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The Split-BREAK model

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Abstract. A special type of the stochastic STOPBREAK process, which behaves properly when applied to time series data with emphatic permanent fluctuations, is presented. A good dynamic behavior is induced by the threshold regime and named the Split-BREAK process. General properties of this threshold STOPBREAK process are investigated, as well as some estimation procedures for the parameters of the process presented. A Monte Carlo simulation of the process is given and its application to the share trading on the Belgrade Stock Exchange illustrated.

1 Introduction

Starting from fundamental results of Engle and Smith (1999), González (2004) and Gonzalo and Martinez (2006), we will introduce a modified version of the well-known STOPBREAK model. Concerning the STOPBREAK process, it is known that it is successfully used to model time series with emphatic permanent fluctuations. In our model, a threshold as a noise indicator will be set, as we have already done in Popović and Stojanović (2005) where we used nonlinear time series of the ARCH type. Therefore, the model which we introduce here will be a *threshold, noise-indicator STOPBREAK process* which we simply name *the Split-BREAK process*.

A definition and main properties of the Split-BREAK process will be described in the next section. In addition to a usual analysis of its stochastic properties, in Section 3, we also give estimates of parameters, above all, *the critical value of reaction* of the Split-BREAK process. Asymptotic properties of those proposed estimates are established, also. In Section 4, the Monte Carlo simulation of innovations of the Split-BREAK process and the application of the estimation procedure explained in Section 3 are given. In Section 5, the Split-BREAK process is applied as a time series analysis for some real trading volume data on the Belgrade Stock Exchange in the period 2002–2006. Finally, in the next section we compared the efficiency of the Split-BREAK model and previously known models.

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2 Definition and general properties of the model

Let (y_t) be the time series with the known values at time $t \in \{0, 1, \dots T\}$, adapted to the filtration $F = (\mathcal{F}_t)$. We say, following Engle and Smith (1999), that (y_t) will be a STOPBREAK process if it satisfies

$$y_t = m_t + \varepsilon_t \tag{2.1}$$

where the sequence (m_t) is a so-called *martingale mean* sequence, and (ε_t) is a white noise, that is, the sequence of independent identically distributed F-adapted random variables with $E(\varepsilon_t|\mathcal{F}_{t-1}) = 0$, $\mathrm{Var}(\varepsilon_t|\mathcal{F}_{t-1}) = \sigma^2$ for each $t = 1, \dots, T$.

Also, let $\mathcal{F}_t = \mathcal{G}\mathrm{en}\{\varepsilon_0, \dots, \varepsilon_t\}$, for any $t = 0, \dots, T$, and let martingale mean m_t be defined by

$$m_t = m_{t-1} + q_{t-1}\varepsilon_{t-1} = m_0 + \sum_{j=0}^{t-1} q_j \varepsilon_j, \qquad t = 1, \dots, T, \tag{2.2}$$

where $m_0 \equiv \text{const}$ and q_t are random variables which depend on white noise (ε_t). In this way y_t will be $\mathcal{F}_t$ measurable, and m_t an $\mathcal{F}_{t-1}$ measurable random variable. In addition, let us suppose that

$$P\{0 \le q_t \le 1\} = 1$$

for each $t = 0, 1, \dots, T$. It means that the sequence (q_t) displays the *(permanent) reaction* of the STOPBREAK process because its values determine the amount of participation of previous elements of the white noise process engaged in the definition of m_t, and consequently in the definition of y_t. In other words, $q_t \approx 0$ gives "small" changes of martingale mean at time t, while in the case of $q_t \approx 1$ an emphatic (permanent) fluctuation is registered. So, the structure of the sequence (q_t) determines the character and properties of the STOPBREAK process.

Let us further suppose that, besides (2.1) and (2.2), the following condition is fulfilled:

$$q_t = I(\varepsilon_{t-1}^2 > c) = \begin{cases} 1, & \varepsilon_{t-1}^2 > c, \\ 0, & \varepsilon_{t-1}^2 \le c, \end{cases} \qquad t = 1, \dots, T \tag{2.3}$$

thus completing the definition of the *threshold STOPBREAK process*, which we named *Split-BREAK process*. The constant $c > 0$ is *the critical value of the reaction*, meaning that it will determine the level of the noise realization which will be statistically significant for the noise to be included in (2.2). According to (2.3), it follows that

$$E(q_t\varepsilon_t|\mathcal{F}_{t-1}) = q_t E(\varepsilon_t|\mathcal{F}_{t-1}) = 0 \tag{2.4}$$

and it can be seen that the sequence $(q_t\varepsilon_t)$ is a martingale difference, that is, the sequence of uncorrelated random variables. Now, the fundamental stochastic properties of the sequence (y_t) can be determined. As, according to (2.1) and (2.2),

$$E(y_t|\mathcal{F}_{t-1}) = m_t + E(\varepsilon_t|\mathcal{F}_{t-1}) = m_t \tag{2.5}$$

we can conclude that realizations of the sequence (y_t) are "close" to the mean sequence (m_t). The mean values of these two sequences are equal and constant because of (2.4) and (2.5):

$$E(y_t) = E(m_t) = \mu(\text{const}), \qquad t = 0, \dots, T. \tag{2.6}$$

The variance of Split-BREAK process can be determined in a similar way. As

$$\operatorname{Var}(y_t|\mathcal{F}_{t-1}) = E(y_t^2|\mathcal{F}_{t-1}) - m_t^2 = \sigma^2 \tag{2.7}$$

we can conclude that the conditional variance (volatility) of the sequence (y_t) is a constant and it is equal to the variance of the noise (ε_t). Let us remark that equalities (2.5) and (2.7) explain the stochastic nature of (2.1). As the sequence (m_t) is predictable, it will be a component which demonstrates the stability of the process itself. On the other hand, the sequence (ε_t) is a factor which represents the deviation from values (m_t). The variance of the sequence (m_t), under the condition of $m_0 \equiv \text{const}$, is

$$\operatorname{Var}(m_t) = \sigma^2 t a_c,$$

where $a_c = P\{\varepsilon_t^2 > c\}$ and, accordingly, we have

$$\operatorname{Var}(y_t) = \operatorname{Var}(m_t) + \sigma^2 = \sigma^2(t a_c + 1).$$

Hence, variances of sequences (y_t) and (m_t) are not constants and they depend on the observation time t. In a similar manner, correlation functions of these sequences can be solved. The correlation function of (y_t) is

$$K(s,t) = \operatorname{Corr}(y_s, y_t) = \begin{cases} \dfrac{a_c[\min(s,t)+1]}{\sqrt{(a_c s+1)\cdot(a_c t+1)}}, & s \neq t; \\ 1, & s = t \end{cases}$$

and the correlation function of martingale means is

$$\widetilde{K}(s,t) = \operatorname{Corr}(m_s, m_t) = \begin{cases} \dfrac{\min(s,t)}{\sqrt{s \cdot t}}, & s \neq t; \\ 1, & s = t. \end{cases}$$

So, correlation functions $K(s,t)$ and $\widetilde{K}(s,t)$ depend on both time variables t and s, and indicate *nonstationarity* of sequences (y_t) and (m_t). However, on the contrary to the correlation function of (y_t), the correlation function of the martingale means is L^2-continuous since

$$\lim_{s \to t} \widetilde{K}(s,t) = \widetilde{K}(t,t) = 1.$$

Finally, in this section, we will discuss properties of the sequence of increments of the Split-BREAK process, defined as $X_t = y_t - y_{t-1}$. The importance of the sequence (X_t) is emphasized by almost all authors who investigated the STOP-BREAK process. For that reason, we will investigate the sequence of increments in case of the Split-BREAK process.

We will use the following representation of elements of (X_t):

$$X_t = \varepsilon_t - \theta_{t-1}\varepsilon_{t-1}, \qquad t = 1, \dots, T \tag{2.8}$$

where $\theta_t = I(\varepsilon_{t-1}^2 \le c)$, and (ε_t) is the white noise that was defined previously. Time series (X_t) will be called the *threshold moving average* (*of order 1*), that is, *Split-MA*(*1*) *process*. It operates in two regimes and it is similar to the threshold integrated moving average (TIMA) model introduced by Gonzalo and Martinez (2006). The main distinction is threshold variables (ε_t) which in the Split-MA(1) case are observed in the "past," that is, in the previous moment $t-1$. If the fluctuation of the white noise in time $t-1$ is large, the equation $X_t = \varepsilon_t$ will describe the sequence (y_t) in a form of the random walk, that is, an increment will be equal to the white noise. On the other hand, the fluctuation of the white noise which does not exceed the critical value c will produce an MA(1) representation of the sequence (X_t).

Engle and Smith (1999) and González (2004) also investigated sufficient conditions for the invertibility of (X_t), which depended on the selection of proper reaction (q_t). Engle and Smith (1999) showed that the invertibility of the increment sequence is equivalent to the fact that the so-called *permanent effect of observation*

$$\lambda_t \stackrel{d}{=} \lim_{k\to\infty} \frac{\partial E(y_{t+k}|\mathcal{F}_t)}{\partial y_t}$$

would satisfy, almost surely, the condition $|\lambda_t| < 2$ for all $t \in \mathbb{Z}$. In our case, it has $\lambda_t \equiv q_t$ and the condition of the invertibility is always satisfied. Furthermore, we will investigate a basic stochastic structure of that sequence and make some generalities in order to get good estimates of parameters and apply this model to real data.

Let us determine basic parameters of the distribution of increments (X_t). Under the previous assumption, the mean value and the variance of the sequence can be easily calculated

$$E(X_t) = 0, \qquad \operatorname{Var}(X_t) = E(X_t^2) = \sigma^2(b_c + 1),$$

where $b_c = 1 - a_c = P\{\varepsilon_t^2 \le c\}$. Also, the covariance function of this time series will be

$$\operatorname{Cov}(X_s, X_t) = \begin{cases} \sigma^2(b_c+1), & s = t, \\ -b_c\sigma^2, & |s-t| = 1, \\ 0, & \text{otherwise,} \end{cases}$$

and the MA(1) structure is evident. This will make it easy to recognize the stationarity of time series (X_t) with the correlation function

$$\rho(h) = \operatorname{Corr}(X_{t+h}, X_t) = \begin{cases} 1, & h = 0, \\ -b_c(b_c+1)^{-1}, & h = \pm 1, \\ 0, & \text{otherwise.} \end{cases}$$

The following theorem asserts an invertibility of Split-MA(1).

Theorem 2.1. *Let time series* (X_t) *be defined by the recurrent relation* (2.8), *where* $t \in \mathbf{Z}$ *and* $b_c = P\{\varepsilon_t^2 \le c\} \in (0, 1)$. *Then, time series* (ε_t) *satisfies*

$$\varepsilon_t = X_t + \sum_{j=1}^{\infty} \alpha_{t,j} X_{t-j}, \qquad t \in \mathbf{Z}, \tag{2.9}$$

where $\alpha_{t,j} = \prod_{k=1}^{j} \theta_{t-k}$. *At the same time, this representation of* (ε_t) *is unique, and the sum converges in the mean square and the almost sure.*

Proof. As, for any $n \in \mathbf{Z}$,

$$A_n = E\left|\varepsilon_t - X_t - \sum_{j=1}^{n} \alpha_{t,j} X_{t-j}\right|^2 = E\left|\left(\prod_{j=1}^{n+1} \theta_{t-j}\right)\varepsilon_{t-n-1}\right|^2 \le \sigma^2 b_c^n$$

it follows that $A_n \to 0$ when $n \to \infty$, the mean square convergence of (2.9) follows. On the other hand, according to

$$E(\alpha_{t,j} X_{t-j}) \le \sigma^2 b_c^{j-1}, \qquad E(\alpha_{t,j} X_{t-j})^2 \le \sigma^2 (b_c^j + b_c^{j-1})$$

and the convergence of the sums $\sum_{j=1}^{\infty} b_c^{j-1}$ and $\sum_{j=1}^{\infty} (b_c^j + b_c^{j-1})$, the almost sure convergence of (2.9) follows.

A uniqueness of the representation (2.9) follows, for instance, from Popović (1992). □

3 Estimates of parameters

Let us first consider an estimate of the (unknown) critical value $c > 0$. The estimation procedure which we will use is analogous to the well-known estimation procedure for coefficients of the linear MA model (Fuller, 1976) but required more investigation and discussion regards the fit.

Let the Split-MA(1) model be defined by the equation (2.8). As we have already shown, the coefficient of the first correlation of the model is

$$\rho(1) = -\frac{b_c}{1 + b_c}, \qquad 0 < b_c < 1$$

and solving the equation with respect to b_c we have

$$\widetilde{b}_c = -\frac{\hat{\rho}_T(1)}{1 + \hat{\rho}_T(1)}, \tag{3.1}$$

where $\hat{\rho}_T(1) = (\sum_{t=0}^{T-1} X_t X_{t+1}) \cdot (\sum_{t=0}^{T-1} X_t^2)^{-1}$ is an estimated value for the first correlation. Using $\widetilde{b}_c$ and solving the equation

$$P\{\varepsilon_t^2 \le c\} = \widetilde{b}_c, \tag{3.2}$$

with respect to c, we can get the proper estimate for the critical value, $\widetilde{c}$. Of course, estimates $\widetilde{b}_c$ and $\widetilde{c}$ are proper estimates if $-0{,}5 < \hat{\rho}_T(1) < 0$. The proof of the consistency of the estimates follows.

Theorem 3.1. *Let $\widetilde{b}_c$ and $\widetilde{c}$, defined in (3.1) and (3.2), be estimates of unknown parameters b and c, respectively. Then,*

$$\widetilde{b}_c \xrightarrow{a.s.} b_c, \qquad T \longrightarrow \infty. \tag{3.3}$$

If also (ε_t) has an absolutely continuous distribution, then

$$\widetilde{c} \xrightarrow{a.s.} c, \qquad T \longrightarrow \infty. \tag{3.4}$$

Proof. The spectral density of the sequence (X_t) is

$$f_X(\omega) = \frac{\sigma^2}{2\pi}(1 - 2\cos\omega b_c + b_c)$$

and it is obviously continuous for $\omega = 0$. According to the ergodicity theorem, we have

$$\frac{1}{T}\sum_{t=0}^{T-1} X_t X_{t+h} \xrightarrow{\text{a.s.}} \gamma(h), \qquad T \longrightarrow \infty, \tag{3.5}$$

where $\gamma(h) = \mathrm{Cov}(X_t, X_{t+h}), h \geq 0$ is the covariance function of the sequence (X_t). The last convergence implies

$$\hat{\rho}_T(1) \xrightarrow{\text{a.s.}} \frac{\gamma(1)}{\gamma(0)} = \rho(1), \qquad T \longrightarrow \infty$$

and, from here, we have (3.3) according to the well-known continuity of the almost sure convergence (see, e.g., Serfling, 1980, page 24), that is, a consistency of the estimate $\widetilde{b}_c$ is fulfilled.

Let us suppose that the distribution of the elements of sequence (ε_t) is absolutely continuous, and $F(x) = P\{\varepsilon_t^2 \leq x\}$ is the distribution function of the sequence (ε_t^2). According to the previously proven convergence (3.3) and the continuity of $F^{-1}(x)$, it will be

$$\widetilde{c} - c = F^{-1}(\widetilde{b}_c) - F^{-1}(b_c) \xrightarrow{\text{a.s.}} 0, \qquad T \longrightarrow \infty$$

which completes the proof. □

The following assertion concerns the asymptotic normality of proposed estimates.

Theorem 3.2. *Let $\widetilde{b}_c$ and $\widetilde{c}$ be estimates of b_c and c defined as in (3.1) and (3.2) where the distribution of elements of (ε_t) is symmetric with respect to zero and*

(i) $E(\theta_t^2\varepsilon_{t-1}^2) = E(\theta_t\varepsilon_{t-1}^2) = \int_0^c x\,dF(x) = L\sigma^2,$

(ii) $E(\varepsilon_t^4) = \eta\sigma^4,$

(iii) $E(\varepsilon_t^6) = \phi\sigma^6,$

where $L = L(c) > 0$, $\eta > 0$, $\phi > 0$ *and* $F(x) = P\{\varepsilon_t^2 \leq x\}$ *is the distribution function of* ε_t^2. *Then, one has*

$$\sqrt{T}(\widetilde{b}_c - b_c) \xrightarrow{d} \mathcal{N}(0, V_1), \qquad T \longrightarrow \infty, \tag{3.6}$$

where $V_1 = (1 + b_c)^2[1 + b_c(1 + \eta + 5L) - 3b_c^2]$. *In addition, in the case of an absolutely continuous distribution of* (ε_t) *one has*

$$\sqrt{T}(\widetilde{c} - c) \xrightarrow{d} \mathcal{N}(0, V_2), \qquad T \longrightarrow \infty \tag{3.7}$$

where $V_2 = V_1/f^2(c)$, *and* $f(x) = F'(x)$ *is the distribution density function of the random variable* ε_t^2.

Proof. First of all, we shall prove the asymptotic normality of $\hat{\rho}_T$. In order to do that, let us consider the sequence

$$Z_t = X_t X_{t+1}, \qquad t = 0, \dots, T - 1$$

which is 1-dependent (see Definition 6.3.1 in Fuller, 1976, page 245). Thanks to (iii) and Cauchy–Swartz and Minkowski's inequalities, it follows that

$$E|Z_t + b_c\sigma^2|^3 \leq [(E|Z_t|^3)^{1/3} + b_c\sigma^2]^3 \leq [\phi^{1/3}(1 + b_c^{1/6})^2 + b_c]^3\sigma^6 < \infty.$$

Then, according to the Hoeffding-Robbins' theorem, we have

$$\frac{1}{\sqrt{T}} \sum_{t=0}^{T-1} [Z_t - \gamma(1)] \xrightarrow{d} \mathcal{N}(0, A), \qquad T \longrightarrow \infty$$

where $A = \mathrm{Var}(Z_t) + 2\,\mathrm{Cov}(Z_t Z_{t+1}) = \sigma^4[1 + b_c(1 + \eta + 5L) - 3b_c^2]$. If we apply the almost sure convergence (3.5), for $h = 0$, we shall have

$$\sqrt{T}[\hat{\rho}_T - \rho(1)] \xrightarrow{d} \mathcal{N}(0, V_0), \qquad T \longrightarrow \infty,$$

where $V_0 = A \cdot \gamma^{-2}(0) = [1 + b_c(1 + \eta + 5L) - 3b_c^2] \cdot (1 + b_c)^{-2}$. Applying convergence in distribution continuity (Serfling, 1980, page 118), we shall conclude that the convergence (3.6) is fulfilled, where $V_1 = (d\rho(1)/db_c)^{-2} \cdot V_0$. We can prove an asymptotic normality of the estimate $\widetilde{c}$ in a similar way. If elements of (ε_t) are continuously distributed, then, because of $\widetilde{c} = F^{-1}(\widetilde{b}_c)$, the random function $\widetilde{c}$ is a continuous function of $\widetilde{b}_c$. Applying the same continuity convergence theorem and the fact that $V_2 = (dF(c)/dc)^{-2} \cdot V_1$, we get the convergence (3.7). □

In spite of good stochastic properties of $\widetilde{b}_c$ and $\widetilde{c}$ discussed in previous two theorems, it can be proven that $\widetilde{b}_c$, as in the case of MA models, is not an efficient estimate of b_c. (The asymptotic efficiency of $\widetilde{b}_c$, described by its variance V_1 for the normally distributed white noise, will be discussed at the end of this section.) In order to get better estimates of unknown parameters we will modify the well-known Gauss–Newton's procedure for nonlinear functions described in details, for instance, in Fuller (1976).

First, let us remark that the equality (2.8) can be expressed in the following form:

$$\varepsilon_t = X_t + \theta_{t-1}\varepsilon_{t-1}, \qquad t = 1, \ldots, T$$

or, in the functional form $\varepsilon_t(X,\theta) = X_t + \theta_{t-1}\varepsilon_{t-1}(X,\theta)$ where the parameter $b_c \in (0,1)$ can be estimated from. Let $\widetilde{b}_c$ be the initial (estimated) value of this parameter which we obtained previously, $\widetilde{\theta}_t = I(\varepsilon_{t-1}^2 \le \widetilde{c})$ and $\varepsilon_0(X,\theta) \equiv 0$. Using this as well as the iterative procedure, we can get values $\varepsilon_t(X,\widetilde{\theta})$.

On the other hand, let us define a sequence of random variables

$$W_t(X,\theta) = \theta_t W_{t-1}(X,\theta) + \varepsilon_{t-1}(X,\theta).$$

It can be easily seen that random variables $W_t(X,\theta)$ are $\mathcal{F}_{t-1}$ adapted for any $t = 1, \ldots, T$, as well as that they are independent of ε_t and θ_{t+1}. According to Theorem 1 of Popović (1992) it follows that the sequence $(W_t(X,\theta))$ is the stationary ergodic sequence of random variables. In this sequence, we can associate the so-called residual sequence using the procedure described in Lawrence and Lewis (1992),

$$R_t(X,\theta) = W_t(X,\theta) - b_c W_{t-1}(X,\theta), \qquad t = 1, \ldots, T. \tag{3.8}$$

It can be easily shown that this is a sequence of uncorrelated random variables. Equation (3.8) defines the sequence $(W_t(X,\theta))$ as a linear AR process with the white noise $(R_t(X,\theta))$, so, we can apply the minimum square procedure and get the estimate of b_c as follows:

$$\hat{b}_c = \left[\sum_{t=0}^{T-1} W_{t+1}(X,\theta) W_t(X,\theta)\right] \cdot \left[\sum_{t=0}^{T-1} W_t^2(X,\theta)\right]^{-1}. \tag{3.9}$$

In the same way as before, we can get the estimate $c = \hat{c}$ of the critical value c based on $\hat{b}_c$ as the solution of equation

$$P\{\varepsilon_t^2 \le c\} = \hat{b}_c \tag{3.10}$$

with respect to c. Let us now prove the consistency of estimates $\hat{b}_c$ and $\hat{c}$.

Theorem 3.3. *Let $\hat{b}_c$ and $\hat{c}$ be the estimators of unknown parameters b_c and c, defined by* (3.9) *and* (3.10), *then*

$$\hat{b}_c \xrightarrow{a.s.} b_c, \qquad T \longrightarrow \infty. \tag{3.11}$$

If, in addition, (ε_t^2) has some absolutely continuous distribution with the distribution function $F(x)$ then also

$$\hat{c} \xrightarrow{a.s.} c, \qquad T \longrightarrow \infty. \tag{3.12}$$

Proof. From (3.9) and the definition of the residual sequence it follows that

$$\hat{b}_c - b_c = \left[\sum_{t=0}^{T-1} R_{t+1}(X,\theta)W_t(X,\theta)\right] \cdot \left[\sum_{t=0}^{T-1} W_t^2(X,\theta)\right]^{-1}. \tag{3.13}$$

If the ergodicity theorem is applied, the almost sure convergence will be as

$$\frac{1}{T}\sum_{t=0}^{T-1} R_{t+1}(X,\theta)W_t(X,\theta) \longrightarrow A_t, \qquad T \longrightarrow \infty,$$

where $A_t = E[R_{t+1}(X,\theta)\cdot W_t(X,\theta)] = 0$. The next almost sure convergence will, similarly, be

$$\frac{1}{T}\sum_{t=0}^{T-1} W_t^2(X,\theta) \longrightarrow B_t, \qquad T \longrightarrow \infty, \tag{3.14}$$

where $B_t = \mathrm{Var}[W_t(X,\theta)] = \sigma^2(1-b_c)^{-1}$. If we now apply these two convergences to (3.13), we will get (3.11).

The second part of this theorem can be proven in the same way as it was done in Theorem 3.1. Particularly, as $\hat{c} = F^{-1}(\hat{b}_c)$ is the continuous function of the consistent estimate $\hat{b}_c$, it can be easily shown that (3.12) is valid. □

At the end of this section we will prove the asymptotic normality of estimates $\hat{b}_c$ and $\hat{c}$.

Theorem 3.4. *Let $\hat{b} = \hat{b}_c$ and $\hat{c}$ be estimates of unknown parameters b_c and c, respectively, and the distribution of the sequence (ε_t) be symmetric with respect to zero and*

$$\text{(i)}\quad E(\theta_t\varepsilon_{t-1}^2) = E(\theta_t^2\varepsilon_{t-1}^2) = \int_0^c x\,dF(x) = L\sigma^2,$$

$$\text{(ii)}\quad E(\varepsilon_t^4) = \eta\sigma^4,$$

where $L = L(c) > 0$, $\eta > 0$ and $F(x)$ is the distribution function of ε_t^2. Then it is valid that

$$\sqrt{T}(\hat{b}_c - b_c) \xrightarrow{d} \mathcal{N}(0, V_3), \qquad T \longrightarrow \infty, \tag{3.15}$$

where $V_3 = (1-b_c)[1 + 6Lb_c + \eta b_c(1-b_c)]$. If, in addition, the distribution of (ε_t^2) is absolutely continuous, then

$$\sqrt{T}(\hat{c} - c) \xrightarrow{d} \mathcal{N}(0, V_4), \qquad T \longrightarrow \infty, \tag{3.16}$$

where $V_4 = V_3/f^2(c)$, and $f(x) = F'(x)$.

Proof. According to equation (3.13), the following separation is valid

$$\sqrt{T}(\hat{b}_c - b_c) = \frac{T^{-1/2} \cdot \mathbf{U}_{T-1}}{T^{-1} \cdot \mathbf{V}_{T-1}}, \tag{3.17}$$

where $\mathbf{U}_{T-1} = \sum_{t=0}^{T-1} R_{t+1}(X,\theta) \cdot W_t(X,\theta)$, $\mathbf{V}_{T-1} = \sum_{t=0}^{T-1} W_t^2(X,\theta)$. As the sequence $(\mathbf{U}_T)$ is martingale, applying the central limit theorem for martingales (see, e.g., Billingsley, 1961, or Nicholls and Quinn, 1982), we will have

$$T^{-1/2} \cdot \mathbf{U}_T \xrightarrow{d} \mathcal{N}(0, D_1), \qquad T \longrightarrow \infty,$$

where $D_1 = \mathrm{Var}[R_{t+1}(X,\theta) \cdot W_t(X,b_c)] = \sigma^4[1 + 6Lb_c + \eta b_c(1-b_c)](1-b_c)^{-1}$. Then, according to the almost sure convergence of (3.14) and equation (3.17), it follows immediately that (3.15) is fulfilled.

The proof of convergence (3.16) is completely analogous to procedure applied in Theorem 3.2. □

Finally, we can note some more facts that follow directly from the above described estimation procedure for the critical value c and the theorems we have just proven.

Remark 1. If we apply estimates $\tilde{b}_c$ and $\hat{b}_c$, we can model values of (ε_t), and thereby, we can estimate the variance σ^2 of the sequence (ε_t). To do this, we can use the sample variance

$$\tilde{\sigma}^2 = \frac{1}{T} \sum_{t=1}^{T} \varepsilon_t^2(X, \tilde{\theta}) \quad \text{or} \quad \hat{\sigma}^2 = \frac{1}{T} \sum_{t=1}^{T} \varepsilon_t^2(X, \hat{\theta}),$$

where $\varepsilon_t(X,\tilde{\theta})$ and $\varepsilon_t(X,\hat{\theta})$ are modeled values of the white noise which we arrived at by applying estimates $\tilde{b}_c$ and $\hat{b}_c$, respectively. In the case of the Gaussian white noise (ε_t), these estimates are identical to those which we can get applying the maximum likelihood procedure (see Section 5 of this paper), as it has been also shown in Stojanović and Popović (2005). The consistency and asymptotic normality of estimates $\tilde{\sigma}$ and $\hat{\sigma}$ can be easily shown. As per the illustration, in Figure 1

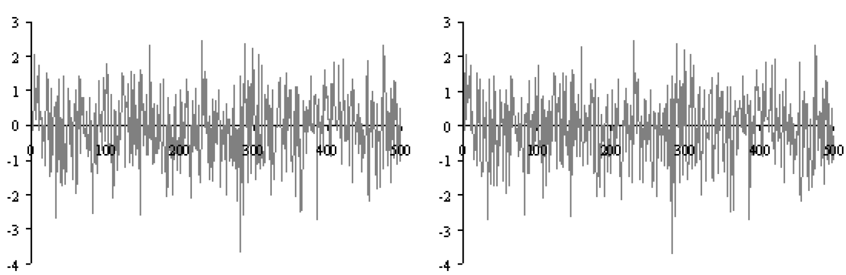

Figure 1 *Comparative values of the white noise (the left side picture) and modeled values (the right side picture).*

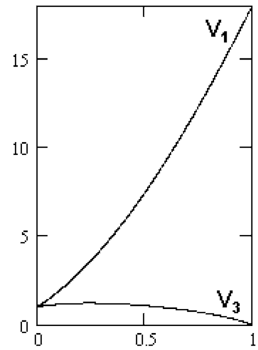

Figure 2 *Comparative values of asymptotic variances of* $\widetilde{b}_c$ *and* $\hat{b}_c$.

we can look at "pretty" coincidental behavior of the Monte Carlo simulation of the white noise (ε_t) with standard $\mathcal{N}(0, 1)$ distribution (see the following section) and modeled values of the white noise $(\varepsilon_t(X, \widetilde{\theta}))$, generated by the estimate $\widetilde{b}_c$.

Remark 2. Asymptotic values of variances V_1 and V_3 are commonly used as a measure of bias of estimates $\widetilde{b}_c$ and $\hat{b}_c$, respectively, compared to the true value of b_c. If we consider them as the functions of $b_c \in (0, 1)$, we will be able to compare values of V_1 and V_3 in order to register the quality of two considered estimates. Figure 2 illustrates the graphs of $V_1 = V_1(b_c)$ and $V_3 = V_3(b_c)$ if $\mathcal{N}(0, 1)$ is the distribution of white noise (ε_t). In that case $V_3(b_c) < V_1(b_c), \forall b_c \in (0, 1)$, that is, $\hat{b}_c$ is more efficient than $\widetilde{b}_c$.

4 Monte Carlo study of the model

In this section we will demonstrate some applications of the above-described estimation procedure of the critical value of Split-MA(1). First of all, using the Monte Carlo simulation for the model

$$X_t = \varepsilon_t - \theta_{t-1}\varepsilon_{t-1}, \qquad t = 1, \dots, T,$$

where $\theta_t = I(\varepsilon_{t-1}^2 \leq 1)$ and $\varepsilon_0 = \varepsilon_{-1} = 0$, we get estimates of the critical value $c = 1$, therefore probabilities $b_c = P\{\varepsilon_t^2 \leq 1\}$. For the white noise we used a simple random sample from $\mathcal{N}(0, 1)$ distribution, so that the elements of the sequence (ε_t^2) were χ_1^2 distributed, which had been used for solving the critical value of the reaction $\widetilde{c}$ and $\hat{c}$. We show these estimates based on 100 independent Monte Carlo simulations for each sample size $T = 50$, $T = 100$ and $T = 500$ in Table 1.

The average values of the estimates are set, together with the correspondent estimating errors (the numeric values set in the brackets) in the rows of Table 1. The second column of Table 1 contains values $\hat{\rho}_T(1)$. The average values of that column are somewhat smaller in the absolute value of the true value which is in

Table 1 *Estimated values of Monte Carlo simulations of the Split-MA(1) process*

	Averages of estimated values						
Sample size	$\hat{\rho}_T(1)$	$\widetilde{b}_c$	$\widetilde{c}$	$\hat{b}_c$	$\hat{c}$	$\widetilde{\sigma}^2$	$\hat{\sigma}^2$
$T = 50$	−0.376	0.614	0.894	0.647	0.944	1.216	1.042
	(0.139)	(0.219)	(0.726)	(0.192)	(0.571)	(0.292)	(0.202)
$T = 100$	−0.386	0.634	0.894	0.671	1.039	1.168	1.016
	(0.097)	(0.156)	(0.444)	(0.141)	(0.427)	(0.184)	(0.124)
$T = 500$	−0.394	0.664	0.916	0.676	0.992	1.135	0.997
	(0.056)	(0.091)	(0.259)	(0.068)	(0.194)	(0.102)	(0.099)
True values	0.406	0.683	1.000	0.683	1.000	1.000	1.000

this case $\rho(1) = -b_c(1 + b_c)^{-1} \approx -0.406$. Next two columns contain estimated values $\widetilde{b}_c$ and $\widetilde{c}$ attained from the correlation coefficient $\hat{\rho}_T$, formula (3.1) and the quantiles of the χ^2_1 distribution. Average values in these columns are smaller than the true values $b_c \approx 0.683$ and $c = 1$ also. In the case of a modified Gauss–Newton procedure two estimated values coincide better to the true values. Realized values of estimates $\hat{b}_c$ and $\hat{c}$ are more often closer to the true values than previously mentioned estimates. Especially, we emphasize the average values of these two estimates.

The dispersion of values $\widetilde{c}$ and $\hat{c}$ is illustrated in Figure 3. It can be seen that $\hat{c}$ has the asymptotically normal distribution even for the sample of a "small" sample size. The gathering of the estimated values of the parameter c around the true value is visible in the figure. Finally, in the last two columns of Table 1, average estimated values of σ^2, based on modeled values of the white noise (ε_t) as mentioned above, are displayed. Their average values differ from the true value $\sigma^2 = 1$ as a consequence of two stage estimating procedure that was used. In spite of that, it can be seen that the average values of $\hat{\sigma}^2$ are closer to the true one than the average values of $\widetilde{\sigma}^2$.

5 Application of the model

Now, we will demonstrate the usage of our STOPBREAK model. We applied it to shares on the Belgrade Stock Exchange. Starting from a similar presumption as Hafner (1998), as a basic financial time series we have considered *log-volume* data

$$y_t = \ln(S_t \cdot H_t), \qquad t = 0, 1, \dots, T, \tag{5.1}$$

where S_t is the share price at time t and H_t is the volume of trading of the same share at time t. (The price is in dinars and the volume is the number of shares that were traded on the certain day. The days of trading are used as successive data.)

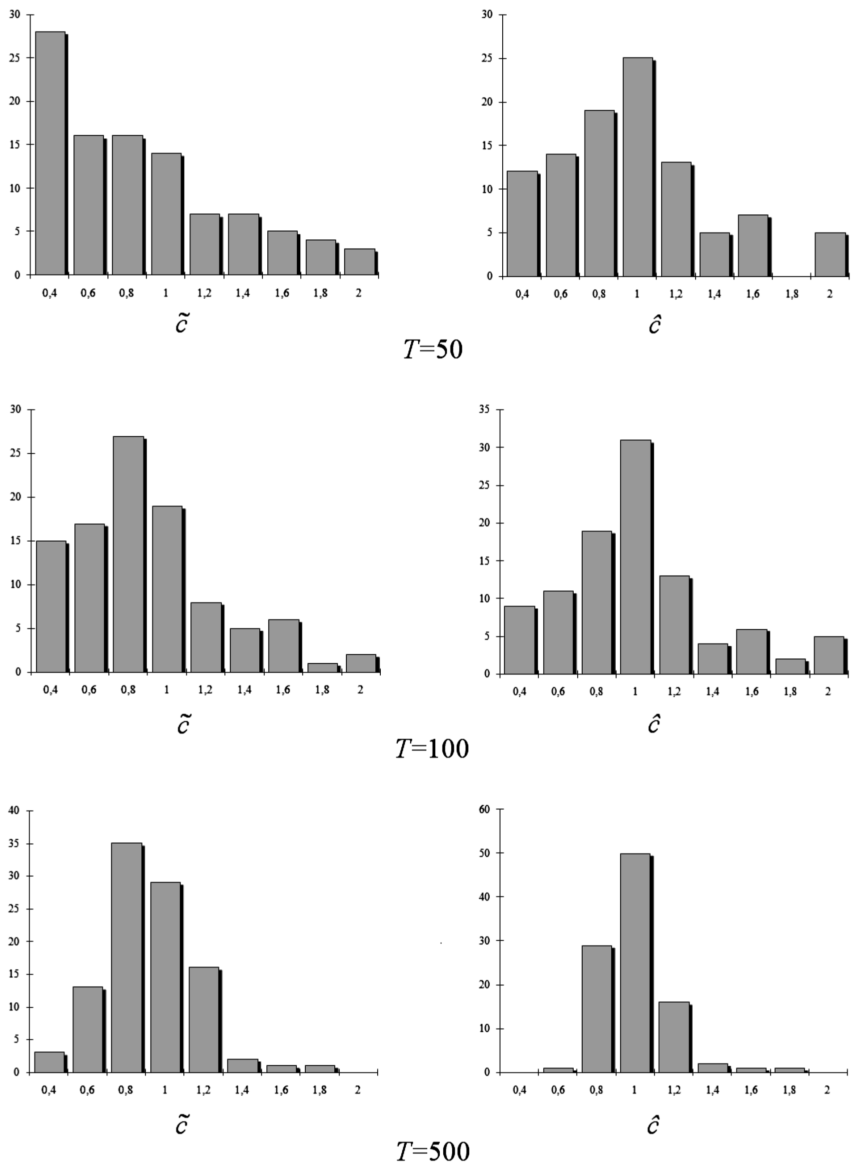

Figure 3 *Empirical densities of $\tilde{c}$ and $\hat{c}$.*

We applied the conditional likelihood method to estimate the conditional variance σ^2 of the series (y_t). Under the assumption that (ε_t) is the Gaussian white noise, the log-likelihood function will be

$$L(y_1, \dots, y_T; \sigma^2) = -\frac{T}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{t=1}^{T}(y_t - m_t)^2.$$

From here, we can see that the estimated value of the variance

$$\hat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{T}(y_t - m_t)^2 = \frac{1}{T}\sum_{t=1}^{T}\varepsilon_t^2$$

is identical to the sample variance of the series (ε_t). Now, we can apply iterative equations

$$\begin{cases} \varepsilon_t = y_t - m_t, \\ m_t = m_{t-1} + \varepsilon_{t-1} I(\varepsilon_{t-2}^2 > \hat{c}), \end{cases} \qquad t = 1, \ldots, T \tag{5.2}$$

to generate the corresponding values of sequences (ε_t) and (m_t). As estimates of the critical value $\hat{c}$, we used the value described above. As initial values of the iterative procedure (5.2) we use

$$m_0 = y_0 = \overline{y}_T, \qquad \varepsilon_0 = \varepsilon_{-1} = 0,$$

where $\overline{y}_T$ is the empirical mean value of (y_t). The basic empirical series are log-volumes defined by (5.1) which we use in solving the series of increments (X_t), that is, the realized values of Split-MA(1) described above.

Table 2 contains the number of observations for the company (T), estimated value of $\rho_T(1)$, estimated values of b_c according to (3.1) and (3.9) and estimated values of the critical value c according to (3.2) and (3.10). Of course, the number of observations (T) will imply the value of the estimating error. The elements of the sequence $\{\theta_t\}$ are independent random variables and the expectation of θ_t is b_c and the variance is $b_c(1-b_c)$. Because of $0 < b_c < 1$, we have $0 < b_c(1-b_c) < 1$. We will be able to apply the central limit theorem and the criterion

$$P\{|\bar{\theta} - b_c| < \delta\} = 1 - \alpha$$

for a certain $\alpha \in (0, 1)$ and a small $\delta > 0$, where $\bar{\theta} = \frac{1}{T}\sum_{t=1}^{T}\theta_t$. So, if the variance of θ_t is limited by a number σ_0^2 $(0 < \sigma_0 \le \frac{1}{2})$ and we chose, for instance, $\sigma_0^2 = 0.25$, $\alpha = 0.05$ and $\delta = 0.15$, it will be sufficient $T = 43$ to fulfill the criterion.

Table 3 contains estimated values of the critical value $(\hat{c})$ for the reaction in Split-BREAK model, mean values and variances of previously defined sequences:

Table 2 *Estimated values of the Split-BREAK parameters of real data*

Companies	Headquarters	T	$\hat{\rho}_T(1)$	$\widetilde{b}_c$	$\widetilde{c}$	$\hat{b}_c$	$\hat{c}$
ALFA PLAM	Vranje	50	−0.337	0.508	0.710	0.417	0.753
DIN	Niš	56	−0.348	0.534	2.660	0.621	3.871
HEMOFARM	Vršac	54	−0.346	0.530	0.582	0.613	0.836
METALAC	G. Milanovac	174	−0.449	0.816	4.929	0.829	5.223
T. MARKOVIĆ	Kikinda	277	−0.351	0.542	1.263	0.469	0.965
SUNCE	Sombor	157	−0.424	0.735	2.836	0.784	3.132

Table 3 *Estimated values of real data*

Companies	Critical values	Log-volumes		Mart.means		Split-MA(1)		Noise	
		Mean	Var	Mean	Var	Mean	Var	Mean	Var
ALFA PLAM	0.753	15.320	1.505	15.354	1.457	−0.011	2.460	−0.034	2.510
DIN	3.871	14.485	4.998	14.628	6.071	0.003	4.116	−0.143	5.016
HEMOFARM	0.836	15.250	0.814	15.310	0.694	0.030	1.741	−0.042	1.576
METALAC	5.223	13.665	2.788	13.798	2.731	0.001	4.002	−0.133	4.376
T. MARKOVIĆ	0.965	13.816	2.295	13.830	1.977	−0.016	2.442	0.026	3.982
SUNCE	3.132	12.748	2.282	12.730	2.151	−0.024	1.978	−0.005	2.052

log-volumes (y_t), martingale means (m_t), the Split-MA(1) process (X_t) and the white noise (ε_t). If we analyze empirical values of these series, we can recognize the relations that could be explained by the above theoretical results. Namely, the empirical mean value of the log-volumes is close to the mean value of martingale means, which is consistent with equations (2.5) and (2.6).

A good match between these two sequences can be seen in Figure 4. Realizations of these sequences confirm a strong correlation among them which concurs with the definition of the STOPBREAK process, that is, equation (2.1). This justifies application of the threshold STOPBREAK process as a proper stochastic model.

On the other hand, Figure 5 shows that the white noise (ε_t) matches increments (X_t). The strong correlation between these two sequences can be explained as it was done in Section 3. Namely, when the fluctuation of (X_t) is strong (in a certain moment t), in the next moment $(t+1)$ it will become equal to the noise (ε_t). It is clear that the concurrence of realizations of these two sequences will be better if, in addition to the great fluctuation of (X_t), the critical value of the reaction c is relatively small. In fact, small values of the parameter c point out to the possibility that the true value of this parameter is $c = 0$, when increments (X_t) match the white noise (ε_t). In that case the basic sequence (y_t) is the sequence with independent increments and the whole statistical analysis is easier. Of course, if the sample size is big, testing the null hypothesis

$$H_0 : c = 0 \qquad (\text{i.e. } H_0 : b_c = 0)$$

will be, in accordance with Theorems 3.2 and 3.4, based on the normal distribution, that is, standard, well-known statistical tests.

6 Comparison of models

In this section we will compare the efficiency of the model that we have purposed and some known models using the same data. The basic model which we are using

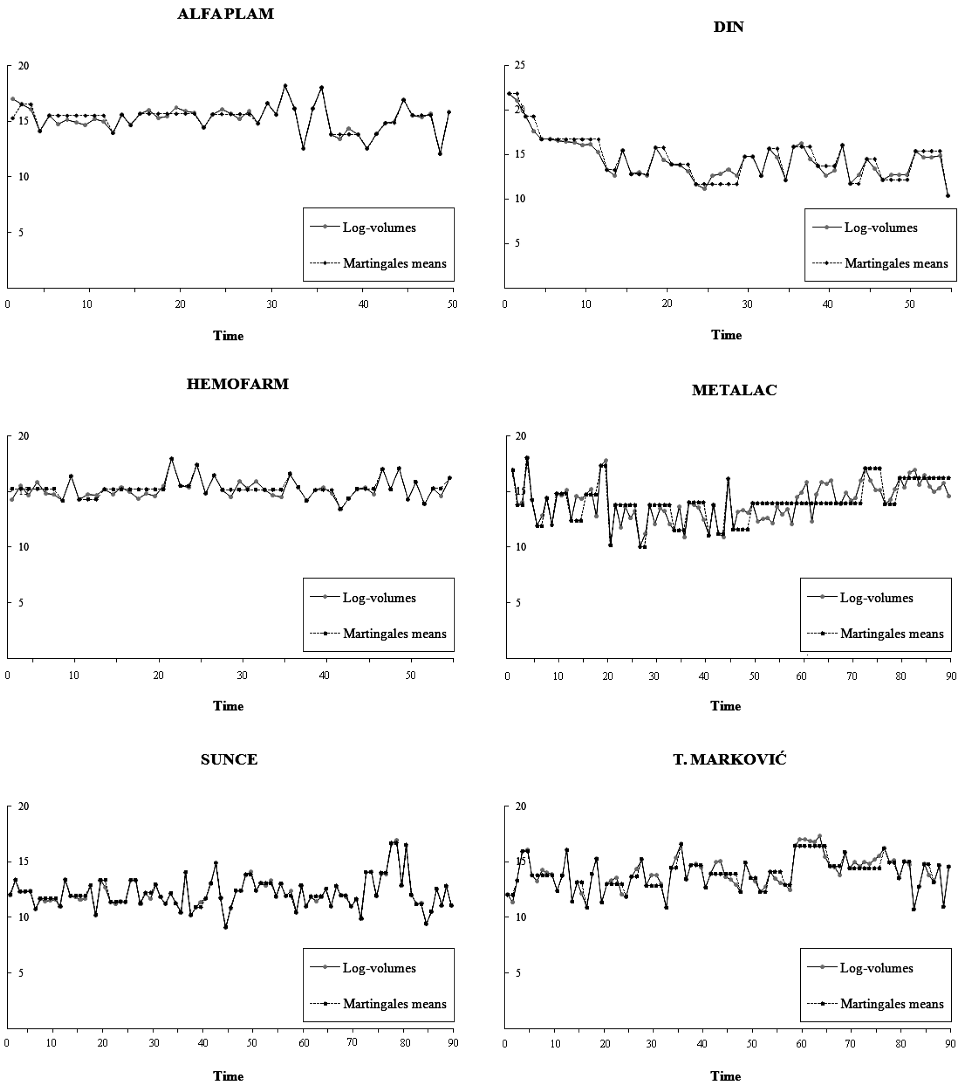

Figure 4 *Comparative graphs of the real and modeled data.*

here is the model

$$y_t = m_t + a_t, \qquad a_t = \sigma_t \varepsilon_t,$$

where m_t is the conditional mean and, in general, it will be a regression on known values of (y_t), σ_t^2 is volatility and ε_t is the (0, 1) i.i.d. white noise.

In order to model the sequence (m_t), we will introduce ARMA models. To capture the possible heteroscedasticity in the volatility of the time series, we will use GARCH models. ARMA models will be determined by using autocorrelation and

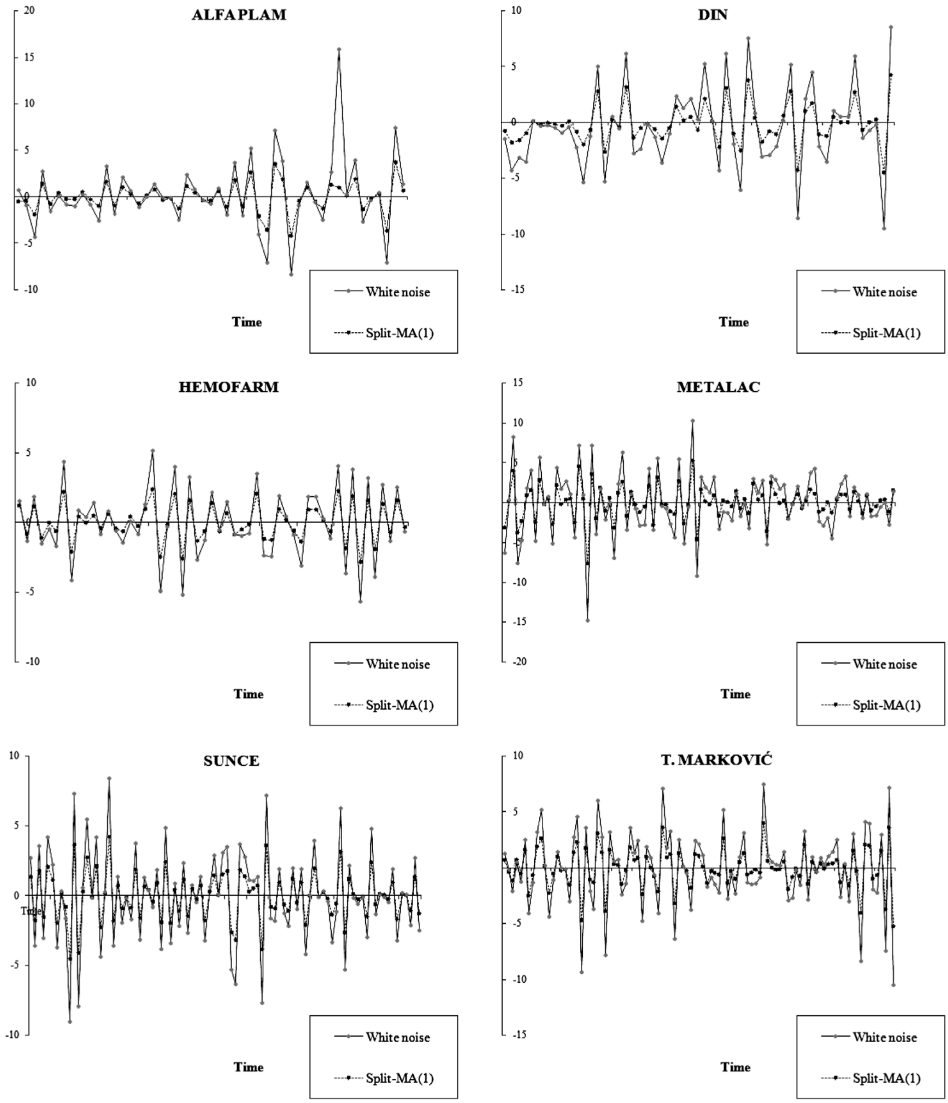

Figure 5 *Comparative graphs of real and modeled data.*

partial autocorrelation functions. After that, we will use a correlogram of squared standardized residuals to check the possible heteroscedasticity. The existence of a significant correlation between squared standardized residuals imply an existence of heteroscedasticity.

Finally, we will test models using Ljung-Box Q statistics. In our case, all calculations and model selections are done in the statistical software EViews 5.0. The above-described method gives results that are shown in Table 4. Regarding the HEMOFARM time series, we can see that has no significant autocorrelation and

Table 4 *Estimated values of real data by the known models*

Companies	Models (m_t/σ_t^2)	
	Type	Equation
ALFA PLAM	AR(0)	$m_t = 15.4656$
	GARCH(2,1)	$\sigma_t^2 = 0.1101 - 0.068a_{t-1}^2 + 0.3524a_{t-2}^2 + 0.6818\sigma_{t-1}^2$
DIN	AR(3)	$m_t = 13.7823 + 0.2258y_{t-1} + 0.3258y_{t-2}$
	–	–
HEMOFARM	–	–
	–	–
METALAC	ARMA(1,1)	$m_t = 13.824 + 0.9697y_{t-1} - 0.7903a_{t-1}$
	GARCH(2,1)	$\sigma_t^2 = 0.0773 + 0.5573a_{t-1}^2 - 0.4208a_{t-2}^2 + 0.8437\sigma_{t-1}^2$
T. MARKOVIĆ	AR(2)	$m_t = 12.5514 + 0.3711y_{t-1} + 0.27919y_{t-2}$
	ARCH(1)	$\sigma_t^2 = 0.8540 + 0.3980a_{t-1}^2$
SUNCE	AR(3)	$m_t = 13.82183 + 0.3670y_{t-1} + 0.2303y_{t-2}$
	ARCH(2)	$\sigma_t^2 = 1.2379 + 0.2287a_{t-1}^2$

also there is no heteroscedasticity. This means that we cannot model this series with well-known models.

If we take a look at the models from Table 4, we can see that a lot of coefficients are required to be estimated. These complex models cannot provide good results, that is, the best results. The option is to apply some other method to estimate these time series. One of these methods is the Split-BREAK, which provides much better results. The standard approach is graphically presented in Figure 6.

7 Conclusion

Nonlinear dynamic systems are very efficient tools for the description of the dynamics of financial time series nowadays. They are, therefore, fundamental for majority of empirical analysis in all segments of a market. Nonlinear stochastic models of financial time series give good results in explaining of many of their features. So, for instance, various modifications of the STOPBREAK process enable successful description of dynamics of financial time series with emphatic permanent fluctuations. In that sense, the original result of this paper, named the Split-BREAK process, better represents these time series and fewer coefficients needed to be estimated than with the well-known models used so far.

The Split-BREAK model can be applied when $-0{,}5 < \hat{\rho}_T(1) < 0$, but many real datasets fulfill that condition. And, whenever is so, the Split-BREAK model is in the advantage comparing with the previously known models. In Section 5, we gave some of the real data which fulfill the condition mentioned above and demonstrated the advantage of the Split-BREAK.

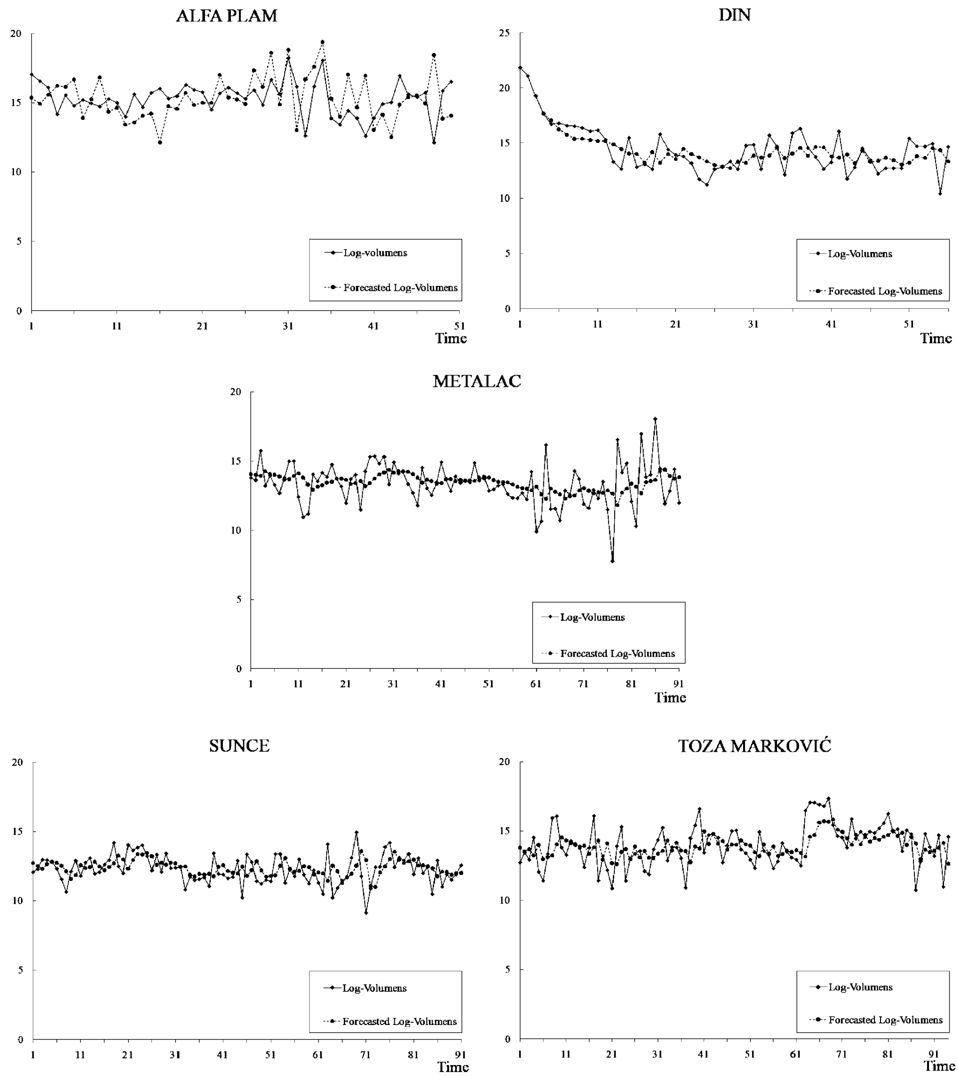

Figure 6 *Comparative graphs of real and modeled data.*

Finally, we underline that the Split-BREAK model was defined to describe the behavior of the volume of trading shares on the stock market, but it can be successfully applied in estimating other financial time series with emphatic permanent fluctuations also.

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