \sup_{x\in C} \left| \int_{[y:\|y-x\|\le\gamma]} \delta_m(x, y)\mu(dy) \right|$$

and the term on the right-hand side can be made smaller than 2ε as $m \to \infty$ by condition (G2). Therefore

$$\lim_{n\to\infty} \sup_{x\in C} |E[f_n(x)] - f(x)| = 0. \tag{2.8}$$

We now prove that

$$\lim_{n\to\infty} \sup_{x\in C} |f_n(x) - E[f_n(x)]| = 0 \quad \text{a.s.}$$

Let $x \in C$. Then

$$f_n(x) - E[f_n(x)] = \frac{1}{n} \sum_{i=1}^{n} Z_{i,x},$$

where

$$Z_{i,x} = \delta_m(x, X_i) - E[\delta_m(x, X_i)].$$

Note that $Z_{i,x}$, $1 \le i \le n$, are independent and identically distributed real-valued bounded random variables bounded by $2c_0 s_m$ by condition (G3). Applying the Bernstein's inequality [see Hoeffding (1963); Prakasa Rao (1983), page 183], we get that, for any $\eta > 0$ and m large,

$$\begin{aligned} P&\left[|f_n(x) - E[f_n(x)]| > \eta\sqrt{\frac{s_m \log m}{m}} \right] \\ &\le 2\exp\left[-\frac{n((s_m \log m)/m)\eta^2}{4c_0 M s_m + 4c_0 s_m \eta\sqrt{(s_m \log m)/m}} \right] \\ &\le 2\exp\left[-\frac{\eta^2 n \log m/m}{8c_0 M} \right]. \end{aligned} \tag{2.9}$$

Since the set $C \subset C_n = \bigcup_{k=1}^{d_n} B(t_k, r_n)$, for any $x \in C$, there exists an index $\tau(x)$ among $t_1, \dots, t_{d_n}$ such that $x \in B(t_{\tau(x)}, r_n)$. Hence

$$\begin{aligned} P&\left[\sup_{x\in C} |f_n(x) - E[f_n(x)]| > 2\eta\sqrt{\frac{s_m \log m}{m}} \right] \\ &\le P\left[\sup_{x\in C} |f_n(x) - E[f_n(x)] \right. \end{aligned} \tag{2.10}$$

$$- f_n\big(t_{\tau(x)}\big) + E\big[f_n\big(t_{\tau(x)}\big)\big]\big| > \eta\sqrt{\frac{s_m \log m}{m}}\Bigg]$$

$$+ P\Bigg[\max_{1\le k\le d_n} |f_n(t_k) - E[f_n(t_k)]| > \eta\sqrt{\frac{s_m \log m}{m}}\Bigg].$$

Conditions (G4) and (G6) imply that

$$\sup_{x\in C}\big|f_n(x) - f_n\big(t_{\tau(x)}\big)\big| \le O(r_n^{\beta_1} s_m^{\beta_2}) = O\Bigg(\sqrt{\frac{s_m \log m}{m}}\Bigg).$$

Hence, for some $\eta > 0$ and for sufficiently large n and large m

$$P\Bigg[\sup_{x\in C}\big|f_n(x) - E[f_n(x)] - f_n\big(t_{\tau(x)}\big) + E\big[f_n\big(t_{\tau(x)}\big)\big]\big| > \eta\sqrt{\frac{s_m \log m}{m}}\Bigg] = 0.$$

Therefore,

$$P\Bigg[\sup_{x\in C}|f_n(x) - E[f_n(x)]| > \eta\sqrt{\frac{s_m \log m}{m}}\Bigg] \le 2d_n \exp\bigg[-\frac{\eta^2 n \log m/m}{8c_0 M}\bigg].$$

Since $m \le n$ and $\log m \ge p \log n$, it follows that there exists $\eta > 0$ such that

$$P\Bigg[\sup_{x\in C}|f_n(x) - E[f_n(x)]| > \eta\sqrt{\frac{s_m \log m}{m}}\Bigg] \le n^{\alpha - p\eta^2/C_1}$$

for some positive constant C_1. Applying the Borel–Cantelli lemma, we get that

$$\sup_{x\in C}|f_n(x) - E[f_n(x)]| = O\Bigg(\sqrt{\frac{s_m \log m}{m}}\Bigg) \qquad \text{a.s.}$$

Since

$$\frac{s_m \log m}{m} \to 0 \qquad \text{as } m \to \infty,$$

we obtain that

$$\sup_{x\in C}|f_n(x) - E[f_n(x)]| \to 0 \qquad \text{a.s. as } n \to \infty. \tag{2.11}$$

Combining equations (2.8) and (2.11), we obtain that

$$\sup_{x\in C}|f_n(x) - f(x)| \to 0 \qquad \text{a.s. as } n \to \infty. \tag{2.12}$$

□

Theorem 2. *Suppose conditions* (G1) *and* (G3)–(G6) *hold. In addition, suppose that the following condition holds: for every* γ, $0 \le \gamma \le \infty$, (G2)′

$$\sup_{x\in C}\left|\int_{[y:\|y-x\|\le\gamma]}\delta_m(x,y)\mu(dy)-1\right|=O(D_m),$$

where $D_m=\sup\{\|y-x\|; x\in C, y\in E, \delta_m(x,y)>0\}=o(1)$ *as* $m\to\infty$. *Further suppose that* f *is Lipschitzian in the sense that there exists a constant* $K>0$ *such that*

$$|f(x)-f(y)|\le K\|x-y\|$$

for any $x\in C$, $y\in E$. *Then, with probability one,*

$$\sup_{x\in C}|f_n(x)-f(x)|=O(D_m)+O\left(\sqrt{\frac{s_m\log m}{m}}\right). \tag{2.13}$$

Proof. Since (G2)′ implies (G2), applying Theorem 1, we get that

$$\sup_{x\in C}|f_n(x)-E[f_n(x)]|=O\left(\sqrt{\frac{s_m\log m}{m}}\right)\qquad\text{a.s.}$$

It is sufficient to prove that

$$\sup_{x\in C}|E[f_n(x)]-f(x)|=O(D_m).$$

Note that

$$\begin{aligned}Ef_n(x)-f(x)&=\int_E\delta_m(x,y)f(y)\mu(dy)-f(x)\\&=\int_E\delta_m(x,y)\big(f(y)-f(x)\big)\mu(dy)\\&\quad+\int_E\delta_m(x,y)f(x)\mu(dy)-f(x)\\&=J+f(x)\left[\int_E\delta_m(x,y)\mu(dy)-1\right],\end{aligned} \tag{2.14}$$

where

$$J=\int_E\delta_m(x,y)\big(f(y)-f(x)\big)\mu(dy).$$

Hence

$$|Ef_n(x)-f(x)-J|=O(D_m) \tag{2.15}$$

by condition (G2)′. Since f satisfies the condition that there exists a constant K such that

$$|f(x)-f(y)|\le K\|y-x\|$$

for all $x \in C$, $y \in E$, it follows that

$$\begin{aligned} |J| &\le \int_E \delta_m(x,y)|f(y)-f(x)|\mu(dy) \\ &\le K D_m \int_E \delta_m(x,y)\mu(dy) \\ &\le O(D_m) \end{aligned} \tag{2.16}$$

by condition (G2)$'$. Combining equations (2.15) and (2.16), we get the relation

$$|Ef_n(x) - f(x)| = O(D_m). \tag{2.17}$$

□

Remarks 1. Let $C_0[0,1]$ be the space of real-valued continuous functions $x(\cdot)$ on the interval [0, 1] with $x(0) = 0$. Suppose the space $C_0[0,1]$ is equipped with uniform topology induced by the norm

$$\|x\| = \sup_{t \in [0,1]} |x(t)|.$$

Let $\mu(\cdot)$ denote the Wiener measure on the space $C_0[0,1]$ induced by the standard Wiener process and B_m^x be the closed ball with center $x \in C_0[0,1]$ and radius $\frac{1}{m}$. Define

$$\delta_m(x,y) = \frac{1}{\mu(B_m^x)} I(y \in B_m^x), \tag{2.18}$$

where $I(A)$ denotes the indicator function of the set A. It is easy to check that the corresponding density estimator $f_n(x)$ is the naive kernel estimator proposed in equation (6) in Dabo-Niang (2004). This sequence clearly satisfies condition (G2).

Suppose a_n^x is a sequence of positive numbers. Let

$$\delta_m(x,y) = \frac{1}{a_n^x} K_n(\|x-y\|), \tag{2.19}$$

where $K_n(\cdot)$ is sequence of functions satisfying conditions (H2)–(H6) in Dabo-Niang, Ferraty and Vieu (2006). As was indicated earlier, the choice of the sequence m may depend on n such that $m \to \infty$ as $n \to \infty$. Then we get the kernel estimator proposed by them. Other estimators using delta sequences can be constructed following the ideas in Example 2.8.3 in Prakasa Rao (1983), page 136, depending on the space E and the measure μ.

Remarks 2. It is true that conditions (G2)–(G6) are patterned on a similar set of conditions for kernel type of estimators but, due to the infinite dimensional nature of the problem, additional conditions are necessary to obtain uniform consistency even over compact sets. In a recent note, Ferrarty and Vieu (2008) comment on the conditions for deriving uniform consistency on compact sets. In particular, they suggest that the conditions on the infinite dimensional spaces should be such that the compact set C satisfies the property

$$C \subset \bigcup_{k=1}^{\tau} B(t_k, \ell),$$

where the number τ of spheres and the radius ℓ satisfy the geometric link condition $\tau \ell^{\alpha} = c$ for some $\alpha > 0$ and $c > 0$. This condition holds trivially for any finite dimensional Euclidean space but it also holds for infinite dimensional projection-based metric spaces. Here $B(t_k, \ell)$ is the open sphere with center t_k and radius ℓ. Condition (G3) relates to the bound on δ_m over $C \times E$ and condition (G4) relates to the uniform Lipschitzian property of $\delta_m(x, y)$. The choice β_1 and β_2 are governed by condition (G6) involving r_n and s_m. Condition (G5) is a condition on the limiting behavior of $\delta_m(x, y)\|x - y\|$ as m tends to infinity and it is not a condition on the bound of $\sup \|x - y\|$ over $C \times C$. Condition (G6) is introduced to get a rate of convergence for uniform consistency.

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