Brazilian Journal of Probability and Statistics
2011, Vol. 25, No. 1, 34–43
DOI: 10.1214/09-BJPS024

Unusual strong laws for arrays of ratios of order statistics

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Abstract. Let $\{X_{n,k}, 1 \le k \le m_n, n \ge 1\}$ be independent random variables from the Pareto distribution. Let $X_{n(k)}$ be the kth largest order statistic from the nth row of our array, where $X_{n(1)}$ denotes the largest order statistic from the nth row. Then set $R_{n,i_n,j_n} = X_{n(j_n)}/X_{n(i_n)}$ where $j_n < i_n$. This paper establishes limit theorems involving weighted sums from the sequence $\{R_{n,i_n,j_n}, n \ge 1\}$, where for the first time we allow $j_n \to \infty$, but only at a slow rate.

1 Introduction

Consider independent random variables $\{X_{n,k_n}, 1 \le k_n \le m_n, n \ge 1\}$ with density $f(x) = p_n x^{-p_n-1} I(x \ge 1)$, where $p_n > 0$. Let $X_{n(k_n)}$ be the k_nth largest order statistic from each row of our array. Hence $X_{n(m_n)} \le X_{n(m_n-1)} \le \cdots \le X_{n(2)} \le X_{n(1)}$. Next define $R_n = R_{n,i_n,j_n} = X_{n(j_n)}/X_{n(i_n)}$ where $j_n < i_n$, which implies that $X_{n(j_n)} \ge X_{n(i_n)}$, or equivalently $R_n \ge 1$. Thus the density of R_n is

$$f_{R_n}(r) = \frac{p_n(i_n-1)!}{(i_n-j_n-1)!(j_n-1)!} r^{-p_n j_n-1}(1-r^{-p_n})^{i_n-j_n-1} I(r \ge 1).$$

It's important to note that the density of R_n is free of m_n. In this paper we will examine strong laws involving weighted sums of $\{R_n, n \ge 1\}$. This paper is a natural extension of Adler [2] and Adler [1]. In Adler [2] all our sequences m_n, j_n and i_n were fixed. In Adler [1], we were allowed to let m_n and i_n grow, but j_n was fixed. Finally, in this paper we allow all our subscripts to grow, but the distance between i_n and j_n is fixed. This case is by far the most difficult, as we will show via the proofs. The growth of j_n cannot be very fast. It turns out that in order to obtain our unusual Strong Laws, which is our goal, we need a logarithmic growth rate for j_n. In some instances we allow i_n to move away from j_n, however that is not the norm. Hence we will set $\Delta = i_n - j_n$, which determines how far apart our order statistics are. We are forced to fix Δ, hence there isn't any Δ_n.

If $p_n j_n$ exceeds one, then ER_n is finite and the associated theorems are straightforward and unremarkable; see Theorems 6, 7 and 8 from Adler [1]. If $p_n j_n < 1$, then these limit theorems fail to exist; see Theorem 5 from Adler [1]. The most interesting case of all occurs when $p_n j_n = 1$. Strange and unusual limit theorems

Key words and phrases. Almost sure convergence, strong law of large numbers.
Received May 2006.

occur when examining random variables that barely do or do not have a first moment which is what happens when $p_n j_n = 1$. These unusual limit theorems are also part of the fair games phenomenon such as the St. Petersburg Game. For more on this and similar topics, see Feller [4], page 251.

The Pareto is a very important distribution. It's used in many settings, such as in modeling in ageing populations. But, the point of this paper is to complete a missing piece in limit theorems for weighted sums of ratios of these type of random variables. It was established in previous papers when laws of large numbers for weighted sums of Pareto random variables exist and also when similar limit theorems for weighted sums of these ratios exist. But in all those cases we were very restricted in the rates of growth of our order statistics. Here we allow much greater freedom in how our order statistics are selected.

In all our theorems we partition $\sum_{n=1}^N a_n R_n / b_N$ into the following three terms:

$$\begin{aligned}\frac{\sum_{n=1}^N a_n R_n}{b_N} &= \frac{\sum_{n=1}^N a_n [R_n I(1 \le R_n \le c_n) - E R_n I(1 \le R_n \le c_n)]}{b_N} \\ &\quad + \frac{\sum_{n=1}^N a_n R_n I(R_n > c_n)}{b_N} \\ &\quad + \frac{\sum_{n=1}^N a_n E R_n I(1 \le R_n \le c_n)}{b_N},\end{aligned}$$

where a_n are our weights, b_n our norming sequence and $c_n = b_n / a_n$. We use the usual Khintchine–Kolmogorov convergence theorem argument; see Chow and Teicher [3], page 113, to show that the first term converges to zero almost surely. The second term is similarly negligible via the Borel–Cantelli lemma. Hence we just need to show that the appropriate sums are finite. However each case greatly differs in the sometimes difficult calculation of our truncated expectation, $E R_n I(1 \le R_n \le c_n)$, which dictates our final limit.

2 Main results

Our first theorem establishes an unusual strong law where $\Delta = 1$. In this case we have complete freedom in our choice of j_n. But, do note that $i_n = j_n + 1$ and $p_n = 1/j_n$. In the event that $j_n = p_n = 1$, for all $n \ge 1$, then our underlying density is just $f(x) = x^{-2} I(x \ge 1)$. And in that case our unusual strong laws will involve the ratio of the two largest order statistics from each row of our array. As always, we define $\lg x = \log(\max\{e, x\})$ and $\lg_2 x = \lg(\lg x)$. Also we use the constant C to denote a generic real number that is not necessarily the same in each appearance.

Theorem 1. *If $p_n j_n = 1$, $\Delta = 1$ and $\alpha > -2$, then*

$$\lim_{N \to \infty} \frac{\sum_{n=1}^N ((\lg n)^\alpha / n) R_n}{(\lg N)^{\alpha+2}} = \frac{1}{\alpha + 2} \qquad \textit{almost surely.}$$

Proof. The density for the ratio of our adjacent order statistics, that is, $\Delta = 1$, is $f_{R_n}(r) = r^{-2} I(r \geq 1)$. Here $a_n = (\lg n)^\alpha / n$, $b_n = (\lg n)^{\alpha+2}$ and $c_n = n(\lg n)^2$. The first term vanishes almost surely since

$$\sum_{n=1}^{\infty} c_n^{-2} E R_n^2 I(1 \leq R_n \leq c_n) = \sum_{n=1}^{\infty} c_n^{-2} \int_1^{c_n} dr < \sum_{n=1}^{\infty} c_n^{-1} = \sum_{n=1}^{\infty} \frac{1}{n(\lg n)^2} < \infty.$$

The second term vanishes almost surely since

$$\sum_{n=1}^{\infty} P\{R_n > c_n\} = \sum_{n=1}^{\infty} \int_{c_n}^{\infty} r^{-2}\, dr = \sum_{n=1}^{\infty} c_n^{-1} = \sum_{n=1}^{\infty} \frac{1}{n(\lg n)^2} < \infty.$$

As for the third term

$$E R_n I(1 \leq R_n \leq c_n) = \int_1^{c_n} r^{-1}\, dr = \lg c_n \sim \lg n.$$

Thus

$$\frac{\sum_{n=1}^{N} a_n E R_n I(1 \leq R_n \leq c_n)}{b_N} \sim \frac{\sum_{n=1}^{N} (\lg n)^{\alpha+1}/n}{(\lg N)^{\alpha+2}} \to \frac{1}{\alpha+2}$$

concluding the proof. □

Next we look at $\Delta > 1$. In this case we need to approximate the coefficient to our density. Using Stirling's formula and letting $p_n j_n = 1$ we have

$$\begin{aligned}
\frac{p_n(i_n-1)!}{(i_n-j_n-1)!(j_n-1)!} &= \frac{(i_n-1)!}{(i_n-j_n-1)!\,j_n!} \\
&= \frac{(j_n+\Delta-1)!}{(\Delta-1)!\,j_n!} \\
&\sim \frac{\sqrt{2\pi}\,(j_n+\Delta-1)^{j_n+\Delta-1/2} e^{-j_n-\Delta+1}}{\sqrt{2\pi}\, j_n^{j_n+1/2} e^{-j_n} (\Delta-1)!} \\
&= \frac{(j_n+\Delta-1)^{j_n+\Delta-1/2} e^{-\Delta+1}}{j_n^{j_n+1/2}(\Delta-1)!} \\
&= \left(\frac{j_n+\Delta-1}{j_n}\right)^{j_n} \left(\frac{(j_n+\Delta-1)^{\Delta-1/2}}{j_n^{1/2}}\right)\left(\frac{e^{-\Delta+1}}{(\Delta-1)!}\right) \\
&= \left(1+\frac{\Delta-1}{j_n}\right)^{j_n} \left(\frac{(j_n+\Delta-1)^{\Delta-1/2}}{j_n^{1/2}}\right)\left(\frac{e^{-\Delta+1}}{(\Delta-1)!}\right) \\
&\sim e^{\Delta-1}\left(\frac{j_n^{\Delta-1/2}}{j_n^{1/2}}\right)\left(\frac{e^{-\Delta+1}}{(\Delta-1)!}\right) \\
&= \frac{j_n^{\Delta-1}}{(\Delta-1)!}.
\end{aligned}$$

The focus of this paper is to explore the growth of our sequence j_n. It turns out that the optimal growth for j_n is $\lg n$; see Theorem 3. By optimal, we mean that we are allowed to obtain these unusual types of Strong Laws when our rates of growth are logarithmic. These are indeed unusual limit theorems since we are able to take ratios of weighted sums of nonintegrable random variables and divide them by a sequence of constants and show that this limit is almost surely a constant. Our next theorem explores what happens when j_n grows slower than $\lg n$.

Theorem 2. *If $p_n j_n = 1$, $j_n = o(\lg n)$, $\Delta \geq 2$ and $\alpha > -2$, then*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} ((\lg n)^\alpha/(n j_n^{\Delta-1}))R_n}{(\lg N)^{\alpha+2}} = \frac{1}{(\alpha+2)(\Delta-1)!} \qquad \text{almost surely.}$$

Proof. Here $a_n = (\lg n)^\alpha/(n j_n^{\Delta-1})$, $b_n = (\lg n)^{\alpha+2}$ and $c_n = n j_n^{\Delta-1}(\lg n)^2$. The first two terms vanish since

$$\begin{aligned}\sum_{n=1}^{\infty} c_n^{-2} E R_n^2 I(1 \leq R_n \leq c_n) &< C \sum_{n=1}^{\infty} \frac{j_n^{\Delta-1}}{c_n^2} \int_1^{c_n} dr \\ &< C \sum_{n=1}^{\infty} \frac{j_n^{\Delta-1}}{c_n} = C \sum_{n=1}^{\infty} \frac{1}{n(\lg n)^2} < \infty\end{aligned}$$

and

$$\sum_{n=1}^{\infty} P\{R_n > c_n\} < C \sum_{n=1}^{\infty} j_n^{\Delta-1} \int_{c_n}^{\infty} r^{-2}\, dr = C \sum_{n=1}^{\infty} \frac{j_n^{\Delta-1}}{c_n} < \infty.$$

Next, we turn our attention to the third term. Thus

$$\begin{aligned}&E R_n I(1 \leq R_n \leq c_n) \\ &\quad\sim \frac{j_n^{\Delta-1}}{(\Delta-1)!} \int_1^{c_n} r^{-1}(1 - r^{-1/j_n})^{\Delta-1}\, dr \\ &\quad= \frac{j_n^{\Delta-1}}{(\Delta-1)!} \int_1^{c_n} r^{-1} \sum_{k=0}^{\Delta-1} \binom{\Delta-1}{k} (-1)^k r^{-k/j_n}\, dr \\ &\quad= \frac{j_n^{\Delta-1}}{(\Delta-1)!} \left[\int_1^{c_n} r^{-1}\, dr + \sum_{k=1}^{\Delta-1} \binom{\Delta-1}{k} (-1)^k \int_1^{c_n} r^{-k/j_n - 1}\, dr \right] \\ &\quad= \frac{j_n^{\Delta-1}}{(\Delta-1)!} \left[\lg c_n + j_n \sum_{k=1}^{\Delta-1} \frac{\binom{\Delta-1}{k}(-1)^{k+1}}{k c_n^{k/j_n}} + j_n \sum_{k=1}^{\Delta-1} \frac{\binom{\Delta-1}{k}(-1)^k}{k} \right] \\ &\quad\sim \frac{j_n^{\Delta-1} \lg n}{(\Delta-1)!}\end{aligned}$$

since $\lg c_n \sim \lg n$, $j_n = o(\lg n)$ and $c_n^{-1/j_n} = o(1)$. Hence

$$\frac{\sum_{n=1}^N a_n E R_n I(1 \le R_n \le c_n)}{b_N} \sim \sum_{n=1}^N \frac{((\lg n)^\alpha/(n j_n^{\Delta-1})) \cdot (j_n^{\Delta-1} \lg n/(\Delta-1)!)}{(\lg N)^{\alpha+2}}$$
$$= \frac{\sum_{n=1}^N (\lg n)^{\alpha+1}/n}{(\Delta-1)!(\lg N)^{\alpha+2}}$$
$$\to \frac{1}{(\alpha+2)(\Delta-1)!}$$

concluding this proof. □

Next we explore the situation of $j_n \sim \lg n$.

Theorem 3. *If* $p_n j_n = 1$, $j_n \sim \lg n$, $\Delta \ge 2$ *and* $\alpha > -2$, *then*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^N ((\lg n)^\alpha/(n j_n^{\Delta-1})) R_n}{(\lg N)^{\alpha+2}} = \frac{\gamma_\Delta}{(\alpha+2)(\Delta-1)!} \qquad \textit{almost surely},$$

where

$$\gamma_\Delta = \sum_{k=1}^{\Delta-1} \frac{\binom{\Delta-1}{k}(-1)^{k+1} e^{-k}}{k} - \sum_{k=2}^{\Delta-1} \frac{1}{k}$$

or, if one wishes

$$\gamma_\Delta = 1 + \sum_{k=1}^{\Delta-1} \frac{\binom{\Delta-1}{k}(-1)^k (1-e^{-k})}{k},$$

where naturally, if $\Delta = 2$ *we have* $\sum_{k=2}^{\Delta-1} \frac{1}{k} = 0$.

Proof. Here $a_n = (\lg n)^{\alpha-\Delta+1}/n$, $b_n = (\lg n)^{\alpha+2}$ and $c_n = n(\lg n)^{\Delta+1}$. The first two terms disappear since

$$\sum_{n=1}^\infty c_n^{-2} E R_n^2 I(1 \le R_n \le c_n) < C \sum_{n=1}^\infty \frac{j_n^{\Delta-1}}{c_n^2} \int_1^{c_n} dr$$
$$< C \sum_{n=1}^\infty \frac{j_n^{\Delta-1}}{c_n} < C \sum_{n=1}^\infty \frac{1}{n(\lg n)^2} < \infty$$

and

$$\sum_{n=1}^\infty P\{R_n > c_n\} < C \sum_{n=1}^\infty j_n^{\Delta-1} \int_{c_n}^\infty r^{-2}\, dr = C \sum_{n=1}^\infty \frac{j_n^{\Delta-1}}{c_n} < \infty.$$

Before we attack the final term in our partition it is important to note that $c_n^{-1/j_n} \to e^{-1}$ as $n \to \infty$. Thus

$$\begin{aligned}
&ER_nI(1 \le R_n \le c_n)\\
&\quad\sim \frac{j_n^{\Delta-1}}{(\Delta-1)!}\int_1^{c_n} r^{-1}(1-r^{-1/j_n})^{\Delta-1}\,dr\\
&\quad= \frac{j_n^{\Delta-1}}{(\Delta-1)!}\int_1^{c_n} r^{-1}\sum_{k=0}^{\Delta-1}\binom{\Delta-1}{k}(-1)^k r^{-k/j_n}\,dr\\
&\quad= \frac{j_n^{\Delta-1}}{(\Delta-1)!}\left[\int_1^{c_n} r^{-1}\,dr + \sum_{k=1}^{\Delta-1}\binom{\Delta-1}{k}(-1)^k\int_1^{c_n} r^{-k/j_n-1}\,dr\right]\\
&\quad= \frac{j_n^{\Delta-1}}{(\Delta-1)!}\left[\lg c_n + j_n\sum_{k=1}^{\Delta-1}\frac{\binom{\Delta-1}{k}(-1)^{k+1}}{kc_n^{k/j_n}} + j_n\sum_{k=1}^{\Delta-1}\frac{\binom{\Delta-1}{k}(-1)^k}{k}\right]\\
&\quad\sim \frac{(\lg n)^{\Delta-1}}{(\Delta-1)!}\left[\lg n + \lg n\sum_{k=1}^{\Delta-1}\frac{\binom{\Delta-1}{k}(-1)^{k+1}e^{-k}}{k} + \lg n\sum_{k=1}^{\Delta-1}\frac{\binom{\Delta-1}{k}(-1)^k}{k}\right]\\
&\quad= \frac{(\lg n)^{\Delta}}{(\Delta-1)!}\left[1 + \sum_{k=1}^{\Delta-1}\frac{\binom{\Delta-1}{k}(-1)^{k+1}e^{-k}}{k} + \sum_{k=1}^{\Delta-1}\frac{\binom{\Delta-1}{k}(-1)^k}{k}\right]\\
&\quad= \frac{(\lg n)^{\Delta}}{(\Delta-1)!}\left[1 + \sum_{k=1}^{\Delta-1}\frac{\binom{\Delta-1}{k}(-1)^{k+1}e^{-k}}{k} - \sum_{k=1}^{\Delta-1}\frac{1}{k}\right]\\
&\quad= \frac{(\lg n)^{\Delta}}{(\Delta-1)!}\left[\sum_{k=1}^{\Delta-1}\frac{\binom{\Delta-1}{k}(-1)^{k+1}e^{-k}}{k} - \sum_{k=2}^{\Delta-1}\frac{1}{k}\right]\\
&\quad= \frac{\gamma_\Delta(\lg n)^{\Delta}}{(\Delta-1)!},
\end{aligned}$$

where we used a combinatorial result from Riordan [5], page 5. Hence

$$\begin{aligned}
\frac{\sum_{n=1}^N a_nER_nI(1 \le R_n \le c_n)}{b_N} &\sim \frac{\sum_{n=1}^N((\lg n)^\alpha/(nj_n^{\Delta-1}))\cdot(\gamma_\Delta(\lg n)^\Delta/(\Delta-1)!)}{(\lg N)^{\alpha+2}}\\
&\sim \frac{\gamma_\Delta\sum_{n=1}^N(\lg n)^{\alpha+1}/n}{(\Delta-1)!(\lg N)^{\alpha+2}}\\
&\to \frac{\gamma_\Delta}{(\alpha+2)(\Delta-1)!}
\end{aligned}$$

concluding this proof. $\square$

Finally, we examine the situation where j_n is larger than $\lg n$. This case proves to be extremely difficult, as we will show via the ensuing and very helpful lemma.

Lemma. *If* $1 < a < 2$, *then*

$$\lim_{x\to\infty} \frac{1 + 3x^{-1}\lg x + x^{a-1}[(e^x x^3)^{-1/x^a} - 1]}{x^{1-a}} = 1/2.$$

Proof. It is easy to see that $(e^x x^3)^{-1/x^a} \to 1$ as $x \to \infty$. Next we need the derivative of $(e^x x^3)^{-1/x^a}$. Using logarithms we obtain

$$\frac{d}{dx}(e^x x^3)^{-1/x^a} = (e^x x^3)^{-1/x^a}[(a-1)x^{-a} - 3x^{-a-1} + 3ax^{-a-1}\lg x].$$

Now we apply L'Hopital's rule twice, but some algebra is necessary in order for our limit to come into view

$$\begin{aligned}
&\lim_{x\to\infty} \frac{1 + 3x^{-1}\lg x + x^{a-1}[(e^x x^3)^{-1/x^a} - 1]}{x^{1-a}} \\
&\quad = \lim_{x\to\infty} \frac{x^{1-a} + 3x^{-a}\lg x + (e^x x^3)^{-1/x^a} - 1}{x^{2-2a}} \\
&\quad = \lim_{x\to\infty} \left\{ \frac{(1-a)x^{-a} + 3x^{-a-1} - 3ax^{-a-1}\lg x}{(2-2a)x^{1-2a}} \right. \\
&\qquad\qquad \left. + \frac{(e^x x^3)^{-1/x^a}[(a-1)x^{-a} - 3x^{-a-1} + 3ax^{-a-1}\lg x]}{(2-2a)x^{1-2a}} \right\} \\
&\quad = \lim_{x\to\infty} \frac{[(e^x x^3)^{-1/x^a} - 1][(a-1)x^{-a} - 3x^{-a-1} + 3ax^{-a-1}\lg x]}{(2-2a)x^{1-2a}} \\
&\quad = \lim_{x\to\infty} \frac{[(e^x x^3)^{-1/x^a} - 1][a - 1 - 3x^{-1} + 3ax^{-1}\lg x]}{(2-2a)x^{1-a}} \\
&\quad = \lim_{x\to\infty} \left\{ \frac{[(e^x x^3)^{-1/x^a} - 1][3x^{-2} + 3ax^{-2} - 3ax^{-2}\lg x]}{(2-2a)(1-a)x^{-a}} \right. \\
&\qquad\qquad + \big((e^x x^3)^{-1/x^a}[(a-1)x^{-a} - 3x^{-a-1} + 3ax^{-a-1}\lg x] \\
&\qquad\qquad\quad \left. \times [(a-1) - 3x^{-1} + 3ax^{-1}\lg x]\big)/\big((2-2a)(1-a)x^{-a}\big) \right\} \\
&\quad = \lim_{x\to\infty} \frac{[o(1)][o(x^{-a})] + [1 + o(1)][(a-1)x^{-a} + o(x^{-a})][(a-1) + o(1)]}{(2-2a)(1-a)x^{-a}} \\
&\quad = \lim_{x\to\infty} \frac{(a-1)^2 x^{-a}}{(2-2a)(1-a)x^{-a}} \\
&\quad = 1/2
\end{aligned}$$

which completes the proof of this lemma. □

Our final theorem only explores the situation of $\Delta = 2$ and $j_n \sim (\lg n)^a$ where $1 < a < 2$. If one wishes to explore larger Δ and j_n then the techniques used in the proof of the following theorem will prove to be quite helpful. And that is the point of this theorem, to show how one can increase either Δ or j_n.

Theorem 4. *If $p_n j_n = 1$, $j_n \sim (\lg n)^a$, where $1 < a < 2$, $\Delta = 2$ and $\alpha > -3$, then*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} ((\lg n)^\alpha / n) R_n}{(\lg N)^{\alpha+3}} = \frac{1}{2(\alpha+3)} \qquad \textit{almost surely.}$$

Proof. Here $a_n = (\lg n)^\alpha / n$, $b_n = (\lg n)^{\alpha+3}$ and $c_n = n(\lg n)^3$. The first two terms vanish since

$$\begin{aligned}\sum_{n=1}^{\infty} c_n^{-2} E R_n^2 I(1 \le R_n \le c_n) &< C \sum_{n=1}^{\infty} \frac{j_n}{c_n^2} \int_1^{c_n} dr \\ &< C \sum_{n=1}^{\infty} \frac{j_n}{c_n} < C \sum_{n=1}^{\infty} \frac{(\lg n)^{a-3}}{n} < \infty\end{aligned}$$

since $1 < a < 2$, while

$$\sum_{n=1}^{\infty} P\{R_n > c_n\} < C \sum_{n=1}^{\infty} j_n \int_{c_n}^{\infty} r^{-2}\, dr = C \sum_{n=1}^{\infty} \frac{j_n}{c_n} < \infty.$$

As for our third term

$$\begin{aligned}E R_n I(1 \le R_n \le c_n) &\sim j_n \int_1^{c_n} r^{-1}(1 - r^{-1/j_n})\, dr \\ &= j_n \int_1^{c_n} (r^{-1} - r^{-1-1/j_n})\, dr \\ &= j_n \lg c_n + j_n^2 (c_n^{-1/j_n} - 1) \\ &\sim (\lg n)^a [\lg n + 3 \lg_2 n] + (\lg n)^{2a} \big[[n(\lg n)^3]^{-1/(\lg n)^a} - 1\big].\end{aligned}$$

At this point we substitute $x = \lg n$ and apply our lemma to obtain

$$\begin{aligned}E R_n I(1 \le R_n \le c_n) &\sim x^a [x + 3 \lg x] + x^{2a} \big[[e^x x^3]^{-1/x^a} - 1\big] \\ &= x^a [x + 3 \lg x] + x^{2a} \big[[e^x x^3]^{-1/x^a} - 1\big] \\ &= x^2 x^{a-1} \bigg[1 + \frac{3 \lg x}{x} + x^{a-1} \big[[e^x x^3]^{-1/x^a} - 1\big]\bigg] \\ &\sim x^2/2 \\ &= (\lg n)^2/2.\end{aligned}$$

Therefore the limit of our partial sums will be

$$\frac{\sum_{n=1}^{N} a_n E R_n I(1 \le R_n \le c_n)}{b_N} \sim \frac{\sum_{n=1}^{N}((\lg n)^\alpha/n)\cdot((\lg n)^2/2)}{(\lg N)^{\alpha+3}}$$

$$= \frac{\sum_{n=1}^{N}(\lg n)^{\alpha+2}/n}{2(\lg N)^{\alpha+3}} \to \frac{1}{2(\alpha+3)}$$

concluding this final proof. □

3 Conclusion

What is quite unusual about Theorem 4 is that the conclusion does not depend on the value of a, as long as our parameter a is between one and two. There are many other directions one could investigate at this point. Naturally, one is where j_n grows faster than $(\lg n)^a$. Another is where Δ_n grows within each row, instead of being fixed. All of these cases involve very delicate computations as one can see from the results in this paper. One should note that by observing the proof of Theorem 4. The possibilities are endless. But it is very important to note that it is extremely difficult to have both

$$\sum_{n=1}^{\infty} P\{R_n > c_n\}$$

and

$$\sum_{n=1}^{\infty} c_n^{-2} E R_n^2 I(1 \le R_n \le c_n)$$

convergent, while have

$$\frac{\sum_{n=1}^{N} a_n E R_n I(1 \le R_n \le c_n)}{b_N}$$

converge to a finite nonzero constant.

Acknowledgment

I would like to thank the referee for his/her careful reading of my work. Those comments lead to significant improvements in this paper.

References

[1] Adler, A. (2005). Limit theorems for arrays of ratios of order statistics. *Bulletin Institute of Mathematics Academia Sinica* **33** 327–344. MR2184113

[2] Adler, A. (2006). Exact laws for sums of ratios of order statistics from the Pareto distribution. *Central European Journal of Mathematics* **4** 1–4. MR2213023

[3] Chow, Y. S. and Teicher, H. (1997). *Probability Theory: Independence, Interchangeability, Martingales*, 3rd ed. Springer, New York. MR1476912

[4] Feller, W. (1968). *An Introduction to Probability Theory and Its Applications* **1**, 3rd ed. Wiley, New York.

[5] Riordan, J. (1979). *Combinatorial Identities*. Krieger, New York. MR0554488

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