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MARKOV LOOPS AND RENORMALIZATION

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We study the Poissonnian ensembles of Markov loops and the associated renormalized self intersection local times.

1. Introduction. The purpose of this paper is to explore some simple relations between Markovian path and loop measures, spanning trees, determinants and Markov fields such as the free field. The main emphasis is put on the study of occupation fields defined by Poissonian ensembles of Markov loops. These were defined in [9] for planar Brownian motion in relation with SLE processes and in [10] for simple random walks. They appeared informally already in [22]. For half integral values $\frac{k}{2}$ of the intensity parameter α, these occupation fields can be identified with the sum of squares of k copies of the associated free field (i.e., the Gaussian field whose covariance is given by the Green function). This fact is related to Dynkin's isomorphism (cf. [2, 13, 16]). We first present the results in the elementary framework of symmetric Markov chains on a finite space, proving also in passing several interesting results such as the relation between loop ensembles and spanning trees. Then we show that some results can be extended to more general Markov processes. There are no essential difficulties when points are not polar but other cases are more problematic. As for the square of the free field, cases for which the Green function is Hilbert Schmidt such as those corresponding to two and three dimensional Brownian motion can be dealt with through appropriate renormalization. In particular, we can show that the renormalized powers of the occupation field (i.e., the self intersection local times of the loop ensemble) converge in the two dimensional case and that they can be identified with higher even Wick powers of the free field when α is a half integer.

2. Symmetric Markov processes on finite spaces. Notation: functions and measures on finite (or countable) spaces are often denoted as vectors and covectors. The multiplication operator defined by a function f acting on functions or on measures is in general simply denoted by f, but sometimes it will be denoted M_f. The function obtained as the density of a measure μ with respect to some other measure ν is simply denoted $\frac{\mu}{\nu}$.

Our basic object will be a finite space X and a set of nonnegative conductances $C_{x,y} = C_{y,x}$, indexed by pairs of distinct points of X.

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We say $\{x, y\}$ is a link or an edge iff $C_{x,y} > 0$ and an oriented edge (x, y) is defined by the choice of an ordering in an edge. We set $-(x, y) = (y, x)$ and if $e = (x, y)$, we denote it also (e^-, e^+).

The points of X together with the set of nonoriented edges E define a graph (X, E). *We assume it is connected*. The set of oriented edges is denoted E^o.

An important example is the case in which conductances are equal to zero or one. Then the conductance matrix is the adjacency matrix of the graph: $C_{x,y} = 1_{\{x,y\}\in E}$.

2.1. *Energy.* Let us consider a nonnegative function κ on X. Set $\lambda_x = \kappa_x + \sum_y C_{x,y} P^x_y = \frac{C_{x,y}}{\lambda_x}$. P is a λ-symmetric (sub) stochastic transition matrix: $\lambda_x P^x_y = \lambda_y P^y_x$ with $P^x_x = 0$ for all x in X. It defines a symmetric irreducible Markov chain ξ_n.

We can define above it a continuous time λ-symmetric irreducible Markov chain x_t, with exponential holding times, of parameter 1. We have $x_t = \xi_{N_t}$, where N_t denotes a Poisson process of intensity 1. The infinitesimal generator is given by $L^x_y = P^x_y - \delta^x_y$.

We denote by P_t its (sub) Markovian semigroup $\exp(Lt) = \sum \frac{t^k}{k!} L^k$. L and P_t are λ-symmetric.

We will consider the Markov chain associated with C, κ, sometimes in discrete time, sometimes in continuous time (with exponential holding times).

Recall that for any complex function $z^x, x \in X$, the "energy"

$$e(z) = \langle -Lz, \overline{z}\rangle_\lambda = \sum_{x\in X} -(Lz)^x \overline{z}^x \lambda_x$$

is nonnegative as it can be written

$$e(z) = \frac{1}{2}\sum_{x,y} C_{x,y}(z^x - z^y)(\overline{z}^x - \overline{z}^y) + \sum_x \kappa_x z^x \overline{z}^x = \sum_x \lambda_x z^x \overline{z}^x - \sum_{x,y} C_{x,y} z^x \overline{z}^y.$$

The Dirichlet space [4] is the space of real functions equipped with the energy scalar product defined by polarization of e.

Note that the nonnegative symmetric "conductance matrix" C and the nonnegative equilibrium or "killing" (or "equilibrium") measure κ are the free parameters of the model.

We have a dichotomy between:

– The recurrent case where 0 is the lowest eigenvalue of $-L$, and the corresponding eigenspace is formed by constants. Equivalently (since X is assumed to be finite), $P1 = 1$ and κ vanishes.
– The transient case where the lowest eigenvalue is positive which means there is a "Poincaré inequality": For some positive ε, the energy $e(f, f)$ dominates $\varepsilon\langle f, f\rangle_\lambda$ for all f. Equivalently, κ does not vanish everywhere.

We will now work in the transient case. We denote by V the associated potential operator $(-L)^{-1} = \int_0^\infty P_t\,dt$. It can be expressed in terms of the spectral resolution of L.

We denote by G the Green function defined on X^2 as $G^{x,y} = \frac{V_y^x}{\lambda_y} = \frac{1}{\lambda_y}[(I - P)^{-1}]_y^x$, that is, $G = (M_\lambda - C)^{-1}$. It induces a linear bijection from measures into functions. We set $(G\mu)^x = \sum_y G^{x,y}\mu_y$.

Note that $e(f, G\mu) = \langle f, \mu\rangle$ (i.e., $\sum_x f^x \mu_x$) for all functions f and measures μ. In particular, $G\kappa = 1$ as $e(1, f) = \sum f^x \kappa_x = \langle f, 1\rangle_\kappa$.

See [4] for a development of this theory in a more general setting.

In the recurrent case, the potential operator V operates on the space $\lambda^\perp$ of functions f such that $\langle f, 1\rangle_\lambda = 0$ as the inverse of the restriction of $I - P$ to $\lambda^\perp$. The Green operator G maps the space of measures of total charge zero onto $\lambda^\perp$. Setting for any signed measure ν of total charge zero $G\nu = V\frac{\nu}{\lambda}$, we have for any function f, $\langle \nu, f\rangle = e(G\nu, f)$ [as $e(G\nu, 1) = 0$] and in particular $f^x - f^y = e(G(\delta_x - \delta_y), f)$.

2.2. *Feynman–Kac formula*. For the continuous time Markov chain x_t (with exponential holding times) and $k(x)$ any nonnegative function, we have the Feynman–Kac formula:

$$\mathbb{E}_x\big(e^{-\int_0^t k(x_s)\,ds} 1_{\{x_t = y\}}\big) = \big[\exp\big(t(L - M_k)\big)\big]_y^x.$$

For any nonnegative measure χ, set $V_\chi = (-L + M_{\chi/\lambda})^{-1}$ and

$$G_\chi = V_\chi M_{1/\lambda} = (M_\lambda + M_\chi - C)^{-1}.$$

It is a symmetric nonnegative function on $X \times X$. G_0 is the Green function G, and G_χ can be viewed as the Green function of the energy form $e_\chi = e + \|\cdot\|^2_{L^2(\chi)}$.

Note that e_χ has the same conductances C as e, but χ is added to the killing measure. Note also that V_χ is not the potential of the Markov chain associated with e_χ when one takes exponential holding times of parameter 1 but the Green function is intrinsic, that is, invariant under a change of time scale. Still, we have by the Feynman–Kac formula

$$\int_0^\infty \mathbb{E}_x\big(e^{-\int_0^t (\chi/\lambda)(x_s)\,ds} 1_{\{x_t = y\}}\big)\,dt = [V_\chi]_y^x.$$

We have also the "resolvent" equation $V - V_\chi = V M_{\chi/\lambda} V_\chi = V_\chi M_{\chi/\lambda} V$. Then

$$G - G_\chi = G M_\chi G_\chi = G_\chi M_\chi G.$$

2.3. *Countable spaces*. The assumption of finiteness of X can be relaxed. On countable spaces, the previous results extend easily under spectral gap conditions. In the transient case, the Dirichlet space $\mathbb{H}$ is the space of all functions f with finite energy $e(f)$ which are limits in energy norm of functions with finite support. The energy of a measure is defined as $\sup_{f\in\mathbb{H}} \frac{\mu(f)^2}{e(f)}$. Finite energy measures include Dirac measures. The potential $G\mu$ is well defined in $\mathbb{H}$ for all finite energy measures μ, by the identity $e(f, G\mu) = \langle f, \mu\rangle$, valid for all f in the Dirichlet space.

Among the many well-known examples of such infinite graphs, note that many important cases can be obtained as a nonramified covering of some finite graph.

3. Loop measures.

3.1. *A measure on based loops.* We denote by $\mathbb{P}_x$ the family of probability laws on piecewise constant paths defined by P_t:

$$\mathbb{P}_x\big(\gamma(t_1) = x_1, \dots, \gamma(t_h) = x_h\big) = P_{t_1}(x, x_1)P_{t_2-t_1}(x_1, x_2) \cdots P_{t_h-t_{h-1}}(x_{h-1}, x_h).$$

Denoting by $p(\gamma)$ the number of jumps and T_i the jump times, we have

$$\mathbb{P}_x\big(p(\gamma) = k, \gamma_{T_1} = x_1, \dots, \gamma_{T_{k-1}} = x_{k-1}, T_1 \in dt_1, \dots, T_k \in dt_k\big)$$
$$= \frac{C_{x,x_1} \cdots C_{x_{k-1},x_k}\kappa_{x_k}}{\lambda_x\lambda_{x_1} \cdots \lambda_{x_k}} 1_{\{0<t_1<\cdots<t_k\}} e^{-t_k}\, dt_1 \cdots dt_k.$$

For any integer $p \geq 2$, let us define a based loop with p points in X as a couple $l = (\xi, \tau) = ((\xi_m, 1 \leq m \leq p), (\tau_m, 1 \leq m \leq p+1))$ in $X^p \times \mathbb{R}_+^{p+1}$, and set $\xi_1 = \xi_{p+1}$ (equivalently, we can parametrize the discrete based loop by $\mathbb{Z}/p\mathbb{Z}$). The integer p represents the number of points in the discrete based loop $\xi = (\xi_1, \dots, \xi_{p(\xi)})$ and will be denoted $p(\xi)$. Note two time parameters are attached to the base point since the based loops do not in general end or start with a jump.

Based loops with one point ($p = 1$) are simply given by a pair (ξ, τ) in $X \times \mathbb{R}_+$.

Based loops have a natural time parametrization $l(t)$ and a time period $T(\xi) = \sum_{i=1}^{p(\xi)+1} \tau_i$. If we denote $\sum_{i=1}^{m} \tau_i$ by T_m: $l(t) = \xi_{m-1}$ on $[T_{m-1}, T_m)$ (with by convention $T_0 = 0$ and $\xi_0 = \xi_p$).

A σ-finite measure μ is defined on based loops by

$$\mu = \sum_{x \in X} \int_0^\infty \frac{1}{t} \mathbb{P}_t^{x,x} \lambda_x\, dt,$$

where $\mathbb{P}_t^{x,y}$ denotes the (nonnormalized) "law" of a path from x to y of duration t: if $t_1 < t_2 < \cdots < t_h < t$,

$$\mathbb{P}_t^{x,y}\big(l(t_1) = x_1, \dots, l(t_h) = x_h\big) = [P_{t_1}]_{x_1}^{x} [P_{t_2-t_1}]_{x_2}^{x_1} \cdots [P_{t-t_h}]_{y}^{x_h} \frac{1}{\lambda_y}.$$

Its mass is $p_t^{x,y} = \frac{[P_t]_y^x}{\lambda_y}$. For any measurable set A of piecewise constant paths indexed by $[0, t]$, we can also write

$$\mathbb{P}_t^{x,y}(A) = \mathbb{P}_x(A \cap \{x_t = y\}) \frac{1}{\lambda_y}.$$

From the first expression, we see that by definition of μ, if $t_1 < t_2 < \cdots < t_h < t$,

$$\mu\big(l(t_1) = x_1, \dots, l(t_h) = x_h, T \in dt\big) \\ = [P_{t_1+t-t_h}]_{x_1}^{x_h} [P_{t_2-t_1}]_{x_2}^{x_1} \cdots [P_{t_h-t_{h-1}}]_{x_h}^{x_{h-1}} \frac{1}{t}\, dt. \tag{1}$$

Note also that for $k > 1$, using the second expression of $\mathbb{P}_t^{x,x}$ and the fact that conditionally on $N_t = k$, the jump times are distributed like an increasingly reordered k-uniform sample of $[0, t]$

$$\lambda_x \mathbb{P}_t^{x,x}(p = k, \xi_2 = x_2, \dots, \xi_k = x_k, T_1 \in dt_1, \dots, T_k \in dt_k) \\ = P_{x_2}^{x} P_{x_3}^{x_2} \cdots P_{x}^{x_k} 1_{\{0<t_1<\cdots<t_k<t\}} e^{-t}\, dt_1 \cdots dt_k.$$

Therefore,

$$\mu(p = k, \xi_1 = x_1, \dots, \xi_k = x_k, T_1 \in dt_1, \dots, T_k \in dt_k, T \in dt) \\ = P_{x_2}^{x_1} \cdots P_{x_1}^{x_k} \frac{1_{\{0<t_1<\cdots<t_k<t\}}}{t} e^{-t}\, dt_1 \cdots dt_k\, dt \tag{2}$$

for $k > 1$.

Moreover, for loops reduced to one point, $\mu\{p(\xi) = 1, \xi_1 = x_1, \tau_1 \in dt\} = \frac{e^{-t}}{t}\, dt$.

3.2. *First properties*. Note that the loop measure is invariant under time reversal.

If D is a subset of X, the restriction of μ to loops contained in D, denoted μ^D is clearly the loop measure induced by the Markov chain killed at the exit of D. This can be called the *restriction property*.

Let us recall that this killed Markov chain is defined by the restriction of λ to D and the restriction P^D of P to D^2 (or equivalently by the restriction e_D of the Dirichlet norm e to functions vanishing outside D).

As $\int \frac{t^{k-1}}{k!} e^{-t}\, dt = \frac{1}{k}$, it follows from (2) that for $k > 1$, on based loops,

$$\mu\big(p(\xi) = k, \xi_1 = x_1, \dots, \xi_k = x_k\big) = \frac{1}{k} P_{x_2}^{x_1} \cdots P_{x_1}^{x_k}. \tag{3}$$

In particular, we obtain that, for $k \geq 2$,

$$\mu(p = k) = \frac{1}{k} \operatorname{Tr}(P^k)$$

and therefore, as $\operatorname{Tr}(P) = 0$,

$$\mu(p > 1) = \sum_2^{\infty} \frac{1}{k} \operatorname{Tr}(P^k) = -\log\big(\det(I - P)\big) = \log\Big(\det(G) \prod_x \lambda_x\Big)$$

since (denoting M_λ the diagonal matrix with entries λ_x), we have

$$\det(I-P)=\frac{\det(M_\lambda-C)}{\det(M_\lambda)}.$$

Moreover,

$$\int p(l)1_{\{p>1\}}\mu(dl)=\sum_2^\infty \mathrm{Tr}(P^k)=\mathrm{Tr}\big((I-P)^{-1}P\big)=\mathrm{Tr}(GC).$$

3.3. *Loops and pointed loops.* It is clear on formula (1) that μ is invariant under the time shift that acts naturally on based loops.

A loop is defined as an equivalence class of based loops for this shift. Therefore, μ induces a *measure on loops also denoted by* μ.

A loop is defined by the discrete loop ξ° formed by the ξ_i in circular order (i.e., up to translation) and the associated scaled holding times. We clearly have

$$\mu\big(\xi^\circ=(x_1,x_2,\dots,x_k)^\circ\big)=P_{x_2}^{x_1}\cdots P_{x_1}^{x_k}.$$

However, loops are not easy to parametrize, that is why we will work mostly with based loops or *pointed loops*. These are defined as based loops ending with a jump, or as loops with a starting point. They can be parametrized by a based discrete loop and by the holding times at each point. Calculations are easier if we work with based or pointed loops, even though we will deal only with functions independent of the base point.

The parameters of the pointed loop naturally associated with a based loop are $\xi_1,\dots,\xi_p$ and

$$\tau_1+\tau_{p+1}=\tau_1^*,\qquad \tau_i=\tau_i^*,\qquad 2\le i\le p.$$

An elementary change of variables, shows the expression of μ on pointed loops and can be written

$$\mu(p=k,\xi_i=x_i,\tau_i^*\in dt_i)=P_{x_2}^{x_1}\cdots P_{x_1}^{x_k}\frac{t_1}{\sum t_i}e^{-\sum t_i}\,dt_1\cdots dt_k. \tag{4}$$

Trivial ($p=1$) pointed loops and trivial based loops coincide.

Note that loop functionals can be written

$$\Phi(l^\circ)=\sum 1_{\{p=k\}}\Phi_k\big((\xi_i,\tau_i^*),i=1,\dots,k\big)$$

with Φ_k invariant under circular permutation of the variables (ξ_i,τ_i^*).

Then, for nonnegative Φ_k,

$$\int \Phi_k(l^\circ)\mu(dl)=\int\sum_i \Phi_k(x_i,t_i)P_{x_2}^{x_1}\cdots P_{x_1}^{x_k}e^{-\sum t_i}\frac{t_1}{\sum t_i}\,dt_1\cdots dt_k$$

and by invariance under circular permutation, the term t_1 can be replaced by any t_i. Therefore, adding up and dividing by k, we get that

$$\int \Phi_k(l^\circ)\mu(dl) = \int \frac{1}{k}\sum_i \Phi_k(x_i, t_i) P^{x_1}_{x_2} \cdots P^{x_k}_{x_1} e^{-\sum t_i}\, dt_1 \cdots dt_k.$$

The expression on the right-hand side, applied to any pointed loop functional defines a different measure on pointed loops, we will denote by μ^*. It induces the same measure as μ on loops. We see from this expression that conditionally on the discrete loop, the holding times of the loop are independent exponential variables:

$$\mu^*(p = k, \xi_i = x_i, \tau_i^* \in dt_i) = \frac{1}{k} \prod_{i \in \mathbb{Z}/p\mathbb{Z}} P^{\xi_i}_{\xi_{i+1}} e^{-t_i}\, dt_i. \tag{5}$$

Conditionally on $p(\xi) = k$, T is a gamma variable of density $\frac{t^{k-1}}{(k-1)!}e^{-t}$ on $\mathbb{R}_+$ and $(\frac{\tau_i^*}{T}, 1 \le i \le k)$ an independent ordered k-sample of the uniform distribution on $(0, T)$ (whence the factor $\frac{1}{t}$). Both are independent, conditionally on the number of points in the discrete loop. We see that μ, on based loops, is obtained from μ on the loops by choosing the based point uniformly. On the other hand, it induces a choice of ξ_1 biased by the size of the τ_i^*'s, different of μ^*. But we will consider only loop functionals.

It will be convenient to rescale the holding time at each ξ_i by λ_{ξ_i} and set

$$\widehat{\tau}_i = \frac{\tau_i^*}{\lambda_{\xi_i}}.$$

The discrete part of the loop is the most important, though we will see that to establish a connection with Gaussian fields it is necessary to consider occupation times. The simplest variables are the number of jumps from x to y, defined for every oriented edge (x, y)

$$N_{x,y} = \#\{i : \xi_i = x, \xi_{i+1} = y\}$$

(recall the convention $\xi_{p+1} = \xi_1$) and

$$N_x = \sum_y N_{x,y}.$$

Note that $N_x = \#\{i \ge 1 : \xi_i = x\}$ except for trivial loops for which it vanishes.

Then, the measure on pointed loops (4) can be rewritten as

$$\mu^*(p = 1, \xi = x, \widehat{\tau} \in dt) = e^{-\lambda_x t}\frac{dt}{t} \quad \text{and} \tag{6}$$

$$\mu^*(p = k, \xi_i = x_i, \widehat{\tau}_i \in dt_i) = \frac{1}{k}\prod_{x,y} C_{x,y}^{N_{x,y}} \prod_x \lambda_x^{-N_x} \prod_{i \in \mathbb{Z}/p\mathbb{Z}} \lambda_{\xi_i} e^{-\lambda_{\xi_i} t_i}\, dt_i. \tag{7}$$

Another *bridge measure* $\mu^{x,y}$ can be defined on paths γ from x to y:

$$\mu^{x,y}(d\gamma) = \int_0^\infty \mathbb{P}_t^{x,y}(d\gamma)\,dt.$$

Note that the mass of $\mu^{x,y}$ is $G^{x,y}$. We also have, with similar notations as the one defined for loops, p denoting the number of jumps

$$\mu^{x,y}\big(p(\gamma)=k, \gamma_{T_1}=x_1, \dots, \gamma_{T_{k-1}}=x_{k-1}, T_1\in dt_1, \dots, T_{k-1}\in dt_{k-1}, T\in dt\big)$$
$$= \frac{C_{x,x_2}C_{x_2,x_3}\cdots C_{x_{k-1},y}}{\lambda_x\lambda_{x_2}\cdots\lambda_y} 1_{\{0<t_1<\cdots<t_k<t\}} e^{-t}\,dt_1\cdots dt_k\,dt.$$

For any $x\neq y$ in X and $s\in[0,1]$, setting $P_v^{(s),u} = P_v^u$ if $(u,v)\neq(x,y)$ and $P_y^{(s),x} = sP_y^x$, we can prove in the same way as to give the expression of $\mu(p>1)$ that

$$\mu\big(s^{N_{x,y}}1_{\{p>1\}}\big) = -\log\big(\det\big(I-P^{(s)}\big)\big).$$

Differentiating in $s=1$, it follows that

$$\mu(N_{x,y}) = [(I-P)^{-1}]_x^y P_y^x = G^{x,y}C_{x,y}$$

and $\mu(N_x) = \sum_y \mu(N_{x,y}) = \lambda_x G^{x,x} - 1$ [as $G(M_\lambda - C) = Id$].

We finally note that if $C_{x,y}>0$, any path segment on the graph starting at x and ending at y can be naturally extended into a loop by adding a jump from y to x. We have the following proposition.

PROPOSITION 1. *For $C_{x,y}>0$, the natural extension of $\mu^{x,y}$ to loops coincides with $\frac{N_{y,x}(l)}{C_{x,y}}\mu(dl)$.*

PROOF. It follows from the expressions of the densities of μ and $\mu^{x,y}$, noticing that a loop l can be associated to $N_{y,x}(l)$ distinct bridges from x to y, obtained by "cutting" one jump from y to x. □

3.4. *Occupation field.* To each loop l° we associate local times, that is, an occupation field $\{\widehat{l_x}, x\in X\}$ defined by

$$\widehat{l^x} = \int_0^{T(l)} 1_{\{\xi(s)=x\}}\frac{1}{\lambda_{\xi(s)}}\,ds = \sum_{i=1}^{p(l)} 1_{\{\xi_i=x\}}\widehat{\tau_i}$$

for any representative $l=(\xi_i,\tau_i^*)$ of l°. For a path γ, $\widehat{\gamma}$ is defined in the same way.

Note that

$$\mu\big((1-e^{-\alpha\widehat{l^x}})1_{\{p=1\}}\big) = \int_0^\infty e^{-t}\big(1-e^{-(\alpha/\lambda_x)t}\big)\frac{dt}{t} = \log\left(1+\frac{\alpha}{\lambda_x}\right) \tag{8}$$

[noticing for example that $\int_a^b (e^{-cx}-e^{-dx})\frac{dx}{x}$ is symmetric in (a,b) and (c,d)].

In particular, $\mu(\widehat{l^x}1_{\{p=1\}}) = \frac{1}{\lambda_x}$.

From formula (4), we get easily that the joint conditional distribution of $(\widehat{l^x}, x\in X)$ given $(N_x, x\in X)$ is a product of gamma distributions. In particular, from the expression of the moments of a gamma distribution, we get that for any function Φ of the discrete loop and $k \geq 1$,

$$\mu\big((\widehat{l^x})^k 1_{\{p>1\}}\Phi\big) = \lambda_x^{-k}\mu\big((N_x+k-1)\cdots(N_x+1)N_x\Phi\big).$$

In particular, $\mu(\widehat{l^x}) = \frac{1}{\lambda_x}[\mu(N_x)+1] = G^{x,x}$.

Note that functions of $\widehat{l}$ are not the only functions naturally defined on the loops. Other such variables of interest are, for $n \geq 2$, the multiple local times, defined as follows:

$$\widehat{l}^{x_1,\dots,x_n} = \sum_{j=0}^{n-1}\int_{0<t_1<\cdots<t_n<T} 1_{\{\xi(t_1)=x_{1+j},\dots,\xi(t_{n-j})=x_n,\dots,\xi(t_n)=x_j\}}\prod \frac{1}{\lambda_{x_i}}\,dt_i.$$

It is easy to check that, when the points x_i are distinct,

$$\widehat{l}^{x_1,\dots,x_n} = \sum_{j=0}^{n-1}\sum_{1\leq i_1<\cdots<i_n\leq p(l)}\prod_{l=1}^{n} 1_{\{\xi_{i_l}=x_{l+j}\}}\widehat{\tau}_{i_l}. \tag{9}$$

Note that in general $\widehat{l}^{x_1,\dots,x_k}$ cannot be expressed in terms of $\widehat{l}$.

If $x_1 = x_2 = \cdots = x_n$, $\widehat{l}^{x_1,\dots,x_n} = \frac{1}{(n-1)!}[\widehat{l^x}]^n$. It can be viewed as a nth self intersection local time.

One can deduce from the defintions of μ the following proposition.

PROPOSITION 2. $\mu(\widehat{l}^{x_1,\dots,x_n}) = G^{x_1,x_2}G^{x_2,x_3}\cdots G^{x_n,x_1}$.

PROOF. Let us denote $\frac{1}{\lambda_y}[P_t]^x_y$ by $p_t^{x,y}$ or $p_t(x,y)$. From the definition of $\widehat{l}^{x_1,\dots,x_n}$ and μ, $\mu(\widehat{l}^{x_1,\dots,x_n})$ equals

$$\sum_x \lambda_x \sum_{j=0}^{n-1}\int\int_{\{0<t_1<\cdots<t_n<t\}} \frac{1}{t} p_{t_1}(x,x_{1+j})\cdots p_{t-t_n}(x_{n+j},x)\prod dt_i\,dt,$$

where sums of indices $k+j$ are computed mod(n). By the semigroup property, it equals

$$\sum_{j=0}^{n-1}\int\int_{\{0<t_1<\cdots<t_n<t\}} \frac{1}{t} p_{t_2-t_1}(x_{1+j},x_{2+j})\cdots p_{t_1+t-t_n}(x_{n+j},x_{1+j})\prod dt_i\,dt.$$

Performing the change of variables $v_2 = t_2 - t_1, \dots, v_n = t_n - t_{n-1}$, $v_1 = t_1 + t - t_n$, and $v = t_1$, we obtain

$$\sum_{j=0}^{n-1}\int_{\{0<v<v_1,0<v_i\}} \frac{1}{v_1+\cdots+v_n} p_{v_2}(x_{1+j},x_{2+j})\cdots p_{v_1}(x_{n+j},x_{1+j})\prod dv_i\,dv$$

$$= \sum_{j=0}^{n-1}\int_{\{0<v_i\}} \frac{v_1}{v_1+\cdots+v_n} p_{v_2}(x_{1+j},x_{2+j})\cdots p_{v_1}(x_{n+j},x_{1+j})\prod dv_i$$

$$= \sum_{j=1}^{n} \int_{\{0<v_i\}} \frac{v_j}{v_1 + \cdots + v_n} p_{v_2}(x_1, x_2) \cdots p_{v_1}(x_n, x_1) \prod dv_i$$

$$= \int_{\{0<v_i\}} p_{v_2}(x_1, x_2) \cdots p_{v_1}(x_n, x_1) \prod dv_i$$

$$= G^{x_1,x_2} G^{x_2,x_3} \cdots G^{x_n,x_1}.$$

Note that another proof can be derived from formula (9). □

Let us come back to the occupation field to compute its Laplace transform. From the Feynman–Kac formula, it comes easily that, denoting $M_{\chi/\lambda}$ the diagonal matrix with coefficients $\frac{\chi_x}{\lambda_x}$

$$\mathbb{P}_t^{x,x}\left(e^{-\langle \widehat{l}, \chi\rangle} - 1\right) = \frac{1}{\lambda_x}\left(\exp\left(t(P - I - M_{\chi/\lambda})\right)_x^x - \exp\left(t(P - I)\right)_x^x\right).$$

Integrating in t after expanding, we get from the definition of μ (first for χ small enough)

$$\begin{aligned}\int \left(e^{-\langle \widehat{l}, \chi\rangle} - 1\right) d\mu(l) &= \sum_{k=1}^{\infty} \int_0^{\infty} \left[\operatorname{Tr}\left((P - M_{\chi/\lambda})^k\right) - \operatorname{Tr}((P)^k)\right] \frac{t^{k-1}}{k!} e^{-t}\, dt \\ &= \sum_{k=1}^{\infty} \frac{1}{k} \left[\operatorname{Tr}\left((P - M_{\chi/\lambda})^k\right) - \operatorname{Tr}((P)^k)\right] \\ &= -\operatorname{Tr}\left(\log(I - P + M_{\chi/\lambda})\right) + \operatorname{Tr}\left(\log(I - P)\right).\end{aligned}$$

Hence, as $\operatorname{Tr}(\log) = \log(\det)$,

$$\int \left(e^{-\langle \widehat{l}, \chi\rangle} - 1\right) d\mu(l) = \log\left[\det\left(-L(-L + M_{\chi/\lambda})^{-1}\right)\right] = -\log\det(I + V M_{\chi/\lambda})$$

which now holds for all nonnegative χ as both members are analytic in χ. Besides, by the resolvent equation,

$$\det(I + G M_\chi)^{-1} = \det(I - G_\chi M_\chi) = \frac{\det(G_\chi)}{\det(G)}. \tag{10}$$

Note that $\det(I + G M_\chi) = \det(I + M_{\sqrt{\chi}} G M_{\sqrt{\chi}})$ and $\det(I - G_\chi M_\chi) = \det(I - M_{\sqrt{\chi}} G_\chi M_{\sqrt{\chi}})$, so we can deal with symmetric matrices. Finally, we have Proposition 3.

PROPOSITION 3.

$$\mu\left(e^{-\langle \widehat{l}, \chi\rangle} - 1\right) = -\log\left(\det(I + M_{\sqrt{\chi}} G M_{\sqrt{\chi}})\right) = \log\left(\frac{\det(G_\chi)}{\det(G)}\right).$$

Note that in particular $\mu(e^{-t\widehat{l}^x} - 1) = -\log(1 + t G^{x,x})$.

Note finally that if χ has support in D, by the restriction property

$$\mu\big(1_{\{\widehat{l}(X\setminus D)=0\}}\big(e^{-\langle\widehat{l},\chi\rangle}-1\big)\big)=-\log\big(\det(I+M_{\sqrt{\chi}}G^D M_{\sqrt{\chi}})\big)=\log\bigg(\frac{\det(G^D_\chi)}{\det(G^D)}\bigg).$$

Here, the determinants are taken on matrices indexed by D and G^D denotes the Green function of the process killed on leaving D.

For paths, we have $\mathbb{P}^{x,y}_t(e^{-\langle\widehat{l},\chi\rangle})=\frac{1}{\lambda_y}\exp(t(L-M_{\chi/\lambda}))_{x,y}$. Hence,

$$\mu^{x,y}\big(e^{-\langle\widehat{\gamma},\chi\rangle}\big)=\frac{1}{\lambda_y}\big((I-P+M_{\chi/\lambda})^{-1}\big)_{x,y}=[G_\chi]^{x,y}.$$

Also $\mathbb{E}^x(e^{-\langle\widehat{\gamma},\chi\rangle})=\sum_y[G_\chi]^{x,y}\kappa_y$, that is, $[G_\chi\kappa]^x$.

Finally, let us note that a direct calculation shows the following.

PROPOSITION 4. *On loops based in* x, $\mu^{x,x}(dl)=\widehat{l}_x\mu(dl)$.

4. Poisson process of loops.

4.1. *Definition.* Still following the idea of [9], the germ of which was implicitly in germ in [22], define, for all positive α, the Poissonian ensemble of loops $\mathcal{L}_\alpha$ with intensity $\alpha\mu$. We denote by $\mathbb{P}$ or $\mathbb{P}_{\mathcal{L}_\alpha}$ its distribution.

Recall it means that for any functional Φ on the loop space, vanishing on loops of arbitrary small length,

$$E\big(e^{i\sum_{l\in\mathcal{L}_\alpha}\Phi(l)}\big)=\exp\bigg(\alpha\int\big(e^{i\Phi(l)}-1\big)\mu(dl)\bigg).$$

Note that by the restriction property, $\mathcal{L}^D_\alpha=\{l\in\mathcal{L}_\alpha, l\subseteq D\}$ is a Poisson process of loops with intensity μ^D, and that $\mathcal{L}^D_\alpha$ is independent of $\mathcal{L}_\alpha\setminus\mathcal{L}^D_\alpha$.

We denote by $\mathcal{DL}_\alpha$ the set of nontrivial discrete loops in $\mathcal{L}_\alpha$. Then

$$\begin{aligned}\mathbb{P}(\mathcal{DL}_\alpha=\{l_1,l_2,\dots,l_k\})&=e^{-\alpha\mu(p>0)}\alpha^k\mu(l_1)\cdots\mu(l_k)\\&=\alpha^k\bigg[\frac{\det(G)}{\prod_x\lambda_x}\bigg]^\alpha\prod_{x,y}C_{x,y}^{N^{(\alpha)}_{x,y}}\prod_x\lambda_x^{-N^{(\alpha)}_x}\end{aligned}$$

with $N^{(\alpha)}_x=\sum_{l\in\mathcal{L}_\alpha}N_x(l)$ and $N^{(\alpha)}_{x,y}=\sum_{l\in\mathcal{L}_\alpha}N_{x,y}(l)$, when these loops are distinct.

We can associate to $\mathcal{L}_\alpha$ a σ-finite measure (in fact as we will see, finite when X is finite, and more generally, if G is trace class) called local time or occupation field

$$\widehat{\mathcal{L}_\alpha}=\sum_{l\in\mathcal{L}_\alpha}\widehat{l}.$$

Then, for any nonnegative measure χ on X,

$$\mathbb{E}(e^{-\langle\widehat{\mathcal{L}_\alpha},\chi\rangle}) = \exp\left(\alpha\int(e^{-\langle\widehat{l},\chi\rangle}-1)\,d\mu(l)\right)$$

and therefore by Proposition 3 we have the following.

COROLLARY 5. $\mathbb{E}(e^{-\langle\widehat{\mathcal{L}_\alpha},\chi\rangle}) = \det(I+M_{\sqrt{\chi}}GM_{\sqrt{\chi}})^{-\alpha} = (\frac{\det(G_\chi)}{\det(G)})^\alpha$.

Many calculations follow from this result.

Note first that $\mathbb{E}(e^{-t\widehat{\mathcal{L}_\alpha}^x}) = (1+tG^{x,x})^{-\alpha}$. We see that $\widehat{\mathcal{L}_\alpha}^x$ follows a gamma distribution $\Gamma(\alpha, G^{x,x})$, with density $1_{\{x>0\}}\frac{e^{-x/G^{xx}}}{\Gamma(\alpha)}\frac{x^{\alpha-1}}{(G^{xx})^\alpha}$ (in particular, an exponential distribution of mean $G^{x,x}$ for $\alpha = 1$). When we let α vary as a time parameter, we get a family of gamma subordinators, which can be called a multivariate gamma subordinator.

We check in particular that $\mathbb{E}(\widehat{\mathcal{L}_\alpha}^x) = \alpha G^{x,x}$ which follows directly from $\mu(\widehat{l_x}) = G^{x,x}$.

Note also that for $\alpha > 1$,

$$\mathbb{E}\left(\left(1-\exp\left(-\frac{\widehat{\mathcal{L}_\alpha}^x}{G^{x,x}}\right)\right)^{-1}\right) = \zeta(\alpha).$$

More generally, for two points:

$$\mathbb{E}(e^{-t\widehat{\mathcal{L}_\alpha}^x}e^{-s\widehat{\mathcal{L}_\alpha}^y}) = \left((1+tG^{x,x})(1+sG^{y,y}) - st(G^{x,y})^2\right)^{-\alpha}.$$

This allows us to compute the joint density of $\widehat{\mathcal{L}_\alpha}^x$ and $\widehat{\mathcal{L}_\alpha}^y$ in terms of Bessel and Struve functions.

We can condition the loops by the set of associated nontrivial discrete loops by using the restricted σ-field $\sigma(\mathcal{DL}_\alpha)$ which contains the variables $N_{x,y}$. We see from (8) and (6) that

$$\mathbb{E}(e^{-\langle\widehat{\mathcal{L}_\alpha},\chi\rangle}|\mathcal{DL}_\alpha) = \prod_x\left(\frac{\lambda_x}{\lambda_x+\chi_x}\right)^{N_x^{(\alpha)}+1}.$$

The distribution of $\{N_x^{(\alpha)}, x\in X\}$ follows easily, from Corollary 5 in terms of generating functions:

$$\mathbb{E}\left(\prod_x s_x^{N_x^{(\alpha)}+1}\right) = \det\left(\delta_{x,y}+\sqrt{\frac{\lambda_x(1-s_x)}{s_x}}G_{x,y}\sqrt{\frac{\lambda_y(1-s_y)}{s_y}}\right)^{-\alpha} \tag{11}$$

so that the vector of components $N_x^{(\alpha)}$ follows a multivariate negative binomial distribution (see, e.g., [24]).

It follows in particular that $N_x^{(\alpha)}$ follows a negative binomial distribution of parameters $-\alpha$ and $\frac{1}{\lambda_x G^{xx}}$. Note that for $\alpha = 1$, $N_x^{(1)}+1$ follows a geometric distribution of parameter $\frac{1}{\lambda_x G^{xx}}$.

4.2. *Moments and polynomials of the occupation field.* It is easy to check (and well known from the properties of the gamma distributions) that the moments of $\widehat{\mathcal{L}_\alpha}^x$ are related to the factorial moments of $N_x^{(\alpha)}$:

$$\mathbb{E}((\widehat{\mathcal{L}_\alpha}^x)^k|\mathcal{D}\mathcal{L}_\alpha) = \frac{(N_x^{(\alpha)}+k)(N_x^{(\alpha)}+k-1)\cdots(N_x^{(\alpha)}+1)}{k!\lambda_x^k}.$$

It is well known that Laguerre polynomials $L_k^{(\alpha-1)}$ with generating function

$$\sum_0^\infty t^k L_k^{(\alpha-1)}(u) = \frac{e^{-ut/(1-t)}}{(1-t)^\alpha}$$

are orthogonal for the $\Gamma(\alpha)$ distribution with density $\frac{u^{\alpha-1}e^{-u}}{\Gamma(\alpha)}1_{\{u>0\}}$. They have mean zero and variance $\frac{\Gamma(\alpha+k)}{k!}$. Hence, if we set $\sigma_x = G^{x,x}$and $P_k^{\alpha,\sigma}(x) = (-\sigma)^k L_k^{(\alpha-1)}(\frac{x}{\sigma})$, the random variables $P_k^{\alpha,\sigma_x}(\widehat{\mathcal{L}_\alpha}^x)$ are orthogonal with mean 0 and variance $\sigma^{2k}\frac{\Gamma(\alpha+k)}{k!}$, for $k>0$.

Note that $P_1^{\alpha,\sigma_x}(\widehat{\mathcal{L}_\alpha}^x) = \widehat{\mathcal{L}_\alpha}^x - \alpha\sigma_x = \widehat{\mathcal{L}_\alpha}^x - \mathbb{E}(\widehat{\mathcal{L}_\alpha}^x)$. *It will be denoted* $\widetilde{\mathcal{L}_\alpha}^x$.

Moreover, we have $\sum_0^\infty t^k P_k^{\alpha,\sigma}(u) = \sum(-\sigma t)^k L_k^{(\alpha-1)}(\frac{u}{\sigma}) = \frac{e^{ut/(1+\sigma t)}}{(1+\sigma t)^\alpha}$.

Note that

$$\begin{aligned}
&\mathbb{E}\left(\frac{e^{\widehat{\mathcal{L}_\alpha}^x t/(1+\sigma_x t)}}{(1+\sigma_x t)^\alpha}\frac{e^{\widehat{\mathcal{L}_\alpha}^y s/(1+\sigma_y s)}}{(1+\sigma_y s)^\alpha}\right)\\
&\quad= \frac{1}{(1+\sigma_x t)^\alpha(1+\sigma_y s)^\alpha}\\
&\qquad\times\left(\left(\left(1-\frac{\sigma_x t}{1+\sigma_x t}\right)\left(1-\frac{\sigma_y s}{1+\sigma_y s}\right)-\frac{t}{1+\sigma_x t}\frac{s}{1+\sigma_y s}(G^{x,y})^2\right)^{-\alpha}\right)\\
&\quad= \left(1-st(G^{x,y})^2\right)^{-\alpha}.
\end{aligned}$$

Therefore, we get, by developing in entire series in (s,t) and identifying the coefficients

$$\mathbb{E}(P_k^{\alpha,\sigma_x}(\widehat{\mathcal{L}_\alpha}^x)P_l^{\alpha,\sigma_y}(\widehat{\mathcal{L}_\alpha}^y)) = \delta_{k,l}(G^{x,y})^{2k}\frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!}. \tag{12}$$

Let us stress the fact that $G^{x,x}$ and $G^{y,y}$ do not appear on the right-hand side of this formula. This is quite important from the renormalization point of view, as we will consider in the last section the two dimensional Brownian motion for which the Green function diverges on the diagonal.

More generally, one can prove similar formulas for products of higher order.

Note that since $G_\chi M_\chi$ is a contraction, from determinant expansions given in [23] and [24], we have

$$\det(I + M_{\sqrt{\chi}} G M_{\sqrt{\chi}})^{-\alpha} = 1 + \sum_{k=1}^{\infty} (-1)^k \sum \chi_{i_1} \cdots \chi_{i_k} \operatorname{Per}_\alpha(G_{i_l,i_m}, 1 \le l, m \le k)$$

and then, from Corollary 5, it follows that

$$\mathbb{E}(\langle \widehat{\mathcal{L}_\alpha}, \chi \rangle^k) = \sum \chi_{i_1} \cdots \chi_{i_k} \operatorname{Per}_\alpha(G_{i_l,i_m}, 1 \le l, m \le k).$$

Here, the α-permanent $\operatorname{Per}_\alpha$ is defined as $\sum_{\sigma \in \mathcal{S}_k} \alpha^{m(\sigma)} G_{i_1,i_{\sigma(1)}} \cdots G_{i_k,i_{\sigma(k)}}$ with $m(\sigma)$ denoting the number of cycles in σ.

Note that from this determinant expansion, an explicit form for the multivariate negative binomial distribution follows directly (see [24]), and consequently a series expansion for the density of the multivariate gamma distribution.

It is actually not difficult to give a direct proof of this result. Thus, the Poisson process of loops provides a natural probabilistic proof and interpretation of this combinatorial identity (see [24] for an historical view of the subject).

We can show in fact the following proposition.

PROPOSITION 6. *For any* $(i_1, \ldots, i_k)$ *in* X^k, $\mathbb{E}(\widehat{\mathcal{L}_\alpha}^{i_1} \cdots \widehat{\mathcal{L}_\alpha}^{i_k}) = \operatorname{Per}_\alpha(G^{i_l,i_m}, 1 \le l, m \le k)$.

PROOF. The cycles of the permutations in the expression of $\operatorname{Per}_\alpha$ are associated with point configurations on loops. We obtain the result by summing the contributions of all possible partitions of the points $i_1, \ldots, i_k$ into a finite set of distinct loops. We can then decompose again the expression according to ordering of points on each loop. We can conclude by using the formula $\mu(\widehat{l}^{x_1,\ldots,x_m}) = G^{x_1,x_2} G^{x_2,x_3} \cdots G^{x_m,x_1}$ and the following property of Poisson measures (cf. formula 3-13 in [6]): For any system of nonnegative loop functionals F_i,

$$\mathbb{E}\Bigl(\sum_{l_1 \ne l_2 \ne \cdots \ne l_k \in \mathcal{L}_\alpha} \prod F_i(l_i)\Bigr) = \prod \alpha\mu(F_i). \tag{13}$$

□

REMARK 7. *We can actually check this formula in the special case* $i_1 = i_2 = \cdots = i_k = x$. *From the moment of the Gamma distribution, we have that* $\mathbb{E}((\widehat{\mathcal{L}_\alpha}^x)^n) = (G^{x,x})^n \alpha(\alpha+1)\cdots(\alpha+n-1)$ *and the* α*-permanent can be written* $\sum_1^n d(n,k)\alpha^k$ *where the coefficients* $d(n,k)$ *are the numbers of* n*-permutations with* k *cycles* (*Stirling numbers of the first kind*). *One checks that* $d(n+1,k) = nd(n,k) + d(n,k-1)$.

PROPOSITION 8. *Given any bounded functionals* Φ *on loops configurations and* F *on loops*, *we have*

$$\mathbb{E}\Bigl(\sum_{l \in \mathcal{L}_\alpha} F(l)\Phi(\mathcal{L}_\alpha)\Bigr) = \int \mathbb{E}\bigl(\Phi(\mathcal{L}_\alpha \cup \{l\})\bigr)\alpha F(l)\mu(dl).$$

PROOF. This is proved by considering first for $\Phi(\mathcal{L}_\alpha)$ the functionals of the form $\sum_{l_1\neq l_2\neq\cdots\neq l_q\in\mathcal{L}_\alpha}\prod_1^q G_j(l_j)$ (with G_j bounded and μ-integrable) which span an algebra separating distinct configurations and applying formula (13): then, the common value of both members is $\alpha^q\sum_1^q\mu(FG_j)\prod_{l\neq j}\mu(G_l)+\alpha^{q+1}\mu(F)\prod_1^q\mu(G_j)$. □

The above proposition applied to $F(l)=\widehat{l}^x, N_{x,y}^{(\alpha)}$ and Propositions 1 and 4 yield the following corollary.

COROLLARY 9. *$\mathbb{E}(\Phi(\mathcal{L}_\alpha)\widehat{\mathcal{L}_\alpha}^x)=\alpha\int\mathbb{E}(\Phi(\mathcal{L}_\alpha\cup\{\gamma\}))\widehat{\gamma}^x\mu(d\gamma)=\alpha\times\int\mathbb{E}(\Phi(\mathcal{L}_\alpha\cup\{\gamma\}))\mu^{x,x}(d\gamma)$ and $\mathbb{E}(\Phi(\mathcal{L}_\alpha)N_{x,y}^{(\alpha)})=\alpha\int\mathbb{E}(\Phi(\mathcal{L}_\alpha\cup\{\gamma\}))N_{x,y}(\gamma)\times\mu(d\gamma)=\alpha C_{x,y}\int\mathbb{E}(\Phi(\mathcal{L}_\alpha\cup\{\gamma\}))\mu^{x,y}(d\gamma)$ if $x\neq y$.*

Let $\mathcal{S}_k^0$ be the set of permutations of k elements without fixed point. They correspond to configurations without isolated points.

Set $\mathrm{Per}_\alpha^0(G^{i_l,i_m},1\leq l,m\leq k)=\sum_{\sigma\in\mathcal{S}_k^0}\alpha^{m(\sigma)}G^{i_1,i_{\sigma(1)}}\cdots G^{i_k,i_{\sigma(k)}}$. Then an easy calculation shows the following.

COROLLARY 10. *$\mathbb{E}(\widetilde{\mathcal{L}_\alpha}^{i_1}\cdots\widetilde{\mathcal{L}_\alpha}^{i_k})=\mathrm{Per}_\alpha^0(G^{i_l,i_m},1\leq l,m\leq k)$.*

PROOF. Indeed, the expectation can be written

$$\sum_{p\leq k}\sum_{I\subseteq\{1,\ldots,k\},|I|=p}(-1)^{k-p}\prod_{l\in I^c}G^{i_l,i_l}\,\mathrm{Per}_\alpha(G^{i_a,i_b},a,b\in I)$$

and

$$\mathrm{Per}_\alpha(G^{i_a,i_b},a,b\in I)=\sum_{J\subseteq I}\prod_{j\in I\setminus J}G^{j,j}\,\mathrm{Per}_\alpha^0(G^{i_a,i_b},a,b\in J).$$

Then, expressing $\mathbb{E}(\widetilde{\mathcal{L}_\alpha}^{i_1}\cdots\widetilde{\mathcal{L}_\alpha}^{i_k})$ in terms of Per_α^0's, we see that if $J\subseteq\{1,\ldots,k\}$, $|J|<k$, the coefficient of $\mathrm{Per}_\alpha^0(G^{i_a,i_b},a,b\in J)$ is $\sum_{I,I\supseteq J}(-1)^{k-|I|}\prod_{j\in J^c}G^{i_j,i_j}$ which vanishes as $(-1)^{-|I|}=(-1)^{|I|}=(-1)^{|J|}(-1)^{|I\setminus J|}$ and $\sum_{I\supseteq J}(-1)^{|I\setminus J|}=(1-1)^{k-|J|}=0$. □

Set $Q_k^{\alpha,\sigma}(u)=P_k^{\alpha,\sigma}(u+\alpha\sigma)$ so that $P_k^{\alpha,\sigma}(\widehat{\mathcal{L}_\alpha}^x)=Q_k^{\alpha,\sigma}(\widetilde{\mathcal{L}_\alpha}^x)$. This quantity will be called the nth renormalized self-intersection local time or the nth renormalized power of the occupation field and denoted $\widetilde{\mathcal{L}}_\alpha^{x,n}$.

From the recurrence relation of Laguerre polynomials

$$nL_n^{(\alpha-1)}(u)=(-u+2n+\alpha-2)L_{n-1}^{(\alpha-1)}-(n+\alpha-2)L_{n-2}^{(\alpha-1)},$$

we get that

$$nQ_n^{\alpha,\sigma}(u)=\bigl(u-2\sigma(n-1)\bigr)Q_{n-1}^{\alpha,\sigma}(u)-\sigma^2(\alpha+n-2)Q_{n-2}^{\alpha,\sigma}(u).$$

In particular, $Q_2^{\alpha,\sigma}(u) = \frac{1}{2}(u^2 - 2\sigma u - \alpha\sigma^2)$.

We have also, from (12)

$$\mathbb{E}(Q_k^{\alpha,\sigma_x}(\widetilde{\mathcal{L}_\alpha}^x) Q_l^{\alpha,\sigma_y}(\widetilde{\mathcal{L}_\alpha}^y)) = \delta_{k,l}(G^{x,y})^{2k} \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!}. \tag{14}$$

The comparison of the identity (14) and Corollary 10 yields a combinatorial result which will be fundamental in the renormalizing procedure presented in the last section.

The identity (14) can be considered as a polynomial identity in the variables σ_x, σ_y and $G^{x,y}$.

If $Q_k^{\alpha,\sigma_x}(u) = \sum_{m=0}^k q_m^{\alpha,k} u^m \sigma_x^{k-m}$, and if we denote $N_{n,m,r,p}$ the number of ordered configurations of n black points and m red points on r nontrivial oriented cycles, such that only $2p$ links are between red and black points, we have

$$\mathbb{E}((\widetilde{\mathcal{L}_\alpha}^x)^n (\widetilde{\mathcal{L}_\alpha}^y)^m) = \sum_r \sum_{p \le \inf(m,n)} \alpha^r N_{n,m,r,p} (G^{x,y})^{2p} (\sigma_x)^{n-p} (\sigma_y)^{m-p}$$

and therefore

$$\sum_r \sum_{p \le m \le k} \sum_{p \le n \le l} \alpha^r q_m^{\alpha,k} q_n^{\alpha,l} N_{n,m,r,p} = 0 \qquad \text{unless } p = l = k, \tag{15}$$

$$\sum_r \alpha^r q_k^{\alpha,k} q_k^{\alpha,k} N_{k,k,r,k} = \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!}. \tag{16}$$

Note that one can check directly that $q_k^{\alpha,k} = \frac{1}{k!}$, and $N_{k,k,1,k} = k!(k-1)!$, $N_{k,k,k,k} = k!$ which confirms the identity (16) above.

4.3. *Hitting probabilities.* Denote by $[H^F]_y^x = \mathbb{P}_x(x_{T_F} = y)$ the hitting distribution of F by the Markov chain starting at x. Set $D = F^c$ and denote by e^D, $P^D = P|_{D\times D}$, $V^D = [(I - P^D)]^{-1}$ and $G^D = [(M_\lambda - C)|_{D\times D}]^{-1}$ the energy, the transition matrix, the potential and the Green function of the process killed at the hitting of F. Recall that $[H^F]_y^x = 1_{\{x=y\}} + \sum_0^\infty \sum_{z\in D} [(P^D)^k]_z^x P_y^z = 1_{\{x=y\}} + \sum_{z\in D} [V^D]_z^x P_y^z$. Moreover, we have by the strong Markov property, $V = V^D + H^F V$ and therefore $G = G^D + H^F G$. (Here, we extend V^D and G^D to $X \times X$ by adding zero entries outside $D \times D$.)

As G and G^D are symmetric, we have $[H^F G]_y^x = [H^F G]_x^y$ so that for any measure ν, $H^F(G\nu) = G(\nu H^F)$.

Therefore, we see that for any function f and measure ν, $e(H^F f, G^D \nu) = e(H^F f, G\nu) - e(H^F f, H^F G\nu) = \langle H^F f, \nu\rangle - e(H^F f, G(H^F \nu)) = 0$ as $(H^F)^2 = H^F$.

Equivalently, we have the following well-known proposition.

PROPOSITION 11. *For any g vanishing on F, $e(H^F f, g) = 0$ so that $I - H^F$ is the e-orthogonal projection on the space of functions supported in D.*

For further developments, see, for example, [12] and its references.

The restriction property holds for $\mathcal{L}_\alpha$ as it holds for μ. The set $\mathcal{L}_\alpha^D$ of loops inside D is associated with μ^D and is independent of $\mathcal{L}_\alpha - \mathcal{L}_\alpha^D$. Therefore, we see from Corollary 5 that

$$\mathbb{E}\big(e^{-\langle \widehat{\mathcal{L}_\alpha} - \widehat{\mathcal{L}_\alpha^D}, \chi\rangle}\big) = \left(\frac{\det(G_\chi)}{\det(G)} \frac{\det(G^D)}{\det(G_\chi^D)}\right)^\alpha.$$

From the support of the Gamma distribution, we see that $\mu(\widehat{l}(F) > 0) = \infty$. But this is clearly due to trivial loops as it can be seen directly from the definition of μ that in this simple framework they cover the whole space X.

Note however that

$$\begin{aligned}\mu\big(\widehat{l}(F) > 0, p > 1\big) &= \mu(p > 1) - \mu\big(\widehat{l}(F) = 0, p > 1\big) = \mu(p > 1) - \mu^D(p > 1)\\ &= -\log\left(\frac{\det(I - P)}{\det_{D\times D}(I - P)}\right) = -\log\left(\frac{\det(G^D)}{\prod_{x\in F} \lambda_x \det(G)}\right).\end{aligned}$$

It follows that the probability that no nontrivial loop (i.e., a loop which is not reduced to a point) in $\mathcal{L}_\alpha$ intersects F equals

$$\exp\big(-\alpha\mu(\{l, p(l) > 1, \widehat{l}(F) > 0\})\big) = \left(\frac{\det(G^D)}{\prod_{x\in F} \lambda_x \det(G)}\right)^\alpha.$$

Recall that by Jacobi's identity, for any $(n + p, n + p)$ invertible matrix A,

$$\det(A^{-1}) \det(A_{ij}, 1 \le i, j \le n) = \det\big((A^{-1})_{k,l}, n \le k, l \le n + p\big).$$

In particular,

$$\det(G^D) = \frac{\det(G)}{\det(G|_{F\times F})}$$

so we have the

PROPOSITION 12. *The probability that no nontrivial loop in $\mathcal{L}_\alpha$ intersects F equals*

$$\left[\prod_{x\in F} \lambda_x \det_{F\times F}(G)\right]^{-\alpha}.$$

Moreover,

$$\mathbb{E}\big(e^{-\langle \widehat{\mathcal{L}_\alpha} - \widehat{\mathcal{L}_\alpha^D}, \chi\rangle}\big) = \left(\frac{\det_{F\times F}(G_\chi)}{\det_{F\times F}(G)}\right)^\alpha.$$

In particular, it follows that the probability that no nontrivial loop in $\mathcal{L}_\alpha$ visits x equals $(\frac{1}{\lambda_x G^{x,x}})^\alpha$ which is also a consequence of the fact that N_x follows a negative binomial distribution of parameters $-\alpha$ and $\frac{1}{\lambda_x G^{x,x}}$.

Also, if F_1 and F_2 are disjoint,

$$\begin{aligned}\mu\big(\widehat{l}(F_1)\widehat{l}(F_2)>0\big)&=\mu\big(\widehat{l}(F_1)>0,\,p>1\big)+\mu\big(\widehat{l}(F_2)>0,\,p>1\big)\\&\quad-\mu\big(\widehat{l}(F_1\cup F_2)>0,\,p>1\big)\\&=\log\left(\frac{\det(G)\det(G^{D_1\cap D_2})}{\det(G^{D_1})\det(G^{D_2})}\right).\end{aligned}$$

Therefore, the probability that no nontrivial loop in $\mathcal{L}_\alpha$ intersects F_1 and F_2 equals

$$\exp\Big(-\alpha\mu\Big(\Big\{l,\,p(l)>1,\prod\widehat{l}(F_i)>0\Big\}\Big)\Big)=\left(\frac{\det(G)\det(G^{D_1\cap D_2})}{\det(G^{D_1})\det(G^{D_2})}\right)^{-\alpha}.$$

It follows that the probability no nontrivial loop in $\mathcal{L}_\alpha$ visits two distinct points x and y equals $(\frac{G^{x,x}G^{y,y}-(G^{x,y})^2}{G^{x,x}G^{y,y}})^\alpha$ and in particular $1-\frac{(G^{x,y})^2}{G^{x,x}G^{y,y}}$ if $\alpha=1$. This formula can be easily generalized to n disjoint sets.

5. The Gaussian free field.

5.1. *Dynkin's isomorphism.* By a well-known calculation, if X is finite, for any $\chi\in\mathbb{R}_+^X$,

$$\frac{\sqrt{\det(M_\lambda-C)}}{(2\pi)^{|X|/2}}\int^{-\langle z,\chi\rangle/2}e^{-e(z)/2}\prod_{u\in X}dz^u=\sqrt{\frac{\det(G_\chi)}{\det(G)}}$$

and

$$\frac{\sqrt{\det(M_\lambda-C)}}{(2\pi)^{|X|/2}}\int z^xz^ye^{-\langle z^2,\chi\rangle/2}e^{-e(z)/2}\prod_{u\in X}dz^u=(G_\chi)^{x,y}\sqrt{\frac{\det(G_\chi)}{\det(G)}}.$$

This can be easily reformulated by introducing on an independent probability space the Gaussian field ϕ defined by the covariance $\mathbb{E}_\phi(\phi^x\phi^y)=G^{x,y}$ (this reformulation cannot be dispensed with when X becomes infinite).

So we have $\mathbb{E}_\phi(e^{-\langle\phi^2,\chi\rangle/2})=\det(I+GM_\chi)^{-1/2}=\sqrt{\det(G_\chi G^{-1})}$ and $\mathbb{E}_\phi(\phi^x\phi^ye^{-\langle\phi^2,\chi\rangle/2})=(G_\chi)^{x,y}\sqrt{\det(G_\chi G^{-1})}$. Then since sums of exponentials of the form $e^{-\langle\cdot,\chi\rangle/2}$ are dense in continuous functions on $\mathbb{R}_+^X$ the following holds.

THEOREM 13. (a) *The fields* $\widehat{\mathcal{L}_{1/2}}$ *and* $\frac{1}{2}\phi^2$ *have the same distribution.*

(b) $\mathbb{E}_\phi(\phi^x\phi^yF(\frac{1}{2}\phi^2))=\int\mathbb{E}(F(\widehat{\mathcal{L}_{1/2}}+\widehat{\gamma}))\mu^{x,y}(d\gamma)$ *for any bounded functional* F *of a nonnegative field.*

REMARKS. (a) *This is a version of Dynkin's isomorphism* (*cf.* [2]). *It can be extended to nonsymmetric generators* (*cf.* [14]).

(b) *Corollary* 9 *and part* (b) *imply that if* $C_{x,y} \neq 0$,

$$\mathbb{E}_\phi\left(\left(\phi^x\phi^y F\left(\frac{1}{2}\phi^2\right)\right)\right) = \frac{2}{C_{x,y}}\int \mathbb{E}\big(F(\widehat{\mathcal{L}_{1/2}})N_{x,y}^{(1/2)}\big).$$

(c) *An analogous result can be given when* α *is any positive half integer, by using real vector valued Gaussian field, or equivalently complex fields for integral values of* α (*in particular,* $\alpha = 1$).

(d) *Note it implies immediately that the process* ϕ^2 *is infinitely divisible. See* [3] *and its references for a converse and earlier proofs of this last fact.*

5.2. *Fock spaces and Wick product.* The Gaussian space $\mathcal{H}$ spanned by $\{\phi^x, x \in X\}$ is isomorphic to the Dirichlet space $\mathbb{H}$ by the linear map mapping ϕ^x on $G^{x,\cdot}$ which extends into an isomorphism between the space of square integrable functionals of the Gaussian fields and the symmetric Fock space obtained as the closure of the sum of all symmetric tensor powers of $\mathbb{H}$ (Bose second quantization: see [17, 21]). We have seen in Theorem 13 that L^2 functionals of $\widehat{\mathcal{L}_1}$ can be represented in this space of Gaussian functionals.

In order to prepare the extension of these isomorphisms to the more difficult framework of continuous spaces (which can often be viewed as scaling limits of discrete spaces), including especially the planar Brownian motion considered in [9], we shall introduce the renormalized (or Wick) powers of ϕ. We set $:(\phi^x)^n: = (G^{x,x})^{n/2}H_n(\phi^x/\sqrt{G^{x,x}})$ where H_n in the nth Hermite polynomial [characterized by $\sum \frac{t^n}{n!}H_n(u) = e^{tu-t^2/2}$]. It is the inverse image of the nth tensor power of $G^{x,\cdot}$ in the Fock space.

Setting as before $\sigma_x = G^{x,x}$, from the relation between Hermite polynomials H_{2n} and Laguerre polynomials $L_n^{-1/2}$,

$$H_{2n}(x) = (-2)^n n! L_n^{-1/2}\left(\frac{x^2}{2}\right)$$

it follows that

$$:(\phi^x)^{2n}: = 2^n n! P_n^{1/2,\sigma}\left(\left(\frac{(\phi^x)^2}{2}\right)\right).$$

More generally, if $\phi_1, \phi_2, \dots, \phi_k$ are k independent copies of the free field, we can define $:\prod_{j=1}^k \phi_j^{n_j}: = \prod_{j=1}^k :\phi_j^{n_j}:$. Then it follows that

$$:\left(\sum_1^k \phi_j^2\right)^n: = \sum_{n_1+\cdots+n_k=n} \frac{n!}{n_1!\cdots n_k!}\prod_{j=1}^k :\phi_j^{2n_j}:.$$

From the generating function of the polynomials $P_n^{k/2,\sigma}$,

$$P_n^{k/2,\sigma}\left(\sum_1^k u_j\right) = \sum_{n_1+\cdots+n_k=n} \frac{n!}{n_1!\cdots n_k!}\prod_{j=1}^k P_{n_j}^{1/2,\sigma}(u_j).$$

Therefore,

$$P_n^{k/2,\sigma}\left(\frac{\sum(\phi_j)^2}{2}\right)=\frac{1}{2^n n!}:\left(\sum_1^k \phi_j^2\right)^n:. \tag{17}$$

Note that $:\sum_1^k \phi_j^2:=\sum_1^k \phi_j^2-\sigma$. These variables are orthogonal in L^2. Let $\widetilde{l}^x=\widehat{l}^x-\sigma$ be the centered occupation field. Note that an equivalent formulation of Theorem 13 is that the fields $\frac{1}{2}:\sum_1^k \phi_j^2:$ and $\widetilde{\mathcal{L}}_{k/2}$ have the same law.

Let us now consider the relation of higher Wick powers with self-intersection local times.

Recall that the renormalized nth self intersections field $\widetilde{\mathcal{L}}_1^{x,n}=P_n^{\alpha,\sigma}(\widehat{\mathcal{L}_\alpha}^x)=Q_n^{\alpha,\sigma}(\widetilde{\mathcal{L}_\alpha}^x)$ have been defined by orthonormalization in L^2 of the powers of the occupation time.

Then comes the following proposition.

PROPOSITION 14. *The fields* $\widetilde{\mathcal{L}}_{k/2}^{\cdot,n}$ *and* $:(\frac{1}{n!2^n}\sum_1^k \phi_j^2)^n:$ *have the same law.*

This follows directly from (17).

REMARK 15. *As a consequence, it can be shown that*

$$\mathbb{E}\left(\prod_{j=1}^r Q_{k_j}^{\alpha,\sigma_{x_j}}(\widetilde{\mathcal{L}_\alpha}^{x_j})\right)=\sum_{\sigma\in\mathcal{S}_{k_1,k_2,\dots,k_j}}(2\alpha)^{m(\sigma)}G^{i_1,i_{\sigma(1)}}\cdots G^{i_k,i_{\sigma(k)}},$$

where $\mathcal{S}_{k_1,k_2,\dots,k_j}$ *is the set of permutations* σ *of* $k=\sum k_j$ *such that* $\sigma(\{\sum_1^{j-1}k_l+1,\dots,\sum_1^{j-1}k_l+k_j\})\cap\{\sum_1^{j-1}k_l+1,\dots,\sum_1^{j-1}k_l+k_j\}$ *is empty for all* j.

The identity follows from Wick's theorem when α is a half integer, then extends to all α since both members are polynomials in α. The condition on σ indicates that no pairing is allowed inside the same Wick power.

6. Energy variation and currents. The loop measure μ depends on the energy e which is defined by the free parameters C,κ. It can be denoted μ_e. We shall denote $\mathcal{Z}_e$ the determinant $\det(G)=\det(M_\lambda-C)^{-1}$. Then $\mu(p>0)=\log(\mathcal{Z}_e)+\sum\log(\lambda_x)$.

$\mathcal{Z}_e^\alpha$ is called the partition function of $\mathcal{L}_\alpha$.

The following result is suggested by an analogy with quantum field theory (cf. [5]).

PROPOSITION 16. (i) $\frac{\partial\mu}{\partial\kappa_x}=-\widehat{l}^x\mu$.

(ii) *If* $C_{x,y}>0$, $\frac{\partial\mu}{\partial C_{x,y}}=-T_{x,y}\mu$ *with* $T_{x,y}(l)=(\widehat{l}^x+\widehat{l}^y)-\frac{N_{x,y}}{C_{x,y}}(l)-\frac{N_{y,x}}{C_{x,y}}(l)$.

PROOF. Recall that by formula (6): $\mu^*(p=1,\xi=x,\widehat{\tau}\in dt)=e^{-\lambda_x t}\frac{dt}{t}$ and $\mu^*(p=k,\xi_i=x_i,\widehat{\tau}_i\in dt_i)=\frac{1}{k}\prod_{x,y}C_{x,y}^{N_{x,y}}\prod_x\lambda_x^{-N_x}\prod_{i\in\mathbb{Z}/p\mathbb{Z}}\lambda_{\xi_i}e^{-\lambda_{\xi_i}t_i}\,dt_i$.

Moreover, we have $C_{x,y}=C_{y,x}=\lambda_x P_y^x$ and $\lambda_x=\kappa_x+\sum_y C_{x,y}$.

The two formulas follow by elementary calculation. □

Recall that $\mu(\widehat{l^x})=G^{x,x}$ and $\mu(N_{x,y})=G^{x,y}C_{x,y}$.

So, we have $\mu(T_{x,y})=G^{x,x}+G^{y,y}-2G^{x,y}$.

Then, the above proposition allows us to compute all moments of T and $\widehat{l}$ relative to μ_e (they could be called Schwinger functions). The above proposition gives the infinitesimal form of the following formula.

PROPOSITION 17. *Consider another energy form e' defined on the same graph. Then we have the following identity*:

$$\frac{\partial\mu_{e'}}{\partial\mu_e}=e^{\sum N_{x,y}\log(C'_{x,y}/C_{x,y})-\sum(\lambda'_x-\lambda_x)\widehat{l^x}}.$$

Consequently,

$$\mu_e\big(\big(e^{\sum N_{x,y}\log(C'_{x,y}/C_{x,y})-\sum(\lambda'_x-\lambda_x)\widehat{l^x}}-1\big)\big)=\log\bigg(\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e}\bigg). \tag{18}$$

PROOF. The first formula is a straightforward consequence of (6). The proof of (18) goes by evaluating separately the contribution of trivial loops, which equals $\sum_x\log(\frac{\lambda_x}{\lambda'_x})$. Indeed,

$$\begin{aligned}&\mu_e\big(\big(e^{\sum N_{x,y}\log(C'_{x,y}/C_{x,y})-\sum(\lambda'_x-\lambda_x)\widehat{l^x}}-1\big)\big)\\&\qquad=\mu_{e'}(p>1)-\mu_e(p>1)+\mu_e\big(1_{\{p=1\}}\big(e^{\sum(\lambda'_x-\lambda_x)\widehat{l^x}}-1\big)\big).\end{aligned}$$

The difference of the first two terms equals $\log(\mathcal{Z}_{e'})+\sum\log(\lambda'_x)-(\log(\mathcal{Z}_e)-\sum\log(\lambda_x))$. The last term equals $\sum_x\int_0^\infty(e^{-((\lambda'_x-\lambda_x)/\lambda_x)t}-1)\frac{e^{-t}}{t}\,dt$ which can be computed as before:

$$\mu_e\big(1_{\{p=1\}}\big(e^{\sum(\lambda'_x-\lambda_x)\widehat{l^x}}-1\big)\big)=-\sum\log\bigg(\frac{\lambda'_x}{\lambda_x}\bigg). \tag{19}$$

□

REMARK 18 (h-transforms). *Note that if $C'_{x,y}=h^xh^yC_{x,y}$ and $\kappa'_x=-h^x(Lh)^x\lambda_x$ for some positive function h on E such that $Lh\le 0$, as $\lambda'=h^2\lambda$ and $[P']_y^x=\frac{1}{h^x}P_y^xh^y$, we have $[G']^{x,y}=\frac{G^{x,y}}{h^xh^y}$ and $\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e}=\frac{1}{\prod(h^x)^2}$.*

REMARK 19. *Note also that $[\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e}]^{1/2}=\mathbb{E}(e^{-(1/2)[e'-e](\phi)})$, if ϕ is the Gaussian free field associated with e.*

Integrating out the holding times, formula (18) can be written equivalently:

$$\mu_e\left(\prod_{(x,y)}\left[\frac{C'_{x,y}}{C_{x,y}}\right]^{N_{x,y}}\prod_x\left[\frac{\lambda_x}{\lambda'_x}\right]^{N_x+1}-1\right)=\log\left(\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e}\right) \tag{20}$$

and therefore

$$\mathbb{E}_{\mathcal{L}_\alpha}\left(\prod_{(x,y)}\left[\frac{C'_{x,y}}{C_{x,y}}\right]^{N^{(\alpha)}_{x,y}}\prod_x\left[\frac{\lambda_x}{\lambda'_x}\right]^{N^{(\alpha)}_x+1}\right)=\mathbb{E}_{\mathcal{L}_\alpha}\left(\prod_{(x,y)}\left[\frac{C'_{x,y}}{C_{x,y}}\right]^{N^{(\alpha)}_{x,y}}e^{-\langle\lambda'-\lambda,\widehat{\mathcal{L}_\alpha}\rangle}\right)$$
$$=\left(\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e}\right)^{\alpha}.$$

Note also that $\prod_{(x,y)}[\frac{C'_{x,y}}{C_{x,y}}]^{N_{x,y}}=\prod_{\{x,y\}}[\frac{C'_{x,y}}{C_{x,y}}]^{N_{x,y}+N_{y,x}}$.

REMARK 20. *These $\frac{\mathcal{Z}_{e'}}{\mathcal{Z}_e}$ determine, when e' varies with $\frac{C'}{C}\leq 1$ and $\frac{\lambda'}{\lambda}=1$, the Laplace transform of the distribution of the traversal numbers of nonoriented links $N_{x,y}+N_{y,x}$.*

Other variables of interest on the loop space are associated with elements of the space $\mathbb{A}^-$ of odd functions ω on oriented links: $\omega^{x,y}=-\omega^{y,x}$. Let us mention a few elementary results.

The operator $[P^{(\omega)}]^x_y=P^x_y\exp(i\omega^{x,y})$ is also self adjoint in $L^2(\lambda)$. The associated loop variable is written $\sum_{x,y}\omega^{x,y}N_{x,y}(l)$. We will denote it $\int_l\omega$. Note it is invariant if $\omega^{x,y}$ is replaced by $\omega^{x,y}+g^y-g^x$ for some g. Set $[G^{(\omega)}]^{x,y}=\frac{[(I-P^{(\omega)})^{-1}]^x_y}{\lambda_y}$. By an argument similar to the one given above for the occupation field, we have: $\mathbb{P}^t_{x,x}(e^{i\int_l\omega}-1)=\exp(t(P^{(\omega)}-I))_{x,x}-\exp(t(P-I))_{x,x}$. Integrating in t after expanding, we get from the definition of μ

$$\int(e^{i\int_l\omega}-1)\,d\mu(l)=\sum_{k=1}^{\infty}\frac{1}{k}\big[\mathrm{Tr}\big((P^{(\omega)})^k\big)-\mathrm{Tr}((P)^k)\big].$$

Hence,

$$\int(e^{i\int_l\omega}-1)\,d\mu(l)=\log\big[\det\big(-L(I-P^{(\omega)})^{-1}\big)\big].$$

Hence, $\int(e^{i\int_l\omega}-1)\,d\mu(l)=\log[\det(-L(I-P^{(\omega)})^{-1})]$ and

$$\int\left(\exp\left(i\int_l\omega\right)-1\right)\mu(dl)=\log\big(\det\big(G^{(\omega)}G^{-1}\big)\big).$$

We can now extend the previous results (18) and (20) to obtain, setting $\det(G^{(\omega)})=\mathcal{Z}_{e,\omega}$

$$\mu_e\big(e^{-\sum N_{x,y}\log(C'_{x,y}/C_{x,y})-\sum(\lambda'_x-\lambda_x)\widehat{l}_x+i\int_l\omega}-1\big)=\log\left(\frac{\mathcal{Z}_{e',\omega}}{\mathcal{Z}_e}\right) \tag{21}$$

and

$$\mathbb{E}\left(\prod_{x,y}\left[\frac{C'_{x,y}}{C_{x,y}}e^{i\omega_{x,y}}\right]^{N^{(\alpha)}_{x,y}} e^{-\sum(\lambda'_x-\lambda_x)\widehat{\mathcal{L}_\alpha}^x}\right) = \left(\frac{\mathcal{Z}_{e',\omega}}{\mathcal{Z}_e}\right)^{\alpha}.$$

Let us now introduce a new definition.

DEFINITION 21. We say that sets Λ_i of nontrivial loops are equivalent when the associated occupation fields are equal and when the total traversal numbers $\sum_{l\in\Lambda_i} N_{x,y}(l)$ are equal for all oriented edges (x, y). Equivalence classes will be called loop networks on the graph. We denote $\overline{\Lambda}$ the loop network defined by Λ.

Similarly, a set L of nontrivial discrete loops defines a discrete network characterized by the total traversal numbers.

Note that these expectations determine the distribution of the network $\overline{\mathcal{L}_\alpha}$ defined by the loop ensemble $\mathcal{L}_\alpha$. We will denote $B^{e,e',\omega}$ the variables

$$\prod_{x,y}\left[\frac{C'_{x,y}}{C_{x,y}}e^{i\omega_{x,y}}\right]^{N^{(\alpha)}_{x,y}} e^{-\sum(\lambda'_x-\lambda_x)\widehat{\mathcal{L}_\alpha}^x}.$$

REMARK 22. *This last formula applies to the calculation of loop indices: If we have for example a simple random walk on an oriented planar graph, and if z' is a point of the dual graph X', $\omega_{z'}$ can be chosen such that $\int_l \omega_{z'}$ is the winding number of the loop around a given point z' of the dual graph X'. Then $e^{i\pi\sum_{l\in\mathcal{L}_\alpha}\int_l \omega'_z}$ is a spin system of interest. We then get for example that*

$$\mu\left(\int_l \omega_{z'} \neq 0\right) = -\frac{1}{2\pi}\int_0^{2\pi} \log\left(\det\left(G^{(2\pi u\omega_{z'})}G^{-1}\right)\right) du$$

and hence

$$\mathbb{P}\left(\sum_{l\in\mathcal{L}_\alpha}\left|\int_l \omega_{z'}\right| = 0\right) = e^{\alpha/(2\pi)\int_0^{2\pi}\log(\det(G^{(2\pi u\omega_{z'})}G^{-1})\,du}.$$

Conditional distributions of the occupation field with respect to values of the winding number can also be obtained.

7. Loop erasure and spanning trees. Recall that an oriented link g is a pair of points (g^-, g^+) such that $C_g = C_{g^-,g^+} \neq 0$. Define $-g = (g^+, g^-)$.

Let $\mu^{\neq}_{x,y}$ be the measure induced by C on discrete self-avoiding paths between x and y: $\mu^{x,y}_{\neq}(x, x_2, \dots, x_{n-1}, y) = C_{x,x_2}C_{x_1,x_3}\cdots C_{x_{n-1},y}$.

Another way to define a measure on discrete self avoiding paths from x to y is loop erasure (see [7, 18] and [8]). In this context, the loops can be trivial as they correspond to a single holding times, and loop erasure produces a discrete path without holding times.

We have the following proposition.

PROPOSITION 23. *The image of* $\mu^{x,y}$ *by the loop erasure map* $\gamma \to \gamma^{\mathrm{BE}}$ *is* $\mu_{\mathrm{BE}}^{x,y}$ *defined on self-avoiding paths by* $\mu_{\mathrm{BE}}^{x,y}(\eta) = \mu_{\neq}^{x,y}(\eta)\frac{\det(G)}{\det(G^{\{\eta\}^c})} = \mu_{\neq}^{x,y}(\eta)\det(G_{|\{\eta\}\times\{\eta\}})$ *(here,* $\{\eta\}$ *denotes the set of points in the path* η*).*

PROOF. If $\eta = (x_1 = x, x_2, \dots, x_n = y)$, and $\eta_m = (x, \dots, x_m)$, with $m > 1$

$$\mu^{x,y}(\gamma^{\mathrm{BE}} = \eta) = \sum_{k=0}^{\infty} [P^k]_x^x P_{x_2}^x \mu_{\{x\}^c}^{x_2,y}(\gamma^{\mathrm{BE}} = \theta\eta),$$

where $\mu_{\{x\}^c}^{x_2,y}$ denotes the bridge measure for the Markov chain killed as it hits x and θ the natural shift on discrete paths. By recurrence, this clearly equals

$$V_x^x P_{x_2}^x [V^{\{x\}^c}]_{x_2}^{x_2} \cdots [V^{\{\eta_{n-1}\}^c}]_{x_{n-1}}^{x_{n-1}} P_y^{x_{n-1}} [V^{\{\eta\}^c}]_y^y \lambda_y^{-1} = \mu_{\neq}^{x,y}(\eta)\frac{\det(G)}{\det(G^{\{\eta\}^c})}$$

as

$$\begin{aligned}
[V^{\{\eta_{m-1}\}^c}]_{x_m}^{x_m} &= \frac{\det([I-P]|_{\{\eta_m\}^c\times\{\eta_m\}^c})}{\det([I-P]|_{\{\eta_{m-1}\}^c\times\{\eta_{m-1}\}^c})} \\
&= \frac{\det(V^{\{\eta_{m-1}\}^c})}{\det(V^{\{\eta_m\}^c})} = \frac{\det(G^{\{\eta_{m-1}\}^c})}{\det(G^{\{\eta_m\}^c})}\lambda^{x_m}
\end{aligned}$$

for all $m \le n-1$. □

Also, by the Feynman–Kac formula, for any self-avoiding path η

$$\begin{aligned}
\int e^{-\langle\widehat{\gamma},\chi\rangle} 1_{\{\gamma^{\mathrm{BE}}=\eta\}} \mu^{x,y}(d\gamma) &= \frac{\det(G_\chi)}{\det(G_\chi^{\{\eta\}^c})}\mu_{\neq}^{x,y}(\eta) = \det(G_\chi)_{|\{\eta\}\times\{\eta\}}\mu_{\neq}^{x,y}(\eta) \\
&= \frac{\det(G_\chi)_{|\{\eta\}\times\{\eta\}}}{\det(G_{|\{\eta\}\times\{\eta\}})}\mu_{\mathrm{BE}}^{x,y}(\eta).
\end{aligned}$$

Therefore, recalling that by the results of Section 4.3 conditionally to η, $\mathcal{L}_1/\mathcal{L}_1^{\{\eta\}^c}$ and $\mathcal{L}_1^{\{\eta\}^c}$ are independent, we see that under $\mu^{x,y}$, the conditional distribution of $\widehat{\gamma}$ given $\gamma^{\mathrm{BE}} = \eta$ is the distribution of $\widehat{\mathcal{L}}_1 - \widehat{\mathcal{L}}_1^{\{\eta\}^c}$, that is, the occupation field of the loops of $\mathcal{L}_1$ which intersect η.

More generally, it can be shown by the following.

PROPOSITION 24. *The conditional distribution of the network* $\overline{\mathcal{L}_\gamma}$ *defined by the loops of* γ*, given that* $\gamma^{\mathrm{BE}} = \eta$*, is identical to the distribution of the network defined by* $\mathcal{L}_1/\mathcal{L}_1^{\{\eta\}^c}$*, that is, the loops of* $\mathcal{L}_1$ *which intersect* η*.*

PROOF. Recall the notation $\mathcal{Z}_e = \det(G)$. First, an elementary calculation using (6) shows that $\mu_{e'}^{x,y}(e^{i\int_\gamma \omega} 1_{\{\gamma^{\mathrm{BE}}=\eta\}})$ equals

$$\mu_e^{x,y}\left(1_{\{\gamma^{\mathrm{BE}}=\eta\}} \prod \left[\frac{C'_{\xi_i,\xi_{i+1}}}{C_{\xi_i,\xi_{i+1}}} e^{i\omega_{\xi_i,\xi_{i+1}}} \frac{\lambda_{\xi_i}}{\lambda'_{\xi_i}}\right]\right) \frac{C'_{x,x_2} C'_{x_1,x_3} \cdots C'_{x_{n-1},y}}{C_{x,x_2} C_{x_1,x_3} \cdots C_{x_{n-1},y}}$$
$$\times e^{i\int_\eta \omega} \mu_e^{x,y}\left(\prod_{u \neq v} \left[\frac{C'_{u,v}}{C_{u,v}} e^{i\omega_{u,v}}\right]^{N_{u,v}(\mathcal{L}_\gamma)} e^{-\langle \lambda' - \lambda, \widehat{\gamma}\rangle} 1_{\{\gamma^{\mathrm{BE}}=\eta\}}\right).$$

[Note the term $e^{-\langle \lambda'-\lambda,\widehat{\gamma}\rangle}$ can be replaced by $\prod_u (\frac{\lambda_u}{\lambda'_u})^{N_u(\gamma)}$.]

Moreover, by the proof of the previous proposition, applied to the Markov chain defined by e' perturbed by ω, we have also

$$\mu_{e'}^{x,y}(e^{i\int_\gamma \omega} 1_{\{\gamma^{\mathrm{BE}}=\eta\}}) = C'_{x,x_2} C'_{x_1,x_3} \cdots C'_{x_{n-1},y} e^{i\int_\eta \omega} \frac{\mathcal{Z}_{e',\omega}}{\mathcal{Z}_{[e']^{\{\eta\}^c},\omega}}.$$

Therefore,

$$\mu_e^{x,y}\left(\prod_{u\neq v}\left[\frac{C'_{u,v}}{C_{u,v}} e^{i\omega_{u,v}}\right]^{N_{u,v}(\mathcal{L}_\gamma)} e^{-\langle \lambda'-\lambda,\widehat{\gamma}\rangle} \parallel \gamma^{\mathrm{BE}} = \eta\right) = \frac{\mathcal{Z}_{e^{\{\eta\}^c}} \mathcal{Z}_{e',\omega}}{\mathcal{Z}_e \mathcal{Z}_{[e']^{\{\eta\}^c},\omega}}.$$

Moreover, by (21) and the properties of the Poisson processes,

$$\mathbb{E}\left(\prod_{u\neq v}\left[\frac{C'_{u,v}}{C_{u,v}} e^{i\omega_{u,v}}\right]^{N_{u,v}(\mathcal{L}_1/\mathcal{L}_1^{\{\eta\}^c})} e^{-\langle \lambda'-\lambda,\widehat{\mathcal{L}}_1-\widehat{\mathcal{L}}_1^{\{\eta\}^c}\rangle}\right) = \frac{\mathcal{Z}_{e^{\{\eta\}^c}} \mathcal{Z}_{e',\omega}}{\mathcal{Z}_e \mathcal{Z}_{[e']^{\{\eta\}^c},\omega}}.$$

It follows that the joint distribution of the traversal numbers and the occupation field are identical for the set of erased loops and $\mathcal{L}_1/\mathcal{L}_1^{\{\eta\}^c}$. □

Similarly, one can define the image of $\mathbb{P}^x$ by BE which is given by

$$\mathbb{P}^x_{\mathrm{BE}}(\eta) = C_{x_1,x_2} \cdots C_{x_{n-1},x_n} \kappa_{x_n} \det(G_{|\{\eta\}\times\{\eta\}})$$

for $\eta = (x_1, \dots, x_n)$, and get the same results.

Wilson's algorithm (see [15]) iterates this construction, starting with $x's$ in arbitrary order. Each step of the algorithm reproduces the first step except it stops when it hits the already constructed tree of self-avoiding paths. It provides a construction of a random spanning tree. Its law is a probability measure $\mathbb{P}^e_{\mathrm{ST}}$ on the set $\mathrm{ST}_{X,\Delta}$ of spanning trees of X rooted at the cemetery point Δ defined by the energy e. The weight attached to each oriented link $g = (x, y)$ of $X \times X$ is the conductance and the weight attached to the link (x, Δ) is κ_x which we can also denote by $C_{x,\Delta}$. As the determinants simplify, the probability of a tree Υ is given by a simple formula

$$\mathbb{P}^e_{\mathrm{ST}}(\Upsilon) = \mathcal{Z}_e \prod_{\xi\in\Upsilon} C_\xi. \tag{22}$$

It is clearly independent of the ordering chosen initially. Now note that, since we get a probability

$$\mathcal{Z}_e \sum_{\Upsilon \in \mathrm{ST}_{X,\Delta}} \prod_{(x,y)\in\Upsilon} C_{x,y} \prod_{x,(x,\Delta)\in\Upsilon} \kappa_x = 1$$

or equivalently

$$\sum_{\Upsilon \in \mathrm{ST}_{X,\Delta}} \prod_{(x,y)\in\Upsilon} P_y^x \prod_{x,(x,\Delta)\in\Upsilon} P_\Delta^x = \frac{1}{\prod_{x\in X} \lambda_x \mathcal{Z}_e}.$$

Then, it follows that, for any e' for which conductances (including κ') are positive only on links of e,

$$\mathbb{E}_{\mathrm{ST}}^e \left(\prod_{(x,y)\in\Upsilon} \frac{P_y'^x}{P_y^x} \prod_{x,(x,\Delta)\in\Upsilon} \frac{P_\Delta'^x}{P_\Delta^x} \right) = \frac{\prod_{x\in X} \lambda_x}{\prod_{x\in X} \lambda_x'} \frac{\mathcal{Z}_e}{\mathcal{Z}_{e'}}$$

and

$$\mathbb{E}_{\mathrm{ST}}^e \left(\prod_{(x,y)\in\Upsilon} \frac{C'_{x,y}}{C_{x,y}} \prod_{x,(x,\Delta)\in\Upsilon} \frac{\kappa'_x}{\kappa_x} \right) = \frac{\mathcal{Z}_e}{\mathcal{Z}_{e'}}. \tag{23}$$

Note also that in the case of a graph (i.e., when all conductances are equal to 1), all spanning trees have the same probability. The expression of their cardinal as the determinant $\mathcal{Z}_e$ is Cayley's theorem (see, e.g., [15]).

COROLLARY 25. *The network defined by the random set of loops $\mathcal{L}_W$ constructed in this algorithm is independent of the random spanning tree, and independent of the ordering. It has the same distribution as the network defined by the loops of $\mathcal{L}_1$.*

This result follows easily from Proposition 24.

8. Decompositions. Note first that with the energy e, we can associate a rescaled Markov chain $\widehat{x}_t$ in which holding times at any point x are exponential times of parameters λ_x: $\widehat{x}_t = x_{\tau_t}$ with $\tau_t = \inf(s, \int_0^s \frac{1}{\lambda_{x_u}} du = t)$. For the rescaled Markov chain, local times coincide with the time spent in a point and the duality measure is simply the counting measure. The Markov loops can be rescaled as well and we did it in fact already when we introduced pointed loops. More generally, we may introduce different holding time parameters but it would essentially be useless as the random variables we are interested in are intrinsic, that is, depend only on e.

If $D \subset X$ and we set $F = D^c$, the orthogonal decomposition of the energy $e(f, f) = e(f)$ into $e^D(f - H^F f) + e(H^F f)$ leads to the decomposition of the Gaussian field mentioned above and also to a decomposition of the rescaled Markov chain into the rescaled Markov chain killed at the exit of D and the trace of the rescaled Markov chain on F, that is, $\widehat{x}_t^{\{F\}} = \widehat{x}_{S_t^F}$, with $S_t^F = \inf(s, \int_0^s 1_F(\widehat{x}_u)\, du = t)$.

PROPOSITION 26. *The trace of the rescaled Markov chain on F is the rescaled Markov chain defined by the energy functional $e^{\{F\}}(f) = e(H^F f)$, for which*

$$C_{x,y}^{\{F\}} = C_{x,y} + \sum_{a,b\in D} C_{x,a} C_{b,y} [G^D]^{a,b},$$

$$\lambda_x^{\{F\}} = \lambda_x - \sum_{a,b\in D} C_{x,a} C_{b,x} [G^D]^{a,b}$$

and

$$\mathcal{Z}_e = \mathcal{Z}_{e^D} \mathcal{Z}_{e^{\{F\}}}.$$

PROOF. For the second assertion, note first that for any $y \in F$,

$$[H^F]_y^x = 1_{x=y} + 1_D(x) \sum_{b\in D} [G^D]^{x,b} C_{b,y}.$$

Moreover, $e(H^F f) = e(f, H^F f)$ and therefore

$$\lambda_x^{\{F\}} = e^{\{F\}}(1_{\{x\}}) = e(1_{\{x\}}, H^F 1_{\{x\}}) = \lambda_x - \sum_{a\in D} C_{x,a} [H^F]_x^a = \lambda_x (1 - p_x^{\{F\}}),$$

where $p_x^{\{F\}} = \sum_{a,b\in D} P_a^x [G^D]^{a,b} C_{b,x} = \sum_{a\in D} P_a^x [H^F]_x^a$ is the probability that the Markov chain starting at x will return to x after an excursion in D.

Then for distinct x and y in F

$$\begin{aligned} C_{x,y}^{\{F\}} &= -e^{\{F\}}(1_{\{x\}}, 1_{\{y\}}) = -e(1_{\{x\}}, H^F 1_{\{y\}}) \\ &= C_{x,y} + \sum_a C_{x,a} [H^F]_y^a = C_{x,y} + \sum_{a,b\in D} C_{x,a} C_{b,y} [G^D]^{a,b}. \end{aligned}$$

Note that the graph defined on F by the nonvanishing conductances $C_{x,y}^{\{F\}}$ has in general more edges than the restriction to F of the original graph.

For the third assertion, note also that $G^{\{F\}}$ is the restriction of G to F as for all $x, y \in F$, $e^{\{F\}}(G\delta_{y|F}, 1_{\{x\}}) = e(G\delta_y, [H^F 1_{\{x\}}]) = 1_{\{x=y\}}$. Hence, the determinant decomposition already used in Section 4.3 yields the final formula. The cases where F has one point was already treated in Section 4.3.

Finally, for the first assertion, note that the transition matrix $[P^{\{F\}}]_y^x$ can be computed directly and equals $P_y^x + \sum_{a,b\in D} P_a^x P_y^b [V^{D\cup\{x\}}]_b^a = P_y^x + \sum_{a,b\in D} P_a^x \times$

$C_{b,y}[G^{D\cup\{x\}}]^{a,b}$. It can be decomposed according to whether the jump to y occurs from x or from D and the number of excursions from x to x:

$$[P^{\{F\}}]^x_y = \sum_{k=0}^{\infty}\Big(\sum_{a,b\in D} P^x_a[V^D]^a_b P^b_x\Big)^k\Big(P^x_y + \sum_{a,b\in D} P^x_a[V^D]^a_b P^b_y\Big)$$
$$= \sum_{k=0}^{\infty}\Big(\sum_{a,b\in D} P^x_a[G^D]^{a,b}C_{b,x}\Big)^k\Big(P^x_y + \sum_{a,b\in D} P^x_a[G^D]^{a,b}C_{b,y}\Big).$$

The expansion of $\frac{C^{\{F\}}_{x,y}}{\lambda^{\{F\}}_x}$ in geometric series yields exactly the same result.

Finally, remark that the holding times of $\widehat{x}^{\{F\}}_t$ at any point $x \in F$ are sums of a random number of independent holding times of $\widehat{x}_t$. This random integer counts the excursions from x to x performed by the chain $\widehat{x}_t$ during the holding time of $\widehat{x}^{\{F\}}_t$. It follows a geometric distribution of parameter $1 - p^{\{F\}}_x$. Therefore, $\frac{1}{\lambda^{\{F\}}_x} = \frac{1}{\lambda_x(1-p_x)}$ is the expectation of the holding times of $\widehat{x}^{\{F\}}_t$ at x. $\square$

If χ is carried by D and if we set $e_\chi = e + \|\cdot\|_{L^2(\chi)}$, and denote $[e_\chi]^{\{F\}}$ by $e^{\{F,\chi\}}$, we have

$$C^{\{F,\chi\}}_{x,y} = C_{x,y} + \sum_{a,b} C_{x,a}C_{b,y}[G^D_\chi]^{a,b}, \qquad p^{\{F,\chi\}}_x = \sum_{a,b\in D} P^x_a[G^D_\chi]^{a,b}C_{b,x}$$

and $\lambda^{\{F,\chi\}}_x = \lambda_x(1 - p^{\{F,\chi\}}_x)$.

More generally, if $e^{\#}$ is such that $C^{\#} = C$ on $F \times F$, and $\lambda = \lambda^{\#}$ on F we have

$$C^{\#\{F\}}_{x,y} = C_{x,y} + \sum_{a,b} C^{\#}_{x,a}C^{\#}_{b,y}[G^{\#D}]^{a,b}, \qquad p^{\#\{F\}}_x = \sum_{a,b\in D} P^{\#x}_a[G^{\#D}]^{a,b}C_{b,x}$$

and $\lambda^{\#\{F\}}_x = \lambda_x(1 - p^{\#\{F\}}_x)$.

A loop in X which hits F can be decomposed into a loop $l^{\{F\}}$ in F and its excursions in D which may come back to their starting point. Let $\mu^{a,b}_D$ denote the bridge measure (with mass $[G^D]^{a,b}$) associated with e^D.

Set

$$\nu^D_{x,y} = \frac{1}{C^{\{F\}}_{x,y}}\Big[C_{x,y}\delta_\varnothing + \sum_{a,b\in D} C_{x,a}C_{b,y}\mu^{a,b}_D\Big],$$

$$\rho^D_x = \sum_{n=1}^{\infty}\frac{1}{\lambda_x p^{\{F\}}_x}\Big(\sum_{a,b\in D} C_{x,a}C_{b,x}\mu^{a,b}_D\Big)$$

and $\nu^D_x = \frac{1}{1-p^{\{F\}}_x}[\delta_\varnothing + \sum_{n=1}^{\infty}[p^{\{F\}}_x\rho^D_x]^{\otimes n}]$.

Note that $\rho^D_x(1) = \nu^D_{x,y}(1) = \nu^D_x(1) = 1$.

A loop l can be decomposed into its restriction $l^{\{F\}} = (\xi_i, \widehat{\tau}_i)$ in F (possibly a one point loop), a family of excursions $\gamma_{\xi_i,\xi_{i+1}}$ attached to the jumps of $l^{\{F\}}$ and systems of i.i.d. excursions $(\gamma^h_{\xi_i}, h \le n_{\xi_i})$ attached to the points of $l^{\{F\}}$. Note the set of excursions can be empty.

We get a decomposition of μ into its restriction μ^D to loops in D (associated to the process killed at the exit of D), the loop measure $\mu^{\{F\}}$ defined on loops of F by the trace of the Markov chain on F, probability measures $\nu^D_{x,y}$ on excursions in D indexed by pairs of points in F and measures ρ^D_x on excursions in D indexed by points of F. Moreover, the integers n_{ξ_i} follow a Poisson distribution of parameter $\lambda^{\{F\}}_{\xi_i}\widehat{\tau}_i$ and the conditional distribution of the rescaled holding times in ξ_i before each excursion $\gamma^l_{\xi_i}$ is the distribution $\beta_{n_{\xi_i},\tau^*_i}$ of the increments of a uniform sample of n_{ξ_i} points in $[0, \widehat{\tau}_i]$ put in increasing order. We denote these holding times by $\widehat{\tau}_{i,h}$ and set $l = \Lambda(l^{\{F\}}, (\gamma_{\xi_i,\xi_{i+1}}), (n_{\xi_i}, \gamma^h_{\xi_i}, \widehat{\tau}_{i,h}))$.

Then $\mu - \mu^D$ is the image measure by Λ of

$$\mu^{\{F\}}(dl^{\{F\}}) \prod (\nu^D_{\xi_i,\xi_{i+1}})(d\gamma_{\xi_i,\xi_{i+1}}) \prod e^{-\lambda^{\{F\}}_{\xi_i}\widehat{\tau}_i} \times \sum \frac{[\lambda^{\{F\}}_{\xi_i}\widehat{\tau}_i]^k}{k!} 1_{n_{\xi_i}=k} [\rho^D_x]^{\otimes k}(d\gamma^h_{\xi_i}) \beta_{k,\tau^*_i}(d\widehat{\tau}_{i,h}).$$

The Poisson process $\mathcal{L}^{\{F\}}_\alpha = \{l^{\{F\}}, l \in \mathcal{L}_\alpha\}$ has intensity $\mu^{\{F\}}$ and is independent of $\mathcal{L}^D_\alpha$.

Note that $\widehat{\mathcal{L}^{\{F\}}_\alpha}$ is the restriction of $\widehat{\mathcal{L}_\alpha}$ to F. If χ is a measure carried by D, we have

$$\begin{aligned}
\mathbb{E}\big(e^{-\langle \widehat{\mathcal{L}_\alpha}, \chi\rangle} \,|\, \mathcal{L}^{\{F\}}_\alpha\big) &= \mathbb{E}\big(e^{-\langle \widehat{\mathcal{L}^D_\alpha}, \chi\rangle}\big) \prod_{x,y\in F} \left[\int e^{-\langle \widehat{\gamma}, \chi\rangle} \nu^D_{x,y}(d\gamma)\right]^{N_{x,y}(\mathcal{L}^{\{F\}}_\alpha)} \\
&\quad \times \prod_{x\in F} e^{\lambda^{\{F\}}_x [\widehat{\mathcal{L}^{\{F\}}_\alpha}]^x \int (e^{-\langle \widehat{\gamma},\chi\rangle} - 1)\rho^D_x(d\gamma)} \\
&= \left[\frac{\mathcal{Z}_{e^D_\chi}}{\mathcal{Z}_{e^D}}\right]^\alpha \prod_{x,y\in F} \left[\frac{C^{\{F,\chi\}}_{x,y}}{C^{\{F\}}_{x,y}}\right]^{N_{x,y}(\mathcal{L}^{\{F\}}_\alpha)} \prod_{x\in F} e^{[\lambda^{\{F,\chi\}}_x - \lambda^{\{F\}}_x]\widehat{\mathcal{L}^x_\alpha}}.
\end{aligned}$$

(Recall that $\widehat{\mathcal{L}^{\{F\}}_\alpha}$ is the restriction of $\widehat{\mathcal{L}_\alpha}$ to F.) Also, if we condition on the set of discrete loops $\mathcal{D}\mathcal{L}^{\{F\}}_\alpha$

$$\mathbb{E}\big(e^{-\langle \widehat{\mathcal{L}_\alpha}, \chi\rangle} \,|\, \mathcal{D}\mathcal{L}^{\{F\}}_\alpha\big) = \left[\frac{\mathcal{Z}_{e^D_\chi}}{\mathcal{Z}_{e^D}}\right]^\alpha \left(\prod_{x,y\in F} \left[\frac{C^{\{F,\chi\}}_{x,y}}{C^{\{F\}}_{x,y}}\right]^{N_{x,y}(\mathcal{L}^{\{F\}}_\alpha)} \prod_{x\in F} \left[\frac{\lambda^{\{F\}}_x}{\lambda^{\{F,\chi\}}_x}\right]^{N_x(\mathcal{L}^{\{F\}}_\alpha)+1} \right),$$

where the last exponent $N_x + 1$ is obtained by taking into account the loops which have a trivial trace on F [see formula (19)].

More generally, we can show in the same way the following.

PROPOSITION 27. *If $C^{\#} = C$ on $F \times F$, and $\lambda = \lambda^{\#}$ on F, we denote $B^{e,e^{\#}}$ the multiplicative functional $\prod_{x,y}[\frac{C^{\#}_{x,y}}{C_{x,y}}]^{N_{x,y}} e^{-\sum_{x\in D}\widehat{l_x}(\lambda^{\#}_x-\lambda_x)}$.*

Then

$$\mathbb{E}(B^{e,e^{\#}}|\mathcal{L}_\alpha^{\{F\}}) = \left[\frac{\mathcal{Z}_{e^{\#D}}}{\mathcal{Z}_{e^D}}\right]^\alpha \prod_{x,y\in F}\left[\frac{C^{\#\{F\}}_{x,y}}{C^{\{F\}}_{x,y}}\right]^{N_{x,y}(\mathcal{L}_\alpha^{\{F\}})} \prod_{x\in F} e^{\lambda_x[p^{\#\{F\}}_x - p^{\{F\}}_x]\widehat{\mathcal{L}^x_\alpha}}$$

and

$$\mathbb{E}(B^{e,e^{\#}}|\mathcal{D}\mathcal{L}_\alpha^{\{F\}}) = \left[\frac{\mathcal{Z}_{e^{\#D}}}{\mathcal{Z}_{e^D}}\right]^\alpha \prod_{x,y\in F}\left[\frac{C^{\#\{F\}}_{x,y}}{C^{\{F\}}_{x,y}}\right]^{N_{x,y}(\mathcal{L}_\alpha^{\{F\}})} \prod_{x\in F}\left[\frac{\lambda^{\{F\}}_x}{\lambda^{\#\{F\}}_x}\right]^{N_x(\mathcal{L}_\alpha^{\{F\}})+1}.$$

These decomposition and conditional expectation formulas extend to include a current ω. Note that $e^{\{F\}}$ will depend on ω unless it is closed (i.e., vanish on every loop) in D. In particular, it allows to define ω^F such that

$$\mathcal{Z}_{e,\omega} = \mathcal{Z}_{e^D}\mathcal{Z}_{e^{\{F\}},\omega^F}.$$

The previous proposition implies the following *Markov property*.

REMARK 28. *If $D = D_1 \cup D_2$ with D_1 and D_2 stongly disconnected* (*i.e., such that for any* $(x, y, z) \in D_1 \times D_2 \times F$, *$C_{x,y}$ and $C_{x,z}C_{y,z}$ vanish*), *the restrictions of the network $\overline{\mathcal{L}_\alpha}$ to $D_1 \cup F$ and $D_2 \cup F$ are independent conditionally on the restriction of $\mathcal{L}_\alpha$ to F.*

PROOF. It follows from the fact that as D_1 and D_2 are disconnected, any excursion measure $\nu^D_{x,y}$ or ρ^D_x from F into $D = D_1 \cup D_2$ is an excursion measure either in D_1 or in D_2. □

Branching processes with immigration. An interesting example can be given after extending slightly the scope of the theory to countable transient symmetric Markov chains: We can take $X = \mathbb{N} - \{0\}$, $C_{n,n+1} = 1$ for all $n \geq 1$ and $\kappa_1 = 1$ and P to be the transfer matrix of the simple symmetric random walk killed at 0.

Then we can apply the previous considerations to check that $\widehat{\mathcal{L}}^n_\alpha$ is a branching process with immigration.

The immigration at level n comes from the loops whose infimum is n and the branching from the excursions of the loops existing at level n to level $n + 1$. Set $F_n = \{1, 2, \ldots, n\}$ and $D_n = F_n^c$.

From the calculations of conditional expectations made above, we get that for any positive parameter γ

$$\mathbb{E}(e^{-\gamma\widehat{\mathcal{L}}^n_\alpha} \parallel \mathcal{L}_\alpha^{\{F_{n-1}\}}) = \mathbb{E}(e^{-\gamma[\widehat{\mathcal{L}}^{D_{n-1}}_\alpha]^n}) e^{[\lambda^{\{F_{n-1},\gamma\delta_n\}}_{n-1} - \lambda^{\{F_{n-1}\}}_{n-1}]\widehat{\mathcal{L}}^{n-1}_\alpha}.$$

From this formula, it is clear that $\widehat{\mathcal{L}}_\alpha^n$ is a branching Markov chain. To be more precise, note that for any $n, m > 0$, $V_m^n = 2(n \wedge m)$ and $\lambda_n = 2$ and that $G^{1,1} = 1$. Moreover, $G_{\gamma\delta_1}^{1,n} = G^{1,n} - G^{1,1}\gamma G_{\gamma\delta_1}^{1,n}$ so that $G_{\gamma\delta_1}^{1,n} = \frac{1}{1+\gamma}$ and for any $n > 0$, the restriction of the Markov chain to D_n is isomorphic to the original Markov chain. Then it comes that for all n, $p_n^{\{F_n\}} = \frac{1}{2}$, $\lambda_n^{\{F_n\}} = 1$, and $\lambda_n^{\{F_n, \gamma\delta_{n+1}\}} = 2 - \frac{1}{1+\gamma} = \frac{2\gamma+1}{1+\gamma}$ so that the Laplace exponent of the convolution semigroup ν_t defining the branching mechanism $\lambda_{n-1}^{\{F_{n-1}, \gamma\delta_n\}} - \lambda_{n-1}^{\{F_{n-1}\}}$ equals $\frac{2\gamma+1}{1+\gamma} - 1 = \frac{\gamma}{1+\gamma} = \int (1 - e^{-\gamma s}) e^{-s}\, ds$. It is the semigroup of a compound Poisson process whose Levy measure is exponential.

The immigration law (on $\mathbb{R}^+$) is a Gamma distribution $\Gamma(\alpha, G^{1,1}) = \Gamma(\alpha, 1)$. It is the law of $\widehat{\mathcal{L}}_\alpha^1$ and also of $[\widehat{\mathcal{L}_\alpha}^{D_{n-1}}]^n$ (it means the occupation field of the trace of $\mathcal{L}_\alpha$ on D_{n-1} evaluated at n) for all $n > 1$.

The conditional law of $\widehat{\mathcal{L}}_\alpha^{n+1}$ given $\widehat{\mathcal{L}}_\alpha^n$ is the convolution of the immigration law $\Gamma(\alpha, 1)$ with $\nu_{\widehat{\mathcal{L}}_\alpha^n}$.

Alternatively, we can consider the integer valued process $N_n(\mathcal{L}_\alpha^{\{F_n\}}) + 1$ which is a Galton Watson process with immigration. In our example, we find the reproduction law $\pi(n) = 2^{-n-1}$ for all $n \geq 0$ (critical binary branching).

If we consider the occupation field defined by the loops going through 1, we get a branching process without immigration: it is the classical relation between random walks local times and branching processes.

9. The case of general Markov processes. We now explain briefly how some of the above results will be extended to a symmetric Markov process on an infinite space X. The construction of the loop measure as well as a lot of computations can be performed quite generally, using Markov processes or Dirichlet space theory (cf., e.g., [4]). It works as soon as the bridge or excursion measures $\mathbb{P}_t^{x,y}$ can be properly defined. The semigroup should have a locally integrable kernel $p_t(x, y)$.

Let us consider more closely the occupation field $\widehat{l}$. The extension is rather straightforward when points are not polar. We can start with a Dirichlet space of continuous functions and a measure m such that there is a mass gap. Let P_t denote the associated Feller semigroup. Then the Green function is well defined as the mutual energy of the Dirac measures δ_x and δ_y which have finite energy. It is the covariance function of a Gaussian free field $\phi(x)$, which will be associated to the field $\widehat{\mathcal{L}}_{1/2}^x$ of local times of the Poisson process of random loops whose intensity is given by the loop measure defined by the semigroup P_t. This will apply to examples related to one-dimensional Brownian motion or to Markov chains on countable spaces.

When we consider Brownian motion on the half line, we get a continuous branching process with immigration, as in the discrete case.

When points are polar, one needs to be more careful. We will consider only the case of the two and three dimensional Brownian motion in a bounded domain D

killed at the boundary, that is, associated with the classical energy with Dirichlet boundary condition. The Green function does not induce a trace class operator but it is still Hilbert–Schmidt which allows us to define renormalized determinants $\det_2$ (cf. [20]).

If A is a symmetric Hilbert Schmidt operator, $\det_2(I + A)$ is defined as $\prod(1 + \lambda_i)e^{-\lambda_i}$ where λ_i are the eigenvalues of A.

The Gaussian field (called free field) whose covariance function is the Green function is now a generalized field: Generalized fields are not defined pointwise but have to be smeared by a test function f. Still $\phi(f)$ is often denoted $\int \phi(x) f(x)\,dx$.

Wick powers $:\phi^n:$ of the free field can be defined as generalized field by approximation as soon as the $2n$th power of the Green function, $G(x, y)^{2n}$ is locally integrable (cf. [21]). This is the case for all n for Brownian motion in dimension two, as the Green function has only a logarithmic singularity on the diagonal, and for $n = 2$ in dimension three as the singularity is of the order of $\frac{1}{\|x-y\|}$. More precisely, taking for example $\pi_\varepsilon^x(dy)$ to be the normalized area measure on the sphere of radius ε around x, $\phi(\pi_\varepsilon^x)$ is a Gaussian field with covariance $\sigma_\varepsilon^x = \int G(z, z')\pi_\varepsilon^x(dz)\pi_\varepsilon^y(dz')$. Its Wick powers are defined with Hermite polynomials as we did previously.

$:\phi(\pi_\varepsilon^x)^n: = (\sigma_\varepsilon^x)^{n/2} H_n(\frac{\phi(\pi_\varepsilon^x)}{\sqrt{\sigma_\varepsilon^x}})$. Then one can see that $\int f(x):\phi(\pi_\varepsilon^x)^n:dx$ converges in L^2 for any bounded continuous function f with compact support toward a limit called the nth Wick power of the free field evaluated on f and denoted $:\phi^n:(f)$. Moreover, $\mathbb{E}(:\phi^n:(f):\phi^n:(h)) = \int G^{2n}(x, y) f(x)h(y)\,dx\,dy$.

In these cases, we can extend the statement of Theorem 13 to the renormalized occupation field $\widetilde{\mathcal{L}}_{1/2}^x$ and the Wick square $:\phi^2:$ of the free field.

Let us explain this in more detail in the Brownian motion case. Let D be an open subset of $\mathbb{R}^d$ such that the Brownian motion killed at the boundary of D is transient and has a Green function. Let $p_t(x, y)$ be its transition density and $G(x, y) = \int_0^\infty p_t(x, y)\,dt$ the associated Green function. The loop measure μ was defined in [9] as

$$\mu = \int_D \int_0^\infty \frac{1}{t} \mathbb{P}_t^{x,x}\,dt,$$

where $\mathbb{P}_t^{x,x}$ denotes the (nonnormalized) excursion measure of duration t such that if $0 \le t_1 \le \cdots \le t_h \le t$,

$$\begin{aligned}
&\mathbb{P}_t^{x,x}\bigl(\xi(t_1) \in dx_1, \ldots, \xi(t_h) \in dx_h\bigr) \\
&\quad = p_{t_1}(x, x_1) p_{t_2-t_1}(x_1, x_2) \cdots p_{t-t_h}(x_h, x)\,dx_1 \cdots dx_h
\end{aligned}$$

[the mass of $\mathbb{P}_t^{x,x}$ is $p_t(x, x)$]. Note that μ is a priori defined on based loops but it is easily seen to be shift-invariant.

For any loop l indexed by $[0, T(l)]$, define the measure $\widehat{l} = \int_0^{T(l)} \delta_{l(s)}\, ds$: for any Borel set A, $\widehat{l}(A) = \int_0^{T(l)} 1_A(l_s)\, ds$. As before, we have the following.

LEMMA 29. *For any nonnegative function* f

$$\mu(\langle \widehat{l}, f\rangle^n) = (n-1)! \int G(x_1, x_2) f(x_2) G(x_2, x_3) f(x_3) \cdots G(x_n, x_1) f(x_1) \prod_1^n dx_i.$$

One can define in a similar way the analogous of multiple local times, and get for their integrals with respect to μ a formula analogous to the one obtained in the discrete case.

Let G denote the operator on $L^2(D, dx)$ defined by G. Let f be a nonnegative continuous function with compact support in D.

Note that $\langle \widehat{l}, f\rangle$ is μ-integrable only in dimension one as then, G is locally trace class. In that case, using for all x an approximation of the Dirac measure at x, local times $\widehat{l^x}$ can be defined in such a way that $\langle \widehat{l}, f\rangle = \int \widehat{l^x} f(x)\, dx$.

$\langle \widehat{l}, f\rangle$ is μ-square integrable in dimensions one, two and three, as G is Hilbert–Schmidt if D is bounded, since $\int\int_{D\times D} G(x, y)^2\, dx\, dy < \infty$, and otherwise locally Hilbert–Schmidt.

N.B. Considering distributions χ such that $\int\int (G(x, y)^2 \chi(dx)\chi(dy) < \infty$, we could see that $\langle \widehat{l}, \chi\rangle$ can be defined by approximation as a square integrable variable and $\mu(\langle \widehat{l}, \chi\rangle^2) = \int (G(x, y)^2 \chi(dx)\chi(dy)$.

Let z be a complex number such that $\operatorname{Re}(z) > 0$.

Note also that $e^{-z\langle \widehat{l}, f\rangle} + z\langle \widehat{l}, f\rangle - 1$ is bounded by $\frac{|z|^2}{2}\langle \widehat{l}, f\rangle^2$ and expands as an alternating series $\sum_2^\infty \frac{z^n}{n!}(-\langle \widehat{l}, f\rangle)^n$, with $|e^{-z\langle \widehat{l}, f\rangle} - 1 - \sum_1^N \frac{z^n}{n!}(-\langle \widehat{l}, f\rangle)^n| \le \frac{|z\langle \widehat{l}, f\rangle|^{N+1}}{(N+1)!}$. Then, for $|z|$ small enough, it follows from the above lemma that

$$\mu\big(e^{-z\langle \widehat{l}, f\rangle} + z\langle \widehat{l}, f\rangle - 1\big) = \sum_2^\infty \frac{z^n}{n} \operatorname{Tr}\big(-(M_{\sqrt{f}} G M_{\sqrt{f}})^n\big).$$

As $M_{\sqrt{f}} G M_{\sqrt{f}}$ is Hilbert–Schmidt $\det_2(I + zM_{\sqrt{f}} G M_{\sqrt{f}})$ is well defined and the second member writes $-\log(\det_2(I + zM_{\sqrt{f}} G M_{\sqrt{f}}))$.

Then the identity

$$\mu\big(e^{-z\langle \widehat{l}, f\rangle} + z\langle \widehat{l}, f\rangle - 1\big) = -\log\big(\det{}_2(I + zM_{\sqrt{f}} G M_{\sqrt{f}})\big)$$

extends, as both sides are analytic as locally uniform limits of analytic functions, to all complex values with positive real part.

The renormalized occupation field $\widetilde{\mathcal{L}_\alpha}$ is defined as the compensated sum of all $\widehat{l}$ in $\mathcal{L}_\alpha$ [formally, $\widetilde{\mathcal{L}_\alpha} = \widehat{\mathcal{L}_\alpha} - \int \int_0^{T(l)} \delta_{l_s}\, ds\, \mu(dl)$]. By a standard argument used for the construction of Levy processes

$$\langle \widetilde{\mathcal{L}_\alpha}, f\rangle = \lim_{\varepsilon\to 0}\langle \widetilde{\mathcal{L}_{\alpha,\varepsilon}}, f\rangle$$

with by definition

$$\langle \widetilde{\mathcal{L}_{\alpha,\varepsilon}}, f\rangle = \sum_{\gamma\in\mathcal{L}_\alpha} 1_{\{T>\varepsilon\}}\int_0^T f(\gamma_s)\,ds - \alpha\mu 1_{\{T>\varepsilon\}}\int_0^T f(\gamma_s)\,ds.$$

The convergence holds a.s. and in L^2, as $\mathbb{E}((\sum_{\gamma\in\mathcal{L}_\alpha}(1_{\{T>\varepsilon\}}\int_0^T f(\gamma_s)\,ds) - \alpha\mu(1_{\{T>\varepsilon\}}\int_0^T f(\gamma_s)\,ds))^2) = \alpha\int(1_{\{T>\varepsilon\}}\int_0^T f(\gamma_s)\,ds)^2\mu(dl)$ and $\mathbb{E}(\langle\widetilde{\mathcal{L}_\alpha}, f\rangle^2) = \mathrm{Tr}((M_{\sqrt{f}}GM_{\sqrt{f}})^2)$. Note that if we fix f, α can be considered as a time parameter and $\langle\widetilde{\mathcal{L}_{\alpha,\varepsilon}}, f\rangle$ as Levy processes with discrete positive jumps approximating a Levy process with positive jumps $\langle\widetilde{\mathcal{L}_\alpha}, f\rangle$. The Levy exponent $\mu(1_{\{T>\varepsilon\}}(e^{-\langle\widehat{l}, f\rangle} + \langle\widehat{l}, f\rangle - 1))$ of $\langle\widetilde{\mathcal{L}_{\alpha,\varepsilon}}, f\rangle$ converges toward the Lévy exponent of $\langle\widetilde{\mathcal{L}_\alpha}, f\rangle$ which is $\mu((e^{-\langle\widehat{l}, f\rangle} + \langle\widehat{l}, f\rangle - 1))$ and, from the identity $E(e^{-\langle\widetilde{\mathcal{L}_\alpha}, f\rangle}) = e^{-\alpha\mu(e^{-\langle\widehat{l}, f\rangle}+\langle\widehat{l}, f\rangle-1)}$, we get the following theorem.

THEOREM 30. *Assume $d \le 3$. Denoting $\widetilde{\mathcal{L}_\alpha}$ the compensated sum of all $\widehat{l}$ in $\mathcal{L}_\alpha$, we have* $\mathbb{E}(e^{-\langle\widetilde{\mathcal{L}_\alpha}, f\rangle}) = \det_2(I + M_{\sqrt{f}}GM_{\sqrt{f}})^{-\alpha}$.

Moreover, $e^{-\langle\widetilde{\mathcal{L}_{\alpha,\varepsilon}}, f\rangle}$ converges a.s. and in L^1 toward $e^{-\langle\widetilde{\mathcal{L}_\alpha}, f\rangle}$.

Considering distributions of finite energy χ [i.e., such that $\int(G(x, y)^2\chi(dx)\times\chi(dy) < \infty$], we can see that $\langle\widetilde{\mathcal{L}_\alpha}, \chi\rangle$ can be defined by approximation as $\lim_{\lambda\to\infty}(\langle\widetilde{\mathcal{L}_\alpha}, \lambda G_\lambda\chi\rangle)$ and

$$\mathbb{E}(\langle\widetilde{\mathcal{L}_\alpha}, \chi\rangle^2) = \alpha\int(G(x, y))^2\chi(dx)\chi(dy).$$

Specializing to $\alpha = \frac{k}{2}$, k being any positive integer we have the following corollary.

COROLLARY 31. *The renormalized occupation field $\widetilde{\mathcal{L}_{k/2}}$ and the Wick square $\frac{1}{2}{:}\sum_1^k\phi_l^2{:}$ have the same distribution.*

If Θ is a conformal map from D onto $\Theta(D)$, it follows from the conformal invariance of the Brownian trajectories that a similar property holds for the Brownian "loop soup" (cf. [9]). More precisely, if $c(x) = \mathrm{Jacobian}_x(\Theta)$ and, given a loop l, if $T^c(l)$ denotes the reparametrized loop l_{τ_s}, with $\int_0^{\tau_s} c(l_u)\,du = s\Theta T^c(\mathcal{L}_\alpha)$ is the Brownian loop soup of intensity parameter α on $\Theta(D)$. Then we have the following proposition.

PROPOSITION 32. *$\Theta(c\widetilde{\mathcal{L}_\alpha})$ is the renormalized occupation field on $\Theta(D)$.*

PROOF. We have to show that the compensated sum is the same if we perform it after or before the time change. For this, it is enough to check that

$$\mathbb{E}\Big(\Big[\sum_{\gamma\in\mathcal{L}_\alpha} 1_{\{\tau_T>\eta\}}1_{\{T\le\varepsilon\}}\int_0^T f(\gamma_s)\,ds - \alpha\int\Big(1_{\{\tau_T>\eta\}}1_{\{T\le\varepsilon\}}\int_0^T f(\gamma_s)\,ds\Big)\mu(d\gamma)\Big]^2\Big) = \alpha\int\Big(1_{\{\tau_T>\eta\}}1_{\{T\le\varepsilon\}}\int_0^T f(\gamma_s)\,ds\Big)^2\mu(d\gamma)$$

and

$$\mathbb{E}\Big(\Big[\sum_{\gamma\in\mathcal{L}_\alpha} 1_{\{T>\varepsilon\}}1_{\tau_T\le\eta}\int_0^T f(\gamma_s)\,ds - \alpha\int\Big(1_{\{T>\varepsilon\}}1_{\tau_T\le\eta}\int_0^T f(\gamma_s)\,ds\Big)\mu(d\gamma)\Big]^2\Big) = \alpha\int\Big(1_{\{T>\varepsilon\}}1_{\tau_T\le\eta}\int_0^T f(\gamma_s)\,ds\Big)^2\mu(d\gamma)$$

converge to zero as ε and η go to zero. It follows from the fact that

$$\int\Big[1_{\{T\le\varepsilon\}}\int_0^T f(\gamma_s)\,ds\Big]^2\mu(d\gamma)$$

and

$$\int\Big[1_{\tau_T\le\eta}\int_0^T f(\gamma_s)\,ds\Big]^2\mu(d\gamma)$$

converge to 0. The second follows easily from the first if c is bounded away from zero. We can always consider the "loop soups" in an increasing sequence of relatively compact open subsets of D to reduce the general case to that situation. □

As in the discrete case (see Corollary 10), we can compute product expectations. In dimensions one and two, for f_j continuous functions with compact support in D:

$$\mathbb{E}(\langle\widetilde{\mathcal{L}_\alpha}, f_1\rangle\cdots\langle\widetilde{\mathcal{L}_\alpha}, f_k\rangle) = \int \mathrm{Per}_\alpha^0\big(G(x_l,x_m), 1\le l,m\le k\big)\prod f_j(x_j)\,dx_j. \tag{24}$$

10. Renormalized powers. In dimension one, powers of the occupation field can be viewed as integrated self-intersection local times. In dimension two, renormalized powers of the occupation field, also called *renormalized self-intersections local times* can be defined as follows.

THEOREM 33. *Assume $d = 2$. Let $\pi_\varepsilon^x(dy)$ be the normalized arc length on the circle of radius ε around x, and set $\sigma_\varepsilon^x = \int G(y,z)\pi_\varepsilon^x(dy)\pi_\varepsilon^x(dz)$. Then, $\int f(x) Q_k^{\alpha,\sigma_\varepsilon^x}(\langle\widetilde{\mathcal{L}}_\alpha, \pi_\varepsilon^x\rangle)\,dx$ converges in L^2 for any bounded continuous function f with compact support toward a limit denoted $\langle\widetilde{\mathcal{L}_\alpha^k}, f\rangle$ and*

$$\mathbb{E}(\langle\widetilde{\mathcal{L}_\alpha^k}, f\rangle\langle\widetilde{\mathcal{L}_\alpha^l}, h\rangle) = \delta_{l,k}\frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!}\int G^{2k}(x,y) f(x)h(y)\,dx\,dy.$$

PROOF. The idea of the proof can be understood by trying to prove that $\mathbb{E}((\int f(x) Q_k^{\alpha,\sigma_x^\varepsilon}(\langle\widetilde{\mathcal{L}}_\alpha, \pi_\varepsilon^x\rangle)\,dx)^2)$ remains bounded as ε decreases to zero. One will expand this expression in terms of sums of integrals of product of Green functions and check that the combinatorial identities (15) imply the cancelation of the logarithmic divergences.

This is done by showing (as done below in the proof of the theorem) that one can modify slightly the products of Green functions appearing in $\mathbb{E}(Q_k^{\alpha,\sigma_\varepsilon^x}(\langle\widetilde{\mathcal{L}}_\alpha, \pi_\varepsilon^x\rangle) Q_k^{\alpha,\sigma_\varepsilon^y}(\langle\widetilde{\mathcal{L}}_\alpha, \pi_\varepsilon^y\rangle))$ to replace them by products of the form $G(x,y)^j(\sigma_\varepsilon^x)^l(\sigma_\varepsilon^y)^h$. The cancelation of terms containing σ_ε^x and/or σ_ε^y then follows directly from the combinatorial identities.

Let us now prove the theorem. Consider first, for any $x_1, x_2, \dots, x_n$, ε small enough and $\varepsilon \le \varepsilon_1, \dots, \varepsilon_n \le 2\varepsilon$, with $\varepsilon_i = \varepsilon_j$ if $x_i = x_j$, an expression of the form

$$\Delta = \Bigg| \prod_{i, x_{i-1}\neq x_i} G(x_{i-1}, x_i)(\sigma_{\varepsilon_i}^{x_i})^{m_i} - \int G(y_1,y_2)\cdots G(y_n,y_1)\pi_{\varepsilon_1}^{x_1}(dy_1)\cdots\pi_{\varepsilon_n}^{x_n}(dy_n)\Bigg|$$

in which we define m_i as $\sup(h, x_{i+h} = x_i)$.

In the integral term, we first replace progressively $G(y_{i-1}, y_i)$ by $G(x_{i-1}, x_i)$ whenever $x_{i-1} \neq x_i$, using triangle, then Schwartz inequalities, to get an upper bound of the absolute value of the difference made by this substitution in terms of a sum Δ' of expressions of the form

$$\prod_l G(x_l, x_{l+1})\biggl(\int (G(y_1,y_2) - G(x_1,x_2))^2 \times \pi_{\varepsilon_1}^{x_1}(dy_1)\pi_{\varepsilon_2}^{x_2}(dy_2)\int \prod G^2(y_k, y_{k+1})\prod \pi_{\varepsilon_k}^{x_k}(dy_k)\biggr)^{1/2}.$$

The expression obtained after these substitutions can be written

$$W = \prod_{i, x_{i-1} \neq x_i} G(x_{i-1}, x_i) \int G(y_1, y_2) \cdots G(y_{m_{i-1}}, y_{m_i}) \pi_{\varepsilon_i}^{x_i}(dy_1) \cdots \pi_{\varepsilon_i}^{x_i}(dy_{m_i})$$

and we see the integral terms could be replaced by $(\sigma_\varepsilon^{x_i})^{m_i}$ if G was translation invariant. But as the distance between x and y tends to 0, $G(x, y)$ is equivalent to $G_0(x, y) = \frac{1}{\pi} \log(\|x - y\|)$ and moreover, $G(x, y) = G_0(x, y) - H^{D^c}(x, dz) G_0(z, y)$, H^{D^c} denoting the Poisson kernel on the boundary of D. As our points lie in a compact inside D, it follows that for some constant C, for $\|y_1 - x\| \leq \varepsilon$, $|\int (G(y_1, y_2) \pi_\varepsilon^x(dy_2) - \sigma_\varepsilon^x| < C\varepsilon$.

Hence, the difference Δ'' between W and $\prod_{i, x_{i-1} \neq x_i} G(x_{i-1}, x_i)(\sigma_\varepsilon^{x_i})^{m_i}$ can be bounded by $\varepsilon W'$, where W' is an expression similar to W.

To get a good upper bound on Δ, using the previous observations, by repeated applications of Hölder inequality, it is enough to show that for ε small enough, C and C' denoting various constants:

(1) $\int (G(y_1, y_2) - G(x_1, x_2))^2 \pi_{\varepsilon_1}^{x_1}(dy_1) \pi_{\varepsilon_2}^{x_2}(dy_2) < C(\varepsilon 1_{\{\|x_1 - x_2\| \geq \sqrt{\varepsilon}\}} + (G(x_1, x_2)^2 + \log(\varepsilon)^2) 1_{\{\|x_1 - x_2\| < \sqrt{\varepsilon}\}})$,
(2) $\int G(y_1, y_2)^k \pi_\varepsilon^x(dy_1) \pi_\varepsilon^x(dy_2) < C|\log(\varepsilon)|^k$,
(3) $\int G(y_1, y_2)^k \pi_{\varepsilon_1}^{x_1}(dy_1) \pi_{\varepsilon_2}^{x_2}(dy_2) < C|\log(\varepsilon)|^k$.

As the main contributions come from the singularities of G, they follow from the following simple inequalities:

$$(1') \qquad \int \left|\log(\varepsilon^2 + 2R\varepsilon\cos(\theta) + R^2) - \log(R)\right|^2 d\theta$$
$$= \int \left|\log((\varepsilon/R)^2 + 2(\varepsilon/R)\cos(\theta) + 1)\right|^2 d\theta$$
$$< C(\varepsilon 1_{\{R \geq \sqrt{\varepsilon}\}} + \log^2(R/\varepsilon) 1_{\{R < \sqrt{\varepsilon}\}})$$

(considering separately the cases where $\frac{\varepsilon}{R}$ is large or small).

(2′) $\int |\log(\varepsilon^2(2 + 2\cos(\theta)))|^k \, d\theta \leq C|\log(\varepsilon)|^k$.

(3′) $\int |\log(\varepsilon_1 \cos(\theta_1) + \varepsilon_2 \cos(\theta_2) + r)^2 + (\varepsilon_1 \sin(\theta_1) + \varepsilon_2 \sin(\theta_2))^2|^k \, d\theta_1 \, d\theta_2 \leq C(|\log(\varepsilon)|)^k$. It can be proved by observing that for $r \leq \varepsilon_1 + \varepsilon_2$, we have near the singularities [i.e., the values $\theta_1(r)$ and $\theta_2(r)$ for which the expression under the log vanishes] to evaluate integrals bounded by $C \int_0^1 (-\log(\varepsilon u))^k \, du \leq C'(-\log(\varepsilon))^k$ for ε small enough.

Let us now show that for $\varepsilon \leq \varepsilon_1, \varepsilon_2 \leq 2\varepsilon$, we have for some integer $N^{n,k}$

$$\begin{aligned}\Big|\mathbb{E}\big(Q_k^{\alpha,\sigma_x^{\varepsilon_1}}(\langle\widetilde{\mathcal{L}_\alpha},\pi_{\varepsilon_1}^x\rangle)Q_l^{\alpha,\sigma_y^{\varepsilon_2}}(\langle\widetilde{\mathcal{L}_\alpha},\pi_{\varepsilon_2}^y\rangle)\big)\\ -\delta_{l,k}G(x,y)^{2k}\frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!}\Big|\\ \leq C\log(\varepsilon)^{N_{l,k}}\big(\sqrt{\varepsilon}+G(x,y)^{2k}1_{\{\|x-y\|<\sqrt{\varepsilon}\}}\big).\end{aligned}\tag{25}$$

Indeed, developing the polynomials and using formula (24) we can express this expectation as a linear combination of integrals under $\prod_i \pi_{\varepsilon_1}^x(dx_i)\prod_j \pi_{\varepsilon_2}^y(dy_j)$ of products of $G(x_i,y_{i'}), G(x_i,x_j)$ and $G(y_j,y_{j'})$ as we did in the discrete case. If we replace each $G(x_i,y_j)$ by $G(x,y)$, each $G(x_i,x_{i'})$ by $\sigma_{\varepsilon_1}^x$ and each $G(y_j,y_{j'})$ by $\sigma_{\varepsilon_2}^y$, we can use the combinatorial identity (15) to get the value $\delta_{l,k}G(x,y)^{2k}\frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!}$. Then, the above results allow us to bound the error made by this replacement.

The bound (25) is uniform in (x,y) only away from the diagonal as $G(x,y)$ can be arbitrarily large, but we conclude from it that for any bounded integrable f and h,

$$\begin{aligned}\Big|\int\Big(\mathbb{E}\big(Q_k^{\alpha,\sigma_x^{\varepsilon_1}}(\langle\widetilde{\mathcal{L}_\alpha},\pi_{\varepsilon_1}^x\rangle)Q_l^{\alpha,\sigma_y^{\varepsilon_2}}(\langle\widetilde{\mathcal{L}_\alpha},\pi_{\varepsilon_2}^y\rangle)\big)\\ -\delta_{l,k}G(x,y)^{2k}\frac{\alpha\cdots(\alpha+k-1)}{k!}\Big)f(x)h(y)\,dx\,dy\Big|\\ \leq C'\sqrt{\varepsilon}\log(\varepsilon)^{N_{l,k}}\end{aligned}$$

[as $\int\int G(x,y)^{2k}1_{\{\|x-y\|<\sqrt{\varepsilon}\}}\,dx\,dy$ can be bounded by $C\varepsilon^{2/3}$, e.g.].

Taking $\varepsilon_n = 2^{-n}$, it is then straightforward to check that $\int f(x)Q_k^{\alpha,\sigma_x^{\varepsilon_n}}(\langle\widetilde{\mathcal{L}_\alpha}, \pi_{\varepsilon_n}^x\rangle)\,dx$ is a Cauchy sequence in L^2. The theorem follows. □

Specializing to $\alpha = \frac{k}{2}$, k being any positive integer as before, Wick powers of $\sum_{j=1}^k \phi_j^2$ are associated with self intersection local times of the loops. More precisely, we have the following proposition.

PROPOSITION 34. *The renormalized self-intersection local times* $\widetilde{\mathcal{L}_{k/2}^n}$ *and the Wick powers* $\frac{1}{2^n n!}{:}(\sum_1^k \phi_l^2)^n{:}$ *have the same joint distributions.*

The proof is similar to the one given in [13] and also to the proof of the above theorem, but simpler. It is just a calculation of the L^2-norm of

$$\int[{:}(\phi^2)^n{:}(x) - Q_n^{1/2,\sigma_x^{\varepsilon}}({:}\phi_x^2{:}(\pi_\varepsilon^x))]f(x)\,dx$$

which converges to zero with ε.

FINAL REMARKS.

(a) These generalized fields have two fundamental properties:

First, they are local fields (or more precisely local functionals of the field $\widetilde{\mathcal{L}_\alpha}$ in the sense that their values on functions supported in an open set D depend only on the trace of the loops on D).

Second, noting we could use different regularizations to define $\widetilde{\mathcal{L}_\alpha^k}$, the action of a conformal transformation Θ on these fields is given by *the kth power of the conformal factor* $c =$ Jacobian(Θ). More precisely, $\Theta(c^k \widetilde{\mathcal{L}_\alpha^k})$ is the renormalized kth power of the occupation field in $\Theta(D)$.

(b) It should be possible to derive from the above remark the existence of exponential moments and introduce nontrivial local interactions as in the constructive field theory derived from the free field (cf. [21]).

(c) Let us also briefly consider currents. We will restrict our attention to the one and two dimensional Brownian case, X being an open subset of the line or plane. Currents can be defined by vector fields, with compact support.

Then, if we now denote by ϕ the complex valued free field (its real and imaginary parts being two independent copies of the free field), $\int_l \omega$ and $\int_X (\overline{\phi}\,\partial_\omega \phi - \phi\,\partial_\omega \overline{\phi})\,dx$ are well-defined square integrable variables in dimension 1 (it can be checked easily by Fourier series). The distribution of the centered occupation field of the loop process "twisted" by the complex exponential $\exp(\sum_{l\in\mathcal{L}_\alpha} \int_l i\omega + \frac{1}{2}\widehat{l}(\|\omega\|^2))$ appears to be the same as the distribution of the field $:\phi\overline{\phi}:$ "twisted" by the complex exponential $\exp(\int_X (\overline{\phi}\,\partial_\omega \phi - \phi\,\partial_\omega \overline{\phi})\,dx)$ (cf. [14]).

In dimension 2, logarithmic divergences occur.

(d) There is a lot of related investigations. The extension of the properties proved here in the finite framework has still to be completed, though the relation with spanning trees should follow from the remarkable results obtained on SLE processes, especially [11]. Note finally that other essential relations between SLE processes, loops and free fields appear in [19, 25] and [1].

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