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KOLMOGOROV EQUATION ASSOCIATED TO THE STOCHASTIC REFLECTION PROBLEM ON A SMOOTH CONVEX SET OF A HILBERT SPACE

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We consider the stochastic reflection problem associated with a self-adjoint operator A and a cylindrical Wiener process on a convex set K with nonempty interior and regular boundary Σ in a Hilbert space H. We prove the existence and uniqueness of a smooth solution for the corresponding elliptic infinite-dimensional Kolmogorov equation with Neumann boundary condition on Σ.

1. Introduction. Let us consider a stochastic differential inclusion in a Hilbert space H,

$$\begin{cases} dX(t) + \big(AX(t) + N_K(X(t))\big)\,dt \ni dW(t), \\ X(0) = x. \end{cases} \tag{1.1}$$

Here $A : D(A) \subset H \to H$ is a self-adjoint operator, $K = \{x \in H : g(x) \le 1\}$, where $g : H \to \mathbb{R}$ is convex and of class C^∞, $N_K(x)$ is the normal cone to K at x and $W(t)$ is a cylindrical Wiener process in H (see Hypothesis 1.1 for more precise assumptions). Obviously the expression in (1.1) is formal and its precise meaning should be defined.

When H is finite-dimensional a solution to (1.1) is a pair of continuous adapted processes (X, η) such that X is K-valued, η is of bounded variation with $d\eta$ concentrated on the set of times where $X(t) \in \Sigma$ (the boundary of K) and

$$X(t) + \int_0^t AX(s)\,ds + \eta(t) = x + W(t), \qquad t \ge 0,\ \mathbb{P}\text{-a.s.},$$

$$\int_0^T \big(d\eta(t), X(t) - z(t)\big) \ge 0, \qquad \mathbb{P}\text{-a.s.},$$

for all $z \in C([0, T]; K)$. The existence and uniqueness of a solution (X, η) to latter equation was first proven by Cépa in [5]. (See also [3] for a slightly different formulation.)

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Therefore, under the assumptions of [3] or [5], one can construct a transition semigroup in $C(K)$ by the usual formula

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))], \qquad t \geq 0,\ \varphi \in C(K).$$

The infinitesimal generator L of P_t is the Kolmogorov operator

$$L\varphi = \frac{1}{2}\Delta\varphi + \langle Ax, D\varphi\rangle$$

equipped with a Neumann condition at the boundary Σ of K. (See, e.g., [3], where the more general case of oblique derivative boundary conditions were also considered.)

Let us go now to the infinite-dimensional situation. In this context (1.1) was first studied by Nualart and Pardoux [18], when $H = L^2(0,1)$, A is the Laplace operator with Dirichlet or Neumann boundary conditions and K is the convex set of all nonnegative functions of $L^2(0,1)$; see also [13].

The Kolmogorov operator in this situation was described by Zambotti [21], in the space $L^2(H,\nu)$ where ν is the law of the $3D$-Bessel Bridge which coincides with the unique invariant measure of (1.1). Zambotti was able to show that the Dirichlet form

$$a(u,v) = \int_K \langle Du, Dv\rangle\, d\nu$$

is closable by proving a suitable integration by parts formula and to construct the corresponding Markov semigroup.

Except the situation mentioned above, no existence and uniqueness results for (1.1) are known for the infinite-dimensional equation (1.1). Also it was so far open the characterization of the of the domain of the corresponding Kolmogorov operator.

In this paper we shall consider a regular convex set K with nonempty interior and, though this does not cover the case considered by [21], we are able, however, to get sharp informations on the Kolmogorov generator for a quite general class of convex sets K. In this way, though we are not able to approach directly the stochastic variational problem (1.1), we can instead find a regular solution of the corresponding infinite-dimensional Kolmogorov equation equipped with the Neumann boundary condition,

$$\begin{cases} \lambda\varphi - \frac{1}{2}\operatorname{Tr}[D^2\varphi] - \langle x, AD\varphi\rangle = f, & x \in K, \\ \langle D\varphi, N_K(x)\rangle = 0, & \forall x \in \Sigma, \end{cases} \tag{1.2}$$

where $\lambda > 0$ and $f \in L^2(K,\nu)$.

In this way we obtain a Markov semigroup P_t which by the results of [16] provides a process corresponding to a martingale solution of (1.1) (see also the forthcoming paper [1]).

A basic tool we are using is a co-area formula from Malliavin; see [17] valid for g of class C^∞. Moreover, in the Appendix we present a direct proof of this formula when g is C^2 and fulfills some additional conditions which are covered in several situations, for instance when K is a ball; in that case the co-area formula was proved (1979) by Hertle [14].

Let us explain the content of this paper. As we said, we take a convex set of the form $K = \{x \in H : g(x) \le 1\}$ where $g : H \to \mathbb{R}$ is of class C^∞ and with second order derivative $D^2 g$ positive definite. Then we consider the probability measure ν given for any Borel set I of K by

$$\nu(I) = \frac{\mu(I)}{\mu(K)},$$

where μ is the Gaussian measure (corresponding to the linear problem without reflection) of mean 0 and covariance $Q = \frac{1}{2}A^{-1}$.

In Section 2, by exploiting a basic infinite-dimensional co-area formula, see [17], we are able to prove an integration by parts formula for ν. This allows us to show in Section 3 that the Dirichlet form

$$a(u, v) = \int_K \langle Du, Dv\rangle \, d\nu$$

is closable (see also [1] for a different approach). In this way, by the usual variational theory, we can define its generator N and construct the corresponding Markov transition semigroup P_t, which is reversible since N is self adjoint.

In Section 4 we study the Kolmogorov equation (1.2) by the classical method of penalization

$$\lambda\varphi_\varepsilon - \frac{1}{2}\operatorname{Tr}[D^2\varphi_\varepsilon] + \langle x, AD\varphi_\varepsilon\rangle + \frac{1}{\varepsilon}\langle x - \Pi_K(x), D\varphi_\varepsilon\rangle = f, \qquad x \in H, \tag{1.3}$$

where $\Pi_K(x)$ is the projection of x on K. We show that $\{\varphi_\varepsilon\}$ strongly converges to the solution $\varphi = (\lambda I - N)^{-1} f$ of (1.2) and that

$$D(N) \subset \left\{\varphi \in W^{2,2}(K, \nu) : \int_K |A^{1/2} D\varphi|^2 \, d\nu < +\infty \right. \\ \left. \text{and } \langle D\varphi, N_K(x)\rangle = 0 \text{ on } \Sigma \right\}. \tag{1.4}$$

These results seem to be new in infinite dimensions; see [2, 3, 7] for the finite-dimensional case.

Finally, Section 5 is devoted to equations of the form

$$\begin{cases} dX(t) + \big(AX(t) + F(X(t)) + N_K(X(t))\big)\, dt \ni dW(t), \\ X(0) = x, \end{cases} \tag{1.5}$$

where $F : H \to H$ is a nonlinear perturbation of A.

In Section 5.1 we assume that $F = DV$ where $V : H \to \mathbb{R}$ is a regular potential. This case is an easy generalization of the previous one (i.e., when $F = 0$), namely measure ν is replaced by the following one:

$$\zeta(dx) = \frac{e^{-2V(x)}}{\int_K e^{-2V(y)}\nu(dy)} \nu(dx).$$

This extension is briefly described in that section.

In Section 5.2 the case of a bounded Borel function F, not necessarily of potential type, is considered. Here we can solve the Kolmogorov equation

$$\begin{cases} \lambda\varphi - \frac{1}{2}\operatorname{Tr}[D^2\varphi] + \langle x, AD\varphi\rangle - \langle F(x), D\varphi\rangle = f, & x \in K, \\ \langle D\varphi, N_K(x)\rangle = 0, & \forall x \in \Sigma \end{cases} \tag{1.6}$$

by a straightforward perturbation argument, taking avantage of the inclusion (1.4). In this way we obtain a solution $\varphi \in D(N)$ of (1.6) only for λ sufficiently large. Also, obviously, measure ν is not invariant for the corresponding semigroup Q_t. However, using the fact that operator Q_t is compact in $L^2(K, \nu)$, we can show the existence of an invariant measure ζ for Q_t so that the extension of Q_t to $L^1(K, \zeta)$ is the natural transition semigroup associated with (1.5). Notice, however, that this semigroup is not reversible (when F is not of potential type).

We conclude this section by precising assumptions and notation which will be used throughout in what follows.

Assumptions. We are given a real separable Hilbert space H (with scalar product $\langle\cdot,\cdot\rangle$ and norm denoted by $|\cdot|$). Concerning A, K and W we shall assume that:

HYPOTHESIS 1.1. (i) $A : D(A) \subset H \to H$ is a linear self-adjoint operator on H such that $\langle Ax, x\rangle \geq \delta|x|^2$, $\forall x \in D(A)$ for some $\delta > 0$. Moreover, A^{-1} is of trace class.

(ii) There exists a convex C^∞ function $g : H \to \mathbb{R}$ with $D^2 g$ positively defined, that is, $\langle D^2 g(x)h, h\rangle \geq \gamma|h|^2$, $\forall h \in H$ where $\gamma > 0$, such that

$$K = \{x \in H : g(x) \leq 1\}, \qquad \Sigma = \{x \in H : g(x) = 1\}.$$

(iii) W is a cylindrical Wiener process on H of the form

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) e_k, \qquad t \geq 0,$$

where $\{\beta_k\}$ is a sequence of mutually independent real Brownian motions on a filtered probability spaces $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ (see, e.g., [8]) and $\{e_k\}$ is an orthonormal basis in H which will be taken as a system of eigen-functions for A for simplicity, that is,

$$Ae_k = \alpha_k e_k \qquad \forall k \in \mathbb{N},$$

where $\alpha_k \geq \delta$.

We notice that the interior $\mathring{K}$ is nonempty since $D^2 g$ is positive definite.

Notation. We denote by $\mathcal{B}(H)$ [resp. $\mathcal{B}(K)$] the σ-field of all Borel subsets of H (resp. K) and by $\mathcal{P}(H)$ [resp. $\mathcal{P}(K)$] the set of all probability measures on

$(H, \mathcal{B}(H))$ [resp. $(K, \mathcal{B}(K))$].

Everywhere in the following $D\varphi$ is the derivative of a function $\varphi : H \to \mathbb{R}$. By $D^2\varphi : H \to L(H, H)$ we shall denote the second derivative of φ. We shall denote also by $C_b(H)$ and $C_b^k(H)$, $k \in \mathbb{N}$, the spaces of all continuous and bounded functions on H and, respectively, of k-times differentiable functions with continuous and bounded derivatives. The space $C^k(K)$, $k \in \mathbb{N}$, is defined as the space of restrictions of functions of $C_b^k(H)$ to the subset K.

The boundary of K will be denoted by Σ. $N_K(x)$ is the normal cone to K at x, that is,

$$N_K(x) = \{z \in H : \langle z, y - x\rangle \le 0, \forall y \in K\}.$$

Moreover, we shall denote by $d_K(x)$ the distance of x from K and by I_K the indicator function of K,

$$I_K(x) = \begin{cases} 0, & \text{if } x \in K, \\ +\infty, & \text{if } x \notin K. \end{cases}$$

For any $\varepsilon > 0$, U_ε will represent the Moreau approximation of I_K given by

$$U_\varepsilon(x) = \inf\left\{ I_K(y) + \frac{1}{2\varepsilon}|x - y|^2, y \in H \right\} = \frac{1}{2\varepsilon} d_K(x)^2, \qquad x \in H.$$

It is well known that

$$DU_\varepsilon(x) = \frac{1}{\varepsilon}\big(x - \Pi_K(x)\big), \qquad x \in H, \varepsilon > 0,$$

where $\Pi_K(x)$ is the projection of x over K. In particular, we have

$$D(d_K^2(x)) = x - \Pi_K(x) \qquad \forall x \in K^c, \tag{1.7}$$

(K^c is the complement of K) which implies

$$Dd_K(x) = \frac{x - \Pi_K(x)}{d_K(x)} \qquad \forall x \in K^c. \tag{1.8}$$

We denote by $\mathbf{n}(\Pi_K(x))$ the exterior normal at $\Pi_K(x)$,

$$\mathbf{n}(\Pi_K(x)) = \frac{x - \Pi_K(x)}{d_K(x)} \qquad \forall x \in K^c.$$

From (1.8) we deduce that

$$D\big(x - \Pi_K(x)\big) = Dd_K(x) \otimes Dd_K(x) + d_K(x) D^2 d_K(x) \qquad \forall x \in K^c. \tag{1.9}$$

Finally, μ will represent the Gaussian measure in H with mean 0 and covariance operator

$$Q := \tfrac{1}{2} A^{-1}.$$

Since A is strictly positive μ is nondegenerate and full. We set

$$\lambda_k = \frac{1}{2\alpha_k} \qquad \forall k \in \mathbb{N},$$

so that

$$Qe_k = \lambda_k e_k \qquad \forall k \in \mathbb{N}.$$

We denote by $\mathcal{E}_A(H)$ the space of all real and imaginary parts of exponential functions $e^{\mathrm{i}\langle h,x\rangle}, h \in D(A)$. Then the operator $D : \mathcal{E}_A(H) \subset L^2(H,\mu) \to L^2(H,\mu; H)$ is closable in $L^2(H,\mu)$ and the domain of its closure is denoted by $W^{1,2}(H,\mu)$ (the Sobolev space).

The following integration by parts formula for the measure μ is well known (see, e.g., [9]). For any $\varphi, \psi \in W^{1,2}(H,\mu)$ and $z \in H$,

$$\int_H \langle D\varphi, Q^{1/2}z\rangle \psi \, d\mu = -\int_H \langle D\psi, Q^{1/2}z\rangle \varphi \, d\mu + \int_H W_z \varphi\psi \, d\mu, \tag{1.10}$$

where W_z represents the *white noise* function,

$$W_z(x) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{\lambda_k}} \langle x, e_k\rangle\langle z, e_k\rangle \qquad \forall z \text{ and } \mu\text{-a.e. } x \in H. \tag{1.11}$$

We recall that W_z is a Gaussian random variable in $L^2(H,\mu)$ with mean 0 and covariance $|z|^2$.

2. The measure μ conditioned to K.

We denote by ν the Gaussian measure μ conditioned to K, that is,

$$\nu(I) = \frac{\mu(K \cap I)}{\mu(K)} \qquad \forall I \in \mathcal{B}(H).$$

Since μ is full and $\mathring{K}$ is nonempty, this definition is meaningful. We notice that, thanks to Hypothesis 1.1(ii) the surface measure μ_Σ is well defined (see [17]).

We want now to prove an integration by parts formula with respect to measure ν which generalizes (1.10). For this it is convenient to introduce a sequence of approximating measures $\{\nu_\varepsilon\}_{\varepsilon>0}$ defined by,

$$\nu_\varepsilon(dx) = \rho_\varepsilon(x)\mu(dx), \qquad x \in H, \tag{2.1}$$

where,

$$\rho_\varepsilon(x) = Z_\varepsilon^{-1} e^{-1/\varepsilon \, d_K^2(x)} \tag{2.2}$$

and

$$Z_\varepsilon = \int_H e^{-1/\varepsilon \, d_K^2(y)} \mu(dy). \tag{2.3}$$

Notice that, by the dominated convergence theorem,

$$\lim_{\varepsilon\to 0} Z_\varepsilon = Z_0 = \mu(K), \tag{2.4}$$

whereas

$$\lim_{\varepsilon\to 0} \rho_\varepsilon(x) = \begin{cases} 1, & \text{if } x \in K, \\ 0, & \text{if } x \notin K. \end{cases} \tag{2.5}$$

So, we have

$$\lim_{\varepsilon\to 0} \nu_\varepsilon = \nu \qquad \text{weakly in } \mathcal{P}(H). \tag{2.6}$$

Moreover

$$D\rho_\varepsilon(x) = -\frac{2}{\varepsilon}\rho_\varepsilon(x)\big(x - \Pi_K(x)\big), \tag{2.7}$$

so that $\rho_\varepsilon \in W^{1,2}(H,\mu)$.

We shall denote by $L^2(K,\nu)$ the space of all ν-square-integrable functions on K with the scalar product

$$\langle u, v\rangle_{L^2(K,\nu)} = \int_K u(x)v(x)\nu(dx)$$

and the norm $|u|^2_{L^2(K,\nu)} = \langle u, u\rangle_{L^2(K,\nu)}$.

2.1. *The integration by parts formula.* Here we are going to derive from (1.10), an integration by parts formula for the measure ν_ε. Let $\varphi \in C_b^1(H)$, $z \in H$, then, since $\rho_\varepsilon \in W^{1,2}(H,\mu)$, we find from (1.10) that

$$\begin{aligned}\int_H \langle D\varphi, Q^{1/2}z\rangle\, d\nu_\varepsilon &= \int_H \langle D\varphi, Q^{1/2}z\rangle \rho_\varepsilon\, d\mu \\ &= -\int_H \varphi\langle D\log\rho_\varepsilon, Q^{1/2}z\rangle\, d\nu_\varepsilon + \int_H W_z\varphi\, d\nu_\varepsilon.\end{aligned}$$

Since,

$$D\log\rho_\varepsilon(x) = -\frac{1}{\varepsilon}(x - \Pi_K x),$$

we find the formula

$$\begin{aligned}\int_H \langle D\varphi, Q^{1/2}z\rangle \nu_\varepsilon(dx) &= \frac{1}{\varepsilon}\int_H \varphi(x)\langle x - \Pi_K(x), Q^{1/2}z\rangle \nu_\varepsilon(dx) \\ &\quad + \int_H W_z(x)\varphi(x)\nu_\varepsilon(dx).\end{aligned} \tag{2.8}$$

LEMMA 2.1. *Let $\varphi \in C_b^1(H)$, $z \in H$. Then there exists the limit*

$$\lim_{\varepsilon\to 0} J_\varepsilon^z(\varphi) := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_H \varphi(x)\langle x-\Pi_K x, Q^{1/2}z\rangle \nu_\varepsilon(dx) \tag{2.9}$$
$$= \int_\Sigma \varphi(y)\langle \mathbf{n}(y), Q^{1/2}z\rangle \mu_\Sigma(dy),$$

where $\mathbf{n}(y) = \nabla g(y)/|\nabla g(y)|$ is the exterior normal to Σ at y and μ_Σ is the surface measure on Σ induced by μ (see [17]*).*

PROOF. First we notice that

$$J_\varepsilon^z(\varphi) = \frac{1}{\varepsilon Z_\varepsilon}\int_{\{d_K(x)>0\}} \varphi(x) d_K(x)\langle \mathbf{n}(\Pi_K(x)), Q^{1/2}z\rangle e^{-d_K^2(x)/\varepsilon}\mu(dx).$$

By the co-area formula (see [17], page 140) (see also Theorem A.5 below) we have

$$\int_H f\mu(dx) = \int_0^\infty \left[\int_{\Sigma_r} f(y)\mu_{\Sigma_r}(dy)\right] dr. \tag{2.10}$$

Notice that the surface measure is defined for all $r \geq 0$ taking into account ([17], Theorem 6.2, Chapter V); moreover ([17], Theorem 1.1, Corollary 6.3.2, Chapter V), give the continuity property in Theorem 6.3.1 of [17], Chapter V. Setting in (2.10)

$$f = (1-\mathbb{1}_K)\varphi(x)\, d_K(x)\langle \mathbf{n}(\Pi_K(x)), Q^{1/2}z\rangle e^{-d_K^2(x)/\varepsilon},$$

we get

$$J_\varepsilon^z(\varphi) = \frac{1}{\varepsilon Z_\varepsilon}\int_0^{+\infty} \xi e^{-\xi^2/\varepsilon}\, d\xi \int_{\Sigma_{\xi+1}} \varphi(y)\langle \mathbf{n}(\Pi_K(x)), Q^{1/2}z\rangle \mu_{\Sigma_\xi}(dy).$$

Hence, setting $\xi = \sqrt{\varepsilon}s$, yields

$$J_\varepsilon^z(\varphi) = \frac{1}{Z_\varepsilon}\int_0^\infty s e^{-s^2}\, ds \int_{\Sigma_{\sqrt{\varepsilon}s+1}} \varphi(y)\langle \mathbf{n}(\Pi_K(y)), Q^{1/2}z\rangle \mu_{\Sigma_{\sqrt{\varepsilon}s}}(dy).$$

So (2.9) follows. □

We are now in position to prove the announced integration by parts formula.

THEOREM 2.2. *Let $\varphi \in C_b^1(H)$, $z \in H$. Then for any $z \in H$ we have*

$$\int_K \langle D\varphi(x), Q^{1/2}z\rangle \nu(dx) = \frac{1}{2\mu(K)}\int_\Sigma \varphi(y)\langle \mathbf{n}(y), Q^{1/2}z\rangle \mu_\Sigma(dy) \tag{2.11}$$
$$+ \int_K W_z(x)\varphi(x)\nu(dx).$$

PROOF. The conclusion of the theorem follows letting $\varepsilon \to 0$ in (2.8) and taking into account Lemma 2.1. □

2.2. *The Sobolev space* $W^{1,2}(K,\nu)$. We shall define space $W^{1,2}(K,\nu)$ by proving, as it is usual, closability of the gradient. For this we need a lemma.

LEMMA 2.3. *The space*

$$C_0^1(K) := \{\varphi \in C^1(K) : \varphi = 0 \text{ on } \Sigma\}$$

is dense in $L^2(K,\nu)$.

PROOF. It is enough to show that if $\varphi \in C^1(K)$ then there exists a sequence $\{\varphi_\alpha\} \subset C_0^1(K)$ such that

$$\lim_{\alpha\to 0} \varphi_\alpha = \varphi \qquad \text{in } L^2(K,\nu). \tag{2.12}$$

Let $\{\chi_\alpha\}_{\alpha\in(0,1)} \subset C^1(\mathbb{R})$ be a sequence such that,

$$\chi_\alpha(r) = \begin{cases} 1, & \text{for } r \in [0, 1-\alpha], \\ 0, & \text{for } r \geq 1. \end{cases}$$

Setting now

$$\varphi_\alpha(x) = \chi_\alpha(g(x))\varphi(x) \qquad \forall \alpha \in (0,1),$$

we see that $\{\varphi_\alpha\}_{\alpha\in(0,1)} \subset C_0^1(K)$ and (2.12) follows from the dominated convergence theorem. □

PROPOSITION 2.4. *The mapping*

$$D : C^1(K) \subset L^2(K,\nu) \to L^2(K,\nu; H), \qquad \varphi \to D\varphi,$$

is closable.

PROOF. Let $(\varphi_n) \subset C^1(K)$ be such that

$$\varphi_n \to 0 \qquad \text{in } L^2(K,\nu), \qquad D\varphi_n \to F \qquad \text{in } L^2(K,\nu; H)$$

as $n \to \infty$. We have to show that $F = 0$. Let $\psi \in C_0^1(K)$ and $z \in H$. Then by (2.11) with $\varphi_n\psi$ replacing φ (see Theorem 2.2) we have that

$$\begin{aligned}
&\int_K \langle D\varphi_n(x), Q^{1/2}z\rangle \psi(x)\nu(dx) \\
&\quad = -\int_K \langle D\psi(x), Q^{1/2}z\rangle \varphi_n(x)\nu(dx) \\
&\qquad + \frac{1}{2\mu(K)} \int_\Sigma \varphi_n(y)\psi(y)\langle \mathbf{n}(y), Q^{1/2}z\rangle \mu_\Sigma(dy) \\
&\qquad + \int_K W_z(x)\varphi_n(x)\psi(x)\nu(dx) \\
&\quad = -\int_K \langle D\psi(x), Q^{1/2}z\rangle \varphi_n(x)\nu(dx) + \int_K W_z(x)\varphi_n(x)\psi(x)\nu(dx),
\end{aligned} \tag{2.13}$$

since ψ vanishes on Σ. Letting $n \to \infty$ we find that

$$\int_H \langle F(x), Q^{1/2}z\rangle \psi(x)\mu(dx) = 0.$$

This implies $F = 0$ in view of the arbitrariness of ψ and z [recall Lemma 2.3 and that $Q^{1/2}(H)$ is dense in H]. $\square$

We shall still denote by D the closure of D and by $W^{1,2}(K,\nu)$ its domain of definition. $W^{1,2}(K,\nu)$ is a Hilbert space with the scalar product

$$\langle \varphi, \psi\rangle_{W^{1,2}(K,\nu)} = \int_K [\varphi\psi + \langle D\varphi, D\psi\rangle]\, d\nu.$$

2.3. *The trace of a function of $W^{1,2}(K,\nu)$.* In order to define the trace of a function $\varphi \in W^{1,2}(K,\nu)$ we need a technical lemma.

LEMMA 2.5. *Assume that $\varphi \in C_b^1(H)$. Then the following estimate holds,*

$$\begin{aligned}(2.14)\qquad &\int_\Sigma |Q^{1/2}\mathbf{n}(y)|^2 \varphi^2(y)\mu_\Sigma(dy) \\ &\quad \le C\left(\int_K \varphi^2(x)\nu(dx) + \int_K |D\varphi(x)|^2 \nu(dx)\right),\end{aligned}$$

where C is a suitable constant.

PROOF. Here we follow [12]. Let $\varphi \in C^1(K)$. Set $F(x) = Dg(x)$. In particular $F(x) = |Dg(x)|\mathbf{n}(x)$ for $x \in \Sigma$. Then, replacing in (2.11) φ with $\lambda_k F_k \varphi^2$ and z with e_k, one gets

$$\begin{aligned}&\int_K \lambda_k D_k F_k \varphi^2\, d\nu + 2\lambda_k \int_K F_k \varphi D_k\varphi\, d\nu \\ &\quad = \frac{1}{2\mu(K)} \int_\Sigma \lambda_k |Dg(y)| \langle \mathbf{n}(y), e_k\rangle^2 \varphi^2(y)\mu_\Sigma(dy) + \int_K x_k F_k \varphi^2 \nu(dx).\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{1}{2\mu(K)} \int_\Sigma \lambda_k |Dg(y)| \langle \mathbf{n}(y), e_k\rangle^2 \varphi^2(y)\mu_\Sigma(dy) \\ &\quad \le \int_K \lambda_k D_k^2 g \varphi^2\, d\nu + \frac{1}{2}\int_K F_k^2 \varphi^2 \nu(dx) + \frac{1}{2}\lambda_k^2 \int_K |D_k\varphi|^2 \nu(dx) \\ &\qquad - \int_K x_k F_k \varphi^2 \nu(dx).\end{aligned}$$

Now the conclusion follows summing up over k, since $|Dg|$ is bounded below on Σ. $\square$

Now we can define the *trace* $\gamma(\varphi)$ on Σ of a function $\varphi \in W^{1,2}(K,\nu)$. Let us consider a sequence $\{\varphi_n\} \subset C^1(K)$ strongly convergent to φ in $W^{1,2}(K,\nu)$. Then by (2.14) it follows that the sequence $\{|Q^{1/2}\mathbf{n}(y)|(\varphi_n)_\Sigma\}$ is convergent in $L^2(\Sigma,\mu_\Sigma)$ to some element $\tilde{\gamma}(\varphi) \in L^2(\Sigma,\mu_\Sigma)$. Then we set

$$\gamma(\varphi)(y) = \frac{1}{|Q^{1/2}\mathbf{n}(y)|}\tilde{\gamma}(\varphi)(y), \qquad \mu_\Sigma\text{-a.s.}$$

By inequality (2.14) it follows that this definition is consistent, that is, is independent of the sequence $\{\varphi_n\}$ and the map $\varphi \to |Q^{1/2}\mathbf{n}(y)|\gamma(\varphi)$ is continuous from $W^{1,2}(K,\nu) \to L^2(\Sigma,\mu_\Sigma)$. Notice also that though $|Q^{1/2}\mathbf{n}(y)| > 0$ for all $y \in \Sigma$ it is not however bounded from below in infinite dimensions. Now the following result is an immediate consequence of Lemma 2.5 and the density of $C_b^1(H)$ in $W^{1,2}(K,\nu)$.

PROPOSITION 2.6. *Assume that* $\varphi \in W^{1,2}(K,\nu)$. *Then*:

(i) $|Q^{1/2}\mathbf{n}(y)|\gamma(\varphi) \in L^2(\Sigma,\mu_\Sigma)$,
(ii) *the following estimate holds*,

$$\begin{aligned}&\int_\Sigma |Q^{1/2}\mathbf{n}(y)|^2\varphi^2(y)\mu_\Sigma(dy)\\ &\quad\le C\biggl(\int_K \varphi^2(x)\nu(dx) + \int_K |D\varphi(x)|^2\nu(dx)\biggr).\end{aligned} \tag{2.15}$$

We notice that if H is finite-dimensional and $Q = I$ formula (2.15) reduces to a classical result since $|Q^{1/2}\mathbf{n}(y)| = 1$ on Σ.

2.4. *Compactness of embedding* $W^{1,2}(K,\nu) \subset L^2(K,\nu)$. We first show the log-Sobolev estimate for ν.

PROPOSITION 2.7. *For all* $\varphi \in W^{1,2}(H,\nu)$ *we have*

$$\int_K \varphi^2\log(\varphi^2)\,d\nu \le \frac{1}{\lambda_1}\int_H |D\varphi|^2\,d\nu + \|\varphi\|^2_{L^2(H,\nu)}\log\bigl(\|\varphi\|^2_{L^2(H,\nu)}\bigr). \tag{2.16}$$

PROOF. It is enough to show (2.16) for $\varphi \in C^1(H)$. By [6] (see also [9] and [10]) we know that the log-Sobolev estimate holds for the measure ν_ε,

$$\int_H \varphi^2\log(\varphi^2)\,d\nu_\varepsilon \le \frac{1}{\lambda_1}\int_H |D\varphi|^2\,d\nu_\varepsilon + \|\varphi\|^2_{L^2(H,\nu_\varepsilon)}\log\bigl(\|\varphi\|^2_{L^2(H,\nu_\varepsilon)}\bigr). \tag{2.17}$$

Now the conclusion follows by (2.6) letting ε tend to 0. □

We can now prove the following result.

PROPOSITION 2.8. *The embedding* $W^{1,2}(K,\nu) \subset L^2(K,\nu)$ *is compact*.

PROOF. Let $\{\varphi_n\}$ be a sequence in $W^{1,2}(K,\nu)$ such that

$$\int_K (\varphi_n^2 + |D\varphi_n|^2)\,d\nu \le C. \tag{2.18}$$

We have to show that there exists a subsequence of $\{\varphi_n\}$ convergent in $L^2(K,\nu)$. For this we proceed as in [6] noticing that, thanks to the log-Sobolev inequality (2.16), $\{\varphi_n\}$ is uniformly integrable and so, it is enough to find a subsequence of $\{\varphi_n\}$ pointwise convergent to an element of $L^2(K,\nu)$. Let $\{\chi_\alpha\}_{\alpha\in(0,1)} \subset C^1(\mathbb{R})$ be such that,

(i) we have

$$\chi_\alpha(r) = \begin{cases} 1, & \text{for } r \in [0, 1-2\alpha], \\ 0, & \text{for } r \ge 1-\alpha. \end{cases}$$

(ii) $|\chi_\alpha'(r)| \le \frac{2}{\alpha}$, $\forall \alpha > 0$.

Set now

$$\varphi_n^\alpha(x) = \chi_\alpha(g(x))\varphi_n(x) \qquad \forall \alpha \in (0, 1/2).$$

We claim that for each $\alpha \in (0,1/2)$ the sequence $\{\varphi_n^\alpha\}_{n\in\mathbb{N}}$ is bounded in $W^{1,2}(H,\mu)$. We have in fact

$$\int_H |\varphi_n^\alpha|^2\,d\mu = \int_H |\varphi_n^\alpha|^2\,d\nu \le C$$

and, since

$$D\varphi_n^\alpha(x) = \chi_\alpha(g(x))D\varphi_n(x) + \chi_\alpha'(g(x))\varphi_n(x)Dg(x),$$

we have

$$|D\varphi_n^\alpha(x)| \le |D\varphi_n(x)| + \frac{2}{\alpha}|Dg|_\infty|\varphi_n(x)|.$$

Therefore, there is a positive constant C_α' such that

$$\int_H |D\varphi_n^\alpha|^2\,d\mu \le C_\alpha'.$$

Recalling that the embedding $W^{1,2}(H,\mu) \subset L^2(H,\mu)$ is compact (see, e.g., [8]), we can construct a subsequence $\{\varphi_{n_k(\alpha)}^\alpha\}$ which is convergent in $L^2(H,\mu)$ and then another subsequence which is pointwise convergent. This implies that for each $\alpha \in (0, \frac{1}{2}]$, $\{\varphi_{n_k(\alpha)}\}$ is μ-a.e. convergent on $K_\alpha = \{x : g(x) \le 1-2\alpha\}$.

Now, by a standard diagonal procedure we can find a subsequence $\{\varphi_{n_k}\}$ pointwisely convergent as required. □

2.5. *The Sobolev space* $W^{2,2}(K,\nu)$. It is easily seen that for all $h,k\in\mathbb{N}$ the linear operator

$$D_hD_k : C^2(K)\subset L^2(K,\nu)\to L^2(K,\nu),\qquad \varphi\mapsto D_hD_k\varphi,$$

is closable. If φ belongs to the domain of the closure of D_hD_k (which we shall still denote by D_hD_k) we shall say that $D_hD_k\varphi$ belongs to $L^2(K,\nu)$. Now we define $W^{2,2}(K,\nu)$ as the space of all functions $\varphi\in W^{1,2}(K,\nu)$ such that $D_hD_k\varphi\in L^2(K,\nu)$ for all $h,k\in\mathbb{N}$ and

$$\sum_{h,k=1}^{\infty}\int_H |D_hD_k\varphi(x)|^2\nu(dx)<+\infty.$$

$W^{2,2}(K,\nu)$ is a Hilbert space with the inner product

$$\langle\varphi,\psi\rangle_{W^{2,2}(K,\nu)}=\langle\varphi,\psi\rangle_{W^{1,2}(K,\nu)}+\sum_{h,k=1}^{\infty}\int_K D_hD_k\varphi(x)D_hD_k\psi(x)\nu(dx).$$

If $\varphi\in W^{2,2}(K,\nu)$ we can define a Hilbert–Schmidt operator $D^2\varphi(x)$ on K for ν-almost all $x\in K$ by setting

$$\langle D^2\varphi(x)y,z\rangle=\sum_{h,k=1}^{\infty}D_hD_k\varphi(x)\langle y,e_h\rangle\langle z,e_k\rangle\qquad\forall y,z\in H.$$

We show now that if $\varphi\in W^{2,2}(K,\nu)$, then one can define the trace on Σ of $D\varphi$. Similarly to the definition of the trace of φ on Σ we define $|Q^{1/2}\mathbf{n}(y)|\gamma(D\varphi)=\lim_{n\to\infty}|Q^{1/2}\mathbf{n}(y)|\gamma(D\varphi_N)$ in $L^2(\Sigma,\mu_\Sigma)$ for all $\{\varphi_n\}\subset C^2(K)$, $\varphi_n\to\varphi$ in $W^{2,2}(K,\nu)$.

Proposition 2.9 below shows that this trace is well defined.

PROPOSITION 2.9. *Assume that* $\varphi\in W^{2,2}(K,\nu)$. *Then*:

(i) $|Q^{1/2}\mathbf{n}(y)||\gamma(D\varphi)|\in L^2(\Sigma,\mu_\Sigma)$,
(ii) *the following estimate holds*,

$$\begin{aligned}&\int_\Sigma |Q^{1/2}\mathbf{n}(y)|^2|\gamma(D\varphi(y))|^2\mu_\Sigma(dy)\\ &\qquad\le C\biggl(\int_K|D\varphi(x)|^2\nu(dx)+\int_K|\operatorname{Tr}[(D^2\varphi(x))^2]|\nu(dx)\biggr).\end{aligned}\tag{2.19}$$

PROOF. Let $\varphi\in W^{2,2}(K,\nu)$ and let $\{\varphi_n\}\subset C^2(K)$ strongly convergent to φ in $W^{2,2}(K,\nu)$. For $i\in\mathbb{N}$ we apply (2.15) to $D_i\varphi_n$. We have

$$\begin{aligned}&\int_\Sigma |Dg(y)||Q^{1/2}\mathbf{n}(y)|^2|D_i\varphi_n(y)|^2\mu_\Sigma(dy)\\ &\qquad\le C\biggl(\int_K|D_i\varphi_n(x)|^2\nu(dx)+\int_K|DD_i\varphi_n(x)|^2\nu(dx)\biggr).\end{aligned}$$

Summing up on i yields

$$\int_\Sigma |Dg(y)||Q^{1/2}\mathbf{n}(y)|^2|D\varphi_n(y)|^2\mu_\Sigma(dy)$$
$$\leq C\left(\int_K |D\varphi_n(x)|^2\nu(dx) + \sum_{i,j=1}^\infty \int_K |D_jD_i\varphi_n(x)|^2\nu(dx)\right).$$

Then letting $n \to \infty$ we see that $\{Q^{1/2}\mathbf{n}(y)|\gamma(D\varphi_n)\}$ is strongly convergent in $L^2(K,\nu)$ and so (i) and (ii) follow. □

When it will be no danger of confusion we shall simply set $D\varphi$ instead of $\gamma(D\varphi)$.

2.6. *The Sobolev space* $W^{1,2}_A(K,\nu)$. We define $W^{1,2}_A(K,\nu)$ as the space of all functions $\varphi \in W^{1,2}(K,\nu)$ such that

$$\sum_h^\infty \lambda_h \int_H |D_h\varphi(x)|^2\nu(dx) < +\infty.$$

It is easy to see that $W^{1,2}_A(K,\nu)$ is a Hilbert space with the inner product

$$\langle\varphi,\psi\rangle_{W^{1,2}_A(K,\nu)} = \int_K \varphi(x)\psi(x)\nu(dx) + \sum_{h=1}^\infty \lambda_h \int_K D_h\varphi(x)D_h\psi(x)\nu(dx).$$

If $\varphi \in W^{1,2}_A(K,\nu)$ we can define an element of K, $A^{1/2}D\varphi(x)$ for ν-almost all $x \in K$ by setting

$$\langle A^{1/2}D\varphi(x), y\rangle = \sum_{h=1}^\infty \lambda_h D_h\varphi(x)\langle y, e_h\rangle \qquad \forall y \in H.$$

3. The Dirichlet form associated to ν. We define the symmetric Dirichlet form

$$a(\varphi,\psi) = \int_K \langle D\varphi, D\psi\rangle\, d\nu \qquad \forall \varphi,\psi \in D(a) = W^{1,2}(K,\nu) \times W^{1,2}(K,\nu).$$

Since, as seen earlier, D is closed in $L^2(K,\nu)$ we infer that the form a is closed in the sense of [15], page 315, and as a matter of fact the form a is the closure of $a_0(\varphi,\psi) = \int_K \langle D\varphi, D\psi\rangle\, d\nu, \forall\varphi,\psi \in C^1_b(H)$.

By the Lax–Milgram theorem there exists an isomorphism

$$\mathcal{N} : W^{1,2}(K,\nu) \to (W^{1,2}(K,\nu))^*$$

[where $(W^{1,2}(K,\nu))^*$ is the dual space of $W^{1,2}(K,\nu)$] such that

$$\langle\varphi,\psi\rangle + a(\varphi,\psi) = \langle\mathcal{N}\varphi,\psi\rangle \qquad \forall\varphi,\psi \in W^{1,2}(K,\nu).$$

(Here $\langle \cdot, \cdot \rangle$ means the duality between $W^{1,2}(K, \nu)$ and $(W^{1,2}(K, \nu))^*$ which coincides with $\langle \cdot, \cdot \rangle_{L^2(K,\nu)}$ on $L^2(K, \nu)$.) We can identify $L^2(K, \nu)$ with its dual and, so, we have the well-known continuous and dense inclusions

$$W^{1,2}(K, \nu) \subset L^2(K, \nu) \subset (W^{1,2}(K, \nu))^*.$$

Now we define a linear operator $N : D(N) \subset L^2(K, \nu) \to L^2(K, \nu)$ as follows. We say that $\varphi \in D(N)$ if it belongs to $W^{1,2}(K, \nu)$ and that there exists $C > 0$ such that

$$\left| \int_K \langle D\varphi, D\psi \rangle \, d\nu \right| \le C|\psi|_{L^2(K,\nu)} \qquad \forall \psi \in W^{1,2}(K, \nu). \tag{3.1}$$

This inequality implies that $\mathcal{N}\varphi \in L^2(K, \nu)$. Finally, if $\varphi \in D(N)$ we set

$$N\varphi = \tfrac{1}{2}(I - \mathcal{N})\varphi.$$

In other words,

$$\langle N\varphi, \psi \rangle = -\tfrac{1}{2} a(\varphi, \psi) \qquad \forall \varphi, \psi \in W^{1,2}(K, \nu). \tag{3.2}$$

THEOREM 3.1. *Operator N is self adjoint in $L^2(K, \nu)$ and ν is an invariant measure for N*,

$$\int_K N\varphi \, d\nu = 0 \qquad \forall \varphi \in D(N). \tag{3.3}$$

PROOF. By the closedness and symmetry of a it follows that N is closed and symmetric. Moreover, by the Lax–Milgram theorem, applied to symmetric bilinear form $(u, v) \to \lambda\langle u, v \rangle + a(u, v)$, we see that the range $R(\lambda I - N)$ of $\lambda I - N$ coincides with $L^2(K, \nu)$ for all $\lambda > 0$. Notice also that by (3.1)

$$\langle N\varphi, \varphi \rangle = -\tfrac{1}{2}|D\varphi|^2_{L^2(K,\nu)} \qquad \forall \varphi \in D(N). \tag{3.4}$$

As regards (3.3) it is immediate by definition of N. □

It is useful to notice also that for each $f \in L^2(K, \nu)$,

$$\begin{aligned}(\lambda I - N)^{-1} f = \{\varphi : \lambda \langle \varphi, \psi \rangle_{L^2(K,\nu)} &+ \tfrac{1}{2} a(\varphi, \psi) \\ &= \langle f, \psi \rangle_{L^2(K,\nu)}, \forall \psi \in W^{1,2}(K, \nu)\}.\end{aligned}$$

4. The penalized problem. We are here concerned with the penalized equation

$$\begin{cases} dX_\varepsilon(t) + \big(AX_\varepsilon(t) + \beta_\varepsilon(X_\varepsilon(t))\big)\, dt = dW_t, \\ X_\varepsilon(0) = x, \end{cases} \tag{4.1}$$

where $\varepsilon > 0$, and

$$\beta_\varepsilon(x) = \frac{1}{\varepsilon}\big(x - \Pi_K(x)\big) \qquad \forall x \in H.$$

Since β_ε is Lipschitz, (4.1) has a unique mild solution $X_\varepsilon(t,x)$.

The corresponding Kolmogorov operator reads as follows,

$$N_\varepsilon\varphi = L\varphi - \langle \beta_\varepsilon(x), D\varphi\rangle, \qquad \varphi \in \mathcal{E}_A(H),\ \varepsilon > 0, \tag{4.2}$$

where L is the Ornstein–Uhlenbeck operator

$$L\varphi = \tfrac{1}{2}\operatorname{Tr}[D^2\varphi] - \langle x, AD\varphi\rangle, \qquad \varphi \in \mathcal{E}_A(H).$$

It is well known that ν_ε [defined in (2.1)–(2.3)] is an invariant measure for N_ε and that

$$\int_H N_\varepsilon\varphi\psi\, d\nu_\varepsilon = -\frac{1}{2}\int_H \langle D\varphi, D\psi\rangle\, d\nu_\varepsilon \qquad \forall \varphi,\psi \in \mathcal{E}_A(H). \tag{4.3}$$

Moreover, since β_ε is Lipschitz continuous, operator N_ε is essentially m-dissipative in $L^2(H,\nu_\varepsilon)$ (we still denote by N_ε its closure) and $\mathcal{E}_A(H)$ is a core for N_ε see [9].

Section 4.1 below is devoted to prove several estimates for the $(\lambda I - N_\varepsilon)^{-1}f$ where $f \in L^2(H,\nu_\varepsilon)$. Then these estimates are used in Section 4.2 to prove that $(\lambda I - N_\varepsilon)^{-1}f$ converges as $\varepsilon \to 0$ for any $f \in L^2(K,\nu)$ to $(\lambda I - N)^{-1}f$. Moreover we shall end up the section giving sharp informations about the domain of N.

4.1. *Estimates for* $(\lambda I - N_\varepsilon)^{-1}f$. We need a lemma.

LEMMA 4.1. *Let* $\lambda > 0$, $\varphi \in \mathcal{E}_A(H)$ *and set*

$$f_\varepsilon = \lambda\varphi - N_\varepsilon\varphi. \tag{4.4}$$

Then the following estimates hold

$$\int_H \varphi^2\, d\nu_\varepsilon \le \frac{1}{\lambda^2}\int_H f_\varepsilon^2\, d\nu_\varepsilon, \tag{4.5}$$

$$\int_H |D\varphi|^2\, d\nu_\varepsilon \le \frac{2}{\lambda}\int_H f_\varepsilon^2\, d\nu_\varepsilon, \tag{4.6}$$

$$\begin{aligned}\lambda\int_H |D\varphi|^2\, d\nu_\varepsilon + \frac{1}{2}\int_H \operatorname{Tr}[(D^2\varphi)^2]\, d\nu_\varepsilon + \int_H |A^{1/2}D\varphi|^2\, d\nu_\varepsilon \\ + \frac{1}{\varepsilon}\int_{K^c}\langle (I - D\Pi_K(x))D\varphi, D\varphi\rangle \nu_\varepsilon \le 4\int_H f_\varepsilon^2\, d\nu_\varepsilon.\end{aligned} \tag{4.7}$$

PROOF. Multiplying both sides of (4.4) by φ, taking into account (4.3) and integrating in ν_ε over H, yields

$$\lambda\int_H \varphi^2\, d\nu_\varepsilon + \frac{1}{2}\int_H |D\varphi|^2\, d\nu_\varepsilon = \int_H \varphi f_\varepsilon\, d\nu_\varepsilon. \tag{4.8}$$

Now (4.5) and (4.6) follow easily from the Hölder inequality. To prove (4.7) let us differentiate in the direction of e_k both sides of (4.4). We obtain

$$\lambda D_k\varphi - N_\varepsilon D_k\varphi + \alpha_k D_k\varphi + \frac{1}{\varepsilon}\sum_{h=1}^{\infty}\big(\delta_{h,k} - \langle \Pi_K(x)e_h, e_k\rangle\big)D_h\varphi = D_k f_\varepsilon.$$

Multiplying both sides of (4.4) by $D_k\varphi$, taking into account (4.3), integrating in ν_ε over H and then summing up over k, yields

$$\begin{aligned}\lambda\int_H |D\varphi|^2\,d\nu_\varepsilon + \frac{1}{2}\int_H \mathrm{Tr}[(D^2\varphi)^2]\,d\nu_\varepsilon + \int_H |A^{1/2}D\varphi|^2\,d\nu_\varepsilon \\ + \frac{1}{\varepsilon}\int_{K^c}\langle\big(I - D\Pi_K(x)\big)D\varphi, D\varphi\rangle\,d\nu_\varepsilon = \int_H \langle D\varphi, Df_\varepsilon\rangle\,d\nu_\varepsilon.\end{aligned} \tag{4.9}$$

Noting finally that, again in view of (4.3),

$$\int_H \langle D\varphi, Df_\varepsilon\rangle\,d\nu_\varepsilon = 2\int_H f_\varepsilon^2\,d\nu_\varepsilon - 2\lambda\int_H f_\varepsilon\varphi\,d\nu_\varepsilon \le 4\int_H f_\varepsilon^2\,d\nu_\varepsilon,$$

the conclusion follows. □

Now we are able to prove the announced estimates.

PROPOSITION 4.2. *Let $\lambda > 0$, $f \in L^2(H, \nu_\varepsilon)$ and let φ_ε be the solution of the equation*

$$\lambda\varphi_\varepsilon - N_\varepsilon\varphi_\varepsilon = f. \tag{4.10}$$

Then $\varphi_\varepsilon \in W^{2,2}(H, \nu_\varepsilon)$, $A^{1/2}D\varphi_\varepsilon \in L^2(H, \nu_\varepsilon)$ and the following estimates hold

$$\int_H \varphi_\varepsilon^2\,d\nu_\varepsilon \le \frac{1}{\lambda^2}\int_H f^2\,d\nu_\varepsilon, \tag{4.11}$$

$$\int_H |D\varphi_\varepsilon|^2\,d\nu_\varepsilon \le \frac{2}{\lambda}\int_H f^2\,d\nu_\varepsilon, \tag{4.12}$$

$$\begin{aligned}\lambda\int_H |D\varphi_\varepsilon|^2\,d\nu_\varepsilon + \frac{1}{2}\int_H \mathrm{Tr}[(D^2\varphi_\varepsilon)^2]\,d\nu_\varepsilon + \int_H |A^{1/2}D\varphi_\varepsilon|^2\,d\nu_\varepsilon \\ + \frac{1}{\varepsilon}\int_{K^c}\langle\big(I - D\Pi_K(x)\big)D\varphi_\varepsilon, D\varphi_\varepsilon\rangle\,d\nu_\varepsilon \le 4\int_H f^2\,d\nu_\varepsilon.\end{aligned} \tag{4.13}$$

PROOF. Inequality (4.11) is obvious since N_ε is dissipative. Let us prove (4.12). Let $\lambda > 0$, $f \in L^2(H, \nu_\varepsilon)$ and let φ_ε be the solution of (4.10). Since $\mathcal{E}_A(H)$ is a core for N_ε there exists a sequence $\{\varphi_{\varepsilon,n}\}_{n\in\mathbb{N}} \subset \mathcal{E}_A(H)$ such that

$$\lim_{n\to\infty}\varphi_{\varepsilon,n} \to \varphi_\varepsilon, \qquad \lim_{n\to\infty} N_\varepsilon\varphi_{\varepsilon,n} \to N_\varepsilon\varphi_\varepsilon \qquad \text{in } L^2(H, \nu_\varepsilon).$$

Set $f_{\varepsilon,n} = \lambda\varphi_{\varepsilon,n} - N_\varepsilon\varphi_{\varepsilon,n}$. Clearly, $f_{\varepsilon,n} \to f$ as $n \to \infty$ in $L^2(H, \nu_\varepsilon)$.

We claim that $\varphi_\varepsilon \in W^{1,2}(H, \nu_\varepsilon)$ and that

$$\lim_{n\to\infty} D\varphi_{\varepsilon,n} \to D\varphi_\varepsilon \qquad \text{in } L^2(H, \nu_\varepsilon; H).$$

Let in fact $m, n \in \mathbb{N}$, then by (4.6) it follows that

$$\int_H |D\varphi_{\varepsilon,n} - D\varphi_{\varepsilon,m}|^2 \, d\nu_\varepsilon \le \frac{1}{\lambda^2} \int_H |f_{\varepsilon,n} - f_{\varepsilon,m}|^2 \, d\nu_\varepsilon.$$

Therefore the sequence $\{\varphi_{\varepsilon,n}\}_{n\in\mathbb{N}}$ is Cauchy in $W^{1,2}(H, \nu_\varepsilon)$ and the claim follows. Estimate (4.13) can be proved similarly. □

We conclude this section with an integration by parts formula needed later. We set

$$V = \{\psi \in W^{1,2}(K, \nu) : |Q^{1/2}\mathbf{n}|\psi \in L^2(\Sigma, \mu_\Sigma)\}.$$

LEMMA 4.3. *Let $\varphi \in D(N_\varepsilon)$ and $\psi \in V$. Then the following identity holds.*

$$\begin{aligned} \int_K N_\varepsilon \varphi \psi \, d\nu &= -\frac{1}{2} \int_K \langle D\varphi, D\psi \rangle \, d\nu \\ &\quad + \frac{1}{\mu(K)} \int_\Sigma \langle \gamma(D\varphi), \mathbf{n}(y) \rangle \psi \, d\mu_\Sigma. \end{aligned} \tag{4.14}$$

PROOF. Taking in account that $\mathcal{E}_A(H)$ is a core for N_ε, it is sufficient to prove (4.14) for $\varphi \in \mathcal{E}_A(H)$. By the basic integration by parts formula we deduce, for any $i \in \mathbb{N}$ and $\psi \in V$ that

$$\begin{aligned} \int_K D_i\varphi D_i\psi \, d\nu &= -\int_K D_i^2 \varphi \psi \, d\nu + \frac{1}{\mu(K)} \int_\Sigma \gamma(D_i\varphi)(\mathbf{n}(y))_i \psi \, d\mu_\Sigma \\ &\quad + \frac{1}{\lambda_i} \int_K x_i D_i \varphi \psi \, d\nu. \end{aligned}$$

Now, summing up on i yields

$$\begin{aligned} \int_K \langle D\varphi, D\psi \rangle \, d\nu &= -\int_K \mathrm{Tr}[D^2\varphi]\psi \, d\nu + \frac{1}{\mu(K)} \int_\Sigma \langle \gamma(D\varphi), \mathbf{n}(y) \rangle \psi \, d\mu_\Sigma \\ &\quad + 2\int_K \langle x, AD\varphi \rangle \psi \, d\nu. \end{aligned}$$

That is nothing else but (4.14). □

4.2. *Convergence of* $\{\varphi_\varepsilon\}$. We are going to show that the sequence $\{\varphi_\varepsilon\}$ is convergent in $L^2(K, \nu)$. We first note that for $f \in C_b(H)$ we have

$$\varphi_\varepsilon(x) = \mathbb{E} \int_0^\infty e^{-\lambda t} f(X_\varepsilon(t, x)) \, dt \qquad \forall x \in H. \tag{4.15}$$

Now, by a standard argument it follows that from (4.15) that if $f \in C_b^1(H)$ we have

$$\sup_{x\in H} |D\varphi_\varepsilon(x)| \le \frac{1}{\lambda}\|Df\|_{C_b(H)} \qquad \forall \varepsilon, \lambda > 0. \tag{4.16}$$

Theorem 4.4 is the main result of this section.

THEOREM 4.4. *Let $\lambda > 0$, $f \in L^2(K,\nu)$ and let φ_ε be the solution of* (4.10). *Then $\{\varphi_\varepsilon\}$ is strongly convergent in $L^2(K,\nu)$ to $\varphi = (\lambda I - N)^{-1} f$ where N is defined by* (3.1).

Moreover, the following statements hold:

(i) $\lim_{\varepsilon\to 0} D\varphi_\varepsilon = D\varphi$ *in* $L^2(K,\nu;H)$,
(ii) $\varphi \in W_A^{1,2}(H,\nu) \cap W^{2,2}(K,\nu)$,
(iii) φ *fulfills the Neumann condition*

$$\frac{d\varphi}{dn}(x) = \langle D\varphi(x), \mathbf{n}(x)\rangle = 0 \qquad \text{on } \Sigma, \tag{4.17}$$

where $\langle D\varphi(x), \mathbf{n}(x)\rangle$ is defined by Proposition 2.9 *and $|Q^{1/2}\mathbf{n}(x)|\langle D\varphi(x), \mathbf{n}(x)\rangle \in L^2(\Sigma, \mu_\Sigma)$.*

PROOF. Without danger of confusion we shall denote again by f the restriction $f|_K$ of f to K. In fact each $f \in L^2(K,\nu)$ can be extended by 0 outside K to a function in $L^2(H,\nu)$. By this convention, everywhere in the sequel $(\lambda I - N)^{-1} f$ for $f \in L^2(H,\nu)$ means $(\lambda I - N)^{-1} f|_K$.

Step 1. We have

$$\lim_{\varepsilon\to 0} \varphi_\varepsilon = (\lambda I - N)^{-1} f \qquad \text{in } L^2(K,\nu). \tag{4.18}$$

In fact by (4.11), (4.12) and the compactness of the embedding of $W^{1,2}(K,\nu)$ in $L^2(K,\nu)$ it follows that there exist a sequence $\{\varepsilon_k\} \to 0$ and $\varphi \in W^{1,2}(K,\nu)$ such that

$$\begin{aligned} \varphi_{\varepsilon_k} &\to \varphi \qquad \text{strongly in } L^2(K,\nu),\\ D\varphi_{\varepsilon_k} &\to D\varphi \qquad \text{weakly in } L^2(K,\nu). \end{aligned}$$

Let $\psi \in C_b^1(H)$ and consider the identity

$$\frac{1}{2}\int_H \langle D\varphi_\varepsilon, D\psi\rangle\, d\nu_\varepsilon = \int_H (f - \lambda\varphi_\varepsilon)\psi\, d\nu_\varepsilon,$$

which is equivalent to

$$\frac{1}{2}\int_K \langle D\varphi_\varepsilon, D\psi\rangle\, d\nu + \frac{1}{2}\int_{K^c} \langle D\varphi_\varepsilon, D\psi\rangle\, d\nu_\varepsilon = \int_H (f - \lambda\varphi_\varepsilon)\psi\, d\nu_\varepsilon. \tag{4.19}$$

Since, we have

$$\left|\int_{K^c} \langle D\varphi_\varepsilon, D\psi\rangle \, d\nu_\varepsilon\right|^2 \le \int_H |D\varphi_\varepsilon|^2 \, d\nu_\varepsilon \int_{K^c} |D\psi|^2 \, d\nu_\varepsilon$$
$$\le \frac{2}{\lambda}\int_H f^2 \, d\nu_\varepsilon \int_{K^c} |D\psi|^2 \, d\nu_\varepsilon \to 0$$

as $\varepsilon \to 0$, we deduce, letting $\varepsilon \to 0$ in (4.19) that

$$\frac{1}{2}\int_K \langle D\varphi, D\psi\rangle \, d\nu = \int_K (f - \lambda\varphi)\psi \, d\nu \qquad \forall \psi \in C_b^1(H).$$

Obviously, this identity extends to all $\psi \in W^{1,2}(H, \nu)$, which implies that $\varphi = (\lambda I - N)^{-1} f$ and that $\varphi_\varepsilon \to \varphi$ strongly in $L^2(K, \nu)$.

Step 2. We have

$$\lim_{\varepsilon\to 0} D\varphi_\varepsilon = D\varphi \qquad \text{in } L^2(K, \nu; K).$$

We first assume that $f \in C_b^1(H)$. Let us start from the identity (4.8),

$$\frac{1}{2}\int_H |D\varphi_\varepsilon|^2 \, d\nu_\varepsilon = \int_K (\lambda\varphi_\varepsilon - f)\varphi_\varepsilon \, d\nu_\varepsilon, \tag{4.20}$$

which implies

$$\begin{aligned}\lim_{\varepsilon\to 0} \frac{1}{2}\int_H |D\varphi_\varepsilon|^2 \, d\nu_\varepsilon &= \int_K (\lambda\varphi - f)\varphi \, d\nu \\ &= -\langle N\varphi, \varphi\rangle = \frac{1}{2}\int_K |D\varphi|^2 \, d\nu.\end{aligned} \tag{4.21}$$

Here we have used the fact that

$$\lim_{\varepsilon\to 0}\int_{K^c} |D\varphi_\varepsilon|^2 \, d\nu_\varepsilon(x) = 0,$$

which follows taking into account (4.16).

Therefore there exists a sequence $\{\varepsilon_k\}$ such that

$$\varphi_{\varepsilon_k} \to \varphi, \qquad \text{strongly in } L^2(K, \nu),$$
$$D\varphi_{\varepsilon_k} \to D\varphi, \qquad \text{weakly in } L^2(K, \nu; H),$$
$$\lim_{k\to\infty}\int_K |D\varphi_{\varepsilon_k}|^2 \, d\nu = \int_K |D\varphi|^2 \, d\nu.$$

This implies that $D\varphi_{\varepsilon_k} \to D\varphi$ strongly in $L^2(K, \nu; H)$.

We finally assume that $f \in L^2(H, \nu)$. Since $C_b^1(H)$ is dense in $L^2(K, \nu)$, there exists a sequence $\{f_n\} \subset C_b^1(H)$ strongly convergent in $L^2(K; \nu)$ to f. Set $\varphi_{n,\varepsilon} = (\lambda I - N_\varepsilon)^{-1} f_n$. By (4.12) we have

$$\int_H |D\varphi_\varepsilon - D\varphi_{n,\varepsilon}|^2 \, d\nu_\varepsilon \le \frac{2}{\lambda}\int_K |f - f_n|^2 \, d\nu,$$

which implies

$$\int_K |D\varphi_\varepsilon - D\varphi_{n,\varepsilon}|^2\,d\nu \le \frac{2}{\lambda}\int_K |f - f_n|^2\,d\nu.$$

So, again $D\varphi_{\varepsilon_k} \to D\varphi$ strongly in $L^2(K,\nu;H)$.

Step 3. We have

$$\varphi \in W_A^{1,2}(K,\nu;H) \cap W^{2,2}(K;\nu). \tag{4.22}$$

By estimate (4.13) we have that $\{\varphi_\varepsilon\}$ is bounded in $W^{2,2}(K,\nu)$. Therefore there is a subsequence, still denoted $\{\varphi_\varepsilon\}$ which converges to φ in $W^{2,2}(K,\nu)$. In the same way we see that $\varphi \in W_A^{1,2}(K,\nu;H)$.

Step 4. Checking the Neumann condition for φ.

From (4.14) we get

$$\int_K N_\varepsilon\varphi_\varepsilon\psi\,d\nu = -\frac{1}{2}\int_K \langle D\varphi_\varepsilon, D\psi\rangle\,d\nu + \frac{1}{\mu(K)}\int_\Sigma \psi\langle\gamma(D\varphi_\varepsilon),\mathbf{n}(y)\rangle\,d\mu_\Sigma.$$

Recalling that $N_\varepsilon\varphi_\varepsilon = \lambda\varphi_\varepsilon - f \longrightarrow \lambda\varphi - f = N\varphi$ in $L^2(K,\nu)$ and that $|Q^{1/2}\mathbf{n}(y)|\langle\gamma(D\varphi_\varepsilon),\mathbf{n}(y)\rangle \to |Q^{1/2}\mathbf{n}(y)|\langle\gamma(D\varphi),\mathbf{n}(y)\rangle$ in $L^2(\Sigma,\mu_\Sigma)$ by Proposition 2.9, by (i) and by (3.4) we obtain

$$\int_\Sigma \langle\gamma(D\varphi),\mathbf{n}(y)\rangle\psi\,d\mu_\Sigma = 0 \qquad \forall\psi\in V,$$

which implies (4.17) as claimed. [The set $\{\gamma(\psi):\psi\in V\}$ is dense in $L^2(\Sigma,\mu_\Sigma)$.] This completes the proof. □

In particular, taking into account that $D(N)$ is equal to the range of $(\lambda I - N)^{-1}$ we derive by Theorem 4.4 the following result, which gives a sharp information on the structure of the domain of N.

COROLLARY 4.5. *We have*

$$D(N) \subset \left\{\varphi \in W_A^{1,2}(H,\nu)\cap W^{2,2}(K,\nu) : \frac{d}{dn}\varphi(x) = 0 \text{ on } \Sigma\right\}. \tag{4.23}$$

We notice also that for $\varphi \in D(N)$ regular $N\varphi$ is the classical elliptic differential operator in H. More precisely, we have

COROLLARY 4.6. *If* $\operatorname{Tr} D^2\varphi \in L^2(K,\nu)$, $\langle x, AD\varphi\rangle \in L^2(K,\nu)$ *and* $\frac{d\varphi}{dn}(y) = 0, \forall y\in\Sigma$ *then* $\varphi\in D(N)$ *and*

$$N\varphi(x) = \tfrac{1}{2}\operatorname{Tr} D^2\varphi - \langle x, AD\varphi\rangle \qquad \forall x \in \mathring{K}. \tag{4.24}$$

PROOF. By integration by parts formula (2.11) we see that

$$\int_K \langle D\varphi, D\psi\rangle \nu(dx) = -\int_K \left(\frac{1}{2}\operatorname{Tr} D^2\varphi - \langle x, AD\varphi\rangle\right)\nu(dx) + \frac{1}{\mu(K)}\int_\Sigma \psi(y)\frac{d\varphi}{dn}(y)\mu_\Sigma(dy) \qquad \forall \psi \in V, \tag{4.25}$$

which in virtue of (iv) and (3.2) implies (4.24) as claimed. □

REMARK 4.7. We conjecture that in Corollary 4.5 one has equality in relation (4.23), but we failed to prove it. This happens when N is replaced by the Ornstein–Uhlenbeck generator L and ν by the Gaussian measure μ (see [9]).

Notice also that if $\varphi \in D(N)$ we cannot conclude that $\operatorname{Tr} D^2\varphi \in L^2(K,\nu)$ and $\langle x, AD\varphi\rangle \in L^2(K,\nu)$. This is obviously true if H is finite-dimensional.

5. Perturbation results.

5.1. *Perturbation by a regular gradient.* Let us consider the stochastic differential inclusion,

$$\begin{cases} dX(t) + \big(AX(t) + DV(X(t)) + N_K(X(t))\big)\,dt \ni dW(t), \\ X(0) = x, \end{cases} \tag{5.1}$$

where A, K and W are as before and $V : H \to \mathbb{R}$ is a C^2 function such that $DV \in C_b^1(H;H)$.

Let us introduce a probability measure $\zeta \in \mathcal{P}(K)$ by setting

$$\zeta(dx) = Z_\zeta^{-1} e^{-2V(x)}\nu(dx),$$

where

$$Z_\zeta = \int_K e^{-2V(y)}\nu(dy).$$

Arguing as in the proof of (2.11), we can show the following integration by parts formula.

THEOREM 5.1. *Let $\varphi \in C_b^1(H)$. Then for any $z \in H$ we have*

$$\begin{aligned} \int_K \langle D\varphi(x), Q^{1/2}z\rangle \zeta(dx) &= \int_K \varphi(x)\langle DV(x), Q^{1/2}z\rangle\zeta(dx) \\ &\quad + \frac{1}{2\mu(K)Z_\zeta}\int_\Sigma \varphi(y)\langle \mathbf{n}(y), Q^{1/2}z\rangle e^{-2U(y)}\mu_\Sigma(dy) \\ &\quad + \int_K W_z(x)\varphi(x)\zeta(dx). \end{aligned} \tag{5.2}$$

Now all considerations of Sections 2, 3 and 4 can be easily generalized. In particular, estimate (4.7) reads as follows

$$
\begin{aligned}
&\lambda \int_H |D\varphi|^2 \, d\zeta_\varepsilon + \frac{1}{2} \int_H \mathrm{Tr}[(D^2\varphi)^2] \, d\zeta_\varepsilon + \int_H |A^{1/2} D\varphi|^2 \, d\zeta_\varepsilon \\
&\quad + \int_H \langle D^2 V \cdot D\varphi, D\varphi \rangle \, d\zeta_\varepsilon + \frac{1}{\varepsilon} \int_{K^c} \langle (I - D\Pi_K(x)) D\varphi, D\varphi \rangle \, d\zeta_\varepsilon \\
&\quad \le 4 \int_H f_\varepsilon^2 \, d\zeta_\varepsilon.
\end{aligned} \tag{5.3}
$$

In conclusion, we arrive at the following result.

THEOREM 5.2. *The operator N (defined as in Section 3 with the Dirichlet form induced by ζ) is self adjoint in $L^2(K, \zeta)$ and ζ is an invariant measure for N,*

$$
\int_K N\varphi \, d\zeta = 0 \qquad \forall \varphi \in D(N). \tag{5.4}
$$

Moreover, we have

$$
D(N) \subset \left\{ \varphi \in W_A^{1,2}(H, \zeta) \cap W^{2,2}(K, \zeta) : \frac{d}{dn} \varphi(x) = 0 \text{ on } \Sigma \right\}. \tag{5.5}
$$

(Details are omitted.)

5.2. *Perturbation by a bounded Borel drift.* Let $F : H \to H$ be bounded and Borel and consider the stochastic differential inclusion,

$$
\begin{cases} dX(t) + \big(AX(t) + F(X(t)) + N_K(X(t))\big)\, dt \ni dW(t), \\ X(0) = x. \end{cases} \tag{5.6}
$$

Let moreover G be the linear operator in $L^2(K, \nu)$ defined as

$$
G\varphi = N\varphi + \langle F(x), D\varphi \rangle, \qquad \varphi \in D(N). \tag{5.7}
$$

PROPOSITION 5.3. *G is the infinitesimal generator of a strongly continuous compact semigroup Q_t on $L^2(K, \nu)$. Moreover its resolvent $(\lambda I - G)^{-1}$ is given by*

$$
(\lambda I - G)^{-1} = (\lambda I - N)^{-1}(1 - T_\lambda)^{-1}, \qquad \lambda > \lambda_0, \tag{5.8}
$$

where

$$
\lambda_0 = 2\|F\|_0^2 = 2 \sup_{x \in H} |F(x)|^2 \tag{5.9}
$$

and

$$
T_\lambda \psi(x) = \langle F(x), D(\lambda I - N)^{-1} \psi(x) \rangle, \qquad \psi \in L^2(K, \nu), \ x \in K. \tag{5.10}
$$

PROOF. Let $\lambda > 0$, $f \in L^2(K, \nu)$. Consider the equation

$$\lambda\varphi - N\varphi - \langle F(x), D\varphi\rangle = f. \tag{5.11}$$

Setting $\psi = \lambda\varphi - N\varphi$ (5.11) becomes

$$\psi - T_\lambda\psi = f, \tag{5.12}$$

where T_λ is defined by (5.10).

On the other hand, by (4.12) it follows that

$$\int_H |D(\lambda I - N)^{-1}\psi|^2 \, d\nu \le \frac{2}{\lambda}\int_H \psi^2 \, d\nu,$$

so that

$$\|T_\lambda\psi\|_{L^2(H,\mu)} \le \sqrt{\frac{2}{\lambda}}\|F\|_0\|\psi\|_{L^2(H,\mu)}.$$

Therefore if $\lambda > \lambda_0$ (5.11) has a unique solution and the conclusion follows.

Finally, the compactness property of Q_t for $t > 0$ follows from (5.9) and the compactness of operator $(\lambda I - N)^{-1}$. □

We want now to show that operator G possesses an invariant measure ζ absolutely continuous with respect to ν. For this let us consider the adjoint semigroup Q_t^*; we denote by G^* its infinitesimal generator, and by Σ^* the set of all its stationary points:

$$\Sigma^* = \{\varphi \in L^2(K, \nu) : Q_t^*\varphi = \varphi, t \ge 0\}.$$

Though the following lemma is standard, we give a proof, however, for reader's convenience. We shall denote by $\mathbb{1}$ the functions identically equal to 1.

LEMMA 5.4. *Q_t^* has the following properties*:

(i) *For all $\varphi \ge 0$ ν-a.e., one has $Q_t^*\varphi \ge 0$ ν-a.e.*
(ii) *Σ^* is a lattice, that is, if $\varphi \in \Sigma^*$ then $|\varphi| \in \Sigma^*$.*

PROOF. Let $\psi_0 \ge 0$ ν-a.e. Then for all $\varphi \ge 0$ ν-a.e. and all $t > 0$ we have

$$\int_K Q_t\varphi\psi_0 \, d\nu = \int_K \varphi Q_t^*\psi_0 \, d\nu \ge 0.$$

This implies that $\psi_0 \ge 0$ ν-a.e., and (i) is proved.

Let us prove (ii). Assume that $\varphi \in \Sigma^*$, so that $\varphi(x) = Q_t^*\varphi(x)$. Then we have

$$|\varphi(x)| = |Q_t^*\varphi(x)| \le Q_t^*(|\varphi|)(x). \tag{5.13}$$

We claim that

$$|\varphi(x)| = Q_t^*(|\varphi|)(x), \qquad x - \nu \text{ a.s.}$$

Assume by contradiction that there is a Borel subset $I \subset K$ such that $\nu(I) > 0$ and

$$|\varphi(x)| < Q_t^*(|\varphi|)(x) \qquad \forall x \in I.$$

Then we have

$$\int_K |\varphi(x)|\nu(dx) < \int_K Q_t^*(|\varphi|)(x)\nu(dx). \tag{5.14}$$

On the other hand,

$$\int_K Q_t^*(|\varphi|)\,d\mu = \langle Q_t^*(|\varphi|), \mathbb{1}\rangle_{L^2(K,\nu)} = \langle|\varphi|, \mathbb{1}\rangle_{L^2(K,\mu)} = \int_K |\varphi|\,d\mu,$$

which is in contradiction with (5.14). □

The following result is a generalization of a similar result concerning the Ornstein–Uhlenbeck semigroup proved in [8].

PROPOSITION 5.5. *There exists an invariant measure ζ of Q_t which is absolutely continuous with respect to ν. Moreover*

$$\rho := \frac{d\zeta}{d\nu} \in L^2(K, \nu).$$

PROOF. Let $\lambda > 0$ be fixed. Clearly $\mathbb{1} \in D(G)$ and we have $G\mathbb{1} = 0$. Consequently $\frac{1}{\lambda}$ is an eigenvalue of $(\lambda I - G)^{-1}$ since

$$(\lambda I - G)^{-1}\mathbb{1} = \frac{1}{\lambda}\mathbb{1}.$$

Moreover $\frac{1}{\lambda}$ has finite multiplicity because $(\lambda I - G)^{-1}$ is compact. Therefore $((\lambda I - G)^{-1})^*$ is compact as well and $\frac{1}{\lambda}$ is an eigenvalue for $((\lambda I - G)^{-1})^*$. Consequently there exists $\rho \in L^2(K, \nu)$ not identically equal to zero such that

$$\big(((\lambda I - G)^{-1})^*\big)\rho = \frac{1}{\lambda}\rho. \tag{5.15}$$

It follows that $\rho \in D(G)$ and $G^*\rho = 0$. Since Σ^* is a lattice, ρ can be chosen to be nonnegative and such that $\int_K \rho\,d\nu = 1$.

Now set

$$\zeta(dx) = \rho(x)\nu(dx), \qquad x \in K.$$

We claim that ζ is an invariant measure for Q_t. In fact taking the inverse Laplace transform in (5.15) we find that

$$Q_t^*\rho = \rho,$$

which implies that for any $\varphi \in L^2(K, \nu)$,

$$\int_K Q_t\varphi\,d\zeta = \int_K Q_t\varphi\rho\,d\nu = \int_K \varphi Q_t^*\rho\,d\nu = \int_K \varphi\,d\zeta.$$

The proof is complete. □

Notice now that, since $\frac{d\zeta}{d\nu} \in L^2(K,\nu)$ there is a natural inclusion of $L^2(K,\nu)$ in $L^1(K,\zeta)$ so, we can introduce the linear operator in $L^1(K,\zeta)$,

$$N_F : D(N) \subset L^2(K,\nu) \to L^1(K,\zeta), \qquad N_F\varphi := G\varphi. \tag{5.16}$$

This is the final result of the paper.

PROPOSITION 5.6. *Operator N_F defined by* (5.16) *is dissipative in $L^1(K,\zeta)$ and its closure is m-dissipative.*

PROOF. The dissipativity of operator N_F in $L^1(K,\zeta)$ follows from the fact that measure ζ is invariant for N_F and a standard argument; see [11]. Moreover the range of $\lambda I - N_F$ contains $L^2(K,\nu)$ for $\lambda > \lambda_0$ which is dense in $L^1(K,\zeta)$. So, the conclusion follows from the Lumer–Phillips theorem. □

6. An example.

EXAMPLE 6.1. Consider the stochastic equation

$$\begin{cases} dX(t) - \Delta X(t)\,dt + N_K(X(t))\,dt \ni dW_t, & \text{in } (0,\infty)\times\mathcal{O}, \\ X(t) = 0, & \text{on } \partial\mathcal{O}, \\ X(0) = x, & \text{in } \mathcal{O}, \end{cases} \tag{6.1}$$

where $\mathcal{O}$ is a bounded and open interval of $\mathbb{R}$, and

$$K = \{x \in L^2(\mathcal{O}) : \|x\|_{L^2(\mathcal{O})} \le \rho\}.$$

Then the previous results apply with $H = L^2(\mathcal{O})$, $A = -\Delta$, $D(A) = H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$.

Thus the Markov semigroup P_t generated by N in this case is given by $(P_t\varphi_0)(x) = \varphi(t,x)$ where

$$\varphi \in C^1([0,\infty); L^2(L^2(\mathcal{O}),\nu)) \cap C([0,\infty); W^{2,2}(K,\nu)) \cap W_A^{1,2}(K,\nu; L^2(\mathcal{O}))$$

is the solution to infinite-dimensional parabolic equation

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_K \varphi(t,x)\psi(x)\nu(dx) \\ \qquad + \displaystyle\int_K \left(\int_{\mathcal{O}} D\varphi(t,x)(\xi) D\psi(x)(\xi)\,d\xi\right)\nu(dx), & \forall t \ge 0, \\ \varphi(0,x) = \varphi_0(x), & x \in L^2(\mathcal{O}) \end{cases}$$

for all $\psi \in W^{1,2}(K,\nu)$.

A more general case is that where

$$K = \left\{x \in L^2(\mathcal{O}) : \int_{\mathcal{O}} j(x(\xi))\,d\xi \le \rho^2\right\}, \tag{6.2}$$

where $j:\mathbb{R}\to\mathbb{R}$ is a C^∞ function such that $0\le j(r)\le C_1r^2$, $j''(r)\ge C_2>0$, $\forall r\in\mathbb{R}$. In this latter case

$$\Sigma=\left\{x:\int_{\mathcal{O}} j(x(\xi))\,d\xi=\rho^2\right\}\quad\text{and}\quad N_K(x)(\xi)=\{\lambda\nabla j(x(\xi))\}_{\lambda>0}\qquad\forall x\in\Sigma.$$

APPENDIX

Here we shall present for the reader's convenience a few results on co-aerea formula used in Section 2.1, under additional conditions on g, Hypothesis A.1.

A.1. The co-area formula. Let H be a separable Hilbert space and $\mu=N_Q$ a nondegenerate Gaussian measure in H. Let (e_k) be the complete orthonormal basis in H corresponding to the eigenvalues (λ_k), a sequence of positive numbers, that is, $Qe_k=\lambda_ke_k$, $k\in\mathbb{N}$.

Let us recall the integration by parts formula

$$\int_H D_h\varphi\psi\,d\mu=-\int_H D_h\psi\varphi\,d\mu+\frac{1}{\lambda_h}\int_H x_h\varphi\psi\,d\mu \tag{A.1}$$

for any φ bounded and Borel.

We are given a Borel bounded mapping $g:H\to\mathbb{R}$ of class C^2 such that

HYPOTHESIS A.1.

$$\begin{aligned}
I_1&:=\int_H\frac{\mathrm{Tr}[QD^2g(x)]}{|Q^{1/2}Dg(x)|^2}\mu(dx)<\infty,\\
I_2&:=\int_H\frac{\langle D^2g(x)\cdot Q^{1/2}g(x),Q^{1/2}g(x)\rangle}{|Q^{1/2}Dg(x)|^4}\mu(dx)<\infty,\\
I_3&:=\int_H\frac{\langle x,Dg(x)\rangle}{|Q^{1/2}Dg(x)|^2}\mu(dx)<\infty.
\end{aligned}\tag{A.2}$$

REMARK A.2. Let ρ be a nonnegative C^2 real function such that for some $c>0$, $m\in\mathbb{N}$

$$|\rho'(r)|+|\rho''(r)|\le c(1+r^m),\qquad |\rho'(r)|\ge c$$

and set $g(x)=\rho(|x|^2)$. Then we have

$$\begin{aligned}
I_1&=\frac{2\rho''(|x|^2)|Q^{1/2}x|^2+\rho'(|x|^2)\,\mathrm{Tr}\,Q}{\rho'(|x|^2)|Q^{1/2}x|^2},\\
I_2&=\frac{4(2\rho''(|x|^2)+\rho'(|x|^2)}{\rho'(|x|^2)|Q^{1/2}x|^2},\\
I_3&=\frac{|x|^2}{\rho'(|x|^2)|Q^{1/2}x|^2}.
\end{aligned}$$

Then it is not difficult to see that Hypothesis A.1 is fulfilled. Let us check for instance that

$$J_1 : \int_H \frac{1}{|Q^{1/2}x|^2} \mu(dx) < +\infty. \tag{A.3}$$

We have in fact

$$\frac{1}{|Q^{1/2}x|^2} = \int_0^{+\infty} e^{-t|Q^{1/2}x|^2}\, dt,$$

so that

$$\begin{aligned} J_1 &= \int_0^{+\infty} dt \int_H e^{-t|Q^{1/2}x|^2} \mu(dx) = \int_0^{+\infty} \prod_{k=1}^{\infty} \frac{1}{\sqrt{1+2t\lambda_k^2}}\, dt \\ &\le \int_0^{+\infty} \prod_{k=1}^{3} \frac{1}{\sqrt{1+2t\lambda_k^2}}\, dt < +\infty. \end{aligned}$$

We denote by $\mu_g := g_\# \mu$ the law of g on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then for any $\varphi : \mathbb{R} \to \mathbb{R}$ it holds

$$\int_{\mathbb{R}} \varphi(r) \mu_g(dr) = \int_H \varphi(g(x)) \mu(dx). \tag{A.4}$$

We are going to show following [4] that, under Hypothesis A.1, $g_\# \mu \ll \ell$, where ℓ is the Lebesgue measure on $\mathbb{R}$, using the well-known sufficient condition

$$\left| \int_{\mathbb{R}} \varphi'(r) \mu_g(dr) \right| = \left| \int_H \varphi'(g(x)) \mu(dx) \right| \le C \|\varphi\|_0 \qquad \forall \varphi \in C_b^1(H). \tag{A.5}$$

PROPOSITION A.3. *Assume that Hypothesis A.1 is fulfilled. Then* $g_\# \mu = \mu_g \ll \ell$.

PROOF. We claim that

$$\begin{aligned} &\int_H \varphi'(g(x)) \mu(dx) \\ &\qquad = - \int_H \varphi(g(x)) \frac{\operatorname{Tr}[QD^2 g(x)]}{|Q^{1/2} Dg(x)|^2} \mu(dx) \\ &\qquad\quad - 2 \int_H \varphi(g(x)) \frac{\langle D^2 g(x) \cdot Q^{1/2} g(x), Q^{1/2} g(x) \rangle}{|Q^{1/2} Dg(x)|^4} \mu(dx) \\ &\qquad\quad + \int_H \varphi(g(x)) \frac{\langle x, Dg(x) \rangle}{|Q^{1/2} Dg(x)|^2} \mu(dx), \end{aligned} \tag{A.6}$$

which will yield the conclusion.

Since

$$\langle D\varphi(g(x)), QDg(x)\rangle = \varphi'(g(x))|Q^{1/2}Dg(x)|^2,$$

we have

$$\begin{aligned}\int_H \varphi'(g(x))\mu(dx) &= \int_H \frac{1}{|Q^{1/2}Dg(x)|^2}\langle D\varphi(g(x)), QDg(x)\rangle\mu(dx)\\ &= \sum_{k=1}^{\infty}\lambda_k\int_H D_k\varphi(g(x))\frac{D_kg(x)}{|Q^{1/2}Dg(x)|^2}\mu(dx).\end{aligned}$$

Using (A.1) yields

$$\begin{aligned}\int_H \varphi'(g(x))\mu(dx) &= \sum_{k=1}^{\infty}\lambda_k\int_H D_k\varphi(g(x))\frac{D_kg(x)}{|Q^{1/2}Dg(x)|^2}\mu(dx)\\ &= -\sum_{k=1}^{\infty}\lambda_k\int_H \varphi(g(x))D_k\left[\frac{D_kg(x)}{|Q^{1/2}Dg(x)|^2}\right]\mu(dx)\\ &\quad+\sum_{k=1}^{\infty}\int_H x_k\varphi(g(x))\frac{D_kg(x)}{|Q^{1/2}Dg(x)|^2}\mu(dx).\end{aligned}$$

But

$$D_k\left[\frac{D_kg(x)}{|Q^{1/2}Dg(x)|^2}\right] = \frac{D_k^2g(x)}{|Q^{1/2}Dg(x)|^2} - 2\frac{\sum_{j=1}^{\infty}\lambda_jD_kD_jg(x)D_jg(x)}{|Q^{1/2}Dg(x)|^4}.$$

Therefore

$$\begin{aligned}&\int_H \varphi'(g(x))\mu(dx)\\ &\quad= \sum_{k=1}^{\infty}\lambda_k\int_H D_k\varphi(g(x))\frac{D_kg(x)}{|Q^{1/2}Dg(x)|^2}\mu(dx)\\ &\quad= -\sum_{k=1}^{\infty}\lambda_k\int_H \varphi(g(x))\frac{D_k^2g(x)}{|Q^{1/2}Dg(x)|^2}\mu(dx)\\ &\quad= -2\sum_{k=1}^{\infty}\lambda_k\int_H \varphi(g(x))D_kg(x)\frac{\sum_{j=1}^{\infty}\lambda_jD_kD_jg(x)D_jg(x)}{|Q^{1/2}Dg(x)|^4}\mu(dx)\\ &\qquad+\sum_{k=1}^{\infty}\int_H x_k\varphi(g(x))\frac{D_kg(x)}{|Q^{1/2}Dg(x)|^2}\mu(dx).\end{aligned}$$

So, (A.5) follows. □

The following result can be proved similarly.

COROLLARY A.4. *Assume that Hypothesis A.1 is fulfilled and let f be bounded and Borel. Then $\mu_{fg} \ll \ell$.*

A.2. Surface measure. We denote by K the closed set $K = \{g(x) \le 1\}$ and set

$$\Sigma_r = \{g(x) = r\}, \qquad \Sigma = \Sigma_1.$$

We recall the disintegration formula, see, for example, [19, 20]. For any $\varphi : H \to \mathbb{R}$ bounded and Borel we have.

$$\int_H \varphi(x)\mu(dx) = \int_0^{+\infty} \left[\int_{\Sigma_r} \varphi(x) m_r(dx)\right] \mu_g(dr), \tag{A.7}$$

where $(m_r)_{r\ge 0}$ is a family of Borel measures on $[0, +\infty)$ such that the support of m_r is included on Σ_r.

Set

$$\alpha(r) = \int_{\{g\le r\}} d\mu = \mu_g([0, r]).$$

By Proposition A.3 α is a.e. differentiable on $(0, \infty)$. We set

$$\sigma_\mu(\Sigma_r) := \alpha'(r) = \lim_{h\to 0} \frac{1}{2h} \int_{r-h\le g(x)\le r+h} \mu(dx).$$

Now let f bounded and Borel and set

$$\alpha_f(r) = \int_{\{g\le r\}} f\, d\mu = (f\mu)_g([0, r]).$$

Then by Corollary A.4 it follows that α_f is a.e. differentiable. We set

$$\int_{\Sigma_r} f(y)\sigma_{\mu_r}(dy) := \alpha_f'(r) = \lim_{h\to 0} \frac{1}{2h} \int_{r-h\le g(x)\le r+h} f(x)\mu(dx), \qquad \text{a.e. } r > 0.$$

We finally prove.

THEOREM A.5. *Let $f \in B_b(H)$. Then we have*

$$\int_H f(x)\mu(dx) = \int_0^{+\infty} \left[\int_{\Sigma_r} f(\sigma)\sigma_{\mu_r}(d\sigma)\right] dr. \tag{A.8}$$

PROOF. Using the disintegration formula (A.7) we have a.e. on $(0, \infty)$

$$\begin{aligned}\int_{\Sigma_r} f(\sigma)\sigma_{\mu_r}(d\sigma) &=: \lim_{h\to 0} \frac{1}{2h} \int_{r-h\le g(x)\le r+h} f(x)\mu(dx)\\ &= \lim_{h\to 0} \frac{1}{2h} \int_{r-h}^{r+h} \left[\int_{g^{-1}(r)} f(x) m_r(dx)\right] \sigma_{\mu_r}(\Sigma_r)\, dr.\end{aligned}$$

By Lebesgue's theorem we deduce that

$$\int_{\Sigma_r} f(\sigma)\sigma_{\mu_r}(d\sigma) = \int_{g^{-1}(r)} f(x)m_r(dx)\sigma_{\mu_r}(\Sigma_r), \qquad \text{a.e. } r > 0,$$

which yields

$$\int_{g^{-1}(r)} f(x)m_r(dx) = \frac{1}{\sigma_{\mu_r}(\Sigma_r)} \int_{\Sigma_r} f(\sigma)\sigma_{\mu_r}(d\sigma), \qquad \text{a.e. } r > 0.$$

Now the conclusion follows by substituting this into (A.7). □

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