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Bayesian analysis of a correlated binomial model

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Abstract. In this paper a Bayesian approach is applied to the correlated binomial model, $CB(n, p, \rho)$, proposed by Luceño (*Comput. Statist. Data Anal.* **20** (1995) 511–520). The data augmentation scheme is used in order to overcome the complexity of the mixture likelihood. MCMC methods, including Gibbs sampling and Metropolis within Gibbs, are applied to estimate the posterior marginal for the probability of success p and for the correlation coefficient ρ. The sensitivity of the posterior is studied taking into account several reference priors and it is shown that the posterior characteristics appear not to be influenced by these prior distributions. The article is motivated by a study of plant selection.

1 Introduction

Data in the form of frequencies are generally analyzed under a conventional binomial distribution or under a Poisson model. If we assume that the variance of response exceeds the nominal variance, it seems wise to consider alternative models for over-dispersion [Hinde and Demetrio (2000)]. The binomial and Poisson distributions have been generalized in several ways to handle the problem of overdispersion, which can usually occur due to the presence of some sort of correlation between events. For instance, as in plant selection study, an example that motivated this article, where there is evidence of the presence of correlation among any two plants in each pot when competing about the quantity of nutrients and the interest resides in the calculation of the probability of selecting a good plant. The binomial distribution has been generalized in various ways. Rudolfer (1990), Madsen (1993) and Luceño and Ceballos (1995) have summarized most of these generalizations. Among these extensions there are the multiplicative and the additive generalized binomial distributions which were derived by Altham (1978). The probability density function (PDF) of the multiplicative case is a multiplication of the PDF of a conventional binomial by a factor. It makes the variance greater or less than the corresponding binomial variance depending on the factor values. On the other hand, the consequential distribution of the additive case is a mixture of three conventional binomials. Kupper and Haseman (1978) developed the

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correlated binomial model. This distribution was derived by correcting the conventional binomial model via a method suggested by Bahadur (1961) to allow for dependency among the Bernoulli variables. A three-parameter binomial distribution was derived by Paul (1985, 1987), which is a generalization of the conventional binomial, the beta-binomial distribution and the correlated binomial distribution proposed by Kupper and Haseman (1978). Ng (1989) developed the modified binomial distributions. In this approach, the conventional binomial distribution is modified sequentially and the resulting distribution becomes more spread out (indicating positive correlation among the Bernoulli variables), or more peaked (indicating negative correlation among the Bernoulli variables), than the conventional binomial distribution. A four-parameter binomial distribution was derived by Fu and Sproule (1995). This new distribution assumes that the underlying Bernoulli trials take on the values α or β where $\alpha < \beta$, rather than the usual values 0 or 1. Luceño (1995) and Luceño and Ceballos (1995) proposed a generalized binomial distribution which is discussed in detail in this paper.

Considering the possibility of introducing prior information concerning the parameters into the model, in this paper the Luceño correlated binomial distribution, $CB(n, p, \rho)$, is studied from the Bayesian point of view using the data augmentation method [Tanner and Wong (1987), Diebolt and Robert (1994)].

2 A Bayesian approach for the correlated binomial model

The generalized binomial distribution derived by Luceño (1995), denoted as correlated binomial distribution or the $CB(n, p, \rho)$ model, is obtained by supposing that the variable Y, the number of successes in n trials of Bernoulli, is the sum of equicorrelated binary responses with probability of success constant p and a correlation coefficient equals ρ.

Formally, let $Y = W_1 + \cdots + W_n$, where W_r, $r = 1, \ldots, n$, is a binary variable with $E(W_r) = p$, $\mathrm{Var}(W_r) = p(1-p)$ and $\mathrm{Corr}(W_r, W_s) = \rho, r \neq s; \rho > 0$.

Theorem 1. *The probability distribution of Y is obtained by the mixture of the distributions of two variables. One of them follows a binomial distribution, $B(n, p)$, with mixing probability $(1-\rho)$, and the other one follows a modified Bernoulli distribution, $MBern(p)$, taking values 0 or n [Fu and Sproule (1995)], rather than the conventional values 0 or 1, with mixing probability ρ. The probability distribution of Y, given n, p, ρ, is, then, given by*

$$P(Y = y|n, p, \rho) = \binom{n}{y} p^y (1-p)^{n-y} (1-\rho) I_{A_1}(y) + p^{y/n} (1-p)^{(n-y)/n} \rho I_{A_2}(y), \tag{1}$$

where $A_1 = \{0, 1, \ldots, n\}$, $A_2 = \{0, n\}$, $y = 0, \ldots, n$ and $0 \leq \rho \leq 1$.

Proof. Given in the Appendix. □

The mean and variance of this model are $E(Y) = np$ and $\mathrm{Var}(Y) = p(1-p)\{n + \rho n(n-1)\}$, which accommodates extra-binomial variation for $\rho \neq 0$. Note that the $CB(n, p, \rho)$ model is equivalent to the binomial model for $\rho = 0$. A natural extension of the model (1) would be to consider coefficient of correlation $\rho_{r,s}$, $0 \leq \rho_{r,s} \leq 1$, for any two binary variables W_r and W_s.

A set of questionable properties associated with the $CB(n, p, \rho)$ model is presented in Luceño (1995) and Luceño and Ceballos (1995), among them, there is one that affects the large properties of the maximum likelihood estimator of the correlation coefficient ρ. The first order derivative, with respect to ρ, of the loglikelihood function provided by the $CB(n, p, \rho)$ model yields an absolute maximum with nonvanishing.

Suppose that $y_1, y_2, \ldots, y_k$ represent a random sample from the $CB(n, p, \rho)$ distribution, then from (1), the likelihood function for p and ρ is given by

$$\begin{aligned} L(p, \rho | n, y_1, y_2, \ldots, y_k) \\ = \prod_{i=1}^{k} \left\{ \binom{n}{y_i} p^{y_i}(1-p)^{n-y_i}(1-\rho) I_{A_1}(y_i) \right. \\ \left. + p^{y_i/n}(1-p)^{(n-y_i)/n} \rho I_{A_2}(y_i) \right\}. \end{aligned} \tag{2}$$

The complexity of the mixture likelihood (2) due to the presence of products of sums is a natural barrier to determine the conditional posterior distributions for the Metropolis-within-Gibbs algorithm. However, the data augmentation scheme [Tanner and Wong (1987), Diebolt and Robert (1994)] overcomes this problem by simplifying the conditional posterior distributions required for the MCMC methods.

2.1 Latent variable

A latent variable Z_i, $i = 1, \ldots, k$, is introduced in the model indicating to which component of the $CB(n, p, \rho)$ model the observation y_i, $i = 1, \ldots, k$, belongs to, that is,

$$Z_i = \begin{cases} 1, & \text{if the observation } y_i \text{ belongs to the } MBern(p) \text{ component,} \\ 0, & \text{if the observation } y_i \text{ belongs to the } B(n, p) \text{ component,} \end{cases}$$

and, conditionally on the observation y_i, the probability of success of the variable Z_i is given by

$$\begin{aligned} \tau_i = P(Z_i = 1 | Y_i = y_i, n, p, \rho) = \frac{P(Y_i = y_i | Z_i = 1) P(Z_i = 1)}{P(Y_i = y_i)} \\ = \frac{\rho p^{y_i/n}(1-p)^{(n-y_i)/n} I_{A_2}(y_i)}{\rho p^{y_i/n}(1-p)^{(n-y_i)/n} I_{A_2}(y_i) + (1-\rho)\binom{n}{y_i} p^{y_i}(1-p)^{n-y_i} I_{A_1}(y_i)}, \end{aligned}$$

where $A_1 = \{0, 1, \ldots, n\}$, $A_2 = \{0, n\}$ and $y_i = 0, \ldots, n; i = 1, \ldots, k$.

Thus,

$$\begin{aligned}
&P(Z_i = z_i | Y_i = y_i, n, p, \rho) \\
&\quad = \tau_i^{z_i}(1-\tau_i)^{1-z_i} \\
&\quad = \left(\rho p^{y_i/n}(1-p)^{(n-y_i)/n} I_{A_2}(y_i)\right)^{z_i} \\
&\qquad \times \left((1-\rho)\binom{n}{y_i} p^{y_i}(1-p)^{n-y_i} I_{A_1}(y_i)\right)^{1-z_i} \\
&\qquad \times \Bigg(\rho p^{y_i/n}(1-p)^{(n-y_i)/n} I_{A_2}(y_i) \\
&\qquad\quad + (1-\rho)\binom{n}{y_i} p^{y_i}(1-p)^{n-y_i} I_{A_1}(y_i)\Bigg)^{-1}.
\end{aligned} \tag{3}$$

Let $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_k)'$ be a vector of latent variables, the joint distribution of $\mathbf{Z}$ is given by

$$\begin{aligned}
&P(\mathbf{Z} = \mathbf{z} | \mathbf{Y} = \mathbf{y}, n, p, \rho) \\
&\quad = \prod_{i=1}^{k} \tau_i^{z_i}(1-\tau_i)^{1-z_i} \\
&\quad = \prod_{i=1}^{k} \Bigg[(\rho p^{y_i/n}(1-p)^{(n-y_i)/n} I_{A_2}(y_i))^{z_i} \\
&\qquad \times \left((1-\rho)\binom{n}{y_i} p^{y_i}(1-p)^{n-y_i} I_{A_1}(y_i)\right)^{1-z_i} \\
&\qquad \times \Bigg(\rho p^{y_i/n}(1-p)^{(n-y_i)/n} I_{A_2}(y_i) \\
&\qquad\quad + (1-\rho)\binom{n}{y_i} p^{y_i}(1-p)^{n-y_i} I_{A_1}(y_i)\Bigg)^{-1}\Bigg].
\end{aligned} \tag{4}$$

The joint distribution of the augmented data (Y_i, Z_i), $i = 1, \ldots, k$, is given by

$$\begin{aligned}
&P(Y_i = y_i, Z_i = z_i | n, p, \rho) \\
&\quad = \rho^{z_i} p^{y_i z_i/n}(1-p)^{(n-y_i)z_i/n}(1-\rho)^{1-z_i} \\
&\qquad \times \binom{n}{y_i}^{1-z_i} p^{y_i(1-z_i)}(1-p)^{(n-y_i)(1-z_i)}.
\end{aligned} \tag{5}$$

Therefore, with the latent variable, the complete data likelihood function is defined as

$$\begin{aligned}
&L(p,\rho|n,\mathbf{y},\mathbf{z})\\
&\quad=\left(\prod_{i=1}^{k}\{P(\mathbf{y}|n,p,\rho)\}\right)\left(\prod_{i=1}^{k}\{P(\mathbf{z}|\mathbf{y},n,p,\rho)\}\right)\\
&\quad=\rho^{\sum_{i=1}^{k} z_i}(1-\rho)^{\sum_{i=1}^{k}(1-z_i)}p^{\sum_{i=1}^{k}\{z_i y_i/n+(1-z_i)y_i\}}\\
&\qquad\times(1-p)^{\sum_{i=1}^{k}\{z_i(n-y_i)/n+(1-z_i)(n-y_i)\}}\\
&\qquad\times\prod_{i=1}^{k}\binom{n}{y_i}^{(1-z_i)}.
\end{aligned}\tag{6}$$

Note that the likelihood function (6) based on the latent data becomes more tractable than the usual likelihood function (2), facilitating the Bayesian analysis.

A Bayesian approach is applied to the correlated binomial model assuming n known and prior independence among the parameters p and ρ. It is considered a prior $Beta(\alpha,\beta)$, with known hyperparameters α and β, for p and a prior $U(0,1)$ for ρ. Hence, the joint prior distribution for (p,ρ) is given by $\pi(p,\rho)\propto p^{\alpha-1}(1-p)^{\beta-1}$.

Combining (6) with $\pi(p,\rho)$, the joint posterior distribution for (p,ρ) given n, α, β, $\mathbf{y}$ and $\mathbf{z}$ is

$$\begin{aligned}
&\pi(p,\rho|n,\alpha,\beta,\mathbf{y},\mathbf{z})\\
&\quad\propto\rho^{\sum_{i=1}^{k} z_i}(1-\rho)^{\sum_{i=1}^{k}(1-z_i)}p^{\sum_{i=1}^{k}\{z_i y_i/n+(1-z_i)y_i\}+(\alpha-1)}\\
&\qquad\times(1-p)^{\sum_{i=1}^{k}\{z_i(n-y_i)/n+(1-z_i)(n-y_i)\}+(\beta-1)}\prod_{i=1}^{k}\left\{\binom{n}{y_i}^{(1-z_i)}\right\}.
\end{aligned}\tag{7}$$

The posterior conditional distributions for the parameters p and ρ are given by

$$\begin{aligned}
&\pi(p|n,\rho,\alpha,\beta,\mathbf{y},\mathbf{z})\\
&\quad\sim Beta\Bigg(\sum_{i=1}^{k}\left\{\frac{z_i y_i}{n}+(1-z_i)y_i\right\}+\alpha;\\
&\qquad\qquad\sum_{i=1}^{k}\left\{\frac{z_i(n-y_i)}{n}+(1-z_i)(n-y_i)\right\}+\beta\Bigg)
\end{aligned}\tag{8}$$

and

$$\pi(\rho|n,p,\alpha,\beta,\mathbf{y},\mathbf{z})\sim Beta\left(1+\sum_{i=1}^{k} z_i;\,1+\sum_{i=1}^{k}(1-z_i)\right).\tag{9}$$

3 Simulation study

In order to verify the sensibility of the posterior distribution with respect to different prior distributions for p, several samples of the marginal posterior distribution for parameters p and ρ are generated from (8) and (9) using the Gibbs sampling algorithm. Four specifications of α and β used to express noninformative prior distributions [see Smith (1991)] are considered. These values are $(\alpha, \beta) = (0, 0)$, $(0, 1)$, $(\frac{1}{2}, \frac{1}{2})$, $(1, 1)$. It is easy to show in these cases that the posterior distribution (7) is a proper distribution even through the prior $\pi(p, \rho)$ is an improper distribution.

The considered data set is a random sample y_i, $i = 1, \ldots, 30$, from a $BC(20, 0.5, 0.8)$. In implementing the Gibbs sampler for this specific data set, chains of 70,000 iterations for p and ρ with a burn-in of 10,000 samples are generated. The posterior samples are based on the sets of 6000 observations obtained by accepting every 10th iteration from the others 60,000 samples in each chain. The characteristics of the marginal posterior distribution obtained via the Gibbs algorithm are shown in Table 1. The convergence diagnostics were performed via CODA [Best, Cowles and Vines (1995)]. The results showed that all the chains converged.

Table 1 shows that there are only minor differences in the results reached using the four reference priors. That is, summaries like the posterior means and posterior medians appear not to be influenced by the prior distributions.

Others data sets were also considered, such as from $BC(20, 0.5, 0.2)$, $BC(20, 0.1, 0.2)$ and $BC(20, 0.9, 0.2)$, and they reached similar conclusions.

Table 1 *Summaries of the marginal posterior of p and ρ, with an artificial data set from a $BC(n = 20, p = 0.5, \rho = 0.8)$*

Priori	Posterior mean	Posterior variance	Posterior median	Credibility region (95%)
$Beta(0, 0)$ for p	0.4994	0.0018	0.4994	[0.4175, 0.5807]
$U(0, 1)$ for ρ	0.7804	0.0051	0.7859	[0.6250, 0.9039]
$Beta(0, 1)$ for p	0.4964	0.0017	0.4963	[0.4157, 0.5773]
$U(0, 1)$ for ρ	0.7824	0.0051	0.7859	[0.6294, 0.9025]
$Beta(\frac{1}{2}, \frac{1}{2})$ for p	0.5001	0.0017	0.5003	[0.4186, 0.5812]
$U(0, 1)$ for ρ	0.7811	0.0052	0.7872	[0.6263, 0.9058]
$Beta(1, 1)$ for p	0.4998	0.0018	0.4999	[0.4183, 0.5812]
$U(0, 1)$ for ρ	0.7818	0.0051	0.7880	[0.6259, 0.9022]

4 Empirical analysis

In this section a numerical example is given to illustrate the proposed Bayesian method. The example involves a commercial soybean selection.

The recommended density for commercial soybean crops is 20 plants per plot of 60×100 cm (i.e., one plant per 360 cm^2). Under greenhouse conditions, an experiment was carried out with the Glycine max (cultivar IAC23) plant in order to investigate the competition effect of nutrients on the contents of lipids, protein, water and carbohydrates. For this purpose, 240 seeds were disinfected with sodium hypochlorite (NaOCl). Groups of ten seeds were placed on a "two-layer" moistened paper in a Petri glass dish and were allowed to germinate. The best 120 one-week-old seedlings of a soybean type were selected and transferred to a 6-liter pot filled with soil and a surface area of 320 cm^2. The set of 120 IAC23 seedlings were divided into 20 subsets of 6 seedlings each, resulting in a final density of 6 seedlings per 320 cm^2. To eradicate soilborne pathogens, 150 liters of red latosoil were sterilized in autoclave for 20 minutes at 120°C. Six soybean seedlings were equidistantly transplanted in each pot and daily irrigated to a 80% field capacity. Fifteen days after transplanting, the best plants of each pot, according to phenotypes of the plants, were selected and the contents of proteins, carbohydrates, water and lipids were determined. In a statistical point of view, the researcher has two main interests in this problem. One of them is to calculate the probability of selecting a "good" plant and the other is to calculate the correlation among any two plants which are competing with each other concerning the nutrients. Table 2 presents the original data set.

The presence of correlation among any two plants in each pot due to the competition of nutrients is a natural problem for the ordinary binomial fit. Thus, to analyze these data it is assumed that the $BC(6, p, \rho)$ model is adequate in this case. Considering the Bayesian methods presented in the previous sections, chains of 80,000 iterations for p and ρ with a burn-in of 15,000 samples were generated. The posterior samples are based on sets of 6500 observations obtained by accepting every 10th iteration from the others 65,000 samples in each chain. The iteration numbers were sufficient to reach convergence in this case.

In Table 3 the posterior summaries for parameters p and ρ are shown. It is apparent that there are only minor differences in the results reached using the four reference priors. The probability of selecting a "good" plant is around to 0.58 and

Table 2 *Number of IAC23 soybean plants selected*

IAC23 soybean selected in each plot after 15 days						
4	4	6	2	3	3	3
5	5	6	6	3	3	4
1	1	5	4	4	2	

Table 3 *Summaries of the marginal posterior of p and ρ for the IAC23 soybean plant*

Priori	Posterior mean	Posterior variance	Posterior median	Credibility region (95%)
$Beta(0, 0)$ for p	0.5844	0.0025	0.5854	[0.4840, 0.6796]
$U(0, 1)$ for ρ	0.1282	0.0068	0.1158	[0.0090, 0.3182]
$Beta(0, 1)$ for p	0.5778	0.0026	0.5787	[0.4768, 0.6746]
$U(0, 1)$ for ρ	0.1307	0.0069	0.1189	[0.0096, 0.3219]
$Beta(\frac{1}{2}, \frac{1}{2})$ for p	0.5836	0.0025	0.5847	[0.4833, 0.6781]
$U(0, 1)$ for ρ	0.1295	0.0068	0.1175	[0.0091, 0.3195]
$Beta(1, 1)$ for p	0.5826	0.0025	0.5833	[0.4841, 0.6773]
$U(0, 1)$ for ρ	0.1296	0.0072	0.1166	[0.0085, 0.3279]

Table 4 *Fitted frequency table including the total observed frequency (O), the fitted ordinary binomial frequency (E_B) and the fitted correlated binomial frequency (E_{CB})*

# of selected plant	O	E_{CB}	E_B
0	0	1.19	0.06
1	2	0.79	0.61
2	2	2.73	2.46
3	5	5.03	5.28
4	5	5.21	6.37
5	3	2.87	4.10
6	3	2.17	1.09

the correlation among any two plants is close to 0.13. In Table 4 the total observed frequency, the fitted ordinary binomial frequency (E_B) and the fitted correlated binomial frequency (E_{CB}) are shown. These results show that the suggested model, that is, the correlated binomial model with noninformative priors has a reasonable overall fit when compared with the ordinary binomial model.

5 Conclusions

In this paper a Bayesian approach is applied to the correlated binomial, $CB(n, p, \rho)$, a model proposed by Luceño (1995). The data augmentation scheme is used in order to overcome the complexity of the mixture likelihood. MCMC methods, including Gibbs sampling and Metropolis within Gibbs, are applied to estimate the posterior marginal for the probability of success p and for the correlation coefficient ρ. The sensitivity of the posterior is studied taking into account several reference priors and it is shown that the posterior characteristics appear not to be influenced in any way by these prior distributions.

Appendix: Proof of Theorem 1

Tallis (1962) presents a probability generating function used to construct a generalized multinomial distribution. Suppose W_j, $j = 1, \ldots, n$, are identically distributed variables with $P(W = i) = \pi_i$, $\sum_{i=0}^{k} \pi_i = 1$, and $\mathrm{Corr}(W_i, W_j) = \rho$, $i \neq j$. A probability generating function for the joint probabilities

$$P(W_1 = a, W_2 = b, \ldots) = \alpha_{ab\ldots} \qquad (a, b, \ldots = 0, 1, 2, \ldots, k),$$

is given by

$$G_n(\mathbf{t}) = \rho \left\{ \sum_{i=0}^{k} \pi_i \left(\prod_{j=1}^{n} t_j \right)^i \right\} + (1-\rho) \prod_{j=1}^{n} P(t_j), \qquad 0 \le \rho \le 1, \tag{10}$$

where $P(t_j) = \sum_{i=0}^{k} \pi_i t_j^i$ and $\mathbf{t} = (t_1, t_2, \ldots, t_n)$.

Considering a variable $Y = W_1 + W_2 + \cdots + W_n$, a probability generating function for Y is given by

$$G_n(t) = \sum_{i=0}^{nk} \alpha_i t^i = \rho \sum_{i=0}^{k} \pi_i t^{ni} + (1-\rho)\{P(t)\}^n, \tag{11}$$

obtained from (10) with $t_j = t$, $j = 1, \ldots, n$. If the random variables W_j, $j = 1, \ldots, n$, are identically distributed variables following Bernoulli distributions, then $k = 1$ and from (11)

$$\begin{aligned} G_n(t) &= \sum_{i=0}^{n} P(Y = i) t^i \\ &= \rho\pi_0 + \rho\pi_1 t^n + (1-\rho)(\pi_0^n + n\pi_0^{n-1}\pi_1 t + \cdots + \pi_1^n t^n). \end{aligned} \tag{12}$$

Thus, by comparing the coefficients of both polynomials,

$$\begin{aligned} P(Y = 0) &= \rho\pi_0 + (1-\rho)\pi_0^n, \\ P(Y = j) &= (1-\rho) \binom{n}{j} \pi_1^j \pi_0^{n-j}, \qquad j = 1, 2, \ldots, n-1, \\ P(Y = n) &= \rho\pi_1 + (1-\rho)\pi_1^n. \end{aligned} \tag{13}$$

From the expressions presented in (13) the probability distribution (1) is obtained.

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