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LARGE FACES IN POISSON HYPERPLANE MOSAICS

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A generalized version of a well-known problem of D. G. Kendall states that the zero cell of a stationary Poisson hyperplane tessellation in $\mathbb{R}^d$, under the condition that it has large volume, approximates with high probability a certain definite shape, which is determined by the directional distribution of the underlying hyperplane process. This result is extended here to typical k-faces of the tessellation, for $k \in \{2, \dots, d-1\}$. This requires the additional condition that the direction of the face be in a sufficiently small neighbourhood of a given direction.

1. Introduction. A well-known problem of D. G. Kendall, popularized in the foreword to the first edition (1987) of the book [17], asked whether the shape of the zero cell of a stationary, isotropic Poisson line process in the plane, under the condition that the cell has large area, must be approximately circular, with high probability. An affirmative answer was given by Kovalenko [11, 12]. Several higher-dimensional versions and variants of Kendall's problem were treated in [4, 5, 7–10]. In [4], the subject of investigation was the zero cell of a stationary Poisson hyperplane process with a general (nondegenerate) directional distribution in d-dimensional Euclidean space, under the condition that the cell has large volume. The asymptotic shape of such cells was found to be that of the so-called Blaschke body of the hyperplane process. This is (up to a dilatation) the convex body, centrally symmetric with respect to the origin, that has the spherical directional distribution of the hyperplane process as its surface area measure. Its existence and uniqueness follow from a celebrated theorem going back to Minkowski.

The purpose of the present paper is an extension of the latter result to k-dimensional faces, for $k \in \{2, \dots, d-1\}$. The natural extension of the zero cell, which is stochastically equivalent to the volume weighted typical cell, is the notion of the (k-volume-)weighted typical k-face. We consider the weighted typical k-face under the condition that it has large k-dimensional volume and that its direction space (the translate of its affine hull passing through the origin) is in a small neighbourhood of a given k-dimensional subspace L^*. We can then again identify an asymptotic shape, namely that of the Blaschke body of the section process of the given hyperplane process with the subspace L^*. The main results, whose precise formulation requires some preparations, are formulated in the theorems at the

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end of the next section. The extension from cells to lower-dimensional faces is not routine; the proof has become possible through a recently established representation for the distribution of the weighted typical k-face ([15], Theorem 1) and a special stability result for the convex bodies obtained from Minkowski's existence theorem, which was proved in [6].

Once the result is proved for weighted typical k-faces (Theorem 2.1), it can be used to derive a variant for typical k-faces (Theorem 2.2). From these theorems, the existence of limit shapes can be deduced (Theorem 7.1).

2. Preliminaries and main results. Fundamental facts about Poisson hyperplane processes and random mosaics, as well as corresponding notions that are not explained here, can be found in the book [16]. For the employed notions and results from convex geometry, we refer to [14].

We denote by $\mathbb{R}^d$ the d-dimensional Euclidean vector space (assuming $d \geq 3$ throughout), with scalar product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Its unit ball and unit sphere are denoted by $\mathbb{B}^d$ and $\mathbb{S}^{d-1}$, respectively. Further, SO_d is the rotation group, $G(d,k)$ is the Grassmannian of k-dimensional linear subspaces of $\mathbb{R}^d$ and $A(d,k)$ is the set of k-flats (k-dimensional affine subspaces) of $\mathbb{R}^d$; all these sets are equipped with their standard topologies.

By $\mathcal{H}^d \subset A(d, d-1)$, we denote the space of hyperplanes in $\mathbb{R}^d$ not passing through the origin $\mathbf{o}$. Every hyperplane in $\mathcal{H}^d$ has a unique representation

$$H(\mathbf{u}, t) = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{u} \rangle = t\}$$

with $\mathbf{u} \in \mathbb{S}^{d-1}$ and $t > 0$, and

$$H^-(\mathbf{u}, t) = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{u} \rangle \leq t\}$$

is the closed halfspace bounded by it that contains $\mathbf{o}$. We write $H^- = H^-(\mathbf{u}, t)$ if $H = H(\mathbf{u}, t)$.

Let $\mathcal{K}$ be the space of convex bodies (nonempty, compact, convex subsets) in $\mathbb{R}^d$, endowed with the Hausdorff metric δ. For $k \in \{2, \dots, d-1\}$ and a subspace $L \in G(d,k)$, we denote by $\mathcal{K}(L)$ the set of convex bodies $K \subset L$ and by $\mathcal{K}_0(L)$ the subset of k-dimensional bodies K with $\mathbf{o} \in \operatorname{relint} K$ (where relint denotes the relative interior).

In the following, measures on a given topological space T, if not further specified, are always positive measures on the Borel σ-algebra $\mathcal{B}(T)$ of the space.

We turn to hyperplane processes. As usual and convenient in the theory of point processes, we often identify a simple counting measure η on a topological space E with its support, so that $\eta(\{x\}) = 1$ and $x \in \eta$ are used synonymously, and $\eta(A)$ and $\operatorname{card}(\eta \cap A)$ both denote the number of elements of η in the subset $A \subset E$.

Let X be a stationary Poisson hyperplane process in $\mathbb{R}^d$. We denote the underlying probability by $\mathbf{P}$ and mathematical expectation by $\mathbf{E}$. The intensity measure $\Theta = \mathbf{E}X(\cdot)$ of X has a representation (equivalent to [16], (4.33))

$$\Theta(A) = 2\gamma \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbf{1}_A(H(\mathbf{u}, t))\, \mathrm{d}t\, \varphi(\mathrm{d}\mathbf{u})$$

for $A \in \mathcal{B}(A(d, d-1))$, where γ is the intensity of X and φ is its spherical directional distribution. This is an even probability measure on the unit sphere; we assume that it is not concentrated on any great subsphere.

Together with the hyperplane process X, the following processes of lower-dimensional flats derived from it will play an essential role. First, let $k \in \{2, \dots, d-1\}$ and $L \in G(d, k)$. The section process $X \cap L$ is obtained by taking all $(k-1)$-dimensional intersections of hyperplanes of X with L; see [16], pages 129 ff. It is a stationary Poisson process of $(k-1)$-flats in L. We denote its intensity by $\gamma_{X \cap L}$ and its spherical directional distribution, defined on $\mathbb{S}^{d-1} \cap L$, by $\varphi_{X \cap L}$. Second, for $k \in \{0, \dots, d-1\}$, the process X_{d-k} is obtained by intersecting any $d-k$ hyperplanes of X which are in general position; see [16], Section 4.4. It is a stationary process of k-flats and is called the intersection process of order $d-k$ of X. We denote its intensity by γ_{d-k} and its directional distribution by $\mathbf{Q}_{d-k}$. The latter is a probability measure on $G(d, k)$.

The hyperplane process X induces a tessellation $X^{(d)}$ of $\mathbb{R}^d$ and with it the process $X^{(k)}$ of its k-dimensional faces, for $k = 0, \dots, d-1$ (for the notation, note the slight digression from [16], where X and $X^{(d)}$ are denoted by $\widehat{X}$ and X, resp.). The zero cell of $X^{(d)}$ is the cell (d-face) containing $\mathbf{o}$ and thus is the random polytope given by

$$Z_0 := \bigcap_{H \in X} H^-.$$

Its counterpart for k-faces can be defined as follows (see [1, 15], e.g.). Let M_k denote the random measure defined by restricting the k-dimensional Hausdorff measure to the union of the k-flats of X_{d-k}. Further, let N_s denote the set of simple counting measures on $A(d, d-1)$ and $\mathcal{N}_s$ the usual σ-algebra of N_s (see [16], Section 3.1). Let $B \subset \mathbb{R}^d$ be a Borel set of Lebesgue measure 1 and let $\mathcal{A} \in \mathcal{N}_s$. Then

$$\mathbf{P}_k^0(\mathcal{A}) := \frac{1}{\mathbf{E} M_k(B)} \mathbf{E} \int_B \mathbf{1}_{\mathcal{A}}(X - \mathbf{x}) M_k(\mathrm{d}\mathbf{x})$$

defines a probability measure $\mathbf{P}_k^0$ (a Palm distribution, independent of B) on the measurable space $(\mathsf{N}_s, \mathcal{N}_s)$. Let Y be a hyperplane process with distribution $\mathbf{P}_k^0$. Then the *weighted typical k-face* $Z_0^{(k)}$ of X (i.e., of the mosaic induced by X) is defined as the a.s. unique k-face in $Y^{(k)}$ containing the origin $\mathbf{o}$. The distribution of the random polytope $Z_0^{(k)}$ is uniquely determined and coincides, up to translations, with that of the typical k-face $Z^{(k)}$ weighted by its k-volume. This is revealed by the relation

$$\mathbf{E} f(Z_0^{(k)}) = \frac{1}{\mathbf{E} V_k(Z^{(k)})} \mathbf{E}\big[f(Z^{(k)}) V_k(Z^{(k)})\big], \tag{2.1}$$

holding for every translation invariant, nonnegative, measurable function f on the space of k-dimensional polytopes (see [15], equation (11)). Here, V_k denotes the

k-dimensional volume, and $Z^{(k)}$ is the typical k-face of the mosaic induced by X, as defined in [16], page 450. An even more intuitive interpretation of the weighted typical k-face, up to translations, is the following. Let s denote the Steiner point (or any other centre function, see [16], page 110), and let $W \in \mathcal{K}$ be an arbitrary convex body with positive volume. Then, for every Borel set A in the space of convex polytopes,

$$\mathbf{P}\{Z_0^{(k)} - s(Z_0^{(k)}) \in A\} = \lim_{r\to\infty} \frac{\mathbf{E}\sum_{F\in X^{(k)}, F\subset rW} \mathbf{1}_A(F - s(F))V_k(F)}{\mathbf{E}\sum_{F\in X^{(k)}, F\subset rW} V_k(F)}.$$

The following integral representation for the distribution of $Z_0^{(k)}$ is proved in [15], Theorem 1. For Borel sets A in the space of convex polytopes,

$$\mathbf{P}\{Z_0^{(k)} \in A\} = \int_{G(d,k)} \mathbf{P}\{Z_0 \cap L \in A\}\mathbf{Q}_{d-k}(\mathrm{d}L). \tag{2.2}$$

Recall ([16], page 162) that the *Blaschke body* of X is the $\mathbf{o}$-symmetric convex body $B(X)$ with surface area measure $S_{d-1}(B(X), \cdot) = \gamma\varphi$. To describe the asymptotic shape of large weighted typical k-faces, we need the Blaschke body $B(X \cap L)$ of the section process $X \cap L$, for L in the support of the measure $\mathbf{Q}_{d-k}$. Since only the homothety class of the Blaschke body plays a role in the following, we may replace it by any dilate. It is convenient here to use the $\mathbf{o}$-symmetric body $B_L \subset L$ with surface area measure on $\mathbb{S}^{d-1} \cap L$ given by the spherical directional distribution $\varphi_{X\cap L}$ of the section process $X \cap L$.

We need some particular notions of distance.

For a rotation $\rho \in SO_d$, let M_ρ be the matrix of ρ with respect to the standard orthonormal basis of $\mathbb{R}^d$. We define the distance of ρ from the identity by

$$|\rho| := \|M_\rho - I\|,$$

where I is the unit matrix and $\|A\| = (\sum_{i,j=1}^d a_{ij}^2)^{1/2}$ is the Frobenius norm of the matrix $A = (a_{ij})_{i,j=1}^d$. Note that $|\rho| = |\rho^{-1}|$, since $M_{\rho^{-1}} - I$ is the transpose of $M_\rho - I$, and that for $\mathbf{x} \in \mathbb{R}^d$ we have

$$\|\mathbf{x} - \rho\mathbf{x}\| \le |\rho|\|\mathbf{x}\|.$$

On $G(d,k)$, we introduce a metric Δ by

$$\Delta(L, E) := \min\{|\rho| : \rho \in SO_d, \rho L = E\}.$$

The triangle inequality follows from

$$\|M_{\rho_1}M_{\rho_2} - I\| \le \|M_{\rho_1}M_{\rho_2} - M_{\rho_1}\| + \|M_{\rho_1} - I\| = \|M_{\rho_2} - I\| + \|M_{\rho_1} - I\|$$

for $\rho_1, \rho_2 \in SO_d$. The metric Δ induces the standard topology of $G(d,k)$ and is particularly convenient for us. For metrics on Grassmannians involving, like this one, a "direct rotation" between subspaces, we refer to [2], the survey article [13], and the references given there.

For $\theta > 0$, the θ-neighbourhood of a subspace $L^* \in G(d,k)$ is defined by

$$N_\theta(L^*) := \{L \in G(d,k) : \Delta(L, L^*) < \theta\}.$$

For $L \in G(d,k)$ and $K, M \in \mathcal{K}_0(L)$ with $M = -M$, let

$$\vartheta(K, M) := \log \min\{\beta/\alpha : \alpha, \beta > 0, \exists \mathbf{z} \in L : \alpha M \subset K + \mathbf{z} \subset \beta M\}. \tag{2.3}$$

The function ϑ measures the deviation of the homothetic shapes of K and M; it is nonnegative, and it vanishes if and only if K and M are homothetic.

For $L, E \in G(d,k)$ and convex bodies $K \in \mathcal{K}_0(L)$ and $M \in \mathcal{K}_0(E)$ with $M = -M$, let

$$\vartheta(K, M) := \min\{\vartheta(\rho K, M) : \rho \in SO_d, \rho L = E, |\rho| = \Delta(L, E)\}. \tag{2.4}$$

Note that this definition is consistent with (2.3), since $|\rho| = \Delta(L, E)$ in the case $L = E$ implies that ρ is the identity. Note also that $\vartheta(K, M)$ is symmetric in K and M if both bodies are **o**-symmetric.

For a k-dimensional convex body K, we denote by $D(K) = \operatorname{lin}(K - K) \in G(d,k)$ its direction space; this is the linear subspace parallel to the affine hull of K.

Throughout the paper, several constants c_i will appear, which may depend on various data. Their possible dependence on the dimension d will not be mentioned, since we work in a space of fixed dimension.

Now, we can formulate our main result.

THEOREM 2.1. *Let X be a stationary Poisson hyperplane process in $\mathbb{R}^d$ with intensity γ and spherical directional distribution φ. Let $k \in \{2, \dots, d-1\}$, and let $Z_0^{(k)}$ be the weighted typical k-face of the mosaic induced by X. Let $\mathbf{Q}_{d-k}$ be the directional distribution of the intersection process of order $d-k$ of X.*

Let $\varepsilon > 0$ be given. Then there exist constants $c_1, c_2 > 0$, depending only on $\varphi, \gamma, \varepsilon$, and a constant $c_3 > 0$, depending only on φ, γ, such that the following is true. If $L^* \in G(d,k)$ is in the support of the measure $\mathbf{Q}_{d-k}$, then

$$\begin{aligned}\mathbf{P}\{\vartheta(Z_0^{(k)}, B_{L^*}) \geq \varepsilon | V_k(Z_0^{(k)}) \geq a, D(Z_0^{(k)}) \in N_\theta(L^*)\}\\ \leq c_2 \exp[-c_3 \varepsilon^{k+1} a^{1/k}]\end{aligned}$$

for all $a \geq 1$ and all $0 < \theta \leq c_1$.

In other words, if a subspace L^* in the support of the distribution $\mathbf{Q}_{d-k}$ and a bound $\varepsilon > 0$ are given, then the probability that the weighted typical cell $Z_0^{(k)}$ deviates in shape from the (dilated) Blaschke body B_{L^*} by at least ε, under the condition that its direction space is contained in a suitable neighbourhood of L^* and its volume is at least $a > 0$, becomes exponentially small for large a. From this, one can deduce that the Blaschke body is the limit shape of $Z_0^{(k)}$ if the volume of $Z_0^{(k)}$ tends to infinity and its direction space tends to L^* (see Theorem 7.1 for a precise formulation). The assumptions of Theorem 2.1 are inevitable: the subspace L^* must be chosen in the support of the measure $\mathbf{Q}_{d-k}$ since, by (2.2), the direction space of $Z_0^{(k)}$ lies almost surely in the support of this measure. (This is also intuitively obvious: the k-faces of the tessellation $X^{(d)}$ are generated by intersections of hyperplanes from the process X.) Further, the Blaschke body B_{L^*} depends on L^*, hence in general only weighted typical cells with a direction space close to L^* can approximate the shape of B_{L^*}. The admissible size of the neighbourhood $N_\theta(L^*)$ in Theorem 2.1 depends heavily on Lemma 3.4 below and thus on the underlying stability theorem for Minkowski's existence theorem. Lemma 3.4 would yield additional information on the dependence of c_1 on ε, but a more explicit specification of the neighbourhood will only be possible for directional distributions φ where the solutions of Minkowski's problem are more explicitly accessible.

There are, however, two simple cases which should be mentioned. If the hyperplane process X is isotropic, that is, its directional distribution φ is invariant under rotations, then all Blaschke bodies B_{L^*} are balls, and the condition on the direction space $D(Z_0^{(k)})$ can be omitted entirely. In fact, if X is isotropic, then it can be deduced from (2.2) (see [15]) that there exists a random rotation ρ such that $\rho Z_0^{(k)}$ has the same distribution as the zero cell of a stationary isotropic Poisson $(k-1)$-flat process in a fixed k-dimensional subspace of $\mathbb{R}^d$. Therefore, one can immediately apply the results from [4] in that subspace.

Another simple case is that of a discrete directional distribution. If the directional distribution φ of X is concentrated in finitely many points, then every body B_{L^*} is a k-dimensional polytope. The distribution $\mathbf{Q}_{d-k}$ is concentrated in finitely many elements of $G(d,k)$. Hence, there exist only finitely many possibilities for the direction space of $Z_0^{(k)}$. If L^* in the support of $\mathbf{Q}_{d-k}$ is given, one can then choose for $N_\theta(L^*)$ in Theorem 2.1 any neighbourhood of L^* containing no other element of the support of $\mathbf{Q}_{d-k}$.

The proof of Theorem 2.1 will be given in Section 5. The next section provides geometric results in preparation for that proof.

The arguments leading to Theorem 2.1 can be modified to yield also a corresponding result for the typical k-face $Z^{(k)}$ of the mosaic induced by X.

THEOREM 2.2. *The assertion of Theorem 2.1 remains true if the weighted typical k-face $Z_0^{(k)}$ is replaced by the typical k-face $Z^{(k)}$ of the mosaic induced by X.*

This is in analogy to the corresponding result for the typical cell, Theorem 2 in [4].

We have restricted ourselves here, in agreement with D. G. Kendall's original question, to the volume functional. For the zero cell, we have investigated in [9] asymptotic shapes when the size of the zero cell is measured by various other functionals. It is a natural question whether such results carry over to k-faces. This is certainly possible in the isotropic case and for rotation invariant size functionals, by the remark made above. However, for non-isotropic distributions and general size functionals, asymptotic shapes are no longer controlled by the Blaschke body, so that the crucial Lemma 3.4 below must be replaced by a different approach.

3. Auxiliary continuity and stability results. Throughout this paper, X is a stationary Poisson hyperplane process in $\mathbb{R}^d$, with intensity γ and spherical directional distribution φ. We assume that φ is not concentrated on a great subsphere and, without loss of generality, that it is even (invariant under reflection in $\mathbf{o}$). The main topic of this section is the dependence of the dilated Blaschke bodies $B_L \subset L$, $L \in G(d,k)$, on the probability measure φ and on L.

Let $L \in G(d,k)$, where $k \in \{2, \dots, d-1\}$. The set $\mathbb{S}_L^{k-1} := \mathbb{S}^{d-1} \cap L$ is the unit sphere in L. The surface area measure of $K \in \mathcal{K}_0(L)$ is denoted by $S_{k-1}^L(K, \cdot)$; this is a measure on $\mathbb{S}_L^{k-1}$. By definition, the Blaschke body $B(X \cap L)$ is the unique convex body in $\mathcal{K}(L)$, centrally symmetric with respect to $\mathbf{o}$, for which

$$S_{k-1}^L\big(B(X \cap L), \cdot\big) = \gamma_{X \cap L} \varphi_{X \cap L},$$

where $\gamma_{X \cap L}$ is the intensity and $\varphi_{X \cap L}$ is the spherical directional distribution of the section process $X \cap L$. Existence and uniqueness of this body follow from Minkowski's theorem (see [14], Section 7.1, e.g.). We work here with a dilate of the Blaschke body, the $\mathbf{o}$-symmetric body B_L defined by

$$S_{k-1}^L(B_L, \cdot) = \varphi_{X \cap L}.$$

The *associated zonoid* Π_X of X is the projection body of $B(X)$ and hence has generating measure $\frac{1}{2}\gamma\varphi$, that is, its support function has the integral representation

$$h(\Pi_X, \mathbf{u}) = V_{d-1}(B(X)|\mathbf{u}^\perp) = \frac{\gamma}{2} \int_{\mathbb{S}^{d-1}} |\langle \mathbf{u}, \mathbf{v} \rangle| \varphi(\mathrm{d}\mathbf{v}), \qquad \mathbf{u} \in \mathbb{S}^{d-1},$$

where $\cdot|L$ denotes the orthogonal projection to L and $\mathbf{u}^\perp$ is the hyperplane through $\mathbf{o}$ orthogonal to $\mathbf{u}$. In the following, the support function $h(K, \cdot)$ of a convex body K is always defined on $\mathbb{R}^d$, also if $K \subset L$, $L \in G(d,k)$.

Let $L \in G(d,k)$. The associated zonoid of the section process $X \cap L$ is given by

$$\Pi_{X \cap L} = \Pi_X | L$$

(see [16], equation (4.61)). From this, we can read off the generating measure of the zonoid $\Pi_{X \cap L}$ and hence the essential parameters of the section process $X \cap L$. We define the spherical projection $\mathrm{pr}_L : \mathbb{S}^{d-1} \setminus L^\perp \to \mathbb{S}_L^{k-1}$ by

$$\mathrm{pr}_L(\mathbf{u}) := \frac{\mathbf{u}|L}{\|\mathbf{u}|L\|} \qquad \text{for } \mathbf{u} \in \mathbb{S}^{d-1} \setminus L^\perp.$$

Let $\mathcal{M}(\mathbb{S}^{d-1})$ denote the cone of finite Borel measures on $\mathbb{S}^{d-1}$. The spherical projection $\pi_L : \mathcal{M}(\mathbb{S}^{d-1}) \to \mathcal{M}(\mathbb{S}^{k-1}_L)$ is defined by

$$\pi_L \mu(A) = \int_{\mathbb{S}^{d-1}\setminus L^\perp} \mathbf{1}_A(\mathrm{pr}_L(\mathbf{u})) \|\mathbf{u}|L\| \mu(\mathrm{d}\mathbf{u}) \tag{3.1}$$

for Borel sets $A \subset \mathbb{S}^{k-1}_L$ and for $\mu \in \mathcal{M}(\mathbb{S}^{d-1})$. (More general spherical projections and their applications are treated in [3].)

For a segment $S = \mathrm{conv}\{-\alpha\mathbf{v}, \alpha\mathbf{v}\}$ with $\mathbf{v} \in \mathbb{S}^{d-1} \setminus L^\perp$ and $\alpha > 0$, we have for $\mathbf{u} \in \mathbb{R}^d$,

$$h(S|L, \mathbf{u}) = \big|\langle \mathbf{v}|L, \mathbf{u}\rangle\big|\alpha = |\langle \mathrm{pr}_L(\mathbf{v}), \mathbf{u}\rangle| \cdot \|\mathbf{v}|L\|\alpha.$$

Hence, if Z is a zonoid with generating measure μ, then the zonoid $Z|L$ has generating measure $\pi_L\mu$. In particular, the generating measure of $\Pi_X|L$ is given by $\frac{1}{2}\gamma\pi_L\varphi$. It follows that the Blaschke body $B(X \cap L)$ has surface area measure $S^L_{k-1}(B(X \cap L), \cdot) = \gamma\pi_L\varphi$. We conclude that

$$\gamma_{X\cap L}\varphi_{X\cap L} = \gamma\pi_L\varphi. \tag{3.2}$$

This is [16], Theorem 4.4.7, for hyperplane processes and in terms of spherical directional distributions.

Since the bodies B_L are obtained from the (nonconstructive) existence theorem of Minkowski, it is not trivial that they depend continuously on L. We need a stronger result, estimating how close B_L and B_E are in a suitable sense if the subspaces $L, E \in G(d,k)$ are close to each other. Such an estimate (Lemma 3.4) is obtained from a stability result for Minkowski's theorem that uses the Prokhorov metric for measures. (Diskant's stability result (see [14], Theorem 7.2.), which is in terms of the total variation norm of the difference, would not be strong enough for this purpose.) For finite measures μ, ν on $\mathbb{S}^{d-1}$, the Prokhorov distance $d_P(\mu, \nu)$ is defined by

$$d_P(\mu,\nu) = \inf\{\varepsilon > 0 : \mu(A) \le \nu(A_\varepsilon) + \varepsilon \text{ and } \nu(A) \le \mu(A_\varepsilon) + \varepsilon \\ \text{for all Borel sets } A \subset \mathbb{S}^{d-1}\},$$

where

$$A_\varepsilon := \{\mathbf{y} \in \mathbb{S}^{d-1} : \|\mathbf{x} - \mathbf{y}\| < \varepsilon \text{ for some } \mathbf{x} \in A\}.$$

Analogous definitions are used for measures on $\mathbb{S}^{k-1}_L$. For a rotation ρ and a measure μ, we denote by $\rho\mu$ the image measure of μ under ρ, defined by $(\rho\mu)(A) = \mu(\rho^{-1}A)$ for all A in the domain of μ.

LEMMA 3.1. *Let $L, E \in G(d,k)$, $\rho \in SO_d$ and $L = \rho E$. If $|\rho| \le 1/8$, then*

$$d_P(\pi_L\varphi, \rho\pi_E\varphi) \le 3|\rho|^{1/3}.$$

PROOF. Put $\mu := \pi_L\varphi$ and $\nu := \rho\pi_E\varphi$, write $\varepsilon := |\rho|^{1/3}$.

We have to show that $\mu(A) \leq \nu(A_{3\varepsilon}) + 3\varepsilon$ and $\nu(A) \leq \mu(A_{3\varepsilon}) + 3\varepsilon$ for all $A \in \mathcal{B}(\mathbb{S}_L^{k-1})$. Let $A \in \mathcal{B}(\mathbb{S}_L^{k-1})$ be given. The first assertion reads

$$\begin{aligned} &\int_{\mathbb{S}^{d-1}\setminus L^\perp} \mathbf{1}_A(\mathrm{pr}_L(\mathbf{u}))\|\mathbf{u}|L\|\varphi(\mathrm{d}\mathbf{u}) \\ &\qquad \leq \int_{\mathbb{S}^{d-1}\setminus E^\perp} \mathbf{1}_{\rho^{-1}A_{3\varepsilon}}(\mathrm{pr}_E(\mathbf{u}))\|\mathbf{u}|E\|\varphi(\mathrm{d}\mathbf{u}) + 3\varepsilon. \end{aligned} \tag{3.3}$$

Writing

$$\begin{aligned} M_1 &= \{\mathbf{u} \in \mathbb{S}^{d-1} : \|\mathbf{u}|L\| < \varepsilon\}, \\ M_2 &= \{\mathbf{u} \in \mathbb{S}^{d-1} : \|\mathbf{u}|L\| \geq \varepsilon\}, \end{aligned}$$

we have

$$\int_{M_1\setminus L^\perp} \mathbf{1}_A(\mathrm{pr}_L(\mathbf{u}))\|\mathbf{u}|L\|\varphi(\mathrm{d}\mathbf{u}) \leq \varepsilon.$$

Let $\mathbf{u}, \mathbf{v} \in \mathbb{S}^{d-1}$ and assume that $\|\mathbf{u} - \mathbf{v}\| \leq \varepsilon^3$ and $\mathbf{u} \in M_2$, hence $\|\mathbf{u}|L\| \geq \varepsilon$. From

$$\big|\|\mathbf{u}|L\| - \|\mathbf{v}|L\|\big| \leq \|(\mathbf{u} - \mathbf{v})|L\| \leq \|\mathbf{u} - \mathbf{v}\| \leq \varepsilon^3$$

we get $\|\mathbf{v}|L\| \geq \|\mathbf{u}|L\| - \varepsilon^3 \geq \varepsilon - \varepsilon^3 \geq \varepsilon/2 > 0$, hence $\mathbf{v} \in \mathbb{S}^{d-1} \setminus L^\perp$.

There are unique representations

$$\begin{aligned} \mathbf{u} &= t\mathbf{u}_0 + \mathbf{u}_1, \qquad \mathbf{u}_0 \in L \cap \mathbb{S}^{d-1}, \mathbf{u}_1 \in L^\perp, t > 0, \\ \mathbf{v} &= \tau\mathbf{v}_0 + \mathbf{v}_1, \qquad \mathbf{v}_0 \in L \cap \mathbb{S}^{d-1}, \mathbf{v}_1 \in L^\perp, \tau > 0. \end{aligned}$$

Here, $\mathbf{u}_0 = \mathrm{pr}_L(\mathbf{u})$ and $\mathbf{v}_0 = \mathrm{pr}_L(\mathbf{v})$. From $|t - \tau| = |\|\mathbf{u}|L\| - \|\mathbf{v}|L\|| \leq \varepsilon^3$ together with $\tau = \|\mathbf{v}|L\| \geq \varepsilon/2$ and $t = \|\mathbf{u}|L\| \geq \varepsilon$, we get

$$\begin{aligned} \|\mathbf{u}_0 - \mathbf{v}_0\| &\leq \left\|\frac{\mathbf{u}}{t} - \frac{\mathbf{v}}{\tau}\right\| = \left\|\frac{\mathbf{u}}{t} - \frac{\mathbf{v}}{t} + \frac{\mathbf{v}}{t} - \frac{\mathbf{v}}{\tau}\right\| \\ &\leq \frac{1}{t}\|\mathbf{u} - \mathbf{v}\| + \left|\frac{1}{t} - \frac{1}{\tau}\right| \leq \varepsilon^{-1} \cdot \varepsilon^3(1 + 2/\varepsilon) \leq 3\varepsilon. \end{aligned}$$

Now let $\mathbf{v} := \rho\mathbf{u}$, then $\|\mathbf{u} - \mathbf{v}\| \leq |\rho| = \varepsilon^3$. Hence, we have $\|\mathbf{u}_0 - \mathbf{v}_0\| \leq 3\varepsilon$.

Suppose that $\mathbf{u} \in M_2$ is such that $\mathbf{u}_0 \in A$. Then $\mathrm{pr}_L(\rho\mathbf{u}) = \mathbf{v}_0 \in A_{3\varepsilon}$, hence $\mathrm{pr}_E(\mathbf{u}) \in \rho^{-1}A_{3\varepsilon}$. From

$$\|\mathbf{u}|L\| - \|\mathbf{u}|E\| = \|\mathbf{u}|L\| - \|\rho\mathbf{u}|L\| = \|\mathbf{u}|L\| - \|\mathbf{v}|L\| \leq \varepsilon^3$$

we see that $\mathbf{u} \notin E^{\perp}$ and conclude that

$$\begin{aligned}\int_{M_2} & \mathbf{1}_A(\mathrm{pr}_L(\mathbf{u}))\|\mathbf{u}|L\|\varphi(\mathrm{d}\mathbf{u}) \\ &\le \int_{\mathbb{S}^{d-1}\setminus E^{\perp}} \mathbf{1}_{\rho^{-1}A_{3\varepsilon}}(\mathrm{pr}_E(\mathbf{u}))\|\mathbf{u}|L\|\varphi(\mathrm{d}\mathbf{u}) \\ &\le \int_{\mathbb{S}^{d-1}\setminus E^{\perp}} \mathbf{1}_{\rho^{-1}A_{3\varepsilon}}(\mathrm{pr}_E(\mathbf{u}))\|\mathbf{u}|E\|\varphi(\mathrm{d}\mathbf{u}) + \varepsilon^3.\end{aligned}$$

Altogether, we obtain

$$\begin{aligned}\int_{\mathbb{S}^{d-1}\setminus L^{\perp}} & \mathbf{1}_A(\mathrm{pr}_L(\mathbf{u}))\|\mathbf{u}|L\|\varphi(\mathrm{d}\mathbf{u}) \\ &\le \varepsilon + \int_{\mathbb{S}^{d-1}\setminus E^{\perp}} \mathbf{1}_{\rho^{-1}A_{3\varepsilon}}(\mathrm{pr}_E(\mathbf{u}))\|\mathbf{u}|E\|\varphi(\mathrm{d}\mathbf{u}) + \varepsilon^3\end{aligned}$$

and hence (3.3).

Since $|\rho| = |\rho^{-1}|$, inequality (3.3) remains true if we interchange L with E and ρ with ρ^{-1} and then replace A by $\rho^{-1}A$. The resulting inequality is $\nu(A) \le \mu(A_{3\varepsilon}) + 3\varepsilon$, which completes the proof of Lemma 3.1. □

In the following, the dependence of some constants c_i on the measure φ is only via the number

$$m(\varphi) := \min_{\mathbf{u}\in\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} |\langle \mathbf{u}, \mathbf{v}\rangle|\varphi(\mathrm{d}\mathbf{v}).$$

This number, which can be considered as a measure of nondegeneracy, is positive, since the support of φ is not contained in a great subsphere.

LEMMA 3.2. *Let B be the* $\mathbf{o}$-*symmetric convex body with $S_{d-1}(B, \cdot) = \varphi$. The inradius r and circumradius R of B can be estimated by*

$$c_4 \le r \le R \le c_5,$$

where $c_4, c_5 > 0$ are constants depending only on $m(\varphi)$ and an upper bound for φ. (*Here, φ can be any finite even measure on $\mathbb{S}^{d-1}$ not concentrated on a great subsphere.*)

PROOF. First, we repeat a known argument ([14], page 303). If $\varphi(\mathbb{S}^{d-1}) \le b$, the isoperimetric inequality gives $V_d(B) \le c(b)$, with a constant $c(b)$ depending only on b (and the dimension, which we do not mention). Let $\mathbf{x} \in B$. Then

$$\begin{aligned}V_d(B) &= \frac{1}{d}\int_{\mathbb{S}^{d-1}} h(B, \mathbf{v})\varphi(\mathrm{d}\mathbf{v}) \ge \frac{1}{d}\int_{\mathbb{S}^{d-1}} \max\{\langle \mathbf{x}, \mathbf{v}\rangle, 0\}\varphi(\mathrm{d}\mathbf{v}) \\ &= \frac{1}{2d}\int_{\mathbb{S}^{d-1}} |\langle \mathbf{x}, \mathbf{v}\rangle|\varphi(\mathrm{d}\mathbf{v}) \ge \frac{1}{2d}\|\mathbf{x}\| m(\varphi).\end{aligned}$$

It follows that $R \le 2dc(b)/m(\varphi)$.

Second, since B is centrally symmetric, an inball of B is touched by two parallel supporting hyperplanes of B. Let $\mathbf{u}$ be a unit vector parallel to these hyperplanes. The projection $B|\mathbf{u}^\perp$ lies between two parallel hyperplanes in $\mathbf{u}^\perp$ which are distance $2r$ apart, and it is contained in $R\mathbb{B}^d \cap \mathbf{u}^\perp$. Hence, $V_{d-1}(B|\mathbf{u}^\perp) \le 2rV_{d-2}(\mathbb{B}^{d-2})R^{d-2}$. On the other hand, using [14], (7.4.1),

$$\begin{aligned} V_{d-1}(B|\mathbf{u}^\perp) &= \frac{1}{2}\int_{\mathbb{S}^{d-1}} |\langle \mathbf{u}, \mathbf{v}\rangle| S_{d-1}(B, \mathrm{d}\mathbf{v}) \\ &= \frac{1}{2}\int_{\mathbb{S}^{d-1}} |\langle \mathbf{u}, \mathbf{v}\rangle| \varphi(\mathrm{d}\mathbf{v}) \ge \frac{1}{2} m(\varphi). \end{aligned}$$

The assertion follows. $\square$

It is technically convenient to consider also the $\mathbf{o}$-symmetric convex body $B(L)$ in L with surface area measure

$$S^L_{k-1}(B(L), \cdot) = \pi_L \varphi.$$

From (3.2), we have

$$B_L = \left(\frac{\gamma}{\gamma_{X\cap L}}\right)^{1/(k-1)} B(L) \tag{3.4}$$

and

$$\frac{\gamma_{X\cap L}}{\gamma} = \int_{\mathbb{S}^{d-1}} \|\mathbf{v}|L\| \varphi(\mathrm{d}\mathbf{v}). \tag{3.5}$$

LEMMA 3.3. *Let $k \in \{2, \dots, d-1\}$. Let r_L, R_L denote the inradius and circumradius, respectively, of either $B(L)$ or B_L, measured in $L \in G(d,k)$. There are constants $c_6, c_7 > 0$, depending only on $m(\varphi)$, such that*

$$c_6 \le r_L \le R_L \le c_7 \qquad \textit{for all } L \in G(d,k).$$

PROOF. Let $L \in G(d,k)$. From (3.1), clearly $\pi_L\varphi(\mathbb{S}^{k-1}_L) \le \varphi(\mathbb{S}^{d-1})$, hence 1 is an upper bound for $\pi_L\varphi$.

Let $\Pi_{\varphi/2}$ be the zonoid with generating measure $\varphi/2$. Then $\Pi_{\varphi/2}|L$ has generating measure $\pi_L\varphi/2$. For $\mathbf{u} \in L$, it follows that

$$\begin{aligned} \int_{\mathbb{S}^{k-1}_L} |\langle \mathbf{u}, \mathbf{v}\rangle| \pi_L\varphi(\mathrm{d}\mathbf{v}) &= 2h(\Pi_{\varphi/2}|L, \mathbf{u}) \\ &= 2h(\Pi_{\varphi/2}, \mathbf{u}) = \int_{\mathbb{S}^{d-1}} |\langle \mathbf{u}, \mathbf{v}\rangle| \varphi(\mathrm{d}\mathbf{v}) \ge m(\varphi). \end{aligned}$$

Now Lemma 3.2, applied in L and to the measure $\pi_L\varphi$, shows that the inradius and circumradius of $B(L)$ can be estimated from both sides by positive constants depending only on $m(\varphi)$. The same fact for B_L follows from (3.4), since (3.5) gives

$$m(\varphi) \le \gamma_{X\cap L}/\gamma \le 1. \tag{3.6}$$

For the left side, note that L contains a unit vector $\mathbf{u}$ and that $\|\mathbf{v}|L\| \ge |\langle \mathbf{v}, \mathbf{u}\rangle|$. □

In the following, we make use of the Hausdorff metric δ and of the deviation function ϑ defined by (2.3).

LEMMA 3.4. *Let $k \in \{2, \dots, d-1\}$. There exist constants c_8, c_9, depending only on $m(\varphi)$, with the following property. If $L, E \in G(d,k)$ and if $\rho \in SO_d$ is a rotation with $L = \rho E$ and $|\rho| \le 1/8$, then*

$$\delta(B(L), \rho B(E)) \le c_8 |\rho|^{1/3k} \tag{3.7}$$

and

$$\vartheta(B_L, \rho B_E) \le c_9 |\rho|^{1/3k}. \tag{3.8}$$

PROOF. By Lemma 3.3, the inradius and circumradius of $B(L')$, $L' \in G(d,k)$, can be bounded from below and from above by positive constants depending only on $m(\varphi)$. We use the stability result of [6], Theorem 3.1, for the solutions of Minkowski's problem and apply it here in the subspace L of the assertion. [Note that $B(L)$, $B(E)$ are centrally symmetric, hence the translations appearing *loc. cit.* can be omitted.] We conclude that

$$\begin{aligned}\delta(B(L), \rho B(E)) &\le c d_P(S^L_{k-1}(B(L), \cdot), S^L_{k-1}(\rho B(E), \cdot))^{1/k}\\ &= c d_P(\pi_L \varphi, \rho \pi_E \varphi)^{1/k}\end{aligned}$$

with a constant c depending only on $m(\varphi)$. By Lemma 3.1,

$$d_P(\pi_L\varphi, \rho\pi_E\varphi) \le 3|\rho|^{1/3},$$

hence (3.7) follows, with suitable c_8.

From (3.7), with $\lambda := c_8|\rho|^{1/3k}$, we get $B(L) \subset \rho B(E) + \lambda \mathbb{B}^k_L$, where $\mathbb{B}^k_L := \mathbb{B}^d \cap L$ is the unit ball in L. Since $c_6 \mathbb{B}^k_L \subset \rho B(E)$ by Lemma 3.3, we get

$$B(L) \subset (1 + \lambda/c_6)\rho B(E).$$

A similar relation holds with $\rho B(E)$ and $B(L)$ interchanged, hence

$$(1+\lambda/c_6)^{-1}\rho B(E) \subset B(L) \subset (1+\lambda/c_6)\rho B(E).$$

This gives

$$\vartheta(B(L), \rho B(E)) \le \log(1+\lambda/c_6)^2 \le 2\lambda/c_6.$$

Here, we may replace $B(L)$, $B(E)$ by B_L, B_E, since ϑ is invariant under dilatations. □

In the following lemma, the deviation function ϑ for convex bodies in different subspaces, as defined by (2.4), is used.

LEMMA 3.5. *Let $k \in \{2, \dots, d-1\}$, let $L, L^* \in G(d,k)$ and $\varepsilon > 0$. If $\Delta(L, L^*) \le \min\{1/8, (\varepsilon/c_9)^{3k}\}$, where c_9 is the constant appearing in Lemma 3.4, then every convex body $K \in \mathcal{K}_0(L)$ with $\vartheta(K, B_L) < \varepsilon$ satisfies $\vartheta(K, B_{L^*}) < 2\varepsilon$.*

PROOF. Let $L, L^* \in G(d,k)$ and $\Delta(L, L^*) \le \min\{1/8, (\varepsilon/c_9)^{3k}\}$. There exists a rotation $\rho \in SO_d$ with $\rho L = L^*$ and $|\rho| = \Delta(L, L^*)$, hence ρ satisfies $|\rho| \le (\varepsilon/c_9)^{3k}$ and $|\rho| \le 1/8$. Lemma 3.4 gives

$$\vartheta(B_{L^*}, \rho B_L) \le \varepsilon. \tag{3.9}$$

Suppose that $K \in \mathcal{K}_0(L)$ and $\vartheta(K, B_L) < \varepsilon$. The definition of ϑ implies the triangle inequality

$$\vartheta(\rho K, B_{L^*}) \le \vartheta(\rho K, \rho B_L) + \vartheta(\rho B_L, B_{L^*}).$$

In fact, if $\vartheta(\rho K, \rho B_L) =: a_1$ and $\vartheta(\rho B_L, B_{L^*}) =: a_2$, there are numbers $\alpha_1, \beta_1, \alpha_2, \beta_2 > 0$ with $\log(\beta_1/\alpha_1) = a_1$, $\log(\beta_2/\alpha_2) = a_2$ and a vector $\mathbf{z} \in \rho L$ such that

$$\alpha_1 \rho B_L \subset \rho K + \mathbf{z} \subset \beta_1 \rho B_L, \qquad \alpha_2 B_{L^*} \subset \rho B_L \subset \beta_2 B_{L^*}.$$

In the second case, we have used that, due to the central symmetry of B_L and B_{L^*}, the translation vector appearing in the definition of ϑ can be omitted. We deduce that

$$\alpha_1 \alpha_2 B_{L^*} \subset \rho K + \mathbf{z} \subset \beta_1 \beta_2 B_{L^*}$$

and hence $\vartheta(\rho K, B_{L^*}) \le \log(\beta_1 \beta_2 / \alpha_1 \alpha_2) = a_1 + a_2$.

From $\vartheta(\rho K, \rho B_L) = \vartheta(K, B_L) < \varepsilon$ and $\vartheta(\rho B_L, B_{L^*}) = \vartheta(B_{L^*}, \rho B_L) \le \varepsilon$ we get $\vartheta(\rho K, B_{L^*}) < 2\varepsilon$. Since $|\rho| = \Delta(L, L^*)$, this yields $\vartheta(K, B_{L^*}) < 2\varepsilon$, which finishes the proof. □

4. Two preparatory probability estimates. The plan is to prove Theorem 2.1 by using (2.2) and applying results for the zero cell Z_0 of X, obtained in [4], to the random polytope $Z_0 \cap L$, for each $L \in G(d,k)$. This is possible since $Z_0 \cap L$ is stochastically equivalent to the zero cell of the section process $X \cap L$, which is a stationary Poisson hyperplane process in L; it has intensity $\gamma_{X \cap L}$ and spherical directional distribution $\varphi_{X \cap L} = (\gamma / \gamma_{X \cap L}) \pi_L \varphi$, by (3.2). In applying the results from [4], we have to ensure that the constants appearing there can be chosen independently of L.

Let $L \in G(d,k)$. For $\mathbf{u} \in \mathbb{S}_L^{k-1}$ and $t > 0$ we write $H_L^-(\mathbf{u}, t) := H^-(\mathbf{u}, t) \cap L$. If $\mathbf{u}_1, \dots, \mathbf{u}_n \in \mathbb{S}_L^{k-1}$ and $t_1, \dots, t_n > 0$, we use the notation

$$\bigcap_{i=1}^{n} H_L^-(\mathbf{u}_i, t_i) =: P_L(\mathbf{u}_1, \dots, \mathbf{u}_n; t_1, \dots, t_n).$$

In the following, we write

$$\tau_L := k V_k(B_L)^{1-1/k}.$$

LEMMA 4.1. *Let $\beta > 0$. There are positive constants c_{10}, h_0, depending only on φ, γ and β, such that for all $L \in G(d,k)$, all $a \ge 1$ and $0 < h \le h_0$,*

$$\mathbf{P}\{V_k(Z_0 \cap L) \in a(1, 1+h)\} \ge c_{10} h \exp[-2(1+\beta)\gamma_{X\cap L}\tau_L a^{1/k}].$$

PROOF. Let $\beta > 0$ and $a \ge 1$ be given.

First, we consider a fixed $L^* \in G(d,k)$. Then $X \cap L^*$ is a stationary Poisson hyperplane process in L^* with intensity $\gamma_{X\cap L^*}$ and spherical directional distribution $\varphi_{X\cap L^*}$. For given $\beta > 0$, Lemma 3.1 in [4], applied to the convex body B_{L^*} in L^*, yields the existence of a number $N \in \mathbb{N}$, of unit vectors $\mathbf{u}_1^0, \dots, \mathbf{u}_N^0 \in \mathbb{S}_{L^*}^{k-1}$ in the support of the measure $\varphi_{X\cap L^*}$ (which is equal to the support of $\pi_{L^*}\varphi$) and of positive numbers $t_1^0, \dots, t_N^0$, all depending only on φ, γ, L^* and β, such that the polytope

$$P^0 := P_{L^*}(\mathbf{u}_1^0, \dots, \mathbf{u}_N^0; t_1^0, \dots, t_N^0)$$

has N facets (in L^*) and satisfies

$$P^0 \subset (1+\beta/4)B_{L^*} \quad \text{and} \quad V_k(P^0) = V_k(B_{L^*}).$$

Next, we can choose neighbourhoods U_i of $\mathbf{u}_i^0$ in $\mathbb{S}_{L^*}^{k-1}$ and a number $\alpha > 0$ with $t_i^0 - \alpha > 0$, $i = 1, \dots, N$, all depending only on φ, γ, L^* and β, such that, for all $\mathbf{u}_1, \dots, \mathbf{u}_N \in \mathbb{S}_{L^*}^{k-1}$ and $t_1, \dots, t_N \in \mathbb{R}$ with

$$\mathbf{u}_i \in U_i, \qquad |t_i - t_i^0| < \alpha, \qquad i = 1, \dots, N, \tag{4.1}$$

the following condition (i) is satisfied.

(i) $P := P_{L^*}(\mathbf{u}_1, \dots, \mathbf{u}_N; t_1, \dots, t_N)$ is a polytope in L^* with N facets and satisfying $P \subset (1+\beta/2)B_{L^*}$.

The set of values

$$V_k(P_{L^*}(\mathbf{u}_1^0, \dots, \mathbf{u}_N^0; t_1^0, \dots, t_{N-1}^0, t)) \qquad \text{with } |t - t_N^0| < \alpha$$

is an interval containing $V_k(B_{L^*})$ in its interior. Therefore, after decreasing $U_1, \dots, U_N, \alpha$, if necessary, we can assume that there exists a number $b > 0$, depending only on φ, γ, L^* and β, with the following property.

(ii) If (4.1) is satisfied, then

$$\begin{aligned}&\big(V_k(B_{L^*}) - b, V_k(B_{L^*}) + b\big)\\ &\qquad \subset \{V_k(P_{L^*}(\mathbf{u}_1, \dots, \mathbf{u}_N; t_1, \dots, t_{N-1}, t)) : |t - t_N^0| < \alpha\}.\end{aligned}$$

We must extend the preceding to the subspaces L in a suitable neighbourhood $N_\theta(L^*)$. The numbers θ, η, h_0 appearing in the following can be chosen to depend only on φ, γ, L^* and β. Let $\theta \in (0, 1/8]$; below it will be specified further. To each $L \in N_\theta(L^*)$, we choose a rotation ρ_L with $L = \rho_L L^*$ and $|\rho_L| \le \theta$.

We choose a number $\eta > 0$ so small that to each $i \in \{1, \dots, N\}$ there exists a neighbourhood U_i' of $\mathbf{u}_i^0$ in $\mathbb{S}_{L^*}^{k-1}$ with $(U_i')_\eta \subset U_i$, where for $A \subset \mathbb{S}_{L^*}^{k-1}$ the set A_η is defined by $A_\eta = \{\mathbf{y} \in \mathbb{S}_{L^*}^{k-1} : \|\mathbf{y} - \mathbf{x}\| < \eta \text{ for some } \mathbf{x} \in A\}$. Decreasing η, if necessary (without changing the sets U_i'), we can also assume that

$$\pi_{L^*}\varphi(U_i') \geq 2\eta \qquad \text{for } i = 1, \dots, N.$$

This is possible since U_i' is a neighbourhood of $\mathbf{u}_i^0 \in \operatorname{supp} \pi_{L^*}\varphi$.

By Lemma 3.1, we can further choose θ so small that

$$d_P(\pi_L\varphi, \rho_L\pi_{L^*}\varphi) \leq \eta \qquad \text{for } L \in N_\theta(L^*).$$

Then,

$$\pi_L\varphi(\rho_L U_i) \geq \pi_L\varphi((\rho_L U_i')_\eta) \geq (\rho_L\pi_{L^*}\varphi)(\rho_L U_i') - \eta = \pi_{L^*}\varphi(U_i') - \eta \geq \eta.$$

Hence, putting $U_i^L := \rho_L U_i$, we have from (3.2)

$$\varphi_{X\cap L}(U_i^L) = \frac{\gamma}{\gamma_{X\cap L}}\pi_L\varphi(U_i^L) \geq \eta > 0 \qquad \text{for } i = 1, \dots, N.$$

Due to (3.8), we can decrease θ, if necessary, such that

$$\rho_L B_{L^*} \subset \frac{1+\beta}{1+\beta/2} B_L \qquad \text{for } L \in N_\theta(L^*). \tag{4.2}$$

Using (ii) above, (3.7) and the fact that $L \mapsto \gamma_{X\cap L} = \gamma\pi_L\varphi(\mathbb{S}_L^{k-1})$ is continuous by Lemma 3.1, we can decrease θ further, if necessary, and choose a number $h_0 > 0$ such that

$$L \in N_\theta(L^*), \qquad \mathbf{u}_i \in U_i^L, \qquad |t_i - t_i^0| < \alpha, \qquad i = 1, \dots, N, \tag{4.3}$$

implies

$$V_k(B_L)(1, 1 + h_0) \subset \{V_k(P_L(\mathbf{u}_1, \dots, \mathbf{u}_N; t_1, \dots, t_{N-1}, t)) : |t - t_N^0| < \alpha\}.$$

Here, we have used that

$$P_L(\mathbf{u}_1, \dots, \mathbf{u}_N; t_1, \dots, t_{N-1}, t) = \rho_L P_{L^*}(\rho_L^{-1}\mathbf{u}_1, \dots, \rho_L^{-1}\mathbf{u}_N; t_1, \dots, t_{N-1}, t).$$

After these choices, the following is true for all $L \in N_\theta(L^*)$. If (4.3) holds, then (i_L) and (ii_L) are satisfied:

(i_L) $P := P_L(\mathbf{u}_1, \dots, \mathbf{u}_N; t_1, \dots, t_N)$ is a polytope with N facets and satisfying $P \subset (1 + \beta)B_L$.

(ii_L)

$$V_k(B_L)(1, 1 + h_0) \subset \{V_k(P_L(\mathbf{u}_1, \dots, \mathbf{u}_N; t_1, \dots, t_{N-1}, t)) : |t - t_N^0| < \alpha\}.$$

In fact, (i_L) follows from (i) and (4.2), since $\rho_L^{-1}\mathbf{u}_i \in U_i$ and therefore

$$\begin{aligned} P &= \rho_L P_{L^*}(\rho_L^{-1}\mathbf{u}_1, \dots, \rho_L^{-1}\mathbf{u}_N; t_1, \dots, t_{N-1}, t) \\ &\subset \rho_L(1+\beta/2)B_{L^*} \subset (1+\beta/2)\frac{1+\beta}{1+\beta/2}B_L = (1+\beta)B_L. \end{aligned}$$

We restate what we have found so far, making explicit the dependence on L^*. For any $L^* \in G(d,k)$, there exist numbers $\theta(L^*) \in (0, 1/8]$, $N(L^*) \in \mathbb{N}$, $\alpha(L^*) > 0$, $t_1^0(L^*), \dots, t_{N(L^*)}^0(L^*) > \alpha(L^*)$, $h_0(L^*) > 0$, $\eta(L^*) > 0$, unit vectors $\mathbf{u}_1^0(L^*), \dots, \mathbf{u}_{N(L^*)}^0(L^*) \in \mathbb{S}_{L^*}^{k-1}$ and neighbourhoods $U_i(L^*)$ of $\mathbf{u}_i^0(L^*)$ in $\mathbb{S}_{L^*}^{k-1}$, $i = 1, \dots, N(L^*)$, such that for all $L \in N_{\theta(L^*)}(L^*)$ and for $\mathbf{u}_i \in U_i^L(L^*)$ and $|t_i - t_i^0(L^*)| < \alpha(L^*)$, $i = 1, \dots, N(L^*)$, the following conditions are satisfied:

$$\varphi_{X\cap L}(U_i^L(L^*)) \ge \eta(L^*) > 0, \qquad i = 1, \dots, N(L^*),$$

(i_L) $P := P_L(\mathbf{u}_1, \dots, \mathbf{u}_{N(L^*)}; t_1, \dots, t_{N(L^*)})$ is a polytope with $N(L^*)$ facets and satisfying $P \subset (1+\beta)B_L$, and

(ii_L)

$$\begin{aligned} &V_k(B_L)\big(1, 1+h_0(L^*)\big) \\ &\quad \subset \big\{V_k\big(P_L\big(\mathbf{u}_1, \dots, \mathbf{u}_{N(L^*)}; t_1, \dots, t_{N(L^*)-1}, t\big)\big) : \big|t - t_{N(L^*)}^0\big| < \alpha(L^*)\big\}. \end{aligned}$$

Since $(G(d,k), \Delta)$ is compact and $\{N_{\theta(L^*)}(L^*) : L^* \in G(d,k)\}$ is an open cover of $G(d,k)$, there are $L_1^*, \dots, L_r^* \in G(d,k)$ such that $\{N_{\theta(L_j^*)}(L_j^*) : j = 1, \dots, r\}$ is a finite subcover of $G(d,k)$. We put

$$\begin{aligned} \eta_0 &:= \min\{\eta(L_j^*) : j = 1, \dots, r\} > 0, \\ h_0 &:= \min\{h_0(L_j^*) : j = 1, \dots, r\} > 0. \end{aligned}$$

Hence, for $L \in G(d,k)$ there is some $j \in \{1, \dots, r\}$ such that $L \in N_{\theta(L_j^*)}(L_j^*)$ and

$$\varphi_{X\cap L}(U_i^L(L_j^*)) \ge \eta(L_j^*) \ge \eta_0 > 0. \tag{4.4}$$

Note that $U_i^L(L_j^*) = \rho_L(L_j^*)U_i(L_j^*)$. For $\mathbf{u}_i \in U_i^L(L_j^*)$ and for $|t_i - t_i^0(L_j^*)| < \alpha(L_j^*)$, $i = 1, \dots, N(L_j^*)$, the set

$$P := P_L\big(\mathbf{u}_1, \dots, \mathbf{u}_{N(L_j^*)}; t_1, \dots, t_{N(L_j^*)}\big)$$

is a polytope with $N(L_j^*)$ facets and satisfying $P \subset (1+\beta)B_L$, and

$$\begin{aligned} &V_k(B_L)(1, 1+h_0) \\ &\quad \subset \big\{V_k\big(P_L\big(\mathbf{u}_1, \dots, \mathbf{u}_{N(L_j^*)}; t_1, \dots, t_{N(L_j^*)-1}, t\big)\big) : \big|t - t_{N(L_j^*)}^0\big| < \alpha(L_j^*)\big\}. \end{aligned}$$

We are now in a situation where we can adjust the second part of the proof of [4], Lemma 3.2, in a fixed linear subspace $L \in G(d,k)$. We choose a corresponding index $j \in \{1, \dots, r\}$ such that $L \in N_{\theta(L_j^*)}(L_j^*)$, as described above.

For the given $a \geq 1$, we choose a number $\varrho > 0$ such that $V_k(\varrho B_L) = a$, that is, $\varrho = a^{1/k} V_k(B_L)^{-1/k}$. Then, for $\mathbf{u}_i \in U_i^L(L_j^*)$ and for $|t_i - t_i^0(L_j^*)| < \alpha(L_j^*)$, $i = 1, \ldots, N(L_j^*)$,

(i$_\varrho$) $P_\varrho := P_L(\mathbf{u}_1, \ldots, \mathbf{u}_{N(L_j^*)}; \varrho t_1, \ldots, \varrho t_{N(L_j^*)})$ is a polytope with $N(L_j^*)$ facets and satisfying $P_\varrho \subset (1+\beta)\varrho B_L$, and,

(ii$_\varrho$) for $0 < h \leq h_0$,

$$V_k(B_L)(1, 1+h) \subset \{v_L(t) : |t - \varrho t^0_{N(L_j^*)}(L_j^*)| < \varrho\alpha(L_j^*)\}$$

with

$$v_L(t) := V_k(P_L(\mathbf{u}_1, \ldots, \mathbf{u}_{N(L_j^*)}; \varrho t_1, \ldots, \varrho t_{N(L_j^*)-1}, t)).$$

Let λ^1 denote 1-dimensional Lebesgue measure. The argument on page 1147, lines -17 to bottom, in [4] now shows that

$$\begin{aligned}&\lambda^1\{t \in \mathbb{R} : |t - \varrho t^0_{N(L_j^*)}(L_j^*)| < \varrho\alpha(L_j^*), v_L(t) \in V_k(B_L)(1, 1+h)\}\\ &\qquad \geq c(\beta, \varphi)\varrho h,\end{aligned}$$

where $c(\beta, \varphi)$ is a constant depending only on β and φ. Here it is implicitly used that $P_\varrho \subset (1+\beta)\varrho B_L$, which implies that the $(k-1)$-dimensional volume of the orthogonal projection of P_ϱ on to the orthogonal complement of $\mathbf{u}_{N(L_j^*)}$ can be bounded from above by a constant depending only on β and $m(\varphi)$. Moreover, it is also used that $V_k(B_L)$ can be bounded from below by a constant depending only on $m(\varphi)$.

Next, we define a sufficiently large set of convex polytopes in L by

$$\begin{aligned}\mathcal{P}_L := \{&P_L(\mathbf{u}_1, \ldots, \mathbf{u}_{N(L_j^*)}; t_1, \ldots, t_{N(L_j^*)}) : \mathbf{u}_i \in U_i^L(L_j^*) \text{ and}\\ &|t_i - \varrho t_i^0(L_j^*)| < \varrho\alpha(L_j^*), \text{ for } i = 1, \ldots, N(L_j^*), \text{ and}\\ &V_k(P_L(\mathbf{u}_1, \ldots, \mathbf{u}_{N(L_j^*)}; t_1, \ldots, t_{N(L_j^*)})) \subset V_k(\varrho B_L)(1, 1+h)\}.\end{aligned}$$

Let $\mathcal{H}_{(1+\beta)\varrho B_L} := \{H \in A(d, k-1) : (1+\beta)\varrho B_L \cap H \neq \varnothing\}$. For a hyperplane process Y in L, we write $Z_0(Y)$ for the induced zero cell in L, and "$\llcorner$" denotes the restriction of a measure. Subsequently, we adapt the argument from [4], page 1148, to the present situation. For the first estimate, we use that any polytope in $\mathcal{P}_L$ is contained in $(1+\beta)\varrho B_L$. Thus, we get

$$\begin{aligned}&\mathbf{P}\{V_k(Z_0 \cap L) \in V_k(\varrho B_L)(1, 1+h)\}\\ &\quad \geq \mathbf{P}\{(X \cap L)(\mathcal{H}_{(1+\beta)\varrho B_L}) = N(L_j^*), Z_0((X \cap L) \llcorner \mathcal{H}_{(1+\beta)\varrho B_L}) \in \mathcal{P}_L\}\\ &\quad = \frac{[2k(1+\beta)\varrho V_k(B_L)\gamma_{X\cap L}]^{N(L_j^*)}}{N(L_j^*)!} \exp[-2k(1+\beta)\varrho V_k(B_L)\gamma_{X\cap L}]\\ &\qquad \times \mathbf{P}\{Z_0((X \cap L) \llcorner \mathcal{H}_{(1+\beta)\varrho B_L}) \in \mathcal{P}_L | (X \cap L)(\mathcal{H}_{(1+\beta)\varrho B_L}) = N(L_j^*)\}.\end{aligned}$$

Using a fundamental property of Poisson processes (cf. [16], Theorem 3.2.2(b)), the relation $\mathbf{E}[(X \cap L)(\mathcal{H}_{(1+\beta)\varrho B_L})] = 2\gamma_{X\cap L} k(1+\beta)\varrho V_k(B_L)$, and the definition of the set $\mathcal{P}_L$, we obtain

$$\begin{aligned}
&\mathbf{P}\{V_k(Z_0 \cap L) \in a(1, 1+h)\} \\
&\quad \geq \frac{(2\gamma_{X\cap L})^{N(L_j^*)}}{N(L_j^*)!} \exp[-2k(1+\beta)\varrho V_k(B_L)\gamma_{X\cap L}] \\
&\qquad \times \int_{U^L_{N(L_j^*)}(L_j^*)} \cdots \int_{U_1^L(L_j^*)} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \\
&\qquad \mathbf{1}\{|t_i - \varrho t_i^0(L_j^*)| < \varrho\alpha(L_j^*), \text{for } i = 1, \dots, N(L_j^*), \text{ and} \\
&\qquad\quad V_k(P(\mathbf{u}_1, \dots, \mathbf{u}_{N(L_j^*)}; t_1, \dots, t_{N(L_j^*)})) \in V_k(\varrho B_L)(1, 1+h)\} \\
&\qquad \mathrm{d}t_1 \cdots \mathrm{d}t_{N(L_j^*)}\, S^L_{k-1}(B_L, \mathrm{d}\mathbf{u}_1) \cdots S^L_{k-1}(B_L, \mathrm{d}\mathbf{u}_{N(L_j^*)}),
\end{aligned}$$

and hence

$$\begin{aligned}
&\mathbf{P}\{V_k(Z_0 \cap L) \in a(1, 1+h)\} \\
&\quad \geq \frac{(2\gamma_{X\cap L})^{N(L_j^*)}}{N(L_j^*)!} \exp[-2(1+\beta)\gamma_{X\cap L}\tau_L a^{1/k}] \\
&\qquad \times c(\beta, \varphi)\varrho h(2\varrho\alpha(L_j^*))^{N(L_j^*)-1} \prod_{i=1}^{N(L_j^*)} S^L_{k-1}(B_L, U_i^L(L_j^*)).
\end{aligned}$$

Since $a \geq 1$, $V_k(B_L) \leq c_7^k \kappa_k$ and $\gamma_{X\cap L} \geq \gamma m(\varphi)$ [cf. (3.6)], we finally get

$$\begin{aligned}
&\mathbf{P}\{V_k(Z_0 \cap L) \in a(1, 1+h)\} \\
&\quad \geq \frac{(2a^{1/k}\gamma_{X\cap L} V_k(B_L)^{-1/k})^{N(L_j^*)}}{N(L_j^*)!} (2\alpha(L_j^*))^{N(L_j^*)-1} c(\beta, \varphi)\eta_0^{N(L_j^*)} \\
&\qquad \times h \exp[-2(1+\beta)\gamma_{X\cap L}\tau_L a^{1/k}] \\
&\quad \geq c_{10} h \exp[-2(1+\beta)\gamma_{X\cap L}\tau_L a^{1/k}],
\end{aligned}$$

which gives the required estimate. $\square$

LEMMA 4.2. *Let $0 < \varepsilon < 1$ and $h \in (0, 1/2)$. There are a constant $c_{11} > 0$, depending only on $m(\varphi), \gamma$, and ε, and a constant $c_{12} > 0$, depending only on $m(\varphi)$, such that, for $L \in G(d, k)$ and $a \geq 1$,*

$$\begin{aligned}
&\mathbf{P}\{\vartheta(Z_0 \cap L, B_L) \geq \varepsilon, V_k(Z_0 \cap L) \in a(1, 1+h)\} \\
&\quad \leq c_{11} h \exp[-2(1 + c_{12}\varepsilon^{k+1})\gamma_{X\cap L}\tau_L a^{1/k}].
\end{aligned}$$

PROOF. The assertion is obtained by applying Proposition 7.1 of [4] in a given subspace $L \in G(d,k)$, again to a stationary Poisson hyperplane process with intensity $\gamma_{X\cap L}$ and spherical directional distribution $\varphi_{X\cap L}$. The slightly different definition of the deviation measure r_B, as opposed to ϑ in the present paper, is inessential for the proof. Where a constant in [4] depends on B, it depends now on B_L. Whenever a constant in [4] depends on B, this dependence is via mixed volumes of B with specific convex bodies, or via the diameter of B, and the constant can, therefore, be estimated from the appropriate side by positive constants for which the dependence on B is only a dependence on the inradius and circumradius of B. Due to the universal bounds for the inradius and circumradius of B_L provided by Lemma 3.3, for the constants appearing in the application of [4] to L, the dependence on B_L is, in fact, a dependence on $m(\varphi)$ only. $\square$

5. Proof of Theorem 2.1. Let $L^* \in G(d,k)$ with $L^* \in \operatorname{supp} \mathbf{Q}_{d-k}$ be given. Let $N^* \subset G(d,k)$ be a neighbourhood of L^*. Then

$$\mathbf{P}\big\{V_k\big(Z_0^{(k)}\big) \geq a, D\big(Z_0^{(k)}\big) \in N^*\big\} > 0.$$

The positivity of this probability follows from (2.2) together with the facts that $\mathbf{Q}_{d-k}(N^*) > 0$ and that, for any $r > 0$,

$$\mathbf{P}\{r\mathbb{B}^d \subset Z_0\} = \mathbf{P}\{H \cap r\mathbb{B}^d = \varnothing \ \ \forall H \in X\} > 0.$$

Let $\varepsilon > 0$ and $a \geq 1$. We have

$$\begin{aligned} &\mathbf{P}\big\{\vartheta\big(Z_0^{(k)}, B_{L^*}\big) \geq \varepsilon | V_k\big(Z_0^{(k)}\big) \geq a, D\big(Z_0^{(k)}\big) \in N^*\big\} \\ &\quad = \frac{\mathbf{P}\{\vartheta(Z_0^{(k)}, B_{L^*}) \geq \varepsilon, V_k(Z_0^{(k)}) \geq a, D(Z_0^{(k)}) \in N^*\}}{\mathbf{P}\{V_k(Z_0^{(k)}) \geq a, D(Z_0^{(k)}) \in N^*\}}. \end{aligned} \tag{5.1}$$

In order to estimate this ratio, we derive an estimate from above for the numerator and an estimate from below for the denominator. As in [4], we first consider the condition $V_k(Z_0^{(k)}) \in a(1, 1+h)$ for $h > 0$, instead of $V_k(Z_0^{(k)}) \geq a$.

For the estimate of the numerator of (5.1), we use (2.2) to get

$$\begin{aligned} &\mathbf{P}\big\{\vartheta\big(Z_0^{(k)}, B_{L^*}\big) \geq \varepsilon, V_k\big(Z_0^{(k)}\big) \in a(1, 1+h), D\big(Z_0^{(k)}\big) \in N^*\big\} \\ &\quad = \int_{G(d,k)} \mathbf{P}\big\{\vartheta(Z_0 \cap L, B_{L^*}) \geq \varepsilon, V_k(Z_0 \cap L) \in a(1, 1+h), \\ &\qquad\qquad\qquad\qquad D(Z_0 \cap L) \in N^*\big\} \mathbf{Q}_{d-k}(\mathrm{d}L) \\ &\quad = \int_{N^*} \mathbf{P}\{\vartheta(Z_0 \cap L, B_{L^*}) \geq \varepsilon, V_k(Z_0 \cap L) \in a(1, 1+h)\} \mathbf{Q}_{d-k}(\mathrm{d}L). \end{aligned}$$

In contrast to the case of the zero cell Z_0 treated in [4], we are here faced with the problem that the random polytope $Z_0 \cap L$, for variable L, must be compared with the fixed Blaschke body B_{L^*}. This explains the necessity of restricting the direction space $D(Z_0^{(k)})$ to a neighbourhood of L^* and of establishing the stability result Lemma 3.4, which allows us the estimate (5.3) and finally (5.4). A similar remark concerns the estimation of the denominator.

We choose numbers $1/2 \le p < 1$ and $q > 1$, depending only on ε and the number c_{12} from Lemma 4.2 (but with ε replaced by $\varepsilon/2$), such that

$$\frac{q}{p} < 1 + \frac{c_{13}}{2}\varepsilon^{k+1} \tag{5.2}$$

with $c_{13} := c_{12}/2^{k+1}$. Then we choose a number $\theta > 0$ satisfying the conditions

$$\theta \le \min\left\{\frac{1}{8}, \left(\frac{\varepsilon}{2c_9}\right)^{3k}\right\},$$

where c_9 is the constant from Lemma 3.4, and

$$p\gamma_{X\cap L^*}\tau_{L^*} \le \gamma_{X\cap L}\tau_L \le q\gamma_{X\cap L^*}\tau_{L^*} \qquad \text{if } \Delta(L, L^*) \le \theta. \tag{5.3}$$

The latter is possible by (3.5) and Lemma 3.4, since $\tau_L = kV_k(B_L)^{1-1/k}$.

If $L \in G(d,k)$ and $\Delta(L, L^*) \le \theta$, then every convex body $K \in \mathcal{K}_0(L)$ with $\vartheta(K, B_L) < \varepsilon/2$ satisfies $\vartheta(K, B_{L^*}) < \varepsilon$, by Lemma 3.5. Now we choose for N^* the neighbourhood $N_\theta := N_\theta(L^*)$. Then

$$L \in N_\theta \quad \text{and} \quad \vartheta(Z_0 \cap L, B_{L^*}) \ge \varepsilon \quad \text{implies} \quad \vartheta(Z_0 \cap L, B_L) \ge \varepsilon/2.$$

This gives

$$\begin{aligned}&\mathbf{P}\{\vartheta(Z_0^{(k)}, B_{L^*}) \ge \varepsilon, V_k(Z_0^{(k)}) \in a(1, 1+h), D(Z_0^{(k)}) \in N_\theta\}\\ &\quad \le \int_{N_\theta} \mathbf{P}\{\vartheta(Z_0 \cap L, B_L) \ge \varepsilon/2, V_k(Z_0 \cap L) \in a(1, 1+h)\}\mathbf{Q}_{d-k}(\mathrm{d}L).\end{aligned}$$

Let $h \in (0, 1/2)$. By Lemma 4.2 (with ε replaced by $\varepsilon/2$),

$$\begin{aligned}&\mathbf{P}\{\vartheta(Z_0 \cap L, B_L) \ge \varepsilon/2, V_k(Z_0 \cap L) \in a(1, 1+h)\}\\ &\quad \le c_{14}h\exp[-2(1 + c_{13}\varepsilon^{k+1})\gamma_{X\cap L}\tau_L a^{1/k}]\end{aligned}$$

with a constant c_{14} depending only on $\varphi, \gamma, \varepsilon$; here c_{13} (defined above) depends only on φ.

By (5.3), we can conclude that

$$\begin{aligned}&\mathbf{P}\{\vartheta(Z_0^{(k)}, B_{L^*}) \ge \varepsilon, V_k(Z_0^{(k)}) \in a(1, 1+h), D(Z_0^{(k)}) \in N_\theta\}\\ &\quad \le \mathbf{Q}_{d-k}(N_\theta)c_{14}h\exp[-2(1 + c_{13}\varepsilon^{k+1})p\gamma_{X\cap L^*}\tau_{L^*}a^{1/k}].\end{aligned} \tag{5.4}$$

Now the argument in [4], pages 1164–1165 (Case 2), leads from (5.4) to the estimate

$$\begin{aligned}&\mathbf{P}\{\vartheta(Z_0^{(k)}, B_{L^*}) \ge \varepsilon, V_k(Z_0^{(k)}) \ge a, D(Z_0^{(k)}) \in N_\theta\}\\ &\quad \le c_{15}\mathbf{Q}_{d-k}(N_\theta)h\exp\left[-2\left(1 + \frac{c_{13}}{2}\varepsilon^{k+1}\right)p\gamma_{X\cap L^*}\tau_{L^*}a^{1/k}\right]\\ &\qquad \times \exp\left[-\frac{c_{13}}{2}\varepsilon^{k+1}p\gamma_{X\cap L^*}\tau_{L^*}a^{1/k}\right],\end{aligned} \tag{5.5}$$

where c_{15} is a positive constant depending only on $\varphi, \gamma, \varepsilon$. Here, we use that $L \mapsto \gamma_{X\cap L}$ and $L \mapsto \tau_L$ are continuous and can be estimated from below by a positive constant independent of L.

For the denominator of (5.1), we obtain similarly

$$\mathbf{P}\{V_k(Z_0^{(k)}) \in a(1, 1+h), D(Z_0^{(k)}) \in N_\theta\}$$
$$= \int_{N_\theta} \mathbf{P}\{V_k(Z_0 \cap L) \in a(1, 1+h)\}\mathbf{Q}_{d-k}(\mathrm{d}L).$$

We define the number β, depending only on φ and ε, by

$$\left(1 + \frac{c_{13}}{2}\varepsilon^{k+1}\right)p = (1+\beta)q. \tag{5.6}$$

It follows from (5.2) that $\beta > 0$. By Lemma 4.1, there are constants c_{10}, $0 < h_0 < 1/2$, depending only on φ, γ and ε, such that, for $L \in G(d, k)$, $a \geq 1$ and $0 < h \leq h_0$,

$$\mathbf{P}\{V_k(Z_0 \cap L) \in a(1, 1+h)\} \geq c_{10}h \exp[-2(1+\beta)\gamma_{X\cap L}\tau_L a^{1/k}].$$

Using (5.3) for $L \in N_\theta$, we deduce that

$$\mathbf{P}\{V_k(Z_0^{(k)}) \in a(1, 1+h), D(Z_0^{(k)}) \in N_\theta\}$$
$$\geq \mathbf{Q}_{d-k}(N_\theta)c_{10}h \exp[-2(1+\beta)q\gamma_{X\cap L^*}\tau_{L^*}a^{1/k}].$$

With β given by (5.6), this yields

$$\mathbf{P}\{V_k(Z_0^{(k)}) \geq a, D(Z_0^{(k)}) \in N_\theta\}$$
$$\geq c_{10}\mathbf{Q}_{d-k}(N_\theta)h \exp\left[-2\left(1 + \frac{c_{13}}{2}\varepsilon^{k+1}\right)p\gamma_{X\cap L^*}\tau_{L^*}a^{1/k}\right].$$

Here and in (5.5), we choose the same number $h \in (0, h_0]$. Then division gives the assertion of Theorem 2.1, since $p \geq 1/2$ and we can estimate $\gamma_{X\cap L^*}\tau_{L^*}$ from below by a constant depending only on φ and γ.

6. Proof of Theorem 2.2. The proof of Theorem 2.2 is based on (2.1), which is applied with different functions f, and on the relation

$$\mathbf{E}V_k(Z^{(k)}) = \frac{d_k^{(k)}}{\gamma^{(k)}} = \frac{V_{d-k}(\Pi_X)}{\binom{d}{k} V_d(\Pi_X)} =: c_{16},$$

which follows from [16], equation (10.3) and Theorem 10.3.3, with c_{16} depending only on φ and γ.

We use definitions and results from the preceding proof of Theorem 2.1. In particular, β is defined by (5.6). Then there are positive constants c_{17}, θ_1 and $h_1 <$

1/2, depending only on φ, γ and ε, such that, for $a \geq 1$, $0 < \theta \leq \theta_1$ and $0 < h \leq h_1$,

$$\mathbf{P}\{V_k(Z_0^{(k)}) \in a(1, 1+h), D(Z_0^{(k)}) \in N_\theta\} \\ \geq \mathbf{Q}_{d-k}(N_\theta) c_{17} h \exp\left[-2\left(1 + \frac{\beta}{2}\right) q \gamma_{X \cap L^*} \tau_{L^*} a^{1/k}\right].$$

For a polytope $K \subset \mathbb{R}^d$, we now define

$$f(K) := \mathbf{1}\{V_k(K) \in a(1, 1+h), D(K) \in N_\theta\} V_k(K)^{-1},$$

if K is k-dimensional, and $f(K) := 0$ otherwise. Clearly, f is translation invariant, and for $a \geq 1$ and $0 < h \leq h_1$, (2.1) gives

$$\begin{aligned} &\mathbf{P}\{V_k(Z^{(k)}) \in a(1, 1+h), D(Z^{(k)}) \in N_\theta\} \\ &\quad = \mathbf{E} V_k(Z^{(k)}) \mathbf{E}[\mathbf{1}\{V_k(Z_0^{(k)}) \in a(1, 1+h), D(Z_0^{(k)}) \in N_\theta\} V_k(Z_0^{(k)})^{-1}] \\ &\quad \geq c_{16} \frac{1}{1+h_1} \frac{1}{a} \mathbf{P}\{V_k(Z_0^{(k)}) \in a(1, 1+h), D(Z_0^{(k)}) \in N_\theta\} \\ &\quad \geq c_{18} \mathbf{Q}_{d-k}(N_\theta) \frac{1}{a} h \exp\left[-2\left(1 + \frac{\beta}{2}\right) q \gamma_{X \cap L^*} \tau_{L^*} a^{1/k}\right] \\ &\quad \geq c_{19} \mathbf{Q}_{d-k}(N_\theta) h \exp[-2(1+\beta) q \gamma_{X \cap L^*} \tau_{L^*} a^{1/k}], \end{aligned}$$

since $\gamma_{X \cap L^*} \tau_{L^*} \geq c_{20} > 0$. Here, c_{18} and c_{19} depend only on $\varphi, \gamma, \varepsilon$, and c_{20} depends only on φ, γ. In particular, recalling the definition of β from (5.6),

$$\begin{aligned} &\mathbf{P}\{V_k(Z^{(k)}) \geq a, D(Z^{(k)}) \in N_\theta\} \\ &\quad \geq c_{19} \mathbf{Q}_{d-k}(N_\theta) h_1 \exp\left[-2\left(1 + \frac{c_{13}}{2} \varepsilon^{k+1}\right) p \gamma_{X \cap L^*} \tau_{L^*} a^{1/k}\right]. \end{aligned} \tag{6.1}$$

For the upper bound, we put

$$f(K) := \mathbf{1}\{\vartheta(K, B_{L^*}) \geq \varepsilon, V_k(K) \geq a, D(K) \in N_\theta\} V_k(K)^{-1},$$

if K is a k-dimensional polytope, and $f(K) := 0$ otherwise, where $0 < \theta \leq \theta_1$, with θ_1 sufficiently small, and $a \geq 1$. Using again (2.1), we obtain

$$\begin{aligned} &\mathbf{P}\{\vartheta(Z^{(k)}, B_{L^*}) \geq \varepsilon, V_k(Z^{(k)}) \geq a, D(Z^{(k)}) \in N_\theta\} \\ &\quad = c_{16} \mathbf{E}[\mathbf{1}\{\vartheta(Z_0^{(k)}, B_{L^*}) \geq \varepsilon, V_k(Z_0^{(k)}) \geq a, D(Z_0^{(k)}) \in N_\theta\} V_k(Z_0^{(k)})^{-1}] \\ &\quad \leq c_{21} \mathbf{Q}_{d-k}(N_\theta) h_1 \exp\left[-2\left(1 + \frac{c_{13}}{2} \varepsilon^{k+1}\right) p \gamma_{X \cap L^*} \tau_{L^*} a^{1/k}\right] \\ &\qquad \times \exp\left[-\frac{c_{13}}{2} \varepsilon^{k+1} p \gamma_{X \cap L^*} \tau_{L^*} a^{1/k}\right], \end{aligned} \tag{6.2}$$

where (5.5) was used in the last estimate and c_{21} depends only on $\varphi, \gamma, \varepsilon$.

From (6.1) and (6.2), we conclude that

$$\begin{aligned}
&\mathbf{P}\{\vartheta(Z^{(k)}, B_{L^*}) \geq \varepsilon | V_k(Z^{(k)}) \geq a, D(Z^{(k)}) \in N_\theta\} \\
&\quad \leq c_{22} \exp\left[-\frac{c_{13}}{2}\varepsilon^{k+1} p\gamma_{X\cap L^*}\tau_{L^*} a^{1/k}\right] \\
&\quad \leq c_{22} \exp[-c_{23}\varepsilon^{k+1} a^{1/k}],
\end{aligned}$$

where c_{22} depends only on $\varphi, \gamma, \varepsilon$ and c_{23} depends only on φ and γ.

7. Limit shapes. Similarly as in [9], Section 4, but with an additional limit procedure referring to direction spaces, we can establish the existence of limit shapes.

For a convex body $K \subset \mathbb{R}^d$, we denote by $s_{\mathsf{H}}(K)$ the equivalence class of all convex bodies homothetic to K; this is the (homothetic) *shape* of K. Let $\mathcal{S}_{\mathsf{H}}$ denote the space of all shapes, equipped with the quotient topology.

Let the assumptions of Theorem 2.2 be satisfied; in particular, $L^* \in G(d,k)$ is contained in the support of the measure $\mathbf{Q}_{d-k}$.

The *conditional law of the shape of* $Z_0^{(k)}$, given the lower bound a for its k-volume and the upper bound θ for the distance of its direction space from L^*, is defined by

$$\mu_{a,\theta}(A) := \mathbf{P}\{s_{\mathsf{H}}(Z_0^{(k)}) \in A | V_k(Z_0^{(k)}) \geq a, \Delta(D(Z_0^{(k)}), L^*) < \theta\}$$

for $A \in \mathcal{B}(\mathcal{S}_{\mathsf{H}})$.

THEOREM 7.1. *The shape $s_{\mathsf{H}}(B_{L^*})$ is the limit shape of the weighted typical cell $Z_0^{(k)}$ with respect to V_k and $\Delta(D(\cdot), L^*)$, in the sense that*

$$\lim_{\substack{a\to\infty \\ \theta\to 0}} \mu_{a,\theta} = \delta_{s_{\mathsf{H}}(B_{L^*})} \qquad \textit{weakly},$$

where $\delta_{s_{\mathsf{H}}(B_{L^*})}$ denotes the Dirac measure concentrated at $s_{\mathsf{H}}(B_{L^*})$.

PROOF. Let $\mathcal{C} \subset \mathcal{S}_{\mathsf{H}}$ be closed. It suffices to show that

$$\limsup_{\substack{a\to\infty \\ \theta\to 0}} \mu_{a,\theta}(\mathcal{C}) \leq \delta_{s_{\mathsf{H}}(B_{L^*})}(\mathcal{C}). \tag{7.1}$$

We assume that $s_{\mathsf{H}}(B_{L^*}) \notin \mathcal{C}$ and that $\mathcal{C}$ contains the shape of at least one k-dimensional body, since otherwise (7.1) holds trivially. For $K \in \mathcal{K}$ with $\dim K = k$, we put $f(K) := \vartheta(K, B_{L^*}) + \Delta(D(K), L^*)$. Let

$$\begin{aligned}
\mathcal{K}^* &:= \{K \in \mathcal{K} : \dim K = k, s_{\mathsf{H}}(K) \in \mathcal{C}, B_{D(K)} \subset K\}, \\
\alpha &:= \inf_{K\in\mathcal{K}^*} f(K),
\end{aligned}$$

and choose $c > \alpha$. There exists $R > 0$ such that every $K \in \mathcal{K}^*$ with $f(K) \leq c$ has a homothetic copy that is contained in $R\mathbb{B}^d$. Hence, if we put

$$\mathcal{K}_c^* := \{K \in \mathcal{K}^* : f(K) \leq c, K \subset R\mathbb{B}^d\},$$

then $\alpha = \inf_{K \in \mathcal{K}_c^*} f(K)$. The function f is continuous and the set $\mathcal{K}_c^*$ is compact (note that the condition $B_{D(K)} \subset K$ in the definition of $\mathcal{K}^*$ ensures that limits of bodies in $\mathcal{K}^*$ still have dimension k). Therefore, the infimum α is attained, say at K_0. If $\alpha = 0$, then K_0 is homothetic to B_{L^*}, hence $s_{\mathsf{H}}(B_{L^*}) = s_{\mathsf{H}}(K_0) \in \mathcal{C}$, a contradiction. It follows that $\alpha > 0$.

Put $\varepsilon := \alpha/2$. To this ε, we can choose constants c_1, c_2, c_3 according to Theorem 2.1, such that

$$\begin{aligned}&\mathbf{P}\{\vartheta(Z_0^{(k)}, B_{L^*}) \geq \varepsilon | V_k(Z_0^{(k)}) \geq a, D(Z_0^{(k)}) \in N_\theta(L^*)\}\\ &\qquad \leq c_2 \exp[-c_3 \varepsilon^{k+1} a^{1/k}]\end{aligned}$$

for $a \geq 1$ and $0 < \theta \leq c_1$.

Every k-dimensional convex body $K \in s_{\mathsf{H}}^{-1}(\mathcal{C})$ with $\Delta(D(K), L^*) \leq \alpha/2$ satisfies $\vartheta(K, B_{L^*}) \geq \varepsilon$. Hence, for $0 < \theta \leq \min\{c_1, \alpha/2\}$ we have

$$\begin{aligned}\mu_{a,\theta}(\mathcal{C}) &= \mathbf{P}\{s_{\mathsf{H}}(Z_0^{(k)}) \in \mathcal{C} | V_k(Z_0^{(k)}) \geq a, D(Z_0^{(k)}) \in N_\theta(L^*)\}\\ &\leq \mathbf{P}\{\vartheta(Z_0^{(k)}, B_{L^*}) \geq \varepsilon | V_k(Z_0^{(k)}) \geq a, D(Z_0^{(k)}) \in N_\theta(L^*)\}\\ &\leq c_2 \exp[-c_3 \varepsilon^{k+1} a^{1/k}].\end{aligned}$$

For $a \to \infty$ this tends to zero, hence (7.1) follows. □

Theorem 2.2 yields a completely analogous result for the typical cell.

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