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CRAMÉR-TYPE MODERATE DEVIATION FOR THE MAXIMUM OF THE PERIODOGRAM WITH APPLICATION TO SIMULTANEOUS TESTS IN GENE EXPRESSION TIME SERIES[1]

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In this paper, Cramér-type moderate deviations for the maximum of the periodogram and its studentized version are derived. The results are then applied to a simultaneous testing problem in gene expression time series. It is shown that the level of the simultaneous tests is accurate provided that the number of genes G and the sample size n satisfy $G = \exp(o(n^{1/3}))$.

1. Introduction. Let $X_1, X_2, \dots$ be a sequence of random variables. Define the periodogram ordinates for $\{X_n\}$ at the standard frequencies $\omega_j = 2\pi j/n$ by

$$I_n(\omega_j) = \frac{1}{n}\left|\sum_{k=1}^{n} X_k e^{ik\omega_j}\right|^2,$$

where $1 \le j \le q$ and $q = [(n-1)/2]$.

The periodogram is a fundamental tool in spectral analysis and is often used to detect periodic patterns in various real applications, such as the analysis of gene expression data and the study of earthquake. Theoretical properties of the periodogram have been extensively studied. An, Chen and Hannan (1983) obtained the logarithm law for the maximum of the periodogram; Davis and Mikosch (1999), Mikosch, Resnick and Samorodnitsky (2000), Lin and Liu (2009a) obtained the asymptotic distribution for the maximum of the periodogram under the i.i.d. and linear process cases, the heavy-tailed case and nonlinear time series case, respectively; Fay and Soulier (2001) proved the central limit theorem for functionals of the periodogram; Shao and Wu (2007) obtained asymptotic distributions for the periodogram and the empirical distribution function of the periodogram for a wide class of nonlinear processes. When $\{X_n\}$ are independent and identically distributed (i.i.d.) random variables with $\operatorname{Var}(X_1) = \sigma^2$ and $\mathrm{E}|X_1|^{2+\delta} < \infty$ for some $\delta > 0$, Davis and Mikosch (1999) show that

$$\lim_{n\to\infty} \mathrm{P}\Bigl(\max_{1\le j\le q} I_n(\omega_j)/\sigma^2 - \log q \le y\Bigr) = \exp(-\exp(-y)). \tag{1.1}$$

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The main purpose of this paper is to study the Cramér-type moderate deviations for the maximum of the periodogram and its studentized version. That is, what is the largest possible a_n so that

$$\frac{\mathrm{P}(\max_{1\le j\le q} I_n(\omega_j)/\sigma^2 - \log q \ge y)}{1-\exp(-\exp(-y))} \to 1 \tag{1.2}$$

uniformly in $y \in [-\log q, a_n]$, or for the studentized periodogram, what is the largest possible b_n so that

$$\frac{\mathrm{P}(\max_{1\le j\le q} I_n(\omega_j)/(q^{-1}\sum_{j=1}^{q} I_n(\omega_j)) - \log q \ge y)}{1-\exp(-\exp(-y))} \to 1 \tag{1.3}$$

uniformly in $y \in [-\log q, b_n]$? We will show that a_n depends on the moment condition of $\{X_n\}$. For example, if $\mathrm{E}|X_1|^{2+\delta} < \infty$, $\delta > 0$, the largest possible value of a_n is $\frac{\delta}{2}\log n$, but a_n can be chosen $o(n^{1/3})$ if the moment generating function of X_1 is finite. However, the situation becomes totally different for the studentized periodogram. We will prove that $b_n = o(n^{1/3})$ provided that $\mathrm{E}X_1^4 < \infty$.

The paper is organized as follows. Our main results, Theorems 2.1–2.3, are stated in Section 2, while proofs of the main results are postponed to Section 4. Our moderate deviation results are motivated by simultaneous tests in gene expression time series. Theoretical results for the simultaneous tests and simulation study are discussed in Section 3.

2. Main results. Throughout this paper, we assume $\{X_n\}$ are i.i.d. random variables. Our first result is the moderate deviation for the maximum of the periodogram for $y \le c\log n$ for some $c > 0$. Such type of moderate deviation for the partial sums of $\{X_n\}$ has been studied in literature, for example, by Michel (1976), Amosova (1982), Petrov (2002) and Wu and Zhao (2008).

THEOREM 2.1. (i) *Suppose that for some $c > 0$,*

$$n^{c+1}\mathrm{P}\big(|X_1| \ge \sqrt{n\log n}\big) = o(1) \tag{2.1}$$

as $n \to \infty$. Then we have

$$\lim_{n\to\infty} \frac{\mathrm{P}(\max_{1\le j\le q} I_n(\omega_j)/\sigma^2 - \log q \ge y)}{1-\exp(-\exp(-y))} = 1 \tag{2.2}$$

uniformly in $y \in [-\log q, c\log n]$, where $\sigma^2 = \mathrm{Var}(X_1)$.

(ii) *If for some $\sigma > 0$ (2.2) holds uniformly in $y \in [-\log q, c\log n]$ with some $c > 0$, then we have*

$$n^{c+1}\mathrm{P}\big(|X_1| \ge \sqrt{n\log n}\big) = O(1). \tag{2.3}$$

Theorem 2.1(ii) shows that condition (2.1) is nearly optimal and hence the range depends on the moment assumption. On the other hand, when the moment generating function exists, the range can be extended to $o(n^{1/3})$.

THEOREM 2.2. *Assume* $\mathrm{Var}(X_1) = \sigma^2$ *and* $\mathrm{E}e^{t_0|X_1|} < \infty$ *for some* $t_0 > 0$. *Then*

$$\lim_{n\to\infty} \frac{\mathrm{P}(\max_{1\le j\le q} I_n(\omega_j)/\sigma^2 - \log q \ge y)}{1 - \exp(-\exp(-y))} = 1$$

uniformly in $y \in [-\log q, o(n^{1/3}))$.

We next consider the maximum of the studentized periodogram. Theorem 2.3 below shows that the moment conditions in Theorems 2.1 and 2.2 can be significantly reduced for the studentized version.

THEOREM 2.3. *If* $\mathrm{E}X_1^4 < \infty$, *then*

$$\lim_{n\to\infty} \frac{\mathrm{P}(\max_{1\le j\le q} I_n(\omega_j)/(q^{-1}\sum_{j=1}^{q} I_n(\omega_j)) - \log q \ge y)}{1 - \exp(-\exp(-y))} = 1 \tag{2.4}$$

uniformly in $y \in [-\log q, o(n^{1/3}))$.

Since the variance σ^2 of X_1 is typically unknown, what used in practice is actually the studentized periodogram. So the result in Theorem 2.3 is more appealing and useful than Theorems 2.1 and 2.2. Theorem 2.3 also shares similar properties with self-normalized partial sums of independent random variables, which usually requires fewer moment assumptions; see Shao (1997, 1999) for self-normalized large deviation without any moment assumption and Cramér moderate deviation under finite third moment, and de la Peña, Lai and Shao (2009) for recent developments in the area of self-normalized limit theory. In view of the moderate deviation for self-normalized partial sums [Shao (1999)], we conjecture that Theorem 2.3 remains true if $\mathrm{E}|X_1|^3 < \infty$. It would also be interesting to see whether a similar result as Theorem 2.3 holds when $\{X_n\}$ is a linear process or a nonlinear process, however, this will be a very challenging question because the moderate deviation result is not clear even for the self-normalized partial sums of this class of $\{X_n\}$.

3. Application to simultaneous tests.

3.1. *Theoretical results.* Periodic phenomena are widely studied in biology. Recently, there are quite a lot of interests in detecting periodic patterns in gene expression time series; see Wichert, Fokianos and Strimer (2004), Ahdesmäki et al. (2005), Chen (2005), Glynn, Chen and Mushegian (2006) and the references therein. Due to modern technology such as microarray experiments, the data are usually high-dimensional and we often need to make many statistical inference simultaneously. Let $Y_{t,g}$ denote the observed expression level of gene g at time t, $1 \le g \le G$ and $1 \le t \le n$, where G is the number of genes. The sample size n is usually much smaller than the number of genes. Consider the following model of periodic gene expression:

$$Y_{t,g} = \mu_g + \beta_g \cos(\omega t + \phi) + \varepsilon_{t,g}, \tag{3.1}$$

where $\beta_g \ge 0$, $\omega \in (0, \pi)$, $\phi \in (-\pi, \pi]$, μ_g is the mean expression level. For each g, $\varepsilon_{1,g}, \dots, \varepsilon_{n,g}$ are i.i.d. noise sequence with mean zero. We wish to test the null hypothesis $H_{0,g} : \beta_g = 0$ against the alternative hypothesis $H_{1,g} : \beta_g \ne 0$. If $H_{0,g}$ is rejected, then we identify gene g with a periodic pattern in its expression. Periodogram is often used to detect periodically expressed gene. Let $q = [(n-1)/2]$ and set

$$I_n^{(g)}(\omega_j) = \frac{1}{n}\left|\sum_{k=1}^{n} Y_{k,g} e^{ik\omega_j}\right|^2, \tag{3.2}$$

where $\omega_j = 2\pi j/n$, $1 \le j \le q$. Define the g-statistic

$$f_g = \frac{\max_{1\le j\le q} I_n^{(g)}(\omega_j)}{\sum_{j=1}^{q} I_n^{(g)}(\omega_j)},$$

and its null distribution $F_{n,g}(x) = \mathrm{P}(f_g \le x | H_{0,g})$. Under the null hypothesis and the assumption that $\varepsilon_{1,g}, \dots, \varepsilon_{n,g}$ are i.i.d. normal random variables, the exact distribution for f_g can be found in Fisher (1929):

$$\mathrm{P}(f_g > x | H_{0,g}) = \sum_{j=1}^{[1/x]} (-1)^{j-1} C_q^j (1 - jx)^{q-1} =: f_n(x). \tag{3.3}$$

Using (3.3), Wichert, Fokianos and Strimer (2004) proposed the following method to identify periodically expressed genes:

Step 1. For each time series calculate Fisher's statistic f_g.

Step 2. For each of the test statistic calculate the corresponding p-value $P_g = f_n(f_g)$.

Step 3. Use the method of Benjamini and Hochberg (1995) to control the False Discovery Rate (FDR) at θ. Let $P_{(1)} \le P_{(2)} \le \cdots \le P_{(G)}$ be the ordered p-values and put

$$i_\theta = \max\{i : P_{(i)} \le i\theta/G\}. \tag{3.4}$$

Reject the null hypothesis for the time series indexed by $\mathcal{S} = \{i : P_i \le P_{(i_\theta)}\}$.

In many applications such as those arising from bioinformatics, the noise can be remarkably non-Gaussian [Ahdesmäki et al. (2005)]. Then the values $P_1, \dots, P_G$ are only the estimators of the true p-values. It is natural to ask:

How large G can be before the accuracy of simultaneous statistical inference becomes poor?

Similar problems have been studied in Fan, Hall and Yao (2007), where they consider $Y_{t,g} = \mu_g + \varepsilon_{t,g}$, the model in (3.1) without the periodic part, and focused on testing $H'_{0,g}: \mu_g = 0$. Let the true p-value be $P_g^{\text{true}} = (1 - F_{n,g}(f_g))$. From (3.4), $P_{(i_\theta)}^{\text{true}}$ may be of the order $O(1/G)$. Hence, $\max_{1\le g\le G} |P_g^{\text{true}} - P_g| = o(1)$, implied by (1.1), is not enough. The required accuracy between the estimated p-value and the true p-value is

$$|P_g - P_g^{\text{true}}| I\{\mathcal{H}_g\} = o(P_g^{\text{true}}) \qquad \text{uniformly in } 1 \le g \le G, \tag{3.5}$$

that is,

$$\max_{1\le g\le G} \left| \frac{P_g}{P_g^{\text{true}}} - 1 \right| I\{\mathcal{H}_g\} = o(1),$$

where $\mathcal{H}_g = \{P_g > \theta/(2G) \text{ or } P_g^{\text{true}} > \theta/(2G)\}$. (On $\mathcal{H}_g^c$, the gene g is always rejected.) Some similar requirements as (3.5) on simultaneous tests have been proposed by Fan, Hall and Yao (2007) and Kosorok and Ma (2007), page 1460.

Recall that $F_{n,g}(x) = \mathrm{P}(f_g \le x | H_{0,g})$. By examining the proof of Theorem 2.3, we have the following corollary.

COROLLARY 3.1. *Suppose that* $\min_{1\le g\le G} \mathrm{Var}(\varepsilon_{1,g}) \ge \kappa$ *for some* $\kappa > 0$ *which does not depend on* G. *Further assume that* $\max_{1\le g\le G} \mathrm{E}\varepsilon_{1,g}^4 = O(1)$. *Then the null distribution* $F_{n,g}(x)$ *satisfies*

$$\max_{1\le g\le G} \left| \frac{1 - F_{n,g}((y + \log q)/q)}{1 - \exp(-\exp(-y))} - 1 \right| = o(1)$$

uniformly in $y \in [-\log q, o(n^{1/3}))$.

In fact, by exactly the same proof as that of Theorem 2.3, it is easy to see that for any $M \ge 0$,

$$\limsup_{n\to\infty} \max_{1\le g\le G} \sup_{y\in[M,o(n^{1/3}))} \left| \frac{1 - F_{n,g}((y + \log q)/q)}{1 - \exp(-\exp(-y))} - 1 \right| \le Ce^{-M}.$$

Also, similarly to (1.1), following the proofs in Davis and Mikosch (1999), we have for any fixed $y \in \mathbf{R}$

$$\limsup_{n\to\infty} \max_{1\le g\le G} \left| F_{n,g}\big((y + \log q)/q\big) - \exp(-\exp(-y)) \right| = 0,$$

which together with the standard discretized approximation argument gives

$$\limsup_{n\to\infty} \max_{1\le g\le G} \sup_{y\in[-\log q, M]} \left| F_{n,g}\big((y + \log q)/q\big) - \exp(-\exp(-y)) \right| = 0.$$

This proves Corollary 3.1.

The following lemma shows that we can replace $1 - \exp(-\exp(-y))$ by $f_n((y + \log q)/q)$.

LEMMA 3.1. *Let f_n be given in* (3.3). *We have*

$$\lim_{n\to\infty}\left|\frac{f_n((y+\log q)/q)}{1-\exp(-\exp(-y))}-1\right|=0$$

uniformly in $y \in [-\log q, o(n^{1/3}))$.

It follows from Corollary 3.1 and Lemma 3.1 the following theorem.

THEOREM 3.1. *Suppose the conditions in Corollary* 3.1 *are satisfied and* $G = \exp(o(n^{1/3}))$. *Then* (3.5) *holds.*

Theorem 3.1 shows that the level of the simultaneous tests is accurate provided that $G = \exp(o(n^{1/3}))$, which seems to be the correct order of asymptotics for microarray experiments with a moderate number of samples.

Using the bootstrap and a refined expansion of t-statistic in Theorem 1.2 of Wang (2005), Fan, Hall and Yao (2007) show that $\exp(o(n^{1/3}))$ can be replaced by $\exp(o(n^{1/2}))$ for the tests $H'_{0,g}: \mu_g = 0$. It would be interesting to investigate whether a similar expansion as Theorem 1.2 of Wang (2005) holds for the maximum of the studentized periodogram.

3.2. *Simulation study.* In this section, we carry out a simple simulation study to assess the finite sample performance. We generate 2000 genes with 100 periodic genes for different sample sizes n. Consider

$$Y_{t,g} = \beta\big(\cos(\omega^{(g)}t) + \sin(\omega^{(g)}t)\big) + \varepsilon_{t,g}, \qquad 1 \le t \le n, 1 \le g \le 100,$$
$$Y_{t,g} = \varepsilon_{t,g}, \qquad 1 \le t \le n, 101 \le g \le 2000.$$

$\varepsilon_{t,g}$ will be taken as $N(0,1)$, $(\sqrt{3/5}) \times t(5)$, EXP(1), $2^{-1} \times \chi^2(2)$, where $t(5)$ has the t distribution with freedom 5, EXP(1) is the exponential random variable with parameter $\mu = 1$, $\chi^2(2)$ is chi square random variable with freedom 2. [The constants on the left-hand side of random variables are chosen so that $\mathrm{Var}(\varepsilon_{t,g}) = 1$.] The FDR level θ is chosen as 0.15 and 0.05. The simulation results are based on 100 replicates.

We only give the simulation study when $\omega^{(g)}$ is of the form of ω_i for some $1 \le i \le q$. To do this, we let $\omega^{(g)} = 2\pi/10$, $n = 20, 50$ and $\beta = 1$. The results are summarized in Table 1, where Tot. = total count identified using FDR; Pos. = the number of true positives identified using FDR; Z = the number of true periodic genes among the smallest 100 p-values genes. We note that when the tails of $\varepsilon_{t,g}$ are heavier than that of Gaussian random variable, the empirical FDR (EFDR) are lower than the target FDR, while most of periodic genes can still be found. There are no significant differences between Gaussian noise and other noises when n is large moderately ($n = 50$). Powers increase as n increases. Overall Fisher's statistic is relatively robust to the noise, as indicated by Theorem 2.3. We refer to Wichert, Fokianos and Strimer (2004) for some real data analysis.

TABLE 1

		Tot. (Pos.)	EFDR	Tot. (Pos.)	EFDR
θ		$n = 20$		$n = 50$	
0.15	Normal	39.9 (33.7)	0.155	116.2 (99.6)	0.143
0.05		12.6 (11.9)	0.059	102.6 (99.2)	0.033
Z		62		97.1	
0.15	EXP(1)	55.3 (49.5)	0.105	104.3 (97.3)	0.067
0.05		33.1 (31.9)	0.036	97.8 (95.8)	0.020
Z		69.5		96.9	
0.15	$\chi^2(2)$	46.0 (43.2)	0.061	105.0 (97.6)	0.071
0.05		29.0 (28.2)	0.028	98.1 (95.6)	0.026
Z		69.7		96.5	
0.15	$t(5)$	43.4 (39.7)	0.085	110.2 (98.2)	0.109
0.05		20.2 (19.6)	0.030	99.9 (96.6)	0.033
Z		67.6		96.6	

4. Proofs. Throughout this section, let C denote a positive constant whose value may be different at each appearance. For two real sequences $\{a_n\}$ and $\{b_n\}$, write $a_n = O(b_n)$ if there exists a constant C such that $|a_n| \le C|b_n|$ holds for large n, $a_n = o(b_n)$ if $\lim_{n\to\infty} a_n/b_n = 0$. Denote by $|\cdot|$ the d-dimensional Euclidean norm in $\mathbf{R}^d$, $d \ge 1$.

PROOF OF THEOREM 2.1. (i) (*Sufficiency*). By $\sum_{k=1}^n e^{ik\omega_j} = 0$ for $1 \le j \le q$, we can assume that $\mathrm{E}X_1 = 0$. Also, for convenience, we assume $\sigma^2 = 1$. For $y \in [-\log q, c\log n]$, set $x = \sqrt{y + \log q}$. We start with truncation of X_k at two levels. Let $\varepsilon_n = (\log n)^{-1}$ and $\varepsilon > 0$ be a small number which will be specified later. Define

$$
\begin{aligned}
\widetilde{X}_k &= X'_k - \mathrm{E}X'_k,\\
X'_k &= X_k I\{|X_k| \le \varepsilon\sqrt{n}x\},\\
\widehat{X}_k &= X''_k - \mathrm{E}X''_k,\\
X''_k &= X_k I\{|X_k| \le \varepsilon_n\sqrt{n}/x\},\\
I'_n(\omega_j) &= \frac{1}{n}\left|\sum_{k=1}^n X'_k e^{ik\omega_j}\right|^2 = \frac{1}{n}\left|\sum_{k=1}^n \widetilde{X}_k e^{ik\omega_j}\right|^2,\\
I''_n(\omega_j) &= \frac{1}{n}\left|\sum_{k=1}^n X''_k e^{ik\omega_j}\right|^2 = \frac{1}{n}\left|\sum_{k=1}^n \widehat{X}_k e^{ik\omega_j}\right|^2.
\end{aligned}
\tag{4.1}
$$

Then we have

$$\left|\mathrm{P}\Big(\max_{1\le j\le q} I_n(\omega_j)\ge x^2\Big)-\mathrm{P}\Big(\max_{1\le j\le q} I_n'(\omega_j)\ge x^2\Big)\right| \le n\mathrm{P}\big(|X_1|\ge \varepsilon\sqrt{n}x\big). \tag{4.2}$$

The independence between X_l and $\{X_k, k\neq l\}$, $1\le l\le n$, implies that

$$\begin{aligned}
&\mathrm{P}\Big(\max_{1\le j\le q} I_n'(\omega_j)\ge x^2\Big)-\mathrm{P}\Big(\max_{1\le j\le q} I_n''(\omega_j)\ge x^2\Big)\\
&\quad\le \mathrm{P}\left(\max_{1\le j\le q}\left|\sum_{k=1}^n X_k' e^{ik\omega_j}\right|\ge\sqrt{n}x, \bigcup_{l=1}^n\{|X_l|>\varepsilon_n\sqrt{n}/x\}\right)\\
&\quad\le \sum_{l=1}^n \mathrm{P}\left(\max_{1\le j\le q}\left|\sum_{k=1,k\neq l}^n X_k' e^{ik\omega_j}\right|\ge(1-\varepsilon)\sqrt{n}x\right)\mathrm{P}\big(|X_l|>\varepsilon_n\sqrt{n}/x\big)\\
&\quad\le n\mathrm{P}\left(\max_{1\le j\le q}\left|\sum_{k=1}^n X_k' e^{ik\omega_j}\right|\ge(1-2\varepsilon)\sqrt{n}x\right)\mathrm{P}\big(|X_1|>\varepsilon_n\sqrt{n}/x\big) \qquad (4.3)\\
&\quad\le n\mathrm{P}\left(\max_{1\le j\le q}\left|\sum_{k=1}^n X_k'' e^{ik\omega_j}\right|\ge(1-2\varepsilon)\sqrt{n}x\right)\mathrm{P}\big(|X_1|>\varepsilon_n\sqrt{n}/x\big)\\
&\qquad+\big(n\mathrm{P}(|X_1|>\varepsilon_n\sqrt{n}/x)\big)^2\\
&\quad=: H_n+\big(n\mathrm{P}(|X_1|>\varepsilon_n\sqrt{n}/x)\big)^2.
\end{aligned}$$

To estimate H_n, we need Lemma 4.2 of Lin and Liu (2009a). The proof and constants $n_0, c_{1,1}, c_{1,2}$, etc. are given in Lin and Liu (2009b), pages 23–25.

Let $d\ge 1$ be a fixed integer. For $z=(x_1,y_1,\dots,x_d,y_d)$ let $\|z\|_d=\min\{(x_k^2+y_k^2)^{1/2}: 1\le k\le d\}$, let I_{2d} denote a $2d\times 2d$ identity matrix, and $\|\cdot\|$ denote the spectral norm of a matrix.

LEMMA 4.1 [Lin and Liu (2009a, 2009b)]. *Let $\xi_{n,1},\dots,\xi_{n,k_n}$ be independent random vectors in $\mathbf{R}^{2d}$ with zero means, and let $S_n=\sum_{i=1}^{k_n}\xi_{n,i}$. Assume that $|\xi_{n,i}|\le c_nB_n^{1/2}$, $1\le i\le k_n$, for some $c_n\to 0$ and $B_n\to\infty$, and that*

$$\|B_n^{-1}\operatorname{Cov}(\xi_{n,1}+\cdots+\xi_{n,k_n})-I_{2d}\|\le C_0c_n^2,$$

where C_0 is a positive constant. If $\beta_n:=B_n^{-3/2}\sum_{k=1}^{k_n}\mathrm{E}|\xi_{n,k}|^3\to 0$, then for all $n\ge n_0$

$$\begin{aligned}
&\big|\mathrm{P}(\|S_n\|_d\ge x)-\mathrm{P}(\|N\|_d\ge x/B_n^{1/2})\big|\\
&\quad\le o(1)\mathrm{P}(\|N\|_d\ge x/B_n^{1/2})\\
&\qquad+C\left(\exp\left(-\frac{\delta_n^2\min(c_n^{-2},\beta_n^{-2/3})}{16d}\right)+\exp\left(-\frac{c_n^2}{C\beta_n^2\log(1/\beta_n)}\right)\right)
\end{aligned}$$

uniformly for $x \in [B_n^{1/2}, \delta_n \min(c_n^{-1}, \beta_n^{-1/3}) B_n^{1/2}]$, *with any* $\delta_n \to 0$ *and* $\delta_n \times \min(c_n^{-1}, \beta_n^{-1/3}) \to \infty$, *where* N *is a centered normal random vector with covariance matrix* I_{2d}, C *is a positive constant which only depends on* d, $o(1)$ *is bounded by* $A(\delta_n + \beta_n + c_n)$, A *is a positive constant depending only on* d,

$$n_0 = \min\left\{n : \forall k \geq n, c_k^2 \leq \frac{\min(C_0^{-1}, 8^{-1})}{2}, \delta_k \leq c_{1,1} \min(C_0^{-2}, 1), \beta_k \leq c_{1,2}\right\},$$

while $c_{1,1}$ *and* $c_{1,2}$ *are some positive constants depending only on* d.

Let

$$\begin{aligned}\mathbf{Y}_k &:= \mathbf{Y}_k(\omega_{i_1}, \dots, \omega_{i_d}) \\ &= \widehat{X}_k(\cos(k\omega_{i_1}), \sin(k\omega_{i_1}), \dots, \cos(k\omega_{i_d}), \sin(k\omega_{i_d}))\end{aligned}$$

for $1 \leq k \leq n$, $1 \leq i_1 < \cdots < i_d \leq q$. By the facts that for $1 \leq j, l \leq q$,

$$\begin{aligned}&\sum_{k=1}^n \cos^2(\omega_j k) = n/2, \qquad \sum_{k=1}^n \sin^2(\omega_j k) = n/2, \\ &\sum_{k=1}^n \cos(\omega_j k)\sin(\omega_l k) = 0,\end{aligned} \tag{4.4}$$

we have

$$\left\| n^{-1} \operatorname{Cov}\left(\sum_{k=1}^n \mathbf{Y}_k\right) - \frac{1}{2}\mathbf{I}_{2d} \right\| \leq \mathrm{E}X_1^2 I\{|X_1| \geq \varepsilon_n \sqrt{n}/x\}. \tag{4.5}$$

It is easy to see that (2.1) implies $\mathrm{E}|X_1|^p < \infty$ for any $2 < p < 2c + 2$. Thus, we have

$$\begin{aligned}&|\mathbf{Y}_k| \leq 2d\varepsilon_n \sqrt{n}/x \quad \text{and} \\ &n^{-3/2} \sum_{k=1}^n \mathrm{E}|\mathbf{Y}_k|^3 \leq C \max(n^{1-p/2} x^{-3+p}, n^{-1/2}).\end{aligned} \tag{4.6}$$

Letting $S_n = \sum_{k=1}^n \mathbf{Y}_k$, $c_n = 2\varepsilon_n (\log q)^{-1/2}$, $B_n = n$ in Lemma 4.1 and $\delta_n \log n \to \infty$ with δ_n being defined in Lemma 4.1, and noting that $\|N\|_d$ is the minimum of d i.i.d. exponential r.v.'s, we have for any $0 \leq \eta < 1$,

$$\frac{\mathrm{P}(\|\sum_{k=1}^n \mathbf{Y}_k\|_d \geq (1-\eta)x\sqrt{n})}{q^{-d(1-\eta)^2} \exp(-d(1-\eta)^2 y)} \to 1, \tag{4.7}$$

uniformly in $x \in [\sqrt{\log q}, \sqrt{(c+1)\log n}]$. Observing that

$$I_n''(\omega_j) = \frac{1}{n} \left\| \sum_{k=1}^n \mathbf{Y}_k(\omega_j) \right\|_1^2,$$

by (4.7),

$$\mathrm{P}\Big(\max_{1\le j\le q} I_n''(\omega_j)\ge x^2\Big)\le \sum_{j=1}^{q}\mathrm{P}\left(\left\|\sum_{k=1}^{n}\mathbf{Y}_k(\omega_j)\right\|_1\ge x\sqrt{n}\right)$$
$$\le \big(1+o(1)\big)\exp(-y)$$

uniformly in $y\in[0,c\log n]$. Combining (4.2)–(4.7) yields

$$\begin{aligned}
&\mathrm{P}\Big(\max_{1\le j\le q} I_n(\omega_j)\ge x^2\Big)\\
&\quad\le \big(1+o(1)\big)\exp(-y)\\
&\qquad + Cn^{1+4\varepsilon}\exp\big(-(1-2\varepsilon)^2 y\big)\mathrm{P}\big(|X_1|>\varepsilon_n\sqrt{n}/x\big)\\
&\qquad + \big(n\mathrm{P}\big(|X_1|>\varepsilon_n\sqrt{n}/x\big)\big)^2 + n\mathrm{P}\big(|X_1|\ge\varepsilon x\sqrt{n}\big)
\end{aligned}\tag{4.8}$$

uniformly in $x\in[\sqrt{\log q},\sqrt{(c+1)\log n}]$. To establish the lower bound, we observe that

$$\begin{aligned}
&\mathrm{P}\Big(\max_{1\le j\le q} I_n(\omega_j)\ge x^2\Big)\\
&\quad\ge \mathrm{P}\left(\max_{1\le j\le q} I_n''(\omega_j)\ge x^2,\ \bigcap_{k=1}^{n}\{|X_k|\le\varepsilon_n\sqrt{n}/x\}\right)\\
&\quad= \mathrm{P}\Big(\max_{1\le j\le q} I_n''(\omega_j)\ge x^2\Big)\\
&\qquad - \mathrm{P}\left(\max_{1\le j\le q} I_n''(\omega_j)\ge x^2,\ \bigcup_{k=1}^{n}\{|X_k|\ge\varepsilon_n\sqrt{n}/x\}\right).
\end{aligned}\tag{4.9}$$

Similarly to (4.3) and by (4.7) again, we have

$$\begin{aligned}
&\mathrm{P}\left(\max_{1\le j\le q} I_n''(\omega_j)\ge x^2,\ \bigcup_{k=1}^{n}\{|X_k|\ge\varepsilon_n\sqrt{n}/x\}\right)\\
&\quad\le n\mathrm{P}\left(\max_{1\le j\le q}\left|\sum_{k=1}^{n}X_k''e^{ik\omega_j}\right|\ge(1-2\varepsilon)x\sqrt{n}\right)\mathrm{P}\big(|X_1|\ge\varepsilon_n\sqrt{n}/x\big)\\
&\quad\le Cn^{1+4\varepsilon}\exp\big(-(1-2\varepsilon)^2y\big)\mathrm{P}\big(|X_1|>\varepsilon_n\sqrt{n}/x\big)
\end{aligned}\tag{4.10}$$

uniformly in $x\in[\sqrt{\log q},\sqrt{(c+1)\log n}]$. For the first term in (4.9), we have

$$\mathrm{P}\Big(\max_{1\le j\le q} I_n''(\omega_j)\ge x^2\Big)\ge\sum_{j=1}^{q}\mathrm{P}(A_j)-\sum_{1\le i<j\le q}\mathrm{P}(A_iA_j),\tag{4.11}$$

where $A_j = \{I_n''(\omega_j) \ge x^2\}$. Applying $d = 1, 2$ in (4.7), respectively, we obtain

$$\mathrm{P}(A_i) = \bigl(1 + o(1)\bigr) q^{-1} \exp(-y),$$
$$\mathrm{P}(A_i A_j) \le C n^{-2} \exp(-2y)$$

uniformly in $1 \le i \ne j \le q$ and $y \in [0, c \log n]$. These two inequalities together with (4.9)–(4.11) yield that

$$\begin{aligned} &\mathrm{P}\Bigl(\max_{1\le j\le q} I_n(\omega_j) \ge x^2\Bigr) \\ &\quad \ge \bigl(1+o(1)\bigr)\exp(-y) - C\exp(-2y) \\ &\qquad - Cn^{1+4\varepsilon}\exp\bigl(-(1-2\varepsilon)^2 y\bigr)\mathrm{P}\bigl(|X_1| > \varepsilon_n\sqrt{n}/x\bigr) \end{aligned} \tag{4.12}$$

uniformly in $y \in [0, c \log n]$. It is easy to see that (2.1) implies that

$$\mathrm{P}\bigl(|X_1| > \varepsilon_n \sqrt{n}/x\bigr) \le C \frac{(\log n)^{4c+4}}{n^{c+1}}$$

for $\varepsilon_n = (\log n)^{-1}$ and $x \in [\sqrt{\log q}, \sqrt{(c+1)\log n}]$. Hence, by (4.8), (4.12) and for ε sufficiently small, we have for any $M > 0$,

$$\limsup_{n\to\infty} \sup_{M \le y \le c\log n} \left| \frac{\mathrm{P}(\max_{1\le j\le q} I_n(\omega_j) - \log q \ge y)}{1 - \exp(-\exp(-y))} - 1 \right| \le Ce^{-M}. \tag{4.13}$$

By (1.1), we have for any fixed $y \in \mathbf{R}$,

$$\mathrm{P}\Bigl(\max_{1\le j\le q} I_n(\omega_j) - \log q \ge y\Bigr) \to 1 - e^{-e^{-y}}.$$

Since the function $1 - e^{-e^{-y}}$ is uniformly continuous, the standard discretized approximation argument shows that

$$\lim_{n\to\infty} \sup_{y\in R} \left| \mathrm{P}\Bigl(\max_{1\le j\le q} I_n(\omega_j) - \log q \ge y\Bigr) - (1 - e^{-e^{-y}}) \right| = 0.$$

Thus, it follows that

$$\limsup_{n\to\infty} \sup_{-\log q \le y \le M} \left| \frac{\mathrm{P}(\max_{1\le j\le q} I_n(\omega_j) - \log q \ge y)}{1 - \exp(-\exp(-y))} - 1 \right| = 0. \tag{4.14}$$

This proves (i) by (4.13) and (4.14).

(ii) (*Necessity*). Applying (2.2) with $y = c \log q$, we have

$$\mathrm{P}\left(\max_{1\le j\le q} \left| \sum_{k=1}^n X_k e^{ik\omega_j} \right| \ge \sigma \sqrt{(1+c) n \log q}\right) \le Cn^{-c}.$$

This implies that

$$\mathrm{P}\left(\max_{1\le j\le q}\left|\sum_{k=1}^{n} X_k^s e^{ik\omega_j}\right| \ge 2^{-1}\sigma\sqrt{(1+c)n\log q}\right) \le Cn^{-c}, \tag{4.15}$$

where $X_n^s = X_n - X_n^c$ and $\{X_n^c\}$ is an independent copy of $\{X_n\}$. For $z = (z_{1,1}, z_{1,2}, \dots, z_{q,1}, z_{q,2}) \in \mathbf{R}^{2q}$, let

$$\|z\|_{\max} = \max_{1\le j\le q}\sqrt{z_{j,1}^2 + z_{j,2}^2}.$$

For $1 \le k \le n$, let

$$\mathbb{D}_k = (X_k^s\cos(k\omega_1), X_k^s\sin(k\omega_1), \dots, X_k^s\cos(k\omega_q), X_k^s\sin(k\omega_q)).$$

Then it is easy to see that $\max_{1\le j\le q}|\sum_{k=1}^n X_k^s e^{ik\omega_j}| = \|\sum_{k=1}^n \mathbb{D}_k\|_{\max}$. By Lévy's inequality in a Banach space [cf. Ledoux and Talagrand (1991), page 47] and (4.15), we have

$$\mathrm{P}\Bigl(\max_{1\le k\le n}\|\mathbb{D}_k\|_{\max} \ge 2^{-1}\sigma\sqrt{(1+c)n\log q}\Bigr) \le Cn^{-c}. \tag{4.16}$$

Observing that $\|\mathbb{D}_k\|_{\max} = |X_k^s|$, we have by (4.16)

$$1 - \bigl(1 - \mathrm{P}\bigl(|X_1^s| \ge 2^{-1}\sigma\sqrt{(1+c)n\log q}\bigr)\bigr)^n \le Cn^{-c},$$

which implies that

$$1 - \exp\bigl(-n\mathrm{P}\bigl(|X_1^s| \ge 2^{-1}\sigma\sqrt{(1+c)n\log q}\bigr)\bigr) \le Cn^{-c}. \tag{4.17}$$

By (4.17), we have

$$n^{c+1}\mathrm{P}\bigl(|X_1^s| \ge 2^{-1}\sigma\sqrt{(1+c)n\log q}\bigr) = O(1). \tag{4.18}$$

Since X_1 and X_1^c are independent, we have for large n

$$\begin{aligned}
&\tfrac{1}{2}\mathrm{P}\bigl(|X_1| \ge \sigma\sqrt{(1+c)n\log q}\bigr)\\
&\quad\le \mathrm{P}\bigl(|X_1^c| \le 2^{-1}\sigma\sqrt{(1+c)n\log q}\bigr)\mathrm{P}\bigl(|X_1| \ge \sigma\sqrt{(1+c)n\log q}\bigr) \qquad (4.19)\\
&\quad\le \mathrm{P}\bigl(|X_1^s| \ge 2^{-1}\sigma\sqrt{(1+c)n\log q}\bigr).
\end{aligned}$$

Now (2.3) follows from (4.18), (4.19) and some elementary calculations. $\square$

To prove Theorems 2.2 and 2.3, we need the following notation and lemma. Let $\lambda := \lambda_n$ be a positive number which will be specified later. Let $0 \le l \le m = [x^2/2]$ and $\mathbf{N}_l = \{j_1, \dots, j_l\} \subset \{1, \dots, n\}$. Define

$$\begin{aligned}
X_k' &= X_k I\{|X_k| \le \lambda\}, \qquad 1 \le k \le n,\\
\mathbf{Y}_k' &= X_k'(\cos(k\omega_{i_1}), \sin(k\omega_{i_1}), \dots, \cos(k\omega_{i_d}), \sin(k\omega_{i_d}))
\end{aligned} \tag{4.20}$$

for $1 \le k \le n, d \ge 1, 1 \le i_1 < \cdots < i_d \le q$, and set

$$S_n^{\mathbf{N}_l} = \sum_{k=1, k \notin \mathbf{N}_l}^{n} \mathbf{Y}'_k, \qquad \overline{S}_n^{\mathbf{N}_l} = \sum_{k=1, k \notin \mathbf{N}_l}^{n} (\mathbf{Y}'_k - \mathrm{E}\mathbf{Y}'_k).$$

LEMMA 4.2. *Suppose that* $\sigma^2 = 1$, $\mathrm{E}|X_1|^3 < \infty$ *and* $0 \le x \le \varepsilon'_n n^{1/6}$, *where* $\varepsilon'_n \to 0$ *is any sequence of constants. Let* $0 < \varepsilon_n \to 0$ *and* $\varepsilon_n \ge \varepsilon_n'^{1/4}$. (i) *If* $\lambda = \varepsilon_n \sqrt{n}/x$, *then we have*

$$\lim_{n\to\infty} \frac{\mathrm{P}(\|S_n^{\mathbf{N}_l}\|_d \ge x\sqrt{n})}{e^{-dx^2}} = 1 \tag{4.21}$$

uniformly in $x \in [4, \varepsilon'_n n^{1/6}]$, $1 \le i_1 < \cdots < i_d \le q$ *and* $0 \le l \le m$. (ii) *If* $\lambda = (\varepsilon_n \sqrt{n}/x)^{3/4}$ *and* $\mathrm{E}X_1^4 < \infty$, *then* (4.21) *holds.*

PROOF. Recall that $\sum_{k=1}^{n} e^{ik\omega_j} = 0$ for $1 \le j \le q$. We have for n large

$$\left| \sum_{k=1, k \notin \mathbf{N}_l}^{n} \mathrm{E}\mathbf{Y}'_k \right| = \left| \sum_{k=1, k \in \mathbf{N}_l}^{n} \mathrm{E}\mathbf{Y}'_k \right| \le dx^2/2 \le \varepsilon_n \sqrt{n}/x.$$

It follows that, for $x \in [4, \varepsilon'_n n^{1/6}]$,

$$\begin{aligned} \mathrm{P}\big(\|\overline{S}_n^{\mathbf{N}_l}\|_d \ge x\sqrt{n} + \varepsilon_n\sqrt{n}/x\big) &\le \mathrm{P}\big(\|S_n^{\mathbf{N}_l}\|_d \ge x\sqrt{n}\big) \\ &\le \mathrm{P}\big(\|\overline{S}_n^{\mathbf{N}_l}\|_d \ge x\sqrt{n} - \varepsilon_n\sqrt{n}/x\big). \end{aligned}$$

Since $\mathrm{E}|X_1|^3 < \infty$, under the conditions of (i) or (ii), we have for large n

$$|\mathbf{Y}'_k| \le 2d\varepsilon_n \sqrt{n}/x \quad \text{and} \quad n^{-3/2} \sum_{k=1}^{n} \mathrm{E}|\mathbf{Y}'_k|^3 \le Cn^{-1/2}.$$

Also, under the conditions of (i) or (ii), simple calculations show that

$$\begin{aligned} \left\| \frac{1}{n} \mathrm{Cov}(\overline{S}_n^{\mathbf{N}_l}) - \mathbf{I}_{2d} \right\| &\le \mathrm{E}X_1^2 I\{|X_1| \ge \lambda\} + C_d n^{-1} x^2 \\ &\le Cxn^{-1/2}\varepsilon_n^{-1} \le C\varepsilon_n^2/x^2 \end{aligned}$$

for $4 \le x \le \varepsilon'_n n^{1/6}$. By taking $c_n = 2d\varepsilon_n/x$, $B_n = \sqrt{n}$ and $\delta_n/\varepsilon_n \to \infty$ in Lemma 4.1, we have

$$\frac{\mathrm{P}(\|\overline{S}_n^{\mathbf{N}_l}\|_d \ge x\sqrt{n} \pm \varepsilon_n\sqrt{n}/x)}{e^{-dx^2}} \to 1$$

uniformly in $x \in [4, \varepsilon'_n n^{1/6}]$. This proves Lemma 4.2. $\square$

PROOF OF THEOREM 2.2. For any $\varepsilon_n' \to 0$, let $x = \sqrt{y + \log q} \in [4, \varepsilon_n' n^{1/6}]$ and $\varepsilon_n = \varepsilon_n'^{1/4}$. Recall X_k' in (4.20) and let $\lambda = \varepsilon_n \sqrt{n}/x$. Define

$$I_n'(\omega_j) = \frac{1}{n}\left|\sum_{k=1}^{n} X_k' e^{ik\omega_j}\right|^2.$$

Then, by Lemma 4.2 (taking $\mathbf{N}_l = \varnothing$ and $d = 1$), we have

$$\begin{aligned} \mathrm{P}\Big(\max_{1\le j\le q} I_n'(\omega_j) \ge x^2\Big) &\le \sum_{j=1}^{q} \mathrm{P}\left(\left|\sum_{k=1}^{n} X_k' e^{ik\omega_j}\right| \ge x\sqrt{n}\right) \\ &\le \big(1 + o(1)\big)e^{-y} \end{aligned}$$

uniformly in $x \in [4, \varepsilon_n' n^{1/6}]$. Since $\mathrm{E}e^{t_0|X_1|} < \infty$, we have $n\mathrm{P}(|X_1| > \varepsilon_n \sqrt{n}/x) = o(1)e^{-x^2}$. Therefore,

$$\begin{aligned} \mathrm{P}\Big(\max_{1\le j\le q} I_n(\omega_j) \ge x^2\Big) &\le \mathrm{P}\Big(\max_{1\le j\le q} I_n'(\omega_j) \ge x^2\Big) + n\mathrm{P}\big(|X_1| > \varepsilon_n\sqrt{n}/x\big) \\ &\le \big(1 + o(1)\big)e^{-y} \end{aligned}$$

uniformly in $x \in [4, \varepsilon_n' n^{1/6}]$. Similarly, we have

$$\mathrm{P}\Big(\max_{1\le j\le q} I_n(\omega_j) \ge x^2\Big) \ge \big(1 + o(1)\big)e^{-y} - Ce^{-2y}$$

uniformly in $x \in [4, \varepsilon_n' n^{1/6}]$. The remaining proof follows similar arguments as in the proof of Theorem 2.1. $\square$

PROOF OF THEOREM 2.3. Recall that $\sum_{k=1}^{n} e^{ik\omega_j} = 0$ for $1 \le j \le q$. Without loss of generality, we can assume that $\mathrm{E}X_1 = 0$ and $\mathrm{E}X_1^2 = 1$. By the fact $e^{ik\omega_{j_1}} = e^{-ik\omega_{j_2}}$ for $j_1 + j_2 = n$, we have

$$\left|\sum_{k=1}^{n} X_k e^{ik\omega_{j_1}}\right|^2 = \left|\sum_{k=1}^{n} X_k e^{ik\omega_{j_2}}\right|^2$$

for $j_1 + j_2 = n$. Hence, when n is odd,

$$\begin{aligned} q^{-1}\sum_{j=1}^{q} I_n(\omega_j) &= (2q)^{-1}\sum_{j=1}^{q}\big(I_n(\omega_j) + I_n(\omega_{n-j})\big) \\ &= (2q)^{-1}\sum_{j=1}^{n} I_n(\omega_j) - \frac{n}{n-1}(\overline{X})^2, \end{aligned} \tag{4.22}$$

where $\overline{X} = n^{-1}\sum_{k=1}^{n} X_k$. Moreover,

$$\sum_{j=1}^{n} I_n(\omega_j) = \sum_{k=1}^{n} X_k^2 + 2n^{-1}\sum_{k=2}^{n} X_k \sum_{i=1}^{k-1} X_i \sum_{j=1}^{n} w_{k,i,j},$$

where $w_{k,i,j} = \cos(\omega_j k)\cos(\omega_j i) + \sin(\omega_j k)\sin(\omega_j i)$. Note that $\omega_j k = \omega_k j$ and $\omega_j i = \omega_i j$. Since $|\sum_{l=1}^n e^{il\lambda}| = |\sin(\lambda n/2)|/|\sin(\lambda/2)|$ when λ/π is not an integer, we get $\sum_{j=1}^n w_{k,i,j} = \sum_{j=1}^n \cos((\omega_k - \omega_i)j) = 0$. So, (4.22) implies that, when n is odd,

$$q^{-1}\sum_{j=1}^q I_n(\omega_j) = (n-1)^{-1}\left(\sum_{k=1}^n X_k^2 - n(\overline{X})^2\right).$$

Similarly, when n is even, we have

$$q^{-1}\sum_{j=1}^q I_n(\omega_j) = (n-2)^{-1}\left(\sum_{k=1}^n X_k^2 - n(\overline{X})^2 - n(\overline{X}')^2\right), \tag{4.23}$$

where $\overline{X}' = n^{-1}\sum_{k=1}^n (-1)^k X_k$. By the self-normalized moderate deviation Theorem 3.1 in Shao (1997), we have

$$\mathrm{P}\left(\left|\sum_{k=1}^n (-1)^k X_k\right| \geq n^{1/3} V_n\right) \leq Ce^{-n^{2/3}/4},$$

$$\mathrm{P}\left(\left|\sum_{k=1}^n X_k\right| \geq n^{1/3} V_n\right) \leq Ce^{-n^{2/3}/4},$$

where $V_n^2 = \sum_{k=1}^n X_k^2$. In view of (4.22) and (4.23), it suffices to show that

$$\lim_{n\to\infty} \frac{\mathrm{P}(\max_{1\leq j\leq q}|\sum_{k=1}^n X_k e^{ik\omega_j}|^2/\sum_{k=1}^n X_k^2 - \log q \geq y)}{1 - e^{-e^{-y}}} = 1 \tag{4.24}$$

uniformly in $y \in [-\log q, o(n^{1/3}))$.

Recall $X_k' = X_k I\{|X_k| \leq \lambda\}$ in (4.20). Let $\mathbf{H}$ be a subset of $\{1, \ldots, n\}$. Put

$$M_n = \max_{1\leq j\leq q}\left|\sum_{k=1}^n X_k e^{ik\omega_j}\right|, \qquad \widetilde{M}_n = \max_{1\leq j\leq q}\left|\sum_{k=1}^n X_k' e^{ik\omega_j}\right|,$$

$$M_n^{(\mathbf{H})} = \max_{1\leq j\leq q}\left|\sum_{k=1,k\notin\mathbf{H}}^n X_k e^{ik\omega_j}\right|,$$

$$\widetilde{M}_n^{(\mathbf{H})} = \max_{1\leq j\leq q}\left|\sum_{k=1,k\notin\mathbf{H}}^n X_k' e^{ik\omega_j}\right|,$$

$$\widetilde{V}_n = \left(\sum_{k=1}^n X_k'^2\right)^{1/2}, \qquad V_n^{(\mathbf{H})} = \left(\sum_{k=1,k\notin\mathbf{H}}^n X_k^2\right)^{1/2},$$

$$\widetilde{V}_n^{(\mathbf{H})} = \left(\sum_{k=1,k\notin\mathbf{H}}^n X_k'^2\right)^{1/2}.$$

Noting that for any real numbers s and t and nonnegative number c and $x \geq 1$,

$$\{s + t \geq x\sqrt{c + t^2}\} \subset \{s \geq \sqrt{x^2 - 1}\sqrt{c}\} \tag{4.25}$$

[see page 2181 in Jing, Shao and Wang (2003)], we have

$$\begin{aligned} \mathrm{P}(M_n \geq xV_n) &\leq \mathrm{P}(\widetilde{M}_n \geq x\widetilde{V}_n) + \sum_{j=1}^{n} \mathrm{P}(M_n \geq xV_n, X_j \neq X'_j) \\ &\leq \mathrm{P}(\widetilde{M}_n \geq x\widetilde{V}_n) + \sum_{j=1}^{n} \mathrm{P}\big(M_n^{(j)} + |X_j| \geq xV_n, X_j \neq X'_j\big) \\ &\leq \mathrm{P}(\widetilde{M}_n \geq x\widetilde{V}_n) + \sum_{j=1}^{n} \mathrm{P}\big(M_n^{(j)} \geq \sqrt{x^2-1}V_n^{(j)}, X_j \neq X'_j\big) \\ &= \mathrm{P}(\widetilde{M}_n \geq x\widetilde{V}_n) + \sum_{j=1}^{n} \mathrm{P}(|X_j| \geq \lambda)\mathrm{P}\big(M_n^{(j)} \geq \sqrt{x^2-1}V_n^{(j)}\big). \end{aligned} \tag{4.26}$$

Repeating the above arguments m times with $m = [x^2/2]$, we have for $x > 4$,

$$\begin{aligned} &\sum_{j_1=1}^{n} \mathrm{P}(|X_{j_1}| \geq \lambda)\mathrm{P}\big(M_n^{(j_1)} \geq \sqrt{x^2-1}V_n^{(j_1)}\big) \\ &\quad \leq \sum_{j_1=1}^{n} \mathrm{P}(|X_{j_1}| \geq \lambda)\mathrm{P}\big(\widetilde{M}_n^{(j_1)} \geq \sqrt{x^2-1}\widetilde{V}_n^{(j_1)}\big) \\ &\qquad + \sum_{j_1=1}^{n}\sum_{j_2=1}^{n} \mathrm{P}(|X_{j_1}| \geq \lambda)\mathrm{P}(|X_{j_2}| \geq \lambda)\mathrm{P}\big(M_n^{(j_1,j_2)} \geq \sqrt{x^2-2}V_n^{(j_1,j_2)}\big) \\ &\quad \leq \sum_{l=1}^{m} \widetilde{Z}_l + Z_{m+1}, \end{aligned} \tag{4.27}$$

where

$$\widetilde{Z}_l = \sum_{j_1=1}^{n} \cdots \sum_{j_l=1}^{n} \left[\prod_{k=1}^{l} \mathrm{P}(|X_{j_k}| \geq \lambda)\right] \times \mathrm{P}\big(\widetilde{M}_n^{(j_1,\dots,j_l)} \geq \sqrt{x^2-l}\,\widetilde{V}_n^{(j_1,\dots,j_l)}\big),$$

$$\begin{aligned} Z_{m+1} = \sum_{j_1=1}^{n} \cdots \sum_{j_{m+1}=1}^{n} \Bigg\{ &\left[\prod_{k=1}^{m+1} \mathrm{P}(|X_{j_k}| \geq \lambda)\right] \\ &\times \mathrm{P}\big(M_n^{(j_1,\dots,j_{m+1})} \geq \sqrt{x^2-m-1}V_n^{(j_1,\dots,j_{m+1})}\big)\Bigg\}. \end{aligned}$$

For $\varepsilon_n' \to 0$ and $4 \le x \le \varepsilon_n' n^{1/6}$, let $\lambda = \varepsilon_n \sqrt{n}/x$, where $\varepsilon_n = \varepsilon_n'^{1/4}$. Then

$$Z_{m+1} \le \bigl(n\mathrm{P}(|X_1| \ge \lambda)\bigr)^{m+1} \le e^{-m \log q_n} = o(1)e^{-x^2}, \tag{4.28}$$

where

$$q_n = (\varepsilon_n'^3 \varepsilon_n^{-3} \mathrm{E}|X_1|^3 I\{|X_1| \ge \lambda\})^{-1} \to \infty$$

as $n \to \infty$. We next estimate $\widetilde{Z}_l$. From Lemma 4.2, we have for $0 \le l \le m = [x^2/2]$,

$$\begin{aligned} &\mathrm{P}\bigl(\widetilde{M}_n^{(j_1,\dots,j_l)} \ge \sqrt{x^2 - l}\,\widetilde{V}_n^{(j_1,\dots,j_l)}\bigr) \\ &\quad \le \mathrm{P}\bigl(\widetilde{M}_n^{(j_1,\dots,j_l)} \ge \sqrt{x^2 - l}\sqrt{n(1-\varepsilon_n x^{-2})}\bigr) \\ &\qquad + \mathrm{P}\bigl(\widetilde{V}_n^{(j_1,\dots,j_l)} \le \sqrt{n(1-\varepsilon_n x^{-2})}\bigr) \\ &\quad \le \bigl(1+o(1)\bigr)qe^{-x^2+l} + \mathrm{P}\bigl(\widetilde{V}_n^{(j_1,\dots,j_l)} \le \sqrt{n(1-\varepsilon_n x^{-2})}\bigr), \end{aligned} \tag{4.29}$$

uniformly in $x \in [4, \varepsilon_n' n^{1/6}]$. By a similar argument as in Hu, Shao and Wang [(2009), page 1193] if $\mathrm{E}X_1^4 < \infty$, we have

$$\begin{aligned} \mathrm{P}\bigl(\widetilde{V}_n^{(j_1,\dots,j_l)} \le \sqrt{n(1-\varepsilon_n x^{-2})}\bigr) &\le \mathrm{P}\bigl(n - m_l - \widetilde{V}_n^{(j_1,\dots,j_l)2} \ge \varepsilon_n n x^{-2}/2\bigr) \\ &\le o(1)e^{-x^2}, \end{aligned}$$

where m_l is the cardinality of $\{j_1, \dots, j_l\}$, and hence

$$\widetilde{Z}_l \le Cn\bigl(n\mathrm{P}(|X_1| \ge \lambda)\bigr)^l e^{-(x^2-l)} \le Cne^{-x^2+l-Cl\log q_n}. \tag{4.30}$$

This together with $q_n \to \infty$ shows that

$$\sum_{l=1}^{m} \widetilde{Z}_l = o(1)ne^{-x^2} \tag{4.31}$$

uniformly in $x \in [4, \varepsilon_n' n^{1/6}]$. Combining (4.26)–(4.28), (4.29) and (4.31) yields

$$\mathrm{P}(M_n \ge xV_n) \le \bigl(1+o(1)\bigr)qe^{-x^2} \tag{4.32}$$

uniformly in $x \in [4, \varepsilon_n' n^{1/6}]$.

We next estimate the lower bound for $\mathrm{P}(M_n \ge xV_n)$. For $\varepsilon_n' \to 0$ and $4 \le x \le \varepsilon_n' n^{1/6}$, let $\varepsilon_n = \max((x/\sqrt{n})^{1/8}, \varepsilon_n'^{1/4})$ and $\lambda = (\varepsilon_n\sqrt{n}/x)^{3/4}$. Then

$$\begin{aligned} \mathrm{P}(M_n \ge xV_n) &\ge \mathrm{P}(\widetilde{M}_n \ge x\widetilde{V}_n) \\ &\quad - \sum_{j=1}^{n} \mathrm{P}\bigl(\widetilde{M}_n^{(j)} \ge \sqrt{x^2-1}\,\widetilde{V}_n^{(j)}\bigr)\mathrm{P}(|X_j| \ge \lambda). \end{aligned} \tag{4.33}$$

Similarly to (4.30), we have

$$\sum_{j=1}^{n} \mathrm{P}\big(\widetilde{M}_n^{(j)} \geq \sqrt{x^2-1}\,\widetilde{V}_n^{(j)}\big)\mathrm{P}(|X_j| \geq \lambda) = o(1)ne^{-x^2} \tag{4.34}$$

uniformly in $x \in [4, \varepsilon_n' n^{1/6}]$. For the first term on the right-hand side of (4.33), we have

$$\begin{aligned}
\mathrm{P}(\widetilde{M}_n \geq x\widetilde{V}_n) &\geq \mathrm{P}\big(\widetilde{M}_n \geq x\widetilde{V}_n, \widetilde{V}_n^2 \leq n(1+\varepsilon_n/x^2)\big)\\
&\geq \mathrm{P}\big(\widetilde{M}_n \geq x\sqrt{n(1+\varepsilon_n/x^2)}\big)\\
&\quad - \mathrm{P}\big(\widetilde{M}_n \geq \sqrt{n}x, \widetilde{V}_n^2 \geq n(1+\varepsilon_n/x^2)\big).
\end{aligned}$$

Define $\mathbf{A} = \{\widetilde{V}_n^2 \geq n(1+\varepsilon_n/x^2)\}$. Set $Y_{k,l}(\theta_1,\theta_2) = X_k'(\theta_1\cos(kw_l) + \theta_2\sin(kw_l))$ for any $\theta_1, \theta_2 \in \mathbf{R}$. Let

$$\begin{aligned}
\Theta_1 &= \{\theta_1 \geq 0, \theta_2 \geq 0; \theta_1^2 + \theta_2^2 = 1\};\\
\Theta_2 &= \{\theta_1 \geq 0, \theta_2 < 0; \theta_1^2 + \theta_2^2 = 1\};\\
\Theta_3 &= \{\theta_1 < 0, \theta_2 \geq 0; \theta_1^2 + \theta_2^2 = 1\};\\
\Theta_4 &= \{\theta_1 < 0, \theta_2 < 0; \theta_1^2 + \theta_2^2 = 1\}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\mathrm{P}\big(\widetilde{M}_n \geq x\sqrt{n}, \mathbf{A}\big)\\
&\quad\leq \sum_{l=1}^{q} \mathrm{P}\left(\sup_{(\theta_1,\theta_2)\in\Theta_1}\left|\sum_{k=1}^{n} Y_{k,l}(\theta_1,\theta_2)\right| \geq x\sqrt{n}, \mathbf{A}\right)\\
&\qquad + \sum_{l=1}^{q} \mathrm{P}\left(\sup_{(\theta_1,\theta_2)\in\Theta_2}\left|\sum_{k=1}^{n} Y_{k,l}(\theta_1,\theta_2)\right| \geq x\sqrt{n}, \mathbf{A}\right)\\
&\qquad + \sum_{l=1}^{q} \mathrm{P}\left(\sup_{(\theta_1,\theta_2)\in\Theta_3}\left|\sum_{k=1}^{n} Y_{k,l}(\theta_1,\theta_2)\right| \geq x\sqrt{n}, \mathbf{A}\right)\\
&\qquad + \sum_{l=1}^{q} \mathrm{P}\left(\sup_{(\theta_1,\theta_2)\in\Theta_4}\left|\sum_{k=1}^{n} Y_{k,l}(\theta_1,\theta_2)\right| \geq x\sqrt{n}, \mathbf{A}\right).
\end{aligned} \tag{4.35}$$

We only deal with the first term above, while other terms can be proved similarly. For fixed $(\theta_1,\theta_2) \in \Theta_1$, following the proof of Theorem 2 in Shao (1999) (see pages 393 and 394), we can obtain a desired exponential bound for $P(|\sum_{k=1}^{n} Y_{k,l}(\theta_1,\theta_2)| \geq x\sqrt{n}, \mathbf{A})$. To this end, we split Θ_1 into n^6 parts so that we can discretize $\sup_{(\theta_1,\theta_2)\in\Theta_1}$ and then apply the proof of Theorem 2 in Shao (1999). Let $\theta_{1,i} = i/n^6$ for $1 \le i \le n^6$ and $\theta_{2,i} = \sqrt{1-\theta_{1,i}^2}$. We have, for $1 \le l \le q$,

$$
\begin{aligned}
&\mathrm{P}\left(\sup_{(\theta_1,\theta_2)\in\Theta_1}\left|\sum_{k=1}^n Y_{k,l}(\theta_1,\theta_2)\right| \ge x\sqrt{n}, \mathbf{A}\right)\\
&\qquad \le \sum_{i=1}^{n^6} \mathrm{P}\left(\left|\sum_{k=1}^n Y_{k,l}(\theta_{1,i},\theta_{2,i})\right| \ge x\sqrt{n} - \sqrt{n}\varepsilon_n x^{-1}, \mathbf{A}\right)\\
&\qquad\quad + \sum_{i=1}^{n^6} \mathrm{P}\Biggl(\sup_{(\theta_1,\theta_2)\in\Theta_1,\theta_{1,i-1}\le\theta_1\le\theta_{1,i}}\Biggl|\sum_{k=1}^n [Y_{k,l}(\theta_1,\theta_2)\\
&\qquad\qquad\qquad - Y_{k,l}(\theta_{1,i},\theta_{2,i})]\Biggr| \ge \sqrt{n}\varepsilon_n x^{-1}\Biggr)\\
&\qquad =: \sum_{i=1}^{n^6} J_{1,i} + \sum_{i=1}^{n^6} J_{2,i}.
\end{aligned}
\tag{4.36}
$$

It is easy to see that $\sup_{(\theta_1,\theta_2)\in\Theta_1,\theta_{1,i-1}\le\theta_1\le\theta_{1,i}} |Y_{k,l}(\theta_1,\theta_2) - Y_{k,l}(\theta_{1,i},\theta_{2,i})| \le n^{-1}$. Hence, $J_{2,i} = 0$ for $x \in [4, \varepsilon_n' n^{1/6}]$. Letting $b = x/\sqrt{n}$ and $\tau = (\sqrt{n}/x)^{1/4}$, we have

$$
\begin{aligned}
J_{1,i} &\le \mathrm{P}\left(\sum_{k=1}^n bY_{k,l}(\theta_{1,i},\theta_{2,i}) + \tau b^2 \widetilde{V}_n^2 \ge x^2 - \varepsilon_n + \tau(x^2+\varepsilon_n)\right)\\
&\quad + \mathrm{P}\left(\sum_{k=1}^n -bY_{k,l}(\theta_{1,i},\theta_{2,i}) + \tau b^2 \widetilde{V}_n^2 \ge x^2 - \varepsilon_n + \tau(x^2+\varepsilon_n)\right)\\
&\le \mathrm{P}\left(\sum_{k=1}^n b\overline{Y}_{k,l}(\theta_{1,i},\theta_{2,i}) + \tau b^2 [\widetilde{V}_n^2 - \mathrm{E}\widetilde{V}_n^2] \ge x^2 - \varepsilon_n + \tau\widetilde{\varepsilon}_n\right)\\
&\quad + \mathrm{P}\left(\sum_{k=1}^n -b\overline{Y}_{k,l}(\theta_{1,i},\theta_{2,i}) + \tau b^2 [\widetilde{V}_n^2 - \mathrm{E}\widetilde{V}_n^2] \ge x^2 - \varepsilon_n + \tau\widetilde{\varepsilon}_n\right)\\
&=: J_{3,i},
\end{aligned}
\tag{4.37}
$$

where

$$
\begin{aligned}
\overline{Y}_{k,l}(\theta_{1,i},\theta_{2,i}) &= Y_{k,l}(\theta_{1,i},\theta_{2,i}) - \mathrm{E}Y_{k,l}(\theta_{1,i},\theta_{2,i}),\\
\widetilde{\varepsilon}_n &= \varepsilon_n + \tau b^2(n - \mathrm{E}\widetilde{V}_n^2)\\
&= \varepsilon_n + o(1)x^3/\sqrt{n}.
\end{aligned}
$$

Let $\eta_k = \overline{Y}_{k,l}(\theta_{1,i}, \theta_{2,i})$ and $\xi_k = X_k'^2 - \mathrm{E}X_k'^2$. Using $|e^s - 1 - s - s^2/2| \le |s|^3 e^{s\vee 0}$, we get

$$\begin{aligned}
\mathrm{E}e^{2b\eta_k + 2\tau b^2\xi_k} &= 1 + 2\mathrm{E}(b\eta_k + \tau b^2\xi_k)^2 + O(1)\mathrm{E}|b\eta_k + \tau b^2\xi_k|^3 e^3 \\
&= 1 + 2b^2\mathrm{E}\eta_k^2 + 4\tau b^3\mathrm{E}(\eta_k\xi_k) + 2\tau^2 b^4\mathrm{E}\xi_k^2 \\
&\quad + O(1)e^3(b^3\mathrm{E}|\eta_k|^3 + \tau^3 b^6\mathrm{E}|\xi_k|^3) \\
&= 1 + 2b^2\big(\mathrm{E}X_k'^2 - (\mathrm{E}X_k')^2\big)[\theta_{1,i}^2\cos^2(kw_l) + \theta_{2,i}^2\sin^2(kw_l) \\
&\qquad\qquad + 2\theta_{1,i}\theta_{2,i}\cos(kw_l)\sin(kw_l)] \\
&\quad + O(1)(1+\tau)b^3 \\
&= 1 + 2b^2\big(\theta_{1,i}^2\cos^2(kw_l) + \theta_{2,i}^2\sin^2(kw_l) \\
&\qquad + 2\theta_{1,i}\theta_{2,i}\cos(kw_l)\sin(kw_l)\big) \\
&\quad + O(1)(1+\tau)b^3
\end{aligned}$$

for $x \in [4, \varepsilon_n' n^{1/6}]$. This, together with (4.4), implies that

$$\begin{aligned}
J_{3,i} &\le 2\exp\big(-2x^2 + 2\varepsilon_n - 2\tau\widetilde{\varepsilon}_n + nb^2 + O(1)(1+\tau)x^3/\sqrt{n}\big) \\
&\le C\exp(-x^2 - b^{-1/8}).
\end{aligned} \tag{4.38}$$

Combining (4.35)–(4.38) gives

$$\mathrm{P}\big(\widetilde{M}_n \ge x\sqrt{n}, \mathbf{A}\big) = o(1)e^{-x^2}. \tag{4.39}$$

Define

$$A_j = \left\{\left|\sum_{k=1}^n X_k' e^{ik\omega_j}\right| \ge x\sqrt{n(1+\varepsilon_n/x^2)}\right\}, \qquad 1 \le j \le q.$$

We have

$$\mathrm{P}\Big(\widetilde{M}_n \ge x\sqrt{n(1+\varepsilon_n/x^2)}\Big) \ge \sum_{j=1}^q \mathrm{P}(A_j) - \sum_{1\le i<j\le q}\mathrm{P}(A_iA_j).$$

By Lemma 4.2(ii), we have

$$\mathrm{P}(A_i) = \big(1+o(1)\big)e^{-x^2}, \qquad \mathrm{P}(A_iA_j) = 2^{-1}\big(1+o(1)\big)e^{-2x^2}$$

uniformly in $x \in [4, \varepsilon_n' n^{1/6}]$ and $1 \le i, j \le q$. This shows that

$$\mathrm{P}\Big(\widetilde{M}_n \ge x\sqrt{n(1+\varepsilon_n/x^2)}\Big) \ge \big(1+o(1)\big)qe^{-x^2}(1 - 2^{-1}qe^{-x^2}). \tag{4.40}$$

It follows from (4.34), (4.39) and (4.40) that

$$\mathrm{P}(M_n \ge xV_n) \ge \big(1+o(1)\big)qe^{-x^2}(1 - 2^{-1}qe^{-x^2}) \tag{4.41}$$

uniformly in $x \in [4, \varepsilon_n' n^{1/6}]$. Let $x = \sqrt{y + \log q}$. Combining (4.32) and (4.41), we have for any fixed $M > 0$,

$$\limsup_{n\to\infty} \sup_{M \le y \le \varepsilon_n' n^{1/3}} \left| \frac{\mathrm{P}(M_n^2/V_n^2 - \log q \ge y)}{1 - \exp(-\exp(-y))} - 1 \right| \le Ce^{-M}. \tag{4.42}$$

For $-\log q \le y \le M$, by (1.1), (4.22) and (4.23),

$$\limsup_{n\to\infty} \sup_{-\log q \le y \le M} \left| \frac{\mathrm{P}(M_n^2/V_n^2 - \log q \ge y)}{1 - \exp(-\exp(-y))} - 1 \right| = 0. \tag{4.43}$$

This completes the proof of Theorem 2.3 by (4.42) and (4.43). $\square$

PROOF OF LEMMA 3.1. This lemma follows immediately by Theorem 2.3 and

$$f_n\big((x + \log q)/q\big) = \mathrm{P}\left(\frac{\max_{1\le j\le q} I_n(\omega_j)}{q^{-1}\sum_{j=1}^q I_n(\omega_j)} - \log q \ge x \right),$$

where $\{X_k\}$ are i.i.d. $N(0, 1)$ random variables. $\square$

PROOF OF THEOREM 3.1. Let $\mathcal{C}_g = \{P_g < \theta/(3G), P_g^{\mathrm{true}} > \theta/(2G)\}$ and define $F(x) = \exp(-\exp(-x))$. Let x_n satisfy $1 - F(x_n) = \theta/(2.5G)$. So $x_n \sim \log G$. Corollary 3.1 yields

$$\max_{1\le g\le G} \left| \frac{1 - F(x_n)}{1 - F_{n,g}((x_n + \log q)/q)} - 1 \right| = o(1). \tag{4.44}$$

By (4.44) and the definition of x_n, we can see that on $\mathcal{C}_g$, it holds $P_g^{\mathrm{true}} > \theta/(2G) > 1 - F_{n,g}((x_n + \log q)/q)$ for n large. By the monotonicity of distribution function, we have $qf_g - \log q \le x_n$. This together with Corollary 3.1 and Lemma 3.1 yields

$$\max_{1\le g\le G} \left| \frac{P_g}{P_g^{\mathrm{true}}} - 1 \right| I\{\mathcal{C}_g\} = o(1).$$

Note that on $\mathcal{H}_g \cap \mathcal{C}_g^c$ we have $P_g \ge \theta/(3G)$. We can show that $qf_g - \log q \le y_n$, where $1 - F(y_n) = \theta/(4G)$, so $y_n \sim \log G$. In fact, by Lemma 3.1,

$$\frac{f_n(q^{-1}(y_n + \log q))}{1 - F(y_n)} - 1 = o(1),$$

and hence, $f_n(f_g) = P_g > f_n(q^{-1}(y_n + \log q))$ for n large, which implies $qf_g - \log q \le y_n$. It follows from Corollary 3.1 and Lemma 3.1 that

$$\max_{1\le g\le G} \left| \frac{P_g}{P_g^{\mathrm{true}}} - 1 \right| I\{\mathcal{H}_g\} I\{\mathcal{C}_g^c\} = o(1).$$

The theorem is now proved. $\square$

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