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Markov chains approximation of jump–diffusion stochastic master equations

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Abstract. *Quantum trajectories* are solutions of stochastic differential equations obtained when describing the random phenomena associated to quantum continuous measurement of open quantum system. These equations, also called *Belavkin equations* or *Stochastic Master equations*, are usually of two different types: diffusive and of Poisson-type. In this article, we consider more advanced models in which jump–diffusion equations appear. These equations are obtained as a continuous time limit of martingale problems associated to classical Markov chains which describe quantum trajectories in a discrete time model. The results of this article goes much beyond those of [*Ann. Probab.* **36** (2008) 2332–2353] and [Existence, uniqueness and approximation for stochastic Schrödinger equation: The Poisson case (2007)]. The probabilistic techniques, used here, are completely different in order to merge these two radically different situations: diffusion and Poisson-type quantum trajectories.

Résumé. *Les trajectoires quantiques* sont des solutions d'équations différentielles stochastiques décrivant des phénomènes aléatoires associés aux prinicpes de mesure (quantique) des systèmes quantiques ouverts. Ces équations, également appelées *équations de Belavkin* ou *équations maîtresses stochastiques* sont habituellement de deux types: soit diffusif soit de type saut. Dans cet article, nous considérons des modèles plus avancés où des équations de type saut-diffusion apparaissent. Ces équations sont obtenues comme solutions de problèmes de martingales. Ces problèmes de martingales sont obtenus comme limites continus (en temps) à partir de chaînes de Markov classiques décrivant des trajectoires quantiques pour des modèles à temps discret. Les résultats de cet article généralisent ceux de [*Ann. Probab.* **36** (2008) 2332–2353] et [Existence, uniqueness and approximation for stochastic Schrödinger equation: The Poisson case (2007)]. Ici, les techniques probabilistes utilisés sont complétement différentes afin de pouvoir mixer les deux types d'évolutions: diffusives et poissoniennes.

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Introduction

In quantum mechanics, an active line of research makes an important use of *Quantum Trajectory* theory with wide applications in *Quantum Optics* or in *Quantum Information* theory (cf. [4,6,16,17,22]). A quantum trajectory is a solution of a stochastic differential equation which describes the random evolution of a quantum system undergoing continuous measurement. These equations are called *Stochastic Master equations* or *Belavkin equations* [2–4,10,12, 14].

More precisely, these equations describe situations where the measurement is *indirect*. The physical setup is the one of an interaction between a small system (atom) and a continuous field (environment). By performing a continuous time quantum measurement on the field, after the interaction, we get a partial information of the evolution of the small system without destroying it. This partial information is governed by stochastic differential equations (Belavkin equations). In the literature, two characteristic equations are usually described as follows:

1. One is described by a diffusive equation

$$\mathrm{d}\rho_t = L(\rho_t)\,\mathrm{d}t + \big(\rho_t C^\star + C\rho_t - \mathrm{Tr}\big[\rho_t\big(C + C^\star\big)\big]\rho_t\big)\,\mathrm{d}W_t, \tag{1}$$

where W_t describes a one-dimensional Brownian motion.

2. The other is given by a stochastic differential equation driven by a counting process

$$\mathrm{d}\rho_t = L(\rho_t)\,\mathrm{d}t + \left(\frac{\mathcal{J}(\rho_t)}{\mathrm{Tr}[\mathcal{J}(\rho_t)]} - \rho_t\right)\big(\mathrm{d}\tilde{N}_t - \mathrm{Tr}\big[\mathcal{J}(\rho_t)\big]\,\mathrm{d}t\big), \tag{2}$$

where $\tilde{N}_t$ is a counting process with stochastic intensity $\int_0^t \mathrm{Tr}[\mathcal{J}(\rho_s)]\,\mathrm{d}s$.

The solutions of these equations are called *continuous quantum trajectories*, they are valued in the set of states of the small system (a *state* or *density matrix* is a positive trace class operator with trace one). Such models describe essentially the interaction between a two-level atom and a spin chain [29,30]. More complicated models, with higher degree of liberty, are in general mixing of these two types, that is, they are driven by both a diffusive and a jump process (jump–diffusion model) ....

Even in the cases (1) and (2), Belavkin equations pose tedious problems in terms of physical and mathematical justifications. First rigorous results are due to Davies [19] which has described the evolution of a two-level atom undergoing a continuous measurement. Heuristic rules can be used to obtain classical Belavkin equations (1) and (2). There exists different ways to justify these models. In [3,9,11,14], *Quantum Filtering* theory is used to obtain stochastic master equations. Such an approach needs high analytic technologies (Von Neumann algebra, conditional expectation in operator algebra) and is based on the quantum stochastic calculus formalism. In [4,5,7,27], jump–diffusion or multi-diffusions models are considered. In these papers, an approach based on classical stochastic differential equations and the notion of *a posteriori state* is used.

Recently, an intuitive approach based on a discrete time model has been used in [29,30] to justify Eqs (1) and (2). The discrete model is called *Quantum Repeated Measurements* and is based on the setup of *Quantum Repeated Interactions*. In this context, instead of considering an interaction with a continuous field, the environment is represented as an infinite chain of identical and independent quantum system. Each piece of the chain interacts with the small system during a time interval of length τ. After each interaction, a quantum measurement of an observable of the field is performed. From the point of view of the small system, according to the laws of quantum mechanics, each result of an observation gives rise to a random modification of its reference state. The evolution of the state of the small system can be described by classical Markov chains called *discrete quantum trajectories*. In [29,30], the discrete quantum trajectories are then expressed as solutions of discrete stochastic differential equations which are approximations of Eqs (1) and (2). Next, by using techniques related to convergence of stochastic differential equations, it has been shown that the solutions of (1) and (2) can be obtained as continuous limit from discrete quantum trajectories (when τ goes to zero). However, the techniques used in [29] and [30] are very different and incompatible (for (1) an abstract result of Kurtz and Protter [28] is used whereas for (2) a random coupling method and a comparison with an Euler scheme is employed). Therefore, such methods can not be adapted in situations of mixing of Brownian evolution and jump evolution.

In this article, in order to prove a similar convergence result for jump–diffusion stochastic master equations, we adopt an approach based on *Markov Chain Approximation* theory. We proceed as follows. As the discrete quantum trajectories are Markov chains, we can naturally define their discrete Markov generators which depend on the time parameter τ. The limit, τ goes to zero, of these generators gives rise to infinitesimal generators. These one are then naturally linked with general *martingale problems* in probability theory [23,25]. Next, we show that such martingale problems are solved by solution of particular jump–diffusion stochastic differential equations, which model continuous time measurement theory. Finally, this approach and these models are physically justified by proving that the solutions of these SDEs can be obtained as continuous time limits (in distribution) of discrete quantum trajectories. This way, for finite-dimensional systems, we get a general description of jump–diffusion stochastic master equations.

This article is organized as follows.

Section 1 is devoted to the description of the discrete model of quantum repeated measurements. We remind the presentation of discrete quantum trajectories developed in [29]. Next, we shall focus on the dependence on τ for these Markov chains and we introduce asymptotic assumptions in order to come into the question of convergence.

In Section 2, we present the Markov chain approximation technique. We compute the Markov generators of discrete quantum trajectories and we investigate the limits of these one. Therefore, we obtain infinitesimal generators linked to general martingale problems. We show how to solve these problems in terms of jump–diffusion stochastic differential equations.

Finally in Section 3, we show that discrete quantum trajectories converge in distribution to the solutions of stochastic differentials equations described in Section 2. This way, the stochastic models of jump–diffusion equations in continuous quantum measurement theory are then justified as limit models of the concrete physical procedure of quantum repeated measurements.

1. Discrete quantum trajectories

1.1. *Quantum repeated measurements*

In this section, we remind the model of quantum repeated measurements and the description of discrete quantum trajectories in terms of Markov chains [29]. As it is mentioned in the Introduction, the physical setup is the one a small system in contact with an infinite chain of quantum systems [1] (each piece of the chain is supposed to be independent and identical).

Mathematically, the small system is represented by a finite-dimensional Hilbert space $\mathcal{H}_0$ equipped with an initial state ρ_0. The set of states of $\mathcal{H}_0$ is denoted by $\mathcal{S}$. The chain is represented by the countable tensor product $T\Phi = \bigotimes_{k\geq 1}\mathcal{H}_k$, where $\mathcal{H}_k = \mathcal{H}$, for all k ($\mathcal{H}$ is also a finite-dimensional Hilbert space and each copy is equipped with the same reference state β). The model of quantum repeated measurements is described as follows. Each copy $\mathcal{H}_k$ of $\mathcal{H}$ interacts, one after the other, with $\mathcal{H}_0$ during a time τ. After each interaction, a measurement is performed on the piece of the chain which has just interacted. The sequence of measurements involves random modifications of the state of $\mathcal{H}_0$. These modifications are described by a random sequence of states of $\mathcal{H}_0$ denoted by (ρ_k) which is called a *discrete quantum trajectory*. Here, we remind the definition and the Markov property of the sequence (ρ_k) (see [29] for a complete details).

In order to define the sequence (ρ_k), we present the description of the first interaction and the first measurement (this allows namely to define the transitions of (ρ_k) in terms of Markov chain). A single interaction between $\mathcal{H}_0$ and a copy of $\mathcal{H}$ is described by a total Hamiltonian H_{tot} acting on the coupling system $\mathcal{H}_0 \otimes \mathcal{H}$. Its general form is given by

$$H_{\text{tot}} = H_0 \otimes I + I \otimes H + H_{\text{int}},$$

where the operators H_0 and H are the free Hamiltonian of each system. The operator H_{int} represents the Hamiltonian of interaction. This defines the unitary-operator

$$U = \mathrm{e}^{\mathrm{i}\tau H_{\text{tot}}}.$$

The evolution of states of $\mathcal{H}_0 \otimes \mathcal{H}$, in the Schrödinger picture, is given by

$$\rho \mapsto U\rho U^{\star}.$$

After this first interaction, a second copy of $\mathcal{H}$ interacts with $\mathcal{H}_0$ in the same fashion and so on. Usually the whole procedure is described by the state space $\boldsymbol{\Gamma} = \mathcal{H}_0 \otimes T\Phi$ and a sequence of unitary operators (U_k). More precisely, the operator U_k describes the kth interaction between $\mathcal{H}_0$ and $\mathcal{H}_k$, it acts as U on $\mathcal{H}_0 \otimes \mathcal{H}_k$ and it acts as the identity operator on the other copies of $\mathcal{H}$. Hence, the result of the k first interactions is described by the operators (V_k) where $V_k = U_k U_{k-1} \cdots U_1$, for all $k > 0$ (this is the usual setup of quantum repeated interactions [1]).

We do not need all the details of the setup of quantum repeated interactions in order to describe the discrete quantum trajectory (ρ_k). We just need to describe the transition probabilities which state the Markov character. To this end, we describe the procedure of measurement after the first interaction. Before the interaction the initial state on $\mathcal{H}_0 \otimes \mathcal{H}$ is $\rho_0 \otimes \beta$ and after the first interaction, the state on $\mathcal{H}_0 \otimes \mathcal{H}$ is given by $\mu_1 = U(\rho_0 \otimes \beta)U^{\star}$.

Now, we shall describe the measurement procedure of an observable of $\mathcal{H}$. Let A be any observable on $\mathcal{H}$, with spectral decomposition $A = \sum_{j=0}^{p}\lambda_j P_j$. According to the laws of quantum mechanics, the result of the measurement of A is random. In an explicit way, the accessible data are its eigenvalues, and the observation of λ_j obeys the probability law

$$\mathrm{P}[\text{to observe } \lambda_j] = \mathrm{Tr}[\mu_1 P_j], \quad j \in \{0, \dots, p\}.$$

Besides, if we have observed the eigenvalue λ_j, the *wave packet reduction* principle imposes that the state after measurement becomes

$$\tilde{\rho}_1(j) = \frac{I \otimes P_j \mu_1 I \otimes P_j}{\mathrm{Tr}[\mu_1 I \otimes P_j]}.$$

As a consequence, depending on the result of the observation, this defines the new reference state of the system $\mathcal{H}_0 \otimes \mathcal{H}$.

In our context, we are only interested in the reduced state of the small system $\mathcal{H}_0$. This state is given by taking a partial trace on $\mathcal{H}_0$ (see the below definition–theorem). If $\mathcal{H}$ is any Hilbert space, we denote by $\mathrm{Tr}_{\mathcal{H}}[W]$ the trace of a trace-class operator W on $\mathcal{H}$.

Definition–Theorem 1. *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces. If α is a state on a tensor product $\mathcal{H} \otimes \mathcal{K}$, then there exists a unique state η on $\mathcal{H}$ which is characterized by the property*

$$\mathrm{Tr}_{\mathcal{H}}[\eta X] = \mathrm{Tr}_{\mathcal{H}\otimes\mathcal{K}}\big[\alpha(X \otimes I)\big]$$

for all $X \in \mathcal{B}(\mathcal{H})$. This unique state η is called the partial trace of α on $\mathcal{H}$ with respect to $\mathcal{K}$.

Let α be a state on $\boldsymbol{\Gamma}$, we denote by $\mathbf{E}_0(\alpha)$ the partial trace of α on $\mathcal{H}_0$ with respect to $\mathcal{H}$. Hence, we define

$$\rho_1(j) = \mathbf{E}_0\big[\tilde{\rho}_1(j)\big] \tag{3}$$

for all $j = 0, \dots, p$. The state $\rho_1(\cdot)$ is a random state which is the new reference state of $\mathcal{H}_0$ after the first interaction and the first measurement. More precisely, each state $\rho_1(j)$ appears with probability $\mathrm{Tr}[\mu_1 I \otimes P_j]$. This describes the transitions between ρ_0 and the possible states $\rho_1(j)$, $j = 0, \dots, p$. Now, we can consider a second copy of $(\mathcal{H}, \beta)$ which interacts with $(\mathcal{H}_0, \rho_1)$ and after the same procedure, we get a new random state ρ_2 on $\mathcal{H}_0$. Recursively, we describe a random sequence (ρ_k) whose the Markov property is expressed in the following theorem.

Theorem 1. *There exists a probability space $(\Omega, \mathcal{F}, P)$ such that the quantum trajectory (ρ_k) is a Markov chain with values in the set of states on $\mathcal{H}_0$. If $\rho_k = \chi_k$, then ρ_{k+1} takes one of the values*

$$\mathbf{E}_0\left[\frac{(I \otimes P_i)U(\chi_k \otimes \beta)U^\star(I \otimes P_i)}{\mathrm{Tr}[U(\chi_k \otimes \beta)U^\star(I \otimes P_i)]}\right], \quad i = 0, \dots, p,$$

with probability $\mathrm{Tr}[U(\chi_k \otimes \beta)U^\star(I \otimes P_i)]$.

As the operator U depends explicitly on the time parameter τ, it is worth noticing that the Markov chain (ρ_k) depends on τ. In the following, we put $\tau = 1/n$. In this way, we write the unitary operator $U(n)$ (with dependence in n) and we define

$$\rho_n(t) = \rho_{[nt]} \tag{4}$$

for all $t > 0$. This defines a sequence of processes $(\rho_n(t))$ and we aim to show next that this sequence of processes converges in distribution, when n goes to infinity. As announced in the Introduction, such a convergence is obtained from the convergence of generators of Markov chains. The following section is then devoted to present these generators for quantum trajectories.

1.2. *Discrete Markov generators*

Let A be an observable and let $(\rho_n(t))$ be the process defined from the quantum trajectory describing the successive measurements of A. In this section, we investigate the explicit computation of the Markov generator $\mathcal{A}_n$ of the process $(\rho_n(t))$ (we will make no distinctions between the infinitesimal generators of the Markov chains (ρ_k) and the process $(\rho_n(t))$ generated by this Markov chain). For instance, let us introduce some notations.

We work with $\mathcal{H}_0 = \mathbb{C}^{K+1}$. The set of operators on $\mathcal{H}_0$ is identified with $\mathbb{R}^P$ for $P = 2(K+1)^2$ (we do not need to give any particular identification). We set $E = \mathbb{R}^P$ and the set of states $\mathcal{S}$ becomes then a compact subset of E, since a state is a positive operator with trace 1. For all states $\rho \in \mathcal{S}$, we define

$$\begin{aligned}
\mathcal{L}_i^{(n)}(\rho) &= \mathbf{E}_0\left[\frac{(I \otimes P_i)U(n)(\rho \otimes \beta)U^\star(n)(I \otimes P_i)}{\operatorname{Tr}[U(n)(\rho \otimes \beta)U^\star(n)(I \otimes P_i)]}\right], \\
p^i(\rho) &= \operatorname{Tr}\bigl[U(n)(\rho \otimes \beta)U^\star(n)I \otimes P_i\bigr], \quad i = 0, \ldots, p.
\end{aligned} \tag{5}$$

Here, we suppose implicitly that $\mathcal{L}_i^{(n)}(\rho) = 0$ if $p^i(\rho) = 0$. The operators $\mathcal{L}_i^{(n)}(\rho)$ represent the transition states of the Markov chains described in Theorem 1; the terms $p^i(\rho)$ are the associated probabilities. The Markov generators of $(\rho_n(t))$ are then expressed as follows.

Definition 1. *Let $(\rho_n(t))$ be the process obtained from the repeated measurements of an observable A of the form $A = \sum_{i=0}^p \lambda_i P_i$. Let us define $P^{(n)}$ the probability measure which satisfies*

$$P^{(n)}\bigl[\rho_n(0) = \rho_0\bigr] = 1, \tag{6}$$

$$P^{(n)}\bigl[\rho_n(s) = \rho_k, k/n \le s < (k+1)/n\bigr] = 1, \tag{7}$$

$$P^{(n)}\bigl[\rho_{k+1} \in \boldsymbol{\Gamma}/\mathcal{M}_k^{(n)}\bigr] = \Pi_n(\rho_k, \boldsymbol{\Gamma}), \tag{8}$$

where $\Pi_n(\rho, \cdot)$ is the transition function of the Markov chain (ρ_k) given by

$$\Pi_n(\rho, \boldsymbol{\Gamma}) = \sum_{i=0}^p p^i(\rho)\delta_{\mathcal{L}_i^{(n)}(\rho)}(\boldsymbol{\Gamma}) \tag{9}$$

for all Borel subsets $\Gamma \in \mathcal{B}(\mathbb{R}^P)$ and $\mathcal{M}_k^{(n)} = \sigma\{\rho_i, i \le k\}$.

For all states $\rho \in \mathcal{S}$ and all functions $f \in C_c^2(E)$ (i.e. C^2 with compact support), we define

$$\begin{aligned}
\mathcal{A}_n f(\rho) &= n \int \bigl(f(\mu) - f(\rho)\bigr)\Pi_n(\rho, \mathrm{d}\mu) \\
&= n \sum_{i=0}^p \bigl(f\bigl(\mathcal{L}_i^{(n)}(\rho)\bigr) - f(\rho)\bigr)p^i(\rho).
\end{aligned} \tag{10}$$

The operator $\mathcal{A}_n$ is called the Markov generator *of the Markov chain (ρ_k) (or of the process $(\rho_n(t))$).*

The complete description of generators $\mathcal{A}_n$ needs the explicit expression of $\mathcal{L}_i^{(n)}(\rho)$, for all ρ and all $i \in \{0, \ldots, p\}$. To this end, we need to compute the partial trace operation $\mathbf{E}_0$ on the tensor product $\mathcal{H}_0 \otimes \mathcal{H}$. A judicious choice of basis for the tensor product allows to simplify the computations. Let $(\Omega_0, \ldots, \Omega_K)$ be any orthonormal basis of $\mathcal{H}_0$ and $(X_0, \ldots, X_N)$ a one of $\mathcal{H}$. For the tensor product, we choose the basis

$$\mathcal{B} = (\Omega_0 \otimes X_0, \ldots, \Omega_K \otimes X_0, \Omega_0 \otimes X_1, \ldots, \Omega_K \otimes X_1, \ldots, \Omega_0 \otimes X_N, \ldots, \Omega_K \otimes X_N).$$

In this basis, any $(N+1)(K+1) \times (N+1)(K+1)$ matrix M on $\mathcal{H}_0 \otimes \mathcal{H}$ can be written by blocks as a $(N+1) \times (N+1)$ matrix $M = (M_{ij})_{0 \le i,j \le N}$, where M_{ij} are operators on $\mathcal{H}_0$. The following easy result justifies the choice of this basis.

Proposition 1. *Let W be a state acting on $\mathcal{H}_0 \otimes \mathcal{H}$. If $W = (W_{ij})_{0\le i,j\le N}$, is the expression of W in the basis $\mathcal{B}$, where the coefficients W_{ij} are operators on $\mathcal{H}_0$, then the partial trace with respect to $\mathcal{H}$ is given by the formula*

$$\mathbf{E}_0[W] = \sum_{i=0}^{N} W_{ii}.$$

Now, we are in the position to compute $\mathcal{L}_i^{(n)}(\rho)$, for all states ρ and all $i \in \{0,\dots,p\}$. We choose the reference state β of $\mathcal{H}$ to be the orthogonal projector on $\mathbb{C}X_0$, that is, with physical notations $\beta = |X_0\rangle\langle X_0|$. This state is called the ground state (or vacuum state) in quantum physics. From general result of G.N.S representation in $C^\star$ algebra, it is worth noticing that it is not a restriction. Indeed, such a representation allows to identify any quantum system $(\mathcal{H},\beta)$ with another system of the form $(\mathcal{K}, |X_0\rangle\langle X_0|)$, where X_0 is the first vector of an orthonormal basis of a particular Hilbert space $\mathcal{K}$ (see [26] for details).

The unitary operator $U(n)$ is described by blocks as $U(n) = (U_{ij}(n))_{0\le i,j\le N}$, where the coefficients U_{ij} are $(K+1)\times(K+1)$ matrices acting on $\mathcal{H}_0$. For $i \in \{0,\dots,p\}$, we denote $P_i = (p^i_{kl})_{0\le k,l\le N}$, the eigen-projectors of the observable A. Hence, the non-normalized states $\mathbf{E}_0[I\otimes P_i U(n)(\rho\otimes\beta)U(n)^\star I\otimes P_i]$ and the probabilities $p^i(\rho)$ satisfy

$$\begin{aligned}
&\mathbf{E}_0\bigl[I\otimes P_i U(n)(\rho\otimes\beta)U(n)^\star I\otimes P_i\bigr] = \sum_{0\le k,l\le N} p^i_{kl} U_{k0}(n)\rho U^\star_{l0}(n),\\
&p^i(\rho) = \sum_{0\le k,l\le N} p^i_{kl}\,\mathrm{Tr}\bigl[U_{k0}(n)\rho U^\star_{l0}(n)\bigr].
\end{aligned} \tag{11}$$

By observing that $\mathcal{L}_i^{(n)}(\rho) = \mathbf{E}_0[I\otimes P_i U(n)(\rho\otimes\beta)U(n)^\star I\otimes P_i]/p^i(\rho)$, for all $i\in\{0,\dots,p\}$, we get the explicit expression of the generator $\mathcal{A}_n$. The next step is to consider the limit of $\mathcal{A}_n$, when n goes to infinity. Such limits need appropriate asymptotic assumptions for the coefficients $U_{ij}(n)$. This is the topic of the following section.

1.3. *Asymptotic assumptions*

The choice of the asymptotic expression of $U(n) = (U_{ij}(n))$ are based on the works of Attal–Pautrat in [1]. They have namely shown that the operator process defined for all $t>0$ by

$$V_{[nt]} = U_{[nt]}(n)\cdots U_1(n),$$

which describes the quantum repeated interactions, strongly converges (in operator sense) to a process $(\tilde V_t)$ satisfying a Quantum Langevin equation. Moreover, this convergence is non-trivial, only if the coefficients $U_{ij}(n)$ are scaled appropriately. When translated in our context, this expresses that there exist operators L_{ij} such that we have for all $(i,j)\in\{0,\dots,N\}^2$

$$\lim_{n\to\infty} n^{\varepsilon_{ij}}\bigl(U_{ij}(n) - \delta_{ij} I\bigr) = L_{ij}, \tag{12}$$

where $\varepsilon_{ij} = \frac{1}{2}(\delta_{0i}+\delta_{0j})$. As the expression (11) of $\mathcal{L}_i^{(n)}(\rho)$ only involves the first column of $U(n)$, we only keep the following asymptotic expressions

$$U_{00}(n) = I + \frac{1}{n}L_{00} + \mathrm{o}\left(\frac{1}{n}\right) \quad\text{and}\quad U_{i0}(n) = \frac{1}{\sqrt{n}}L_{i0} + \mathrm{o}\left(\frac{1}{\sqrt{n}}\right) \quad \text{for } i>0.$$

Another fact, which will be important in the computation of limit generators, is the following result (cf. [1]).

Proposition 2. *In order that $U = (U_{ij})$ is an unitary-operator, there exists a self-adjoint operator H_0 such that*

$$L_{00} = -\left(iH_0 + \frac{1}{2}\sum_{i=1}^{N} L^\star_{i0}L_{i0}\right).$$

Furthermore we have for all $\rho \in \mathcal{S}$

$$\mathrm{Tr}\left[L_{00}\rho + \rho L_{00}^\star + \sum_{1\le k\le N} L_{k0}\rho L_{k0}^\star\right] = 0,$$

because $\mathrm{Tr}[U(n)\rho U^\star(n)] = 1$, *for all* n.

We can now apply these considerations to give the asymptotic expression of non-normalized states and probabilities given by the expression (11). For the non-normalized states, we have

$$\begin{aligned}
&\mathbf{E}_0\big[I\otimes P_i U(n)(\rho\otimes\beta)U(n)^\star I\otimes P_i\big] \\
&\quad = p_{00}^i\rho + \frac{1}{\sqrt{n}}\sum_{1\le k\le N}\big(p_{k0}^i L_{k0}\rho + p_{0k}^i\rho L_{k0}^\star\big) \\
&\qquad + \frac{1}{n}\left[p_{00}^i\big(L_{00}\rho + \rho L_{00}^\star\big) + \sum_{1\le k,l\le N} p_{kl}^i L_{k0}\rho L_{l0}^\star\right] + \mathrm{o}\left(\frac{1}{n}\right).
\end{aligned} \tag{13}$$

Moreover, as $p^i(\rho) = \mathrm{Tr}[\mathbf{E}_0[I\otimes P_i U(n)(\rho\otimes\beta)U(n)^\star I\otimes P_i]]$, we get

$$\begin{aligned}
p^i(\rho) &= p_{00}^i + \frac{1}{\sqrt{n}}\mathrm{Tr}\left[\sum_{1\le k\le N}\big(p_{k0}^i L_{k0}\rho + p_{0k}^i\rho L_{k0}^\star\big)\right] \\
&\quad + \frac{1}{n}\mathrm{Tr}\left[\left(p_{00}^i\big(L_{00}\rho + \rho L_{00}^\star\big) + \sum_{1\le k,l\le N} p_{kl}^i L_{k0}\rho L_{l0}^\star\right)\right] + \mathrm{o}\left(\frac{1}{n}\right).
\end{aligned} \tag{14}$$

The asymptotic expression of $\mathcal{L}_i^{(n)}(\rho)$ then follows from (13) and (14). Depending on the fact that p_{00}^i is equal to zero or not, we consider three cases:

1. If $p_{00}^i = 0$, then $p_{0k}^i = 0$, for all $k > 0$ and we have

$$\mathcal{L}_i^{(n)}(\rho) = \frac{\sum_{1\le k,l\le N} p_{kl}^i L_{k0}\rho L_{l0}^\star + \mathrm{o}(1)}{\mathrm{Tr}[\sum_{1\le k,l\le N} p_{kl}^i L_{k0}\rho L_{l0}^\star + \mathrm{o}(1)]}\mathbf{1}_{\mathrm{Tr}[\sum_{1\le k,l\le N} p_{kl}^i L_{k0}\rho L_{l0}^\star]\neq 0} + \mathrm{o}(1). \tag{15}$$

1. If $p_{00}^i = 1$, then we have

$$\begin{aligned}
\mathcal{L}_i^{(n)}(\rho) &= \rho + \frac{1}{n}\left[\big(L_{00}\rho + \rho L_{00}^\star\big) + \sum_{1\le k,l\le N} p_{kl}^i L_{k0}\rho L_{l0}^\star\right] \\
&\quad - \frac{1}{n}\mathrm{Tr}\left[\big(L_{00}\rho + \rho L_{00}^\star\big) + \sum_{1\le k,l\le N} p_{kl}^i L_{k0}\rho L_{l0}^\star\right]\rho + \mathrm{o}\left(\frac{1}{n}\right).
\end{aligned} \tag{16}$$

1. If $p_{00}^i \notin \{0, 1\}$, then we have

$$\begin{aligned}
\mathcal{L}_i^{(n)}(\rho) = \rho + \frac{1}{\sqrt{n}}&\left[\frac{1}{p_{00}^i}\sum_{1\le k\le N}\big(p_{k0}^i L_{k0}\rho + p_{0k}^i\rho L_{k0}^\star\big)\right. \\
&\left. - \frac{1}{p_{00}^i}\mathrm{Tr}\left(\sum_{1\le k\le N}\big(p_{k0}^i L_{k0}\rho + p_{0k}^i\rho L_{k0}^\star\big)\right)\times\rho\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{n}\Bigg[\frac{1}{p^i_{00}}\Bigg(p^i_{00}\big(L_{00}\rho+\rho L^\star_{00}\big)+\sum_{1\le k,l\le N}p^i_{kl}L_{k0}\rho L^\star_{l0}\Bigg)\\
&+\frac{1}{(p^i_{00})^2}\operatorname{Tr}\Bigg(\sum_{1\le k\le N}\big(p^i_{k0}L_{k0}\rho+p^i_{0k}\rho L^\star_{k0}\big)\Bigg)^2\times\rho\\
&-\frac{1}{p^i_{00}}\operatorname{Tr}\Bigg(p^i_{00}\big(L_{00}\rho+\rho L^\star_{00}\big)+\sum_{1\le k,l\le N}p^i_{kl}L_{k0}\rho L^\star_{l0}\Bigg)\times\rho\\
&-\frac{1}{(p^i_{00})^2}\sum_{1\le k\le N}\big(p^i_{k0}L_{k0}\rho+p^i_{0k}\rho L^\star_{k0}\big)\times\operatorname{Tr}\Bigg(\sum_{1\le k\le N}\big(p^i_{k0}L_{k0}\rho+p^i_{0k}\rho L^\star_{k0}\big)\Bigg)\Bigg]\\
&+\mathrm{o}\bigg(\frac{1}{n}\bigg).
\end{aligned}
\tag{17}
$$

It is worth noticing that all the o are uniform in ρ since $\mathcal{S}$ is a compact set.

The next section is dedicated to the convergence of $\mathcal{A}_n$ and the presentation of the continuous models.

2. Jump–diffusion models of quantum measurement

In this section, we show that the limit ($n\to\infty$) of generators $\mathcal{A}_n$ gives rise to explicit infinitesimal generators. We interpret these generators as Markov generators of continuous time processes associated with martingale problems. Besides, we show that these processes are solution of jump–diffusion stochastic differential equations which are a generalization of the Belavkin equations (1) and (2) presented in the Introduction.

In our framework, the notion of martingale problem is expressed as follows (see [18,20,23,25] for complete references). We still consider the identification of the set of states as a compact subset of $E=\mathbb{R}^P$. Let Π be a transition kernel on E, let $a(\cdot)=(a_{ij}(\cdot))$ be a measurable mapping on E with values in the set of positive semi-definite symmetric $P\times P$ matrices and let $b(\cdot)=(b_i(\cdot))$ be a measurable function from E to E. Let f be any $C^2_c(E)$ and let $\rho\in E$. In this article, we consider infinitesimal generators $\mathcal{A}$ of the form

$$
\begin{aligned}
\mathcal{A}f(\rho)={}&\sum_{i=1}^P b_i(\rho)\frac{\partial f(\rho)}{\partial\rho_i}+\frac{1}{2}\sum_{i,j=1}^P a_{ij}(\rho)\frac{\partial f(\rho)}{\partial\rho_i\,\partial\rho_j}\\
&+\int_E\Bigg[f(\rho+\mu)-f(\rho)-\sum_{i=1}^P\mu_i\frac{\partial f(\rho)}{\partial\rho_i}\Bigg]\Pi(\rho,\mathrm{d}\mu).
\end{aligned}
\tag{18}
$$

The notion of martingale problem associated with such generators is expressed as follows.

Definition 2. *Let $\rho_0\in E$. We say that a measurable stochastic process (ρ_t) on some probability space $(\Omega,\mathcal{F},P)$ is a solution of the martingale problem for $(\mathcal{A},\rho_0)$, if for all $f\in C^2_c(E)$,*

$$
\mathcal{M}^f_t=f(\rho_t)-f(\rho_0)-\int_0^t\mathcal{A}f(\rho_s)\,\mathrm{d}s,\quad t\ge 0,
\tag{19}
$$

is a martingale with respect to $\mathcal{F}^\rho_t=\sigma(\rho_s,s\le t)$.

It is worth noticing that we must also define a probability space $(\Omega,\mathcal{F},P)$ to make explicit a solution of a problem of martingale.

In the following section, we show that the Markov generators of discrete quantum trajectories converges to infinitesimal generators of the form (18).

2.1. *Limit infinitesimal generators*

Before expressing the theorem which gives the limit infinitesimal generators of $\mathcal{A}_n$, we define particular functions on $\mathcal{S}$ which appear in the limit. For all i and all states $\rho \in \mathcal{S}$, set

$$\begin{aligned} g_i(\rho) &= \left(\frac{\sum_{1\le k,l\le N} p^i_{kl} L_{k0}\rho L^\star_{l0}}{\operatorname{Tr}[\sum_{1\le k,l\le N} p^i_{kl} L_{k0}\rho L^\star_{l0}]} - \rho \right), \\ v_i(\rho) &= \operatorname{Tr}\left[\sum_{1\le k,l\le N} p^i_{kl} L_{k0}\rho L^\star_{l0} \right], \\ h_i(\rho) &= \frac{1}{\sqrt{p^i_{00}}} \left[\sum_{1\le k\le N} \left(p^i_{k0} L_{k0}\rho + p^i_{0k}\rho L^\star_{k0} \right) - \operatorname{Tr}\left[\sum_{1\le k\le N} \left(p^i_{k0} L_{k0}\rho + p^i_{0k}\rho L^\star_{k0} \right) \right] \rho \right], \\ L(\rho) &= L_{00}\rho + \rho L^\star_{00} + \sum_{1\le k\le N} L_{k0}\rho L^\star_{k0}. \end{aligned} \tag{20}$$

The next theorem concerning the limit generators follows from Eqs (15)–(17) described in Section 1. In this theorem, the term $D_\rho f$ denotes the first differential in ρ of a function $f \in C^2_c(E)$ and $D^2_\rho f$ the second differential.

Theorem 2. *Let A be an observable with spectral decomposition $A = \sum_{i=0}^p \lambda_i P_i$, where $P_i = (p^i_{kl})_{0\le k,l\le N}$ are its eigen-projectors. Up to permutation of eigen-projectors, we can suppose that $p^0_{00} \neq 0$. We define the sets*

$$I = \left\{ i \in \{1,\dots,p\} / p^i_{00} = 0 \right\} \quad \text{and} \quad J = \{1,\dots,p\} \setminus I.$$

Let $(\rho^J_n(t))$ be the corresponding quantum trajectory obtained from the measurement of A and let $\mathcal{A}^J_n$ be its infinitesimal generator (cf. Definition 1). The limit generators $\mathcal{A}^J$ of $\mathcal{A}^J_n$ exist and are described as follows:

1. *If $I = \{1,\dots,p\}$, that is, $p^0_{00} = 1$ and $J = \varnothing$, we have for all $f \in C^2_c(E)$*

$$\lim_{n\to\infty} \sup_{\rho\in\mathcal{S}} \left| \mathcal{A}^J_n f(\rho) - \mathcal{A}^J f(\rho) \right| = 0, \tag{21}$$

where $\mathcal{A}^J$ satisfies

$$\mathcal{A}^J f(\rho) = D_\rho f\big(L(\rho)\big) + \int_E \left[f(\rho+\mu) - f(\rho) - D_\rho f(\mu) \right] \Pi(\rho, \mathrm{d}\mu), \tag{22}$$

the transition kernel Π being defined as $\Pi(\rho, \mathrm{d}\mu) = \sum_{i=1}^p v_i(\rho)\delta_{g_i(\rho)}(\mathrm{d}\mu)$.

2. *If $I \neq \{1,\dots,p\}$, that is, $p^0_{00} \neq 1$ and $J \neq \varnothing$, we have for all $f \in C^2_c(E)$*

$$\lim_{n\to\infty} \sup_{\rho\in\mathcal{S}} \left| \mathcal{A}^J_n f(\rho) - \mathcal{A}^J f(\rho) \right| = 0, \tag{23}$$

where $\mathcal{A}^J$ satisfies

$$\begin{aligned} \mathcal{A}^J f(\rho) &= D_\rho f\big(L(\rho)\big) + \frac{1}{2} \sum_{i\in J\cup\{0\}} D^2_\rho f\big(h_i(\rho), h_i(\rho)\big) \\ &\quad + \int_E \left[f(\rho+\mu) - f(\rho) - D_\rho f(\mu) \right] \Pi(\rho, \mathrm{d}\mu), \end{aligned} \tag{24}$$

the transition kernel Π being defined as $\Pi(\rho, \mathrm{d}\mu) = \sum_{i\in I} v_i(\rho)\delta_{g_i(\rho)}(\mathrm{d}\mu)$.

Proof. Recall that $\mathcal{S}$ is the set of states and it is a compact subset of E. For any $i \in \{0, \dots, p\}$ and for any $\rho \in \mathcal{S}$, let us compute $\lim_{n\to\infty} \mathcal{A}_n^J f(\rho)$. To this aim, we use the asymptotic results of Section 1 (Eqs (15)–(17)). There are three different types of limit to consider:

1. Suppose $p_{00}^i = 0$, we have,

$$\begin{aligned}
&\lim_{n\to\infty} n\big(f\big(\mathcal{L}_i^{(n)}(\rho)\big) - f(\rho)\big)p^i(\rho) \\
&\quad= \left[f\left(\frac{\sum_{1\le k,l\le N} p_{kl}^i L_{k0}\rho L_{l0}^\star}{\operatorname{Tr}[\sum_{1\le k,l\le N} p_{kl}^i L_{k0}\rho L_{l0}^\star]}\right) - f(\rho)\right]\operatorname{Tr}\left[\sum_{1\le k,l\le N} p_{kl}^i L_{k0}\rho L_{l0}^\star\right]
\end{aligned}$$

   for all $\rho \in \mathcal{S}$. This defines a uniformly continuous function on $\mathcal{S}$ since $f \in C_c^2$ and $\mathcal{S}$ is compact. As a consequence, the asymptotic concerning this case (and the fact that all the o are uniform on $\mathcal{S}$ cf. Section 1) implies the uniform convergence.
2. Suppose $p_{00}^i = 1$, by using the Taylor formula of order one, we have

$$\begin{aligned}
&\lim_{n\to\infty} n\big(f\big(\mathcal{L}_i^{(n)}(\rho)\big) - f(\rho)\big)p^i(\rho) \\
&\quad= D_\rho f\left(\left[\big(L_{00}\rho + \rho L_{00}^\star\big) + \sum_{1\le k,l\le N} p_{kl}^i L_{k0}\rho L_{l0}^\star\right]\right. \\
&\qquad \left. - \operatorname{Tr}\left[\big(L_{00}\rho + \rho L_{00}^\star\big) + \sum_{1\le k,l\le N} p_{kl}^i L_{k0}\rho L_{l0}^\star\right]\rho\right).
\end{aligned}$$

   To obtain the uniform result, we use the asymptotic expressions of Section 1 and the uniform continuity of Df on $\mathcal{S}$.
3. Suppose $p_{00}^i \notin \{0, 1\}$. By applying the Taylor formula of order two, we get the convergence

$$\begin{aligned}
&\sum_{i/p_{00}^i\notin\{0,1\}} \lim_{n\to\infty} n\big(f\big(\mathcal{L}_i^{(n)}(\rho)\big) - f(\rho)\big)p^i(\rho) \\
&\quad= \sum_{i/p_{00}^i\notin\{0,1\}} \left[D_\rho f\left(\left(p_{00}^i\big(L_{00}\rho + \rho L_{00}^\star\big) + \sum_{1\le k,l\le N} p_{kl}^i L_{k0}\rho L_{l0}^\star\right)\right.\right. \\
&\qquad \left. - \operatorname{Tr}\left[p_{00}^i\big(L_{00}\rho + \rho L_{00}^\star\big) + \sum_{1\le k,l\le N} p_{kl}^i L_{k0}\rho L_{l0}^\star\right]\rho\right) \\
&\qquad + \frac{1}{2p_{00}^i} D_\rho^2 f\left(\sum_{1\le k\le N}\big(p_{k0}^i L_{k0}\rho + p_{0k}^i \rho L_{k0}^\star\big) - \operatorname{Tr}\left[\sum_{1\le k\le N}\big(p_{k0}^i L_{k0}\rho + p_{0k}^i\rho L_{k0}^\star\big)\right]\rho\right. \\
&\qquad \left.\left. + \sum_{1\le k\le N}\big(p_{k0}^i L_{k0}\rho + p_{0k}^i \rho L_{k0}^\star\big) - \operatorname{Tr}\left[\sum_{1\le k\le N}\big(p_{k0}^i L_{k0}\rho + p_{0k}^i\rho L_{k0}^\star\big)\right]\rho\right)\right].
\end{aligned} \tag{25}$$

   This last equality needs further explanation. In the Taylor formula, terms of the form

$$G_i(\rho) = \frac{1}{\sqrt{n}} D_\rho f\left(\sum_{1\le k\le N}\big(p_{k0}^i L_{k0}\rho + p_{0k}^i \rho L_{k0}^\star\big) - \operatorname{Tr}\left[\sum_{1\le k\le N}\big(p_{k0}^i L_{k0}\rho + p_{0k}^i\rho L_{k0}^\star\big)\right]\rho\right)$$

   appear for each i such that $p_{00}^i \notin \{0, 1\}$. We have $\sum_{i/p_{00}^i\notin\{0,1\}} G_i(\rho) = 0$, since $\sum_{i/p_{00}^i\notin\{0,1\}} p_{k0}^i = \sum_{i/p_{00}^i\notin\{0,1\}} p_{0k}^i = \sum_{i=0}^p p_{0k} = \sum_{i=0}^p p_{k0} = 0$, for all $k > 0$ (indeed we have $\sum_{i=0}^p P_i = I$). Furthermore, this convergence is uniform for the same reasons as previously.

The different cases of the theorem follows from these three limits. The first case of Proposition 2 follows from the first two limits described above, the second case follows from the first and the third limits above. Before to describe this in details, we have to notice that

$$\sum_{i=0}^{p}\sum_{1\le k,l\le N} p_{kl}^{i} L_{k0}\rho L_{l0}^{\star} = \sum_{1\le k\le N} L_{k0}\rho L_{k0}^{\star}$$

since $\sum_{i=0}^{p} P_i = I$. Moreover, we have $\mathrm{Tr}[L(\rho)] = \mathrm{Tr}[L_{00}\rho + \rho L_{00}^{\star} + \sum_{1\le k\le N} L_{k0}\rho L_{k0}^{\star}] = 0$ (see Proposition 2). By using these facts, in the case $p_{00}^{i} = 0$, the limit can be written as

$$\int_E \big[f(\rho+\mu) - f(\rho) - D_\rho f(\mu)\big] v_i(\rho)\delta_{g_i(\rho)}(\mathrm{d}\mu) + D_\rho f\big(g_i(\rho)\big)v_i(\rho).$$

Besides, we have

$$D_\rho f\big(g_i(\rho)\big)v_i(\rho) = D_\rho f\left(\sum_{1\le k,l\le N} p_{kl}^{i} L_{k0}\rho L_{l0}^{\star} - \mathrm{Tr}\left[\sum_{1\le k,l\le N} p_{kl}^{i} L_{k0}\rho L_{l0}^{\star}\right]\rho\right).$$

Hence, it implies the first case of Theorem 2, that is for $I = \{1,\dots,p\}$, we get indeed

$$\mathcal{A}_f^J(\rho) = D_\rho f\big(L(\rho)\big) + \int_E \big[f(\rho+\mu) - f(\rho) - D_\rho f(\mu)\big]\Pi(\rho,\mathrm{d}\mu).$$

A similar reasoning gives the expression of the infinitesimal generator in the second case, where $I \ne \{1,\dots,p\}$ and the proposition is proved. □

It is worth noticing that the generators $\mathcal{A}^J$ are generators of type (18), it suffices to expand the differential terms $D_\rho f$ and $D_\rho^2 f$ in terms of partial derivatives $\frac{\partial f}{\partial \rho_j}$ and $\frac{\partial^2 f}{\partial \rho_i\,\partial \rho_j}$. In the next section, we present the continuous time stochastic models.

2.2. *Solutions of martingale problems*

In all this section, we consider an observable A with spectral decomposition

$$A = \sum_{i\in I}\lambda_i P_i + \sum_{j\in J\cup 0}\lambda_j P_j, \tag{26}$$

where I and J are the subsets of $\{1,\dots,p\}$ defined in Theorem 2. Let $\mathcal{A}^J$ be the associated limit generator and let ρ_0 be a state. In order to solve the martingale problem for $(\mathcal{A}^J,\rho_0)$, by Definition 2, we have to define a probability space $(\Omega,\mathcal{F},P)$ and a stochastic process (ρ_t^J) such that the process

$$\mathcal{M}_t^f = f\big(\rho_t^J\big) - f(\rho_0) - \int_0^t \mathcal{A}^J f\big(\rho_s^J\big)\,\mathrm{d}s \tag{27}$$

is a martingale for the natural filtration of (ρ_t^J). In a usual way, such martingale problems are solved in terms of stochastic differential equations [18,21].

Concerning the suitable probability space, let us consider $(\Omega,\mathcal{F},P)$ a probability space which supports a $(p+1)$-dimensional Brownian motion $W = (W_0,\dots,W_p)$ and p independent Poisson point processes $(N_i)_{1\le i\le p}$ on $\mathbb{R}^2$, independent of the Brownian motion.

As there are two types of limit generators in Theorem 2, we define two types of stochastic differential equations in the following way. Let ρ_0 be an initial deterministic state:

1. In case $J = \varnothing$, we define the following stochastic differential equation on $(\Omega, \mathcal{F}, P)$,

$$\rho_t^J = \rho_0 + \int L(\rho_{s-}^J)\,\mathrm{d}s + \sum_{i=1}^{p} \int_0^t \int_{\mathbb{R}} g_i(\rho_{s-}^J)\mathbf{1}_{0<x<v_i(\rho_{s-}^J)}\big[N_i(\mathrm{d}x, \mathrm{d}s) - \mathrm{d}x\,\mathrm{d}s\big]. \tag{28}$$

2. In case $J \neq \varnothing$, we define

$$\rho_t^J = \rho_0 + \int L(\rho_{s-}^J)\,\mathrm{d}s + \sum_{i\in J\cup\{0\}} \int_0^t h_i(\rho_{s-}^J)\,\mathrm{d}W_i(s) + \sum_{i\in I} \int_0^t \int_{\mathbb{R}} g_i(\rho_{s-}^J)\mathbf{1}_{0<x<v_i(\rho_{s-}^J)}\big[N_i(\mathrm{d}x, \mathrm{d}s) - \mathrm{d}x\,\mathrm{d}s\big]. \tag{29}$$

In this way of writing, these equations are rigorously defined only if the solution takes values in the set of states (in general the term $v_i(\rho)$ is not real for all operators ρ). In order to consider such equations for all processes, we introduce some modifications. For all i, we define when it has a meaning

$$\tilde{g}_i(\rho) = \frac{\sum_{1\le k,l\le N} p_{kl}^i L_{k0}\rho L_{l0}^\star}{\mathrm{Re}(v_i(\rho))} - \rho.$$

Hence, we consider the modified stochastic differential equations

$$\rho_t^J = \rho_0 + \int L(\rho_{s-}^J)\,\mathrm{d}s + \sum_{i=1}^{p} \int_0^t \int_{\mathbb{R}} \tilde{g}_i(\rho_{s-}^J)\mathbf{1}_{0<x<\mathrm{Re}(v_i(\rho_{s-}^J))}\big[N_i(\mathrm{d}x, \mathrm{d}s) - \mathrm{d}x\,\mathrm{d}s\big] \tag{30}$$

and

$$\rho_t^J = \rho_0 + \int L(\rho_{s-}^J)\,\mathrm{d}s + \sum_{i\in J\cup\{0\}} \int_0^t h_i(\rho_{s-}^J)\,\mathrm{d}W_i(s) + \sum_{i\in I} \int_0^t \int_{\mathbb{R}} \tilde{g}_i(\rho_{s-}^J)\mathbf{1}_{0<x<\mathrm{Re}(v_i(\rho_{s-}^J))}\big[N_i(\mathrm{d}x, \mathrm{d}s) - \mathrm{d}x\,\mathrm{d}s\big]. \tag{31}$$

The fact that $\mathrm{Re}(v_i(\rho)) = v_i(\rho)$ and $\tilde{g}_i(\rho) = g_i(\rho)$, for all states ρ, implies that a solution (ρ_t^J) of Eq. (30) (resp. (31)) is a solution of Eq. (28) (resp. (29)) if the process (ρ_t^J) takes values in the set of states.

We proceed in the following way to solve the martingale problem (27). Firstly, we show that the modified Eqs (30) and (31) admit a unique solution. Secondly, we show that the solutions of (30) and (31) can be obtained as limits (in distribution) of discrete quantum trajectories (cf. Section 3). Finally, we show that the property of being a process valued in the set of states follows from convergence (cf. Section 3) and we conclude that the solutions of (30) and (31) take values in the set of states. Moreover, we show that they are solutions of martingale problems of type (27).

The fact that if the solutions of (30) and (31) take values in the set of states, they are solutions of martingale problems is expressed in the following theorem.

Theorem 3. *Let ρ_0 be any initial state.*

If the modified stochastic differential equation (30) admits a solution (ρ_t^J) which takes values in the set of states, then it is a solution of the martingale problem $(\mathcal{A}^J, \rho_0)$ in the case $I = \{1, \ldots, p\}$.

If the modified stochastic differential equation (31) admits a solution (ρ_t^J) which takes values in the set of states, then it is a solution of the martingale problem $(\mathcal{A}^J, \rho_0)$ in the case $J \neq \varnothing$.

As a consequence, if $\tilde{\mathcal{A}}^J$ designs the infinitesimal generator of a solution of (31) or (30), then we have $\tilde{\mathcal{A}}^J f(\rho) = \mathcal{A}^J f(\rho)$, for all states ρ and all functions $f \in C_c^2$.

Proof. Assume that the processes take values in the set of states. For any state ρ, we have $\mathrm{Re}(v_i(\rho)) = v_i(\rho)$ and $\tilde{g}_i(\rho) = g_i(\rho)$ and the last part concerning the generators follows. Concerning the martingale problem, this is a consequence of the Itô formula. Let $\rho_t^J = (\rho_1^J(t), \dots, \rho_P^J(t))$ denote the coordinates of a solution of (28) or (29) (with identification between the set of operators on $\mathcal{H}_0$ and $\mathbb{R}^P$), we have for all $f \in C_c^2$

$$\begin{aligned} f(\rho_t^J) - f(\rho_0) &= \sum_{i=1}^P \int_0^t \frac{\partial f(\rho_{s-}^J)}{\partial \rho_i^J} \mathrm{d}\rho_i^J(s) + \frac{1}{2} \sum_{i,j=1}^P \int_0^t \frac{\partial f(\rho_{s-}^J)}{\partial \rho_i^J \partial \rho_j^J} \mathrm{d}\big[\rho_i^J(s), \rho_j^J(s)\big]^c \\ &\quad + \sum_{0 \le s \le t} \left[f(\rho_s^J) - f(\rho_{s-}^J) - \sum_{i=1}^P \frac{\partial f(\rho_{s-}^J)}{\partial \rho_i^J} \Delta \rho_i^J(s) \right], \end{aligned} \tag{32}$$

where $[\rho_i^J(\cdot), \rho_j^J(\cdot)]^c$ denotes the continuous part of $[\rho_i^J(\cdot), \rho_j^J(\cdot)]$ [31].

Let us deal with the case $J \neq \varnothing$. If $(e_i)_{1 \le i \le P}$ denotes the canonical basis of $\mathbb{R}^P$, then we have $\rho_i^J(t) = \langle \rho_t^J, e_i \rangle$, for all $t \neq 0$. Hence, we have $\mathrm{d}\rho_i^J(t) = \langle \mathrm{d}\rho_t^J, e_i \rangle$. As a consequence, we have for all $i \in \{1, \dots, P\}$

$$\begin{aligned} \rho_i^J(t) &= \rho_0 + \int \big\langle L(\rho_{s-}^J), e_i \big\rangle \mathrm{d}s + \sum_{k \in J \cup \{0\}} \int_0^t \big\langle h_k(\rho_{s-}^J), e_i \big\rangle \mathrm{d}W_k(s) \\ &\quad + \sum_{k \in I} \int_0^t \int_{\mathbb{R}} \big\langle g_k(\rho_{s-}^J), e_i \big\rangle \mathbf{1}_{0 < x < v_k(\rho_{s-}^J)} \big[N_k(\mathrm{d}x, \mathrm{d}s) - \mathrm{d}x\, \mathrm{d}s\big]. \end{aligned} \tag{33}$$

It implies that

$$\big[\rho_i^J(t), \rho_j^J(t)\big]^c = \sum_{k \in J \cup \{0\}} \int_0^t \big\langle h_k(\rho_{s-}^J), e_i \big\rangle \big\langle h_k(\rho_{s-}^J), e_i \big\rangle \mathrm{d}s,$$

since $[W_i(t), W_j(t)] = \delta_{ij} t$. Furthermore, if we set $g_k^i(\rho) = \langle g_k(\rho), e_i \rangle$, then the process

$$\begin{aligned} &\sum_{0 \le s \le t} \left[f(\rho_s^J) - f(\rho_{s-}^J) - \sum_{i=1}^P \frac{\partial f(\rho_{s-}^J)}{\partial \rho_i} \Delta \rho_i^J(s) \right] \\ &\quad - \sum_{k \in J} \int_0^t \int_{\mathbb{R}} \left[f\big(\rho_{s-}^J + g_k(\rho_{s-}^J)\big) - f(\rho_{s-}^J) - \sum_{i=1}^P \frac{\partial f(\rho_{s-}^J)}{\partial \rho_i} g_k^i(\rho_{s-}^J) \right] \mathbf{1}_{0 < x \le v_k(\rho_{s-}^J)} N_k(\mathrm{d}x, \mathrm{d}s) \end{aligned}$$

is a martingale. Hence, we have

$$\begin{aligned} &\sum_{k \in J} \int_0^t \int_{\mathbb{R}} \left[f\big(\rho_{s-}^J + g_k(\rho_{s-}^J)\big) - f(\rho_{s-}^J) - \sum_{i=1}^P \frac{\partial f(\rho_{s-}^J)}{\partial \rho_i^J} g_k^i(\rho_{s-}^J) \right] \mathbf{1}_{0 < x \le v_k(\rho_{s-}^J)} N_k(\mathrm{d}x, \mathrm{d}s) \\ &\quad - \sum_{k \in J} \int_0^t \int_{\mathbb{R}} \left[f\big(\rho_{s-}^J + g_k(\rho_{s-}^J)\big) - f(\rho_{s-}^J) - \sum_{i=1}^P \frac{\partial f(\rho_{s-}^J)}{\partial \rho_i^J} g_k^i(\rho_{s-}^J) \right] \mathbf{1}_{0 < x \le v_k(\rho_{s-}^J)} \mathrm{d}x\, \mathrm{d}s \end{aligned} \tag{34}$$

is a martingale because each N_k is a Poisson point process with intensity measure $\mathrm{d}x \otimes \mathrm{d}s$. Furthermore, we can notice that

$$\begin{aligned} &\sum_{k \in J} \int_0^t \int_{\mathbb{R}} \left[f\big(\rho_{s-}^J + g_k(\rho_{s-}^J)\big) - f(\rho_{s-}^J) - \sum_{i=1}^P \frac{\partial f(\rho_{s-}^J)}{\partial \rho_i^J} g_k^i(\rho_{s-}^J) \right] \mathbf{1}_{0 < x \le v_k(\rho_{s-}^J)} \mathrm{d}x\, \mathrm{d}s \\ &\quad = \int_0^t \big[f(\rho_{s-}^J + \mu) - f(\rho_{s-}^J) - D_{\rho_{s-}^J} f(\mu) \big] \Pi(\rho_{s-}^J, \mathrm{d}\mu). \end{aligned} \tag{35}$$

As the Lebesgue measure of the set of times where $\rho^J_{s-} \neq \rho^J_s$ is equal to zero, we get that

$$f(\rho^J_t) - f(\rho_0) - \int_0^t \mathcal{A}^J f(\rho^J_{s-})\, \mathrm{d}s = f(\rho^J_t) - f(\rho_0) - \int_0^t \mathcal{A}^J f(\rho^J_s)\, \mathrm{d}s.$$

This defines a martingale with respect to the natural filtration of (ρ_t) and the proposition is proved. □

As announced, the first step consists in proving that Eqs (30) and (31) admit a unique solution. By regrouping the term in $\mathrm{d}t$, we consider the following way of writing

$$\begin{aligned}\rho^J_t = \rho_0 &+ \int_0^t \Bigl(L(\rho^J_{s-}) - \sum_{i=1}^p \tilde{g}_i(\rho^J_{s-})\operatorname{Re}\bigl(v_i(\rho^J_{s-})\bigr)\Bigr)\mathrm{d}s \\ &+ \sum_{i=1}^p \int_0^t \int_{\mathbb{R}} \tilde{g}_i(\rho^J_{s-})\mathbf{1}_{0<x<\operatorname{Re}(v_i(\rho^J_{s-}))} N_i(\mathrm{d}x, \mathrm{d}s)\end{aligned} \tag{36}$$

and

$$\begin{aligned}\rho^J_t = \rho_0 &+ \int_0^t \Bigl(L(\rho^J_{s-}) - \sum_{i\in I} \tilde{g}_i(\rho^J_{s-})\operatorname{Re}\bigl(v_i(\rho^J_{s-})\bigr)\Bigr)\mathrm{d}s + \sum_{i\in J\cup\{0\}} \int_0^t h_i(\rho^J_{s-})\,\mathrm{d}W_i(s) \\ &+ \sum_{i\in I} \int_0^t \int_{\mathbb{R}} \tilde{g}_i(\rho^J_{s-})\mathbf{1}_{0<x<\operatorname{Re}(v_i(\rho^J_{s-}))} N_i(\mathrm{d}x, \mathrm{d}s).\end{aligned} \tag{37}$$

Sufficient conditions (see [24]), in order to prove that Eqs (36) and (37) admit a unique solution can be expressed as follows. On the one hand, the functions $L(\cdot)$, $h_i(\cdot)$ and $\tilde{g}_i(\cdot)\operatorname{Re}(v_i(\cdot))$ must be Lipschitz, for all i. On the other hand, the functions $\operatorname{Re}(v_i(\cdot))$ must satisfy that there exists a constant K such that

$$\sup_i \sup_{\rho\in\mathbb{R}^P} \bigl|\operatorname{Re}\bigl(v_i(\rho)\bigr)\bigr| \le K. \tag{38}$$

Actually such conditions (Lipschitz and (38)) are not satisfied by the functions $L(\cdot)$, $h_i(\cdot)$, $\operatorname{Re}(v_i(\cdot))$ and $\tilde{g}_i(\cdot)\operatorname{Re}(v_i(\cdot))$. However, these functions are C^∞, then these conditions are in fact locally satisfied. Therefore, a truncature method can be used to make the functions $L(\cdot)$, $h_i(\cdot)$ and $\tilde{g}_i(\cdot)\operatorname{Re}(v_i(\cdot))$ Lipschitz and functions $\operatorname{Re}(v_i(\cdot))$ bounded. This is described as follows.

Fix $k > 0$. We consider a truncature function ϕ^k of the form

$$\begin{aligned}&\phi^k(x) = (\psi^k(x_i))_{i=1,\dots,P}, \quad \text{where} \\ &\psi^k(x_i) = -k\mathbf{1}_{x_i\le -k} + x_i\mathbf{1}_{|x_i|<k} + k\mathbf{1}_{x_i\neq k}\end{aligned} \tag{39}$$

for all $x = (x_i) \in \mathbb{R}^P$. Hence, if F is any function defined on $\mathbb{R}^P$, we define the function F^k on $\mathbb{R}^P$ by

$$F^k(x) = F\bigl(\phi^k(x)\bigr)$$

for all $x \in \mathbb{R}^P$. By extension, we will denote $F^k(\rho)$ when we deal with operators on $\mathcal{H}_0$. As a consequence, the functions $L^k(\cdot)$, $h^k_i(\cdot)$ and $\tilde{g}^k_i(\cdot)\operatorname{Re}(v^k_i(\cdot)$ become Lipschitz. Furthermore, as ϕ^k is a bounded function, we have

$$\sup_i \sup_{\rho\in\mathbb{R}^P} \bigl|\operatorname{Re}\bigl(v^k_i(\rho)\bigr)\bigr| \le K.$$

This theorem follows from these conditions.

Theorem 4. *Let $k \in \mathbb{R}^+$ and let ρ_0 be any operator on $\mathcal{H}_0$. The following stochastic differential equations, in case $J = \varnothing$,*

$$\rho_t^J = \rho_0 + \int_0^t \left(L^k(\rho_{s-}^J) - \sum_{i=1}^p \tilde{g}_i^k(\rho_{s-}^J) \operatorname{Re}\big(v_i^k(\rho_{s-}^J)\big) \right) \mathrm{d}s$$
$$+ \sum_{i=1}^p \int_0^t \int_{\mathbb{R}} \tilde{g}_i^k(\rho_{s-}^J) \mathbf{1}_{0<x<\operatorname{Re}(v_i^k(\rho_{s-}^J))} N_i(\mathrm{d}x, \mathrm{d}s), \tag{40}$$

and in case $J \neq \varnothing$,

$$\rho_t^J = \rho_0 + \int_0^t \left(L^k(\rho_{s-}^J) - \sum_{i \in I} \tilde{g}_i^k(\rho_{s-}^J) \operatorname{Re}\big(v_i^k(\rho_{s-}^J)\big) \right) \mathrm{d}s + \sum_{i \in J \cup \{0\}} \int_0^t h_i^k(\rho_{s-}^J) \, \mathrm{d}W_i(s)$$
$$+ \sum_{i \in I} \int_0^t \int_{\mathbb{R}} \tilde{g}_i^k(\rho_{s-}^J) \mathbf{1}_{0<x<\operatorname{Re}(v_i^k(\rho_{s-}^J))} N_i(\mathrm{d}x, \mathrm{d}s) \tag{41}$$

admit a unique solution.

Let $\overline{\mathcal{A}}_k^J$ be the infinitesimal generator of the solution of an equation of the form (40) or (41). For all $f \in C_c^2$, all states ρ, and all $k > 1$, we have $\overline{\mathcal{A}}_k^J f(\rho) = \mathcal{A}^J f(\rho)$.

Furthermore in all cases, the processes defined by

$$\overline{N}_t^i = \int_0^t \int_{\mathbb{R}} \mathbf{1}_{0<x<\operatorname{Re}(v_i^k(\rho_{s-}^J))} N_i(\mathrm{d}x, \mathrm{d}s) \tag{42}$$

are counting processes with stochastic intensity $t \to \int_0^t [\operatorname{Re}(v_i(\rho_{s-}^J))]^+ \, \mathrm{d}s$, where $(x)^+ = \max(0, x)$.

Proof. The part of this theorem concerning generators is the equivalent of Theorem 3. This follows from Theorem 3 and from the fact that $\phi^k(\rho) = \rho$, for all states ρ and all $k > 1$. Indeed, if $\rho = (\rho_i)_{i=1,\dots,P}$ is a state, we have $|\rho_i| \le 1$ for all i.

The last part of this theorem follows from properties of random Poisson measure theory and is treated in details in [30] for Eq. (2). The proof of Theorem 3 follows from Lipschitz character and works of Jacod and Protter in [24]. In a sake of completeness, we describe how to construct the solutions. We concentrate on the case $J \neq \varnothing$ (the case $J = \varnothing$ is easy to adapt to this case with a similar proof).

Let us prove that Eq. (41) admits a unique solution (we suppress the index J and the index k of truncature to lighten the notations). As we have $\sup_i \sup_{\rho \in \mathbb{R}^P} |\operatorname{Re}(v_i(\rho))| \le K$, we can consider Poisson point processes defined on $\mathbb{R} \times [0, K]$. Hence, for all $i \in I$, the process $(\tilde{N}_i(t))$ defined by $\mathcal{N}_i(t) = \operatorname{card}\{N_i(\cdot, [0, t] \times [0, K])\}$ is a classical Poisson process of intensity K [15]. As a consequence, for all t, it defines a random sequence $\{(\tau_k^i, \xi_k^i), k \in \{1, \dots, \mathcal{N}_i(t)\}\}$, where τ_k^i designs the jump time of $\mathcal{N}_i(\cdot)$ and the ξ_k^i's are independent uniform random variables on $[0, K]$. Consequently, the solution of the stochastic differential equation is given by

$$\rho_t = \rho_0 + \int_0^t L(\rho_{s-}) \, \mathrm{d}s - \sum_{i \in I} \int_0^t \tilde{g}_i(\rho_{s-}) \operatorname{Re}\big(v_i(\rho_{s-})\big) \, \mathrm{d}s$$
$$+ \sum_{i \in j \cup \{0\}} \int_0^t h_i(\rho_{s-}) \, \mathrm{d}W_i(s) + \sum_{i \in I} \sum_{k=1}^{\mathcal{N}_i(t)} \tilde{g}_i(\rho_{\tau_k^i -}) \mathbf{1}_{0<\xi_k^i \le \operatorname{Re}(v_i(\rho_{\tau_k^i -}))}. \tag{43}$$

More precisely, the solution (43) is described as follows. According to the Lipschitz property, there exists a unique solution (ρ_t^1) of the equation

$$\rho_t^1 = \rho_0 + \int_0^t L(\rho_{s-}^1) \, \mathrm{d}s - \sum_{i \in I} \int_0^t \tilde{g}_i(\rho_{s-}^1) \operatorname{Re}\big(v_i(\rho_{s-}^1)\big) \, \mathrm{d}s + \sum_{i \in j \cup \{0\}} \int_0^t h_i(\rho_{s-}^1) \, \mathrm{d}W_i(s). \tag{44}$$

The first jump time is then defined by

$$T_1 = \inf\left\{ t \Big/ \sum_{i\in I} \int_0^t \int_{[0,K]} \mathbf{1}_{0<x<\mathrm{Re}(v_i(\rho^1_{s-}))} N_i(\mathrm{d}x,\mathrm{d}s) > 1 \right\}.$$

By definition of Poisson point processes and by independence, we have for all $i \neq j$,

$$P\left[\exists t \Big/ \int_0^t \int_{[0,K]} \mathbf{1}_{0<x<\mathrm{Re}(v_i(\rho^1_{s-}))} N_i(\mathrm{d}x,\mathrm{d}s) = \int_0^t \int_{[0,K]} \mathbf{1}_{0<x<\mathrm{Re}(v_i(\rho^1_{s-}))} N_j(\mathrm{d}x,\mathrm{d}s) \right] = 0.$$

As a consequence at T_1, there exists a unique index i_{T_1} such that

$$\int_0^{T_1} \int_{[0,K]} \mathbf{1}_{0<x<\mathrm{Re}(v_{i_{T_1}}(\rho^1_{s-}))} N_{i_{T_1}}(\mathrm{d}x,\mathrm{d}s) = 1,$$

and all the other terms concerning the other Poisson point processes (for different indexes of i_{T_1}) are equal to zero. Moreover, we have almost surely

$$\int_0^{T_1} \int_{[0,K]} \mathbf{1}_{0<x<\mathrm{Re}(v_{i_{T_1}}(\rho^1_{s-}))} N_{i_{T_1}}(\mathrm{d}x,\mathrm{d}s) = \sum_{k=1}^{\mathcal{N}_{i_{T_1}}} \mathbf{1}_{0<\xi_k^{i_{T_1}}<\mathrm{Re}(v_{i_{T_1}}(\rho^1_{T_1-}))}.$$

We define then the solution of (40) on $[0, T_1]$ in the following way

$$\begin{cases} \rho_t = \rho^1_t & \text{on } [0, T_1[, \\ \rho_{T_1} = \tilde{g}_{i_{T_1}}(\rho_{T_1-}) + \rho^1_{T_1-}. \end{cases} \tag{45}$$

The operator ρ_{T_1} can then be considered as a new initial condition of Eq. (44). Therefore, we consider for $t > T_1$ the process (ρ^2_t) defined by

$$\rho^2_t = \rho_{T_1} + \int_{T_1}^t L(\rho^2_{s-})\,\mathrm{d}s - \sum_{i\in I} \int_{T_1}^t \tilde{g}_i(\rho^2_{s-})\,\mathrm{Re}\big(v_i(\rho^2_{s-})\big)\,\mathrm{d}s + \sum_{i\in j\cup\{0\}} \int_{T_1}^t h_i(\rho^2_{s-})\,\mathrm{d}W_i(s). \tag{46}$$

In the same fashion as the definition of T_1, we can define the random jump time T_2 as

$$T_2 = \inf\left\{ t > T_1 \Big/ \sum_{i\in I} \int_{T_1}^t \int_{[0,K]} \mathbf{1}_{0<x<\mathrm{Re}(v_i(\rho^2_{s-}))} N_i(\mathrm{d}x,\mathrm{d}s) > 1 \right\}.$$

By adapting the expression (45), we can define the solution on $[T_1, T_2]$ and so on. By induction, we define then the solution of (40). The uniqueness comes from the uniqueness of solution for diffusive equations of type (46). Moreover, the boundness property of the intensity implies that there is no exploding time and the solution is defined for all t (see [30] or [24] for complete details). □

Equations (40) or (41) (with truncature) admit then a unique solution (ρ_t). As we have already mentioned, it remains to prove that these solutions are valued in $\mathcal{S}$ to prove that they are solutions of (28) and (29). In Section 3, we show that it is provided by the convergence result.

Before tackling the problem of convergence in Section 3, we state a proposition concerning martingale problems $(\overline{\mathcal{A}}^J, \rho_0)$ and uniqueness of the solutions for such problems (cf. [20]). This will be namely useful in Section 3.

Proposition 3. *Let ρ_0 be any operator. Let $\overline{\mathcal{A}}^J_k$ be the infinitesimal generator of the process (ρ^J_t), solution of a truncated equation of the form (40) or (41).*

The process (ρ^J_t) is then the unique solution in distribution of the martingale problem $(\overline{\mathcal{A}}^J_k, \rho_0)$.

The fact that the solution of a stochastic differential equation (40) or (41) is a solution of the martingale problem for the corresponding infinitesimal generator follows from Itô formula as in Theorem 3. In other words, this proposition expresses that all other solutions of a martingale problem $(\overline{\mathcal{A}}^J, \rho_0)$ have the same distribution of the solution (ρ_t^J) of the associated stochastic differential equation. This result is classical in Markov process generator theory, it follows from the pathwise uniqueness of the solutions of Eqs (40) and (41) (see [20] for a complete reference about existence and uniqueness of solutions for martingale problems).

3. Convergence of discrete quantum trajectories

In this section, we show that the discrete quantum trajectories $(\rho_n^J(t))$ converge in distribution to the solutions of the martingale problem for $(\overline{\mathcal{A}}_k^J, \rho_0)$ related to Eqs (40) or (41). Next, we show that such convergence results allow to conclude that the solutions of (40) or (41) are valued in the set of states.

Let ρ_0 be any initial state. In order to prove that a discrete trajectory starting from ρ_0 converges in distribution, we show at first that the finite-dimensional distributions of the associated discrete process $(\rho_n^J(t))$ converge to the finite-dimensional distributions of the associated solution of the martingale problem $(\overline{\mathcal{A}}_k^J, \rho_0)$. Secondly, we show that the discrete process $(\rho_n^J(t))$ is tight and the convergence in distribution follows. For the weak convergence of finite-dimensional distributions, we use the following theorem of Ethier and Kurtz [20] translated in the context of quantum trajectories.

Theorem 5. *Let $\overline{\mathcal{A}}_k^J$ be the infinitesimal generator of the solution of the corresponding equation* (40) *or* (41). *Let $(\mathcal{F}_t^n)$ be a filtration and let $(\rho_n^J(\cdot))$ be a càdlàg $\mathcal{F}_t^n$ adapted-process which is relatively compact* (*or tight*). *Let ρ_0 be any state.*

Suppose that:

1. *The martingale problem $(\overline{\mathcal{A}}_k^J, \rho_0)$ has a unique solution* (*in distribution*),
2. $\rho_n^J(0) = \rho_0$,
3. *For all $m \geq 0$, for all $0 \leq t_1 < t_2 < \cdots < t_m \leq t < t+s$, for all function $(\theta_i)_{i=1,\dots,m}$ and for all f in C_c^2 we have*

$$\lim_{n\to\infty} \mathbf{E}\left[\left(f\left(\rho_n^J(t+s)\right) - f\left(\rho_n^J(t)\right) - \int_t^{t+s} \overline{\mathcal{A}}_k^J f\left(\rho_n^J(s)\right) \mathrm{d}s\right) \prod_{i=1}^m \theta_i\left(\rho_n^J(t_i)\right)\right] = 0. \tag{47}$$

Then $(\rho_n^J(\cdot))$ converges in distribution to the solution of the martingale problem for $(\overline{\mathcal{A}}_k^J, \rho_0)$.

In our context, recall that uniqueness of solution of the martingale problem follows from Proposition 3 of Section 2. Theorem 5 expresses the fact that if a subsequence of $(\rho_n^J(t))$ converges in distribution to a stochastic process (Y_t), necessarily this process is a solution of the martingale problem associated with $(\overline{\mathcal{A}}_k^J, \rho_0)$. Indeed, from the convergence property (47), the process (Y_t) satisfies

$$\mathbf{E}\left[\left(f(Y_{t+s}) - f(Y_t) - \int_t^{t+s} \overline{\mathcal{A}}_k^J f(Y_s)\,\mathrm{d}s\right) \prod_{i=1}^m \theta_i\left(\rho_n^J(t_i)\right)\right] = 0. \tag{48}$$

As this equality is satisfied for all $m \geq 0$, for all $0 \leq t_1 < t_2 < \cdots < t_m \leq t < t+s$, for all functions $(\theta_i)_{i=1,\dots,m}$ and for all f in C_c^2, this implies the martingale property of the process

$$t \to f(Y_t) - f(Y_0) - \int_0^t \overline{\mathcal{A}}_k^J (Y_s)\,\mathrm{d}s.$$

Hence, the uniqueness of the solution of the martingale problem allows to conclude to the convergence of finite-dimensional distributions and the tightness property allows to conclude to the convergence in distribution for stochastic processes.

Let us deal with the application of Theorem 5 in the context of quantum trajectories. Concerning the definition of a filtration $(\mathcal{F}_t^n)$, we consider the natural filtration of the discrete quantum trajectory $(\rho_n^J(t))$, that is, if $r/n \leq t < (r+1)/n$, we have

$$\mathcal{F}_t^n = \sigma\big(\rho_n^J(s), s \leq t\big) = \sigma\big(\rho_p^J, p \leq r\big).$$

It is obvious that $\mathcal{F}_t^n = \mathcal{F}_{r/n}^n$.

Let us first assume tightness. In order to conclude, it suffices to prove the condition (47). To this end, we make the following consideration. As k is supposed to be strictly larger than 1, recall that infinitesimal generators of quantum trajectories $\mathcal{A}^J$ satisfy $\overline{\mathcal{A}}_k^J f(\rho) = \mathcal{A}^J f(\rho)$, for all $f \in C_c^2$ and for all states ρ. This fact is then valid for $(\rho_n^J(\cdot))$ and the condition (47) follows from this proposition.

Proposition 4. *Let ρ_0 be any state. Let $(\rho_n^J(\cdot))$ be a quantum trajectory starting from ρ_0. Let $\mathcal{F}_t^n$ be the natural filtration of $(\rho_n^J(\cdot))$. We have*

$$\lim_{n\to\infty} \mathbf{E}\left[\left(f\big(\rho_n^J(t+s)\big) - f\big(\rho_n^J(t)\big) - \int_t^{t+s} \mathcal{A}^J f\big(\rho_n^J(s)\big)\,\mathrm{d}s\right)\prod_{i=1}^m \theta_i\big(\rho_n^J(t_i)\big)\right] = 0 \tag{49}$$

for all $m \geq 0$, for all $0 \leq t_1 < t_2 < \cdots < t_m \leq t < t+s$, for all functions $(\theta_i)_{i=1,\dots,m}$ and for all f in C_c^2.

Proof. In order to treat the limit (49), we make the following observations. Let n be fixed, from definition of infinitesimal generators for Markov chains (see [32]), we have that

$$f\big(\rho_n^J(k/n)\big) - f(\rho_0) - \sum_{j=0}^{k-1} \frac{1}{n}\mathcal{A}_n^J f\big(\rho_n^J(j/n)\big) \tag{50}$$

is a $(\mathcal{F}_{k/n}^n)$ martingale (this is the discrete equivalent of solutions for martingale problems for discrete processes). Furthermore, suppose $r/n \leq t < (r+1)/n$ and $l/n \leq t+s < (l+1)/n$. Then we have $\mathcal{F}_t^n = \mathcal{F}_{r/n}^n$. Moreover, the random states $\rho_n^J(t)$ and $\rho_n^J(t+s)$ satisfy $\rho_n^J(t) = \rho_n^J(r/n)$ and $\rho_n^J(t+s) = \rho_n^J(l/n)$. The martingale property (50) implies then

$$\begin{aligned}
&\mathbf{E}\big[f\big(\rho_n^J(t+s)\big) - f\big(\rho_n^J(t)\big)/\mathcal{F}_t^n\big]\\
&\quad= \mathbf{E}\big[f\big(\rho_n^J(l/n)\big) - f\big(\rho_n^J(k/n)\big)/\mathcal{F}_{r/n}^n\big]\\
&\quad= \mathbf{E}\left[\sum_{j=k}^{l-1}\frac{1}{n}\mathcal{A}_n^J f\big(\rho_n(j/n)\big)\Big/\mathcal{F}_{r/n}^n\right]\\
&\quad= \mathbf{E}\left[\int_t^{t+s}\mathcal{A}_n^J f\big(\rho_n^J(s)\big)\,\mathrm{d}s\Big/\mathcal{F}_t^n\right]\\
&\qquad+ \mathbf{E}\left[\left(t - \frac{r}{n}\right)\mathcal{A}_n^J f\big(\rho_n^J(t)\big) + \left(\frac{l}{n} - (t+s)\right)\mathcal{A}_n^J f\big(\rho_n^J(t+s)\big)\Big/\mathcal{F}_t^n\right].
\end{aligned} \tag{51}$$

Now, we are in position to treat the limit (49). Let $m \geq 0$, let $0 \leq t_1 < t_2 < \cdots < t_m \leq t < t+s$, let $(\theta_i)_{i=1,\dots,m}$ and let f be functions in C_c^2, we have

$$\begin{aligned}
&\left|\mathbf{E}\left[\left(f\big(\rho_n^J(t+s)\big) - f\big(\rho_n^J(t)\big) - \int_t^{t+s}\mathcal{A}^J f\big(\rho_n^J(s)\big)\right)\prod_{i=1}^m\theta_i\big(\rho_n^J(t_i)\big)\right]\right|\\
&\quad= \left|\mathbf{E}\left[\mathbf{E}\left[\left(f\big(\rho_n^J(t+s)\big) - f\big(\rho_n^J(t)\big) - \int_t^{t+s}\mathcal{A}^J f\big(\rho_n^J(s)\big)\right)\Big/\mathcal{F}_t^n\right]\prod_{i=1}^m\theta_i\big(\rho_n^J(t_i)\big)\right]\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathbf{E}\left[\left|\int_t^{t+s}\Big(\mathcal{A}_n^J f\big(\rho_n(s)\big)-\mathcal{A}^J f\big(\rho_n^J(s)\big)\Big)\,\mathrm{d}s\right|\right]\prod_{i=1}^m\|\theta_i\|_\infty \\
&\quad+\mathbf{E}\left[\left|\left(t-\frac{[nt]}{n}\right)\mathcal{A}_n f\big(\rho_n^J(t)\big)+\left(\frac{[n(t+s)]}{n}-(t+s)\right)\mathcal{A}_n^J f\big(\rho_n^J(t+s)\big)\right|\right]\prod_{i=1}^m\|\theta_i\|_\infty \\
&\leq M\sup_{\rho\in\mathcal{S}}\big|\mathcal{A}_n^J f(\rho)-\mathcal{A}^J f(\rho)\big|+\frac{L}{n}\sup_{\rho\in\mathcal{S}}\big|\mathcal{A}_n^J f(\rho)\big|,
\end{aligned}\tag{52}
$$

where M and L are constants depending on $\|h_i\|$ and s. Thanks to the condition of uniform convergence of Proposition 2, we obtain

$$
\lim_{n\to\infty}\left|\mathbf{E}\left[\left(f\big(\rho_n^J(t+s)\big)-f\big(\rho_n^J(t)\big)-\int_t^{t+s}\mathcal{A}^J f\big(\rho_n^J(s)\big)\right)\prod_{i=1}^m\theta_i\big(\rho_n^J(t_i)\big)\right]\right|=0 \tag{53}
$$

and the result holds. □

Finally, in order to apply Theorem 5 of Ethier and Kurtz, it suffices to prove the tightness of discrete quantum trajectories. To this end, we need the following lemma.

Lemma 1. *Recall that $\mathcal{M}_r^{(n)}=\mathcal{F}_{r/n}^n=\sigma\{\rho_j^J, j\leq r\}$, for all $r\in\mathbb{N}^\star$. There exists a constant K_J such that we have almost surely*

$$
\mathbf{E}\big[\|\rho_l^J-\rho_r^J\|^2/\mathcal{M}_r^{(n)}\big]\leq K_J\frac{l-r}{n} \tag{54}
$$

for all $(r,l)\in(\mathbb{N}^\star)^2$, with $r<l$.

Proof. Let us deal with the case $J\neq\varnothing$ and $I\neq\varnothing$. For all integers $r<l$, we have $\mathbf{E}[\|\rho_l^i-\rho_r^i\|^2/\mathcal{M}_r^{(n)}]=\mathbf{E}[\mathbf{E}[\|\rho_l^i-\rho_r^i\|^2/\mathcal{M}_{l-1}^{(n)}]/\mathcal{M}_r^{(n)}]$. This way, we need to treat the term $\mathbf{E}[\|\rho_l^i-\rho_r^i\|^2/\mathcal{M}_{l-1}^{(n)}]$. By remarking that $\rho_l^J=\sum_{j=0}^p\mathcal{L}_j^{(n)}(\rho_{l-1}^J)\mathbf{1}_j^l$, where $\mathbf{1}_j^l$ corresponds to the indicator function which describes the observation of the eigenvalue λ_j during the lth measurement, we have

$$
\begin{aligned}
\mathbf{E}\big[\|\rho_l^J-\rho_r^J\|^2/\mathcal{M}_{l-1}^{(n)}\big]&=\mathbf{E}\left[\left\|\sum_{j=0}^p\mathcal{L}_j^{(n)}\big(\rho_{l-1}^J\big)\mathbf{1}_j^l-\rho_r^J\right\|^2\Big/\mathcal{M}_{l-1}^{(n)}\right] \\
&=\mathbf{E}\left[\sum_{j=0}^p\big\|\mathcal{L}_j^{(n)}\big(\rho_{l-1}^J\big)-\rho_r^J\big\|^2p_l^j\big(\rho_{l-1}^J\big)\Big/\mathcal{M}_{l-1}^{(n)}\right] \\
&=\sum_{j\in I}\mathbf{E}\big[\big\|\mathcal{L}_j^{(n)}\big(\rho_{l-1}^J\big)-\rho_r^J\big\|^2p_l^j\big(\rho_{l-1}^J\big)\big/\mathcal{M}_{l-1}^{(n)}\big] \\
&\quad+\sum_{j\in J\cup\{0\}}\mathbf{E}\big[\big\|\mathcal{L}_j^{(n)}\big(\rho_{l-1}^J\big)-\rho_{l-1}^i+\rho_{l-1}^i-\rho_k^i\big\|^2p_l^j(\rho_{l-1})\big/\mathcal{M}_{l-1}^{(n)}\big].
\end{aligned}\tag{55}
$$

As I is supposed to be not empty, for the first term of (55) we have for all $i\in I$,

$$
p_l^i\big(\rho_{l-1}^J\big)=\frac{1}{n}\big(v_i\big(\rho_{l-1}^J\big)+\mathrm{o}(1)\big),
$$

where the functions $v_i(\cdot)$ are defined in Section 2. As $\mathcal{L}_i^{(n)}(\rho)$ converges uniformly in $\rho \in \mathcal{S}$, set

$$R = \sup_{j \in I} \sup_{n} \sup_{(\rho,\mu) \in \mathcal{S}^2} \bigl\{ \bigl\| \mathcal{L}_j^{(n)}(\rho) - \mu \bigr\|^2 \bigl(v_i(\rho) + \mathrm{o}(1) \bigr) \bigr\}.$$

This constant is finite since all the o$'s$ are uniform in ρ. We have then almost surely

$$\sum_{j \in I} \mathbf{E}\bigl[\bigl\| \mathcal{L}_j^{(n)}\bigl(\rho_{l-1}^J\bigr) - \rho_r^J \bigr\|^2 p_l^j\bigl(\rho_{l-1}^J\bigr) / \mathcal{M}_{l-1}^{(n)} \bigr] \le \frac{\mathrm{card}(I) \times R}{n}.$$

For the second term of (55), we have

$$\begin{aligned}
&\sum_{j \in J \cup \{0\}} \mathbf{E}\bigl[\bigl\| \mathcal{L}_j^{(n)}\bigl(\rho_{l-1}^J\bigr) - \rho_{l-1}^J + \rho_{l-1}^J - \rho_r^J \bigr\|^2 p_l^j\bigl(\rho_{l-1}^J\bigr) / \mathcal{M}_{l-1}^{(n)} \bigr] \\
&\quad = \sum_{j \in J \cup \{0\}} \mathbf{E}\bigl[\bigl\| \mathcal{L}_j^{(n)}\bigl(\rho_{l-1}^J\bigr) - \rho_{l-1}^J \bigr\|^2 p_l^j\bigl(\rho_{l-1}^J\bigr) / \mathcal{M}_{l-1}^{(n)} \bigr] \\
&\qquad + \sum_{j \in J \cup \{0\}} \mathbf{E}\bigl[2\,\mathrm{Re}\bigl(\bigl\langle \mathcal{L}_j^{(n)}\bigl(\rho_{l-1}^J\bigr) - \rho_{l-1}^J, \rho_{l-1}^J - \rho_r^J \bigr\rangle \bigr) p_l^j\bigl(\rho_{l-1}^J\bigr) / \mathcal{M}_{l-1}^{(n)} \bigr] \\
&\qquad + \sum_{j \in J \cup \{0\}} \mathbf{E}\bigl[\bigl\| \rho_{l-1}^J - \rho_r^J \bigr\|^2 p_l^j\bigl(\rho_{l-1}^J\bigr) / \mathcal{M}_{l-1}^{(n)} \bigr].
\end{aligned} \tag{56}$$

Concerning the indexes $j \in J \cup \{0\}$, we have

$$\mathcal{L}_j^{(n)}\bigl(\rho_{l-1}^J\bigr) - \rho_{l-1}^J = \frac{1}{\sqrt{n}} \bigl(h_j\bigl(\rho_{l-1}^J\bigr) + \mathrm{o}(1) \bigr).$$

Similar to the constant R, we define

$$S = \sup_{j \in J \cup \{0\}} \sup_{n} \sup_{\rho \in \mathcal{S}} \bigl\| h_j(\rho) + \mathrm{o}(1) \bigr\|^2 p_j(\rho).$$

For the first term of (56), we then have almost surely

$$\sum_{j \in J \cup \{0\}} \mathbf{E}\bigl[\bigl\| \mathcal{L}_j^{(n)}\bigl(\rho_{l-1}^J\bigr) - \rho_{l-1}^J \bigr\|^2 p_l^j\bigl(\rho_{l-1}^J\bigr) / \mathcal{M}_{l-1}^{(n)} \bigr] \le \frac{(\mathrm{card}(J) + 1) \times S}{n}.$$

Furthermore, as we have $\sum_{j \in J \cup \{0\}} p_l^j(\rho_{l-1}^J) \le 1$ almost surely, we have almost surely

$$\sum_{j \in J \cup \{0\}} \mathbf{E}\bigl[\bigl\| \rho_{l-1}^J - \rho_r^J \bigr\|^2 p_l^j(\rho_{l-1}) / \mathcal{M}_{l-1}^{(n)} \bigr] \le \mathbf{E}\bigl[\bigl\| \rho_{l-1}^J - \rho_r^J \bigr\|^2 / \mathcal{M}_{l-1}^{(n)} \bigr].$$

For the second term of (56), we have

$$\begin{aligned}
&\sum_{j \in J \cup \{0\}} \mathbf{E}\bigl[2\,\mathrm{Re}\bigl(\bigl\langle \mathcal{L}_j^{(n)}\bigl(\rho_{l-1}^J\bigr) - \rho_{l-1}^J, \rho_{l-1}^J - \rho_r^J \bigr\rangle \bigr) p_l^j\bigl(\rho_{l-1}^J\bigr) / \mathcal{M}_{l-1}^{(n)} \bigr] \\
&\quad = \mathbf{E}\biggl[2\,\mathrm{Re}\biggl(\biggl\langle \sum_{j \in J \cup \{0\}} \bigl(\mathcal{L}_j^{(n)}\bigl(\rho_{l-1}^J\bigr) - \rho_{l-1}^J \bigr) p_l^j\bigl(\rho_{l-1}^J\bigr), \rho_{l-1}^J - \rho_r^J \biggr\rangle \biggr) \Big/ \mathcal{M}_{l-1}^{(n)} \biggr].
\end{aligned}$$

Let us treat this term. As in the proof of Proposition 2 concerning infinitesimal generators, with the asymptotic of $\mathcal{L}_i^{(n)}$ in this situation, we have uniformly in $\rho \in \mathcal{S}$,

$$\sum_{j \in J \cup \{0\}} \bigl(\mathcal{L}_j^{(n)}(\rho) - \rho \bigr) p_l^j(\rho) = \frac{1}{n} \bigl(H(\rho) + \mathrm{o}(1) \bigr),$$

since the terms in $1/\sqrt{n}$ disappear by summing over $j \in J \cup \{0\}$. As a consequence, by defining the finite constant W as

$$W = \sup_n \sup_{(\rho,\mu)\in\mathcal{S}^2} \left\{ \left| 2\,\mathrm{Re}\left(\left\langle \sum_{j\in J\cup\{0\}} n\big(\mathcal{L}_j^{(n)}(\rho) - \rho\big) p_l^j(\rho), \rho - \mu \right\rangle \right) \right| \right\},$$

we have then almost surely

$$\sum_{j\in J\cup\{0\}} \mathbf{E}\big[2\,\mathrm{Re}\big(\langle \mathcal{L}_j^{(n)}\big(\rho_{l-1}^J\big) - \rho_{l-1}^J, \rho_{l-1}^J - \rho_r^J\rangle\big) p_l^j\big(\rho_{l-1}^J\big)/\mathcal{M}_{l-1}^{(n)}\big] \leq \frac{W}{n}.$$

Let us stress that the constant W are independent of l and r. Therefore, we can conclude that there exists a constant K_J such that for all $r < l$, we have almost surely

$$\mathbf{E}\big[\big\|\rho_l^J - \rho_r^J\big\|^2/\mathcal{M}_{l-1}^{(n)}\big] \leq \frac{K_J}{n} + \mathbf{E}\big[\big\|\rho_{l-1}^J - \rho_r^J\big\|^2/\mathcal{M}_{l-1}^{(n)}\big]. \tag{57}$$

It implies that almost surely

$$\mathbf{E}\big[\big\|\rho_l^J - \rho_r^J\big\|^2/\mathcal{M}_r^{(n)}\big] \leq \frac{K_J}{n} + \mathbf{E}\big[\big\|\rho_{l-1}^J - \rho_r^J\big\|^2/\mathcal{M}_r^{(n)}\big]. \tag{58}$$

Thus, by conditioning successively by $\mathcal{M}_{l-i}^{(n)}$, with $i \in \{2, \ldots, l-r\}$ and by induction, we can show

$$\mathbf{E}\big[\big\|\rho_l^J - \rho_r^J\big\|^2/\mathcal{M}_r^{(n)}\big] \leq \frac{K_J(l-r)}{n}.$$

The same results holds when $J = \varnothing$ or $I = \varnothing$ by similar computations. □

This lemma implies the following proposition which expresses the tightness property of discrete quantum trajectories.

Proposition 5. *Let $(\rho_n^J(t))$ be a quantum trajectory. There exists some constant Z_J such that*

$$\mathbf{E}\big[\big\|\rho_n(t_2) - \rho_n(t)\big\|^2 \big\|\rho_n(t) - \rho_n(t_1)\big\|^2\big] \leq Z_J(t_2 - t_1)^2 \tag{59}$$

for all $t_1 < t < t_2$. Therefore, the discrete quantum trajectory $(\rho_n^J(t))$ is tight.

Proof. The inequality (59) implies the tightness of $(\rho_n(t))$ (see [13]). Let us prove (59). It is worth noticing that $\mathcal{M}_k^{(n)} = \mathcal{F}_{k/n}^n$, where $\mathcal{F}_t^n$ is the natural filtration of (ρ_n^J). Thanks to the previous lemma, we then have

$$\begin{aligned}
\mathbf{E}\big[\big\|\rho_n^J(t_2) - \rho_n^J(t)\big\|^2 \big\|\rho_n^J(t) - \rho_n(t_1)\big\|^2\big] &= \mathbf{E}\big[\mathbf{E}\big[\big\|\rho_n^J\big([nt_2]\big) - \rho_n^J\big([nt]\big)\big\|^2/\mathcal{F}_{[nt]/n}^n\big]\big\|\rho_n^J\big([nt]\big) - \rho_n\big([nt_1]\big)\big\|^2\big] \\
&\leq \frac{K_J([nt_2] - [nt])}{n}\mathbf{E}\big[\mathbf{E}\big[\big\|\rho_n^J\big([nt]\big) - \rho_n^J\big([nt_1]\big)\big\|^2/\mathcal{F}_{[nt_1]/n}^n\big]\big] \\
&\leq \frac{K_J([nt_2] - [nt])}{n}\frac{K_J([nt] - [nt_1])}{n} \leq Z_J(t_2 - t_1)^2,
\end{aligned} \tag{60}$$

with $Z_J = 4(K_J)^2$ and the result follows. □

Now, we are in position to express the final theorem.

Theorem 6. *Let A be an observable on $\mathcal{H}=\mathbb{C}^{N+1}$, with spectral decomposition*

$$A=\sum_{i=0}^{p}\lambda_i P_i=\sum_{i\in I}\lambda_i P_i+\sum_{j\in J\cup\{0\}}\lambda_j P_j, \tag{61}$$

where:

1. *For $i\in\{0,\dots,p\}$ the operators $P_i=(p^i_{kl})_{0\leq k,l\leq N}$ are the eigen-projectors of A (satisfying $p^0_{00}\neq 0$).*
2. *The sets I and J satisfy that $I=\{i\in\{1,\dots,p\}/p^i_{00}=0\}$ and $J=\{1,\dots,p\}\setminus I$.*

Let ρ_0 be a state on $\mathcal{H}_0$. Let $(\rho^J_n(t))$ be the discrete quantum trajectory describing the repeated quantum measurements of A and starting with ρ_0 as initial state.

1. *Suppose $J=\varnothing$. Then the discrete quantum trajectory $(\rho^J_n(t))$ converges in distribution in $\mathcal{D}[0,T)$ for all T to the solution (ρ^J_t) of the stochastic differential equation (40). Therefore, the process (ρ^J_t) takes values in the set of states on $\mathcal{H}_0$. The discrete quantum trajectory $(\rho^J_n(t))$ converges in distribution to the unique solution of the following jump–diffusion Belavkin equation*

$$\rho^J_t=\rho_0+\int L\big(\rho^J_{s-}\big)\,\mathrm{d}s+\sum_{i=1}^{p}\int_0^t\int_{\mathbb{R}} g_i\big(\rho^J_{s-}\big)\mathbf{1}_{0<x<v_i(\rho^J_{s-})}\big[N_i(\mathrm{d}x,\mathrm{d}s)-\mathrm{d}x\,\mathrm{d}s\big]. \tag{62}$$

2. *Suppose $J\neq\varnothing$. Then the discrete quantum trajectory $(\rho^J_n(t))$ converges in distribution in $\mathcal{D}[0,T)$ for all T to the solution (ρ^J_t) of the stochastic differential equation (41). The process (ρ^J_t) takes values in the set of states on $\mathcal{H}_0$. The discrete quantum trajectory $(\rho^J_n(t))$ converges in distribution to the unique solution of the following jump–diffusion Belavkin equation*

$$\begin{aligned}\rho^J_t&=\rho_0+\int L\big(\rho^J_{s-}\big)\,\mathrm{d}s+\sum_{i\in J\cup\{0\}}h_i\big(\rho^J_{s-}\big)\,\mathrm{d}W_i(s)\\&\quad+\sum_{i\in I}\int_0^t\int_{\mathbb{R}} g_i\big(\rho^J_{s-}\big)\mathbf{1}_{0<x<v_i(\rho^J_{s-})}\big[N_i(\mathrm{d}x,\mathrm{d}s)-\mathrm{d}x\,\mathrm{d}s\big].\end{aligned} \tag{63}$$

   Furthermore the processes defined by

$$\tilde{N}^i_t=\int_0^t\int_{\mathbb{R}}\mathbf{1}_{0<x<v_i(\rho^J_{s-})}N_i(\mathrm{d}x,\mathrm{d}s) \tag{64}$$

are counting processes with stochastic intensities $t\to\int_0^t v_i(\rho^J_{s-})\,\mathrm{d}s$.

As in Theorem 3, the last assertion concerning the counting processes of Theorem 6 follows from properties of Poisson point processes N_i. It means actually that the processes defined by

$$\int_0^t\int_{\mathbb{R}}\mathbf{1}_{0<x<v_i(\rho^J_{s-})}N_i(\mathrm{d}x,\mathrm{d}s)-\int_0^t v_i\big(\rho^J_{s-}\big)\,\mathrm{d}s \tag{65}$$

are martingale with respect to the natural filtration of (ρ^J_t) (see [8,23,24]). Let us prove the convergence results of Theorem 5.

Proof of Theorem 6. In all cases, the convergence result follows from Theorem 5 and Proposition 4 for the finite-dimensional distributions convergence and from Proposition 5 for the tightness. In order to finish the proof of this theorem, we have to prove that the solutions of stochastic differential equations (40) and (41) take values in the set of states. This follows from the convergence in distribution.

Indeed, let $(\rho_n^J(t))$ be converging to the corresponding solution (ρ_t^J) of Eq. (40) or (41), we have to prove that this solution is self-adjoint, positive with trace 1. By using the convergence in distribution, we have for all $z \in \mathbb{C}^2$

$$\rho_n^J(t) - \left(\rho_n^J(t)\right)^\star \underset{n\to\infty}{\overset{\mathcal{D}}{\longmapsto}} \rho_t^J - \left(\rho_t^J\right)^\star,$$

$$\mathrm{Tr}\left[\rho_n^J(t)\right] \underset{n\to\infty}{\overset{\mathcal{D}}{\longmapsto}} \mathrm{Tr}\left[\rho_t^J\right],$$

$$\left\langle z, \rho_n^J(t) z\right\rangle \underset{n\to\infty}{\overset{\mathcal{D}}{\longmapsto}} \left\langle z, \rho_t^J z\right\rangle,$$

where $\mathcal{D}$ denotes the convergence in distribution for processes. As $(\rho_n^J(t))$ takes values in the set of states, we have almost surely for all t and all $z \in \mathbb{C}^2$,

$$\rho_n^J(t) - \left(\rho_n^J(t)\right)^\star = 0, \qquad \mathrm{Tr}\left[\rho_t^J\right] = 1, \qquad \left\langle z, \rho_n^J(t) z\right\rangle \geq 0.$$

These properties are conserved in the limit in distribution and the process (ρ_t^J) therefore takes values in the set of states. The proof of Theorem 6 is then complete. □

This theorem is a mathematical and physical justification of stochastic models of continuous time quantum measurement theory. Let us stress that in general it is difficult to prove that stochastic master equations admit a unique solution which takes values in the set of states. In [2,7,27], results of existence and uniqueness are obtained by using auxiliary linear stochastic master equations and Girsanov transformations. Actually, the difficulty consists in showing that these equations preserve the positivity of the solution. In our approach, this property is naturally implied by the convergence theorem.

Let us conclude this article with some remarks concerning these continuous stochastic models.

The first remark concerns the average of solutions of (62) or (63). Let (ρ_t^J) be a solution of (62) or (63). In all cases, we have

$$\mathbf{E}\left[\rho_t^J\right] = \int_0^t L\left(\mathbf{E}\left[\rho_s^J\right]\right) \mathrm{d}s. \tag{66}$$

Namely, this follows from martingale property of the Brownian motion and counting processes. The identity (66) means that the function

$$t \to \mathbf{E}\left[\rho_t^J\right],$$

is the solution of the ordinary differential equation

$$\mathrm{d}\mu_t = L(\mu_t)\,\mathrm{d}t.$$

This equation is called the "Master equation" in quantum mechanics and describes the evolution of the reference state of the small system $\mathcal{H}_0$ without measurement. On average, continuous quantum trajectories evolve thus as the solution of the Master equation (for all measurement experiences).

The second remark concerns the classical Belavkin equations (1) and (2). In [29] and [30], it was shown that such continuous models are justified from convergence of stochastic integral and random coupling method (it does not use infinitesimal generators theory). With Theorem 5, we recover these equations by considering the case where the measured observable A is of the form $A = \lambda_0 P_0 + \lambda_1 P_1$. Indeed in this case, we just have one noise process at the limit as in the classical case.

The last remark concerns the uniqueness of a solution of the martingale problems. In this article, we have identified the set of operators on $\mathcal{H}_0$ with $\mathbb{R}^P$ in order to introduce definition of infinitesimal generators and notion of martingale problem (see Section 2, Definition 2). As observed, the infinitesimal generators of quantum trajectories can be written in term of the partial derivatives in the following way

$$\sum_{i\in J\cup\{0\}} D_\rho^2 f\left(h_i(\rho), h_i(\rho)\right) = \sum_{i,j=1}^P a_{ij}(\rho)\frac{\partial f(\rho)}{\partial \rho_i\,\partial\rho_j} \quad \text{and} \quad D_\rho f\left(L(\rho)\right) = \sum_{i=1}^P b_i(\rho)\frac{\partial f(\rho)}{\partial \rho_i}, \tag{67}$$

by expanding the differential terms. The matrix $a(\cdot) = (a_{ij}(\cdot))$ is a semi definite matrix. Let W be a P-dimensional Brownian motion, the solution of the martingale problem can thus be expressed as the solution of

$$\begin{aligned}\rho_t^J &= \rho_0 + \int_0^t L\big(\rho_{s-}^J\big)\,\mathrm{d}s + \int_0^t \sigma\big(\rho_s^J\big)\,\mathrm{d}W_s \\ &\quad + \sum_{k\in I}\int_0^t \int_{\mathbb{R}} \tilde{g}_k\big(\rho_{s-}^J\big)\mathbf{1}_{0<x<\mathrm{Re}(v_k(\rho_{s-}^J))}\big[N_k(\mathrm{d}x,\mathrm{d}s) - \mathrm{d}x\,\mathrm{d}s\big],\end{aligned} \tag{68}$$

where $\sigma(\cdot)$ is as matrix defined by $\sigma(\cdot)\sigma^t(\cdot) = a(\cdot)$ (see [32]). Let us stress that, in this description we deal with a P-dimensional Brownian motion corresponding to the dimension of $\mathbb{R}^P$ (which depends only on the dimension of $\mathcal{H}_0$) whereas in Theorem 6 we consider a $(p+1)$-dimensional Brownian motion corresponding to the number of eigenvalues (which only depends on the dimension of the interacting quantum system $\mathcal{H}$). As a consequence from uniqueness of martingale problem (Proposition 3) we have two different descriptions of continuous quantum trajectories, but they are the same as regards their distributions.

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