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A VECTOR-VALUED ALMOST SURE INVARIANCE PRINCIPLE FOR HYPERBOLIC DYNAMICAL SYSTEMS

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We prove an almost sure invariance principle (approximation by d-dimensional Brownian motion) for vector-valued Hölder observables of large classes of nonuniformly hyperbolic dynamical systems. These systems include Axiom A diffeomorphisms and flows as well as systems modeled by Young towers with moderate tail decay rates.

In particular, the position variable of the planar periodic Lorentz gas with finite horizon approximates a two-dimensional Brownian motion.

1. Introduction. The scalar almost sure invariance principle (ASIP), or approximation by one-dimensional Brownian motion, is a strong statistical property of sequences of random variables introduced by Strassen [40, 41]. It implies numerous other statistical limit laws including the central limit theorem, the functional central limit theorem, and the law of the iterated logarithm. See [22, 37] and references therein for a survey of consequences of the ASIP.

The scalar ASIP has been shown to hold for large classes of dynamical systems [13, 16, 17, 20, 23, 24, 29, 30, 34]. Chernov and Dolgopyat [9], Problem 1, asked for a proof of the ASIP for $\mathbb{R}^d$-valued observables, and it is this problem that is solved in this paper. Our main result applies to a large variety of dynamical systems, as surveyed in Section 4.

As a secondary matter, we obtain explicit error estimates that depend on the dimension d and the lack of hyperbolicity. Even for $d = 1$, this estimate is better than those in almost all of the above references. The exception is [20] which gives the best available estimate for scalar ASIPs for a restricted class of systems.

1.1. *Statement of the main results.* Throughout, we use "big O" and $\ll$ notation interchangeably, writing $a_N = O(b_N)$ or $a_N \ll b_N$ (as $N \to \infty$) if there is a constant $C \geq 1$ such that $a_N \leq C b_N$ for all $N \geq 1$.

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DEFINITION 1.1. A sequence $\{S_N\}$ of random variables with values in $\mathbb{R}^d$ satisfies a *d-dimensional almost sure invariance principle* (ASIP) if there exists $\lambda > 0$ and a probability space supporting a sequence of random variables S_N^* and a d-dimensional Brownian motion $W(t)$ such that:

(a) $\{S_N; N \geq 1\} =_d \{S_N^*; N \geq 1\}$, and
(b) $S_N^* = W(N) + O(N^{1/2-\lambda})$ as $N \to \infty$ almost everywhere.

For brevity, we write $S_N = W(N) + O(N^{1/2-\lambda})$ a.e. The ASIP for a one-parameter family S_T of $\mathbb{R}^d$-valued random variables is defined similarly, and denoted $S_T = W(T) + O(T^{1/2-\lambda})$ a.e.

REMARK 1.2. The ASIP is said to be *nondegenerate* if the Brownian motion $W(t)$ has nonsingular covariance matrix Σ. For the classes of dynamical systems considered in this paper, the ASIP is nondegenerate for *typical* observables. More precisely, there is a closed subspace Z of infinite codimension in the space of all (piecewise) Hölder $\mathbb{R}^d$-valued observables such that Σ is nonsingular whenever $\phi \notin Z$. (By considering all one-dimensional projections it suffices to consider the case $d = 1$. This is done explicitly in, e.g., [24], Section 4.3.)

AXIOM A (Diffeomorphisms and flows). Our results are most easily stated in the uniformly hyperbolic (Axiom A) context. Let $f : M \to M$ be a C^1 diffeomorphism on a manifold M. A compact f-invariant set $X \subset M$ is *uniformly hyperbolic* if there is a continuous Df-invariant splitting $E^s \oplus E^u$ of the tangent bundle $T_X M$ and constants $C_1 > 0$, $\lambda \in (0, 1)$ such that for all $n \geq 0$ and $x \in X$,

$$\|(Df^n)_x v\| \leq C_1 \lambda^n \|v\|, \qquad v \in E^s, \qquad \|(Df^{-n})_x v\| \leq C_1 \lambda^n \|v\|, \qquad v \in E^u.$$

The subset X is a *uniformly hyperbolic basic set* if in addition (i) it is transitive (there is a dense orbit $\{f^n x_0 : n \geq 0\}$ in X) and (ii) it is locally maximal (there exists a neighborhood U of X such that every f-invariant subset of U is contained in X). Such a basic set is *nontrivial* if it is not a periodic orbit. Analogous definitions hold for flows $f_t : M \to M$ allowing for the one-dimensional direction along the flow. (See [39] or [35], Appendix III, for further details on uniformly hyperbolic diffeomorphisms and flows.)

THEOREM 1.3. *Let $f : M \to M$ be a diffeomorphism with a (nontrivial) uniformly hyperbolic basic set $X \subset M$, and suppose that μ is an equilibrium measure corresponding to a Hölder potential. Let $\phi : X \to \mathbb{R}^d$ be a mean zero Hölder observable with partial sums $S_N = \sum_{n=1}^N \phi \circ f^n$. Then for any $\varepsilon > 0$,*

$$S_N = W(N) + O(N^{\beta+\varepsilon}) \text{ a.e.}, \qquad \text{where } \beta = \frac{2d+3}{4d+7}.$$

(*The improved estimate $\beta = \frac{1}{4}$ holds when $d = 1$* [20].)

An immediate consequence (see, e.g., [16, 31]) is the corresponding result for Axiom A flows.

COROLLARY 1.4. *Let $f_t : M \to M$ be a smooth flow with a (nontrivial) uniformly hyperbolic basic set $X \subset M$, and suppose that μ is an equilibrium measure corresponding to a Hölder potential. Let $\phi : X \to \mathbb{R}^d$ be a mean zero Hölder observable with partial sums $S_T = \int_0^T \phi \circ f_t \, dt$. Then for any $\varepsilon > 0$,*

$$S_T = W(T) + O(T^{\beta+\varepsilon}) \text{ a.e.}, \qquad \text{where } \beta = \frac{2d+3}{4d+7}.$$

(*The improved estimate $\beta = \frac{1}{4}$ holds when $d = 1$* [20, 31].)

REMARK 1.5. Denker and Philipp [16] proved Theorem 1.3 and Corollary 1.4 in the case $d = 1$ (though with a weaker error term).

Nonuniformly hyperbolic systems. Our results apply also to maps $f : M \to M$ that are nonuniformly expanding/hyperbolic in the sense of Young [44, 45]. Roughly speaking, such maps possess a subset $\Lambda \subset M$ and a return time $R : \Lambda \to \mathbb{Z}^+$ such that the induced map $f^R : \Lambda \to \Lambda$ is uniformly hyperbolic. In particular, f satisfies conditions (A1)–(A4) in Section 4.2 and possesses an SRB measure m.

THEOREM 1.6. *Let $f : M \to M$ be a diffeomorphism (possibly with singularities) that is nonuniformly hyperbolic in the sense of Young* [44, 45]. *Assume that the return time function R lies in L^p, $p > 2$. Let $\phi : M \to \mathbb{R}^d$ be a mean zero Hölder observable with partial sums $S_N = \sum_{n=1}^N \phi \circ f^n$. Then there exists $\lambda > 0$ such that*

$$S_N = W(N) + O(N^{1/2-\lambda}) \qquad \text{a.e.}$$

REMARK 1.7. Again, we obtain explicit estimates for the error term in the form $O(N^{\beta+\varepsilon})$, where $\varepsilon > 0$ is arbitrarily small and β depends on $d \geq 1$ and $p > 2$ as follows: for $d \geq 1$, we have

$$\beta = \frac{1/p + B/2}{1+B}, \qquad B = \begin{cases} \dfrac{4pd+6p-4}{p-2}; & 2 < p \leq 3 - \dfrac{1}{2d+3}, \\ 12d+18; & 3 - \dfrac{1}{2d+3} \leq p \leq \dfrac{20d+29}{6d+9}, \\ \dfrac{4d+4}{p-3}; & \dfrac{20d+29}{6d+9} \leq p \leq 4 - \dfrac{1}{2d+3}, \\ 4d+6; & p \geq 4 - \dfrac{1}{2d+3}. \end{cases}$$

In particular, $\beta = \frac{1/p+2d+3}{4d+7}$ for $p \geq 4 - \frac{1}{2d+3}$.

For $d = 1$, we have the improved result $\beta = \frac{1}{2p} + \frac{1}{4}$ for $2 < p \leq 4$ and $\beta = \frac{3}{8}$ for $p \geq 4$.

Again, there is an immediate extension to nonuniformly hyperbolic flows. Suppose that $f : M \to M$ satisfies the assumptions of Theorem 1.6 with $R \in L^p$, $p > 2$, and that f_t is a suspension flow over f with a (uniformly bounded) Hölder roof function. By [31], $\mathbb{R}^d$-valued Hölder observables of the suspension flow satisfy an ASIP of the form $S_T = W(T) + O(T^{\beta+\varepsilon})$ a.e. where β is as in Remark 1.7.

Application to Lorentz gases. The planar periodic Lorentz gas was studied by Sinai [38]. This is a three-dimensional flow with phase space $(\mathbb{R}^2 - \Omega) \times S^1$, where $\Omega \subset \mathbb{R}^2$ is a periodic array of disjoint convex regions with C^3 boundaries. The coordinates are position $q \in \mathbb{R}^2 - \Omega$ and velocity $v \in S^1$. The flow satisfies the finite horizon condition if the time between collisions with $\partial\Omega$ is uniformly bounded.

Let $q(t) \in \mathbb{R}^2$ denote the position at time t of a particle starting at position $q(0)$ pointing in direction $v(0)$. Bunimovich and Sinai [5], see also [6], proved that $q(t)$ satisfies a two-dimensional functional central limit theorem (weak invariance principle) supporting the view of such flows as a deterministic model for Brownian motion. We complete this circle of ideas by proving the strong version of this result.

THEOREM 1.8. *Consider a planar periodic Lorentz gas satisfying the finite horizon condition. Let $\varepsilon > 0$. There is a two-dimensional Brownian motion $W(t)$ with nonsingular covariance matrix such that for almost every initial condition, $q(T) = W(T) + O(T^{7/15+\varepsilon})$.*

REMARK 1.9. (a) A number of authors [8, 29, 34] have independently established scalar ASIPs for one-dimensional projections of $q(t)$. In hindsight, the scalar ASIP for the Lorentz gas follows from earlier work of [20], again with $\beta = \frac{1}{4}$. (We note that the methods of [20] apply in the first place only to the time-reversal of the dynamical system under study. Their applicability here is due to the fact that the class of systems is closed under time-reversal.)

(b) The finite horizon condition is crucial. For infinite horizons, Szász and Varjú [42] prove that $q(t)$ lies in the nonstandard domain of the normal distribution. In particular, the central limit theorem fails, hence the ASIP fails.

1.2. *Consequences of the vector-valued ASIP.* For convenience, we suppose that the Brownian motion in the ASIP is nondegenerate. Coordinates can be chosen on $\mathbb{R}^d$ so that $W(t)$ is a standard d-dimensional Brownian motion with $\Sigma = I_d$. Throughout, the norm on $\mathbb{R}^d$ is taken to be the usual Euclidean norm. The following consequences of the ASIP are summarized in [33], page 233. Here, LIL stands for *law of the iterated logarithm* and the functional LIL stated below is a far-reaching generalization, due to Strassen, of the classical LIL.

PROPOSITION 1.10. *For the dynamical systems to which the results in this paper apply, the following consequences hold (after normalization so that $\Sigma = I_d$):*

- Functional LIL. *Let $C = C([0,1], \mathbb{R}^d)$ be the Banach space of continuous maps $f : [0,1] \to \mathbb{R}^d$ with the supremum norm. Let K be the (compact) set of $f \in C$ absolutely continuous with $f(0) = 0$, $\int_0^1 |f'(t)|^2\, dt \le 1$. Define $f_n(i/n) = S_i/\sqrt{2n \log\log n}$, $i = 0, \dots, n$, and linearly interpolate to obtain $f_n \in C$. Then a.s. the sequence $\{f_n\}$ is relatively compact in C and its set of limit points is precisely K.*
- Upper and lower class refinement of the LIL. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a positive nondecreasing function. Then*

$$P\big(|S_N| > N^{1/2}\phi(N) \text{ i.o.}\big) = 0 \quad \text{or} \quad 1$$

  according to whether $\int_1^\infty \frac{\phi^d(u)}{u} \exp(-\frac{1}{2}\phi^2(u))\, du$ converges or diverges.
- Upper and lower class refinement of Chung's LIL. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a positive nondecreasing function. Then there is a constant c (depending only on d) such that*

$$P\Big(\max_{n \le N} |S_n| < cN^{1/2}\phi^{-1}(N) \text{ i.o.}\Big) = 0 \quad \text{or} \quad 1$$

  according to whether $\int_1^\infty \frac{\phi^2(u)}{u} \exp(-\phi^2(u))\, du$ converges or diverges.
- Central limit theorem and functional central limit theorem.

REMARK 1.11. (a) Berger [3] gives a unified approach to the ASIP for weakly dependent sequences of random variables with values in a real separable Banach space, but with error term $o(\sqrt{N \log\log N})$. It follows from Berger [3], Corollary 4.1, part A.5, and Melbourne and Nicol [29] that the Banach space-valued ASIP formulated in [3], Theorem 3.2, holds for all dynamical systems considered in this paper. In particular, the $\mathbb{R}^d$-valued ASIP holds with error term $o(\sqrt{N \log\log N})$. This error term suffices for the functional LIL, but is inadequate for the upper and lower class refinements and for the (functional) central limit theorem; whereas the error term established in this paper suffices. Indeed this was the original motivation of Jain, Jogdeo and Stout [25] to improve the error term in Strassen's scalar ASIP.

(b) The $\mathbb{R}^d$-valued functional central limit theorem, being a distributional result, can be proved directly under the more general condition $R \in L^2$ in Theorem 1.6: reduce as in this paper to the setting in Section 3 and then apply the method of [20], Section 3.3.

We end this section by discussing briefly the probabilistic methods used in this paper. Strassen's original proof of the scalar ASIP for IIDs and martingales [40, 41] relies heavily on the Skorokhod embedding theorem for scalar stochastic processes. This method was extended to *weakly dependent* sequences of random variables by a number of authors, using *blocking arguments* to reduce to the

martingale case, see [37]. In particular, Philipp and Stout [37], Theorem 7.1, formulated a version of the scalar ASIP which is particularly useful for dynamical systems [16, 23, 29].

Attempts to extend Strassen's proof to the $\mathbb{R}^d$-valued situation were only partially successful [26], and the same is true for the completely different *quantile transform method* of Csörgő and Révész [14]. Eventually, Berkes and Philipp [4] introduced a third method which works in any number of dimensions, and the applicability of this method was extended to weakly dependent sequences by Kuelbs and Philipp [27].

The remainder of the paper is organized as follows: in Section 2, we combine the blocking argument in [37] with the results of [4, 27] to prove a vector-valued ASIP for $\mathbb{R}^d$-valued random variables satisfying certain hypotheses. In Section 3, we verify these hypotheses for Gibbs–Markov maps and derive Theorem 1.3 as a consequence. In Section 4, we first prove the ASIP for nonuniformly expanding maps and then prove Theorems 1.6 and 1.8. We also list numerous other situations to which our results apply, and we mention some open problems regarding time-one maps of flows.

2. A vector-valued ASIP for functions of mixing sequences. In this section, we prove a vector-valued ASIP for $\mathbb{R}^d$-valued random variables satisfying certain hypotheses. This is the vector-valued analogue of [37], Theorem 7.1, though with hypotheses tailored to the dynamical systems setting. (A result of this type is hinted at in Kuelbs and Philipp [27], but it is necessary to work through the details to determine the hypotheses, which were left unstated. In any case, the estimates in (2.2) and (2.4) are not so natural in the probabilistic setting in [27], and partly account for our strong error term.)

2.1. *Statement of the ASIP.* Let $\xi_1, \xi_2, \dots$ be a sequence of real-valued random variables and let $\mathcal{F}_a^b = \sigma\{\xi_n; a \le n \le b\}$. We assume the strong-mixing hypothesis

$$|P(AB) - P(A)P(B)| \le C\tau^n \qquad \text{for all } A \in \mathcal{F}_1^k \text{ and } B \in \mathcal{F}_{k+n}^\infty, \tag{2.1}$$

where $\tau \in (0, 1)$.

Let $p > 2$, and let η_n be a strictly stationary sequence of $\mathcal{F}_n^\infty$-measurable $\mathbb{R}^d$-valued random variables in L^p satisfying $E\eta_n = 0$. We assume the following (backward) Burkholder-type inequality:

$$\left| \max_{1\le\ell\le N} \left| \sum_{n=\ell}^{N} \eta_n \right| \right|_p \le CN^{1/2}. \tag{2.2}$$

Define $\eta_{\ell,n} = \eta_{\ell n} = E(\eta_n | \mathcal{F}_n^{n+\ell})$. We require that

$$|\eta_n - \eta_{\ell n}|_p \le C\tau^\ell. \tag{2.3}$$

Let Σ be a symmetric positive semidefinite $d \times d$ covariance matrix. Given $u \in \mathbb{R}^d$, define $f_N(u) = E \exp(i\langle u, \sum_{n\le N} \eta_n/\sqrt{N}\rangle)$ and $g(u) = \exp(-\frac{1}{2}\langle u, \Sigma u\rangle)$. Assume that there exists $\varepsilon > 0$ and $a, b \in (0, \frac{1}{2}]$ such that

$$|f_N(u) - g(u)| \le CN^{-a} \qquad \text{for all } |u| \le \varepsilon N^b. \tag{2.4}$$

THEOREM 2.1. *Assume hypotheses* (2.1)–(2.4). *Set*

$$\beta = \frac{1/p + B/2}{1+B}, \qquad B = \max\left\{\frac{2d+2}{a}, \frac{1}{b}, 4d+6\right\}$$

and let $\varepsilon > 0$. *Then there is a* d*-dimensional Brownian motion* $W(t)$ *with covariance matrix* Σ *such that* $\sum_{n\le N} \eta_n = W(N) + O(N^{\beta+\varepsilon})$ *a.e.*

In particular, if $a = b = \frac{1}{2}$, *then* $\beta = \frac{1/p+2d+3}{4d+7}$.

Next, we consider the alternative hypothesis:

$$E\left\{\left(\sum_{n\le N} \eta_n\right)\left(\sum_{n\le N} \eta_n\right)^T\right\} = N\Sigma + O(N^{1/2}). \tag{2.5}$$

THEOREM 2.2. *The statement of Theorem* 2.1 *goes through with hypothesis* (2.4) *replaced by hypothesis* (2.5) *and*

$$\beta = \frac{1/p + B/2}{1+B}, \qquad B = \begin{cases} \dfrac{4pd+6p-4}{p-2}; & p \le 3 - \dfrac{1}{2d+3}, \\ 12d+18; & p \ge 3 - \dfrac{1}{2d+3}. \end{cases}$$

REMARK 2.3. (a) For $d = 1$, under the hypotheses of Theorem 2.2, we obtain the improved error estimate

$$\beta = \begin{cases} \dfrac{1}{2p} + \dfrac{1}{4}; & 2 < p \le 4, \\ \dfrac{3}{8}; & p \ge 4. \end{cases}$$

This is proved using a different method, see Appendix.

(b) It is evident from the proof of Theorems 2.1 and 2.2 that the exponential rates in (2.1) and (2.3) can be replaced by sufficiently high polynomial rates. Further relaxing of the assumptions is possible at the cost of obtaining a weaker error estimate.

2.2. *Preliminaries.* The following result of [15, 43] is stated as [37], Lemma 7.2.1.

LEMMA 2.4. *Let $\mathcal{F}, \mathcal{G}$ be σ-fields and $\beta \geq 0$ such that $|P(AB) - P(A) \times P(B)| \leq \beta$ for all $A \in \mathcal{F}$, $B \in \mathcal{G}$. Let $p, r, s > 1$ satisfy $\frac{1}{p} + \frac{1}{r} + \frac{1}{s} = 1$. Suppose that $\xi \in L^r(\mathcal{F})$, $\eta \in L^s(\mathcal{G})$. Then $|E(\xi\eta) - E(\xi)E(\eta)| \leq 10\beta^{1/p}\|\xi\|_r\|\eta\|_s$.*

The next result is due to Dvoretsky [18], see [27], Lemma 2.2.

LEMMA 2.5. *Let $\mathcal{F}, \mathcal{G}$ be σ-fields. Let ξ be a complex-valued $\mathcal{F}$-measurable random variable with $|\xi| \leq 1$. Then $E|E(\xi|\mathcal{G}) - E\xi| \leq 2\pi \sup_{A\in\mathcal{F}, B\in\mathcal{G}} |P(AB) - P(A)P(B)|$.*

The following Gal–Koksma strong law [21] is stated in [37], Theorem A1.

LEMMA 2.6. *Let ξ_j be a sequence of random variables with $E\xi_j = 0$, and let $q > 0$. Suppose that $E|\sum_{j=m}^n \xi_j|^2 \leq n^q - m^q$ for all $n \geq m \geq 1$. For any $\varepsilon > 0$, $\sum_{j=1}^M \xi_j \ll M^{q/2+\varepsilon}$ a.e.*

2.3. *Introduction of the blocks.* In this subsection we assume that hypotheses (2.1)–(2.3) are satisfied. Fix $Q > 2$, $\alpha \in (0, 1)$. Define random variables $y_1, z_1, y_2, z_2, \ldots$ consisting of sums of consecutive $\eta_{\ell(n)n}$ where the jth blocks y_j and z_j consist of $[j^Q]$ and $[j^\alpha]$ such terms, respectively, and throughout the jth blocks $\ell(n) = [\frac{1}{2}j^\alpha]$.

In other words, $y_j = \sum_n \eta_{\ell n}$, where $\ell = [\frac{1}{2}j^\alpha]$ and the sum ranges over $\sum_{i=1}^{j-1}([i^Q] + [i^\alpha]) < n \leq \sum_{i=1}^{j-1}([i^Q] + [i^\alpha]) + [j^Q]$. Similarly for z_j.

Let $\mathcal{L}_a^b = \sigma\{y_j; a \leq j \leq b\}$ and $\tilde{\mathcal{L}}_a^b = \sigma\{z_j; a \leq j \leq b\}$.

LEMMA 2.7. *There exists (a modified) $\tau \in (0, 1)$ such that for all $k, n \geq 1$,*

$$|P(AB) - P(A)P(B)| \ll \tau^{(k+n)^\alpha} \qquad \text{for all } A \in \mathcal{L}_1^k \text{ and } B \in \mathcal{L}_{k+n}^\infty.$$

The same is true for all $A \in \tilde{\mathcal{L}}_1^k$ and $B \in \tilde{\mathcal{L}}_{k+n}^\infty$.

PROOF. Note that $\mathcal{L}_1^k$ is defined using $y_1, \ldots, y_k$, which are defined using $\eta_{\ell n}$ with $\ell \leq [\frac{1}{2}k^\alpha]$, $n \leq \sum_{i=1}^{k-1}([i^Q] + [i^\alpha]) + [k^Q]$. This involves conditioning on ξ_n with $n \leq \sum_{i=1}^{k-1}([i^Q] + [i^\alpha]) + [k^Q] + [\frac{1}{2}k^\alpha]$. Similarly for $\mathcal{L}_{k+n}^\infty$ and we obtain

$$\mathcal{L}_1^k \subset \mathcal{F}_1^{\sum_{i=1}^{k-1}([i^Q]+[i^\alpha])+[k^Q]+[k^\alpha/2]}, \qquad \mathcal{L}_{k+n}^\infty \subset \mathcal{F}_{\sum_{i=1}^{k+n-1}([i^Q]+[i^\alpha])+1}^\infty.$$

Hence by (2.1), $|P(AB) - P(A)P(B)| \leq \tau^N$ where $N = \sum_{i=k+1}^{k+n-1}([i^Q] + [i^\alpha]) + [k^\alpha] - [\frac{1}{2}k^\alpha] + 1$. For all $k, n \geq 1$, we compute that $N \gg (k + n)^\alpha$ as required for the first statement. (Note that the details for the cases $n = 1$ and $n \geq 2$ are slightly different.) The second statement is proved in the same way. □

For $N \ge 1$, let y_{M_N}, z_{M_N} be the pair of blocks that contains $\eta_{\ell(N)N}$. Write

$$y_{M_N} + z_{M_N} = \sum_{j=P_{M_N-1}+1}^{P_{M_N}} \eta_{\ell j}, \qquad \ell = [\tfrac{1}{2} M_N^\alpha].$$

In particular, $P_{M_N-1} < N \le P_{M_N}$, and $P_{M_N} - P_{M_N-1} = [M_N^Q] + [M_N^\alpha] \sim M_N^Q$. The next result is immediate.

PROPOSITION 2.8. *Writing $M = M_N$, we have $M^{1+Q} \sim \sum_{j \le M} j^Q \sim N$. In particular, $P_M - P_{M-1} \sim N^{Q/(1+Q)}$.*

PROPOSITION 2.9. $\sum_{n \ge 1} |\eta_n - \eta_{\ell n}|_p < \infty$.

PROOF. Focusing on the Mth block, and applying (2.3), we obtain $\sum_{P_{M-1} < n \le P_M} |\eta_n - \eta_{\ell(n)n}|_p \ll M^Q \tau^{M^\alpha/2}$ which is summable. □

PROPOSITION 2.10. $|y_j|_p \ll j^{Q/2}$ *and* $|z_j|_p \ll j^{\alpha/2}$.

PROOF. Write $y_j = \sum^* \eta_{\ell n}$, where $\sum^*$ is a sum of length $[j^Q]$. By Proposition 2.9, (2.2) and stationarity, $|y_j|_p \le |\sum^* (\eta_{\ell n} - \eta_n)|_p + |\sum^* \eta_n|_p \ll 1 + j^{Q/2} \ll j^{Q/2}$. Similarly for z_j. □

2.4. *Approximation result.* In this subsection we continue to assume that hypotheses (2.1)–(2.3) are satisfied.

THEOREM 2.11. *Let $\beta = (\frac{1}{p} + \frac{1}{2}Q)/(1+Q)$. For any $\varepsilon > 0$, there exists $\alpha > 0$ such that $\sum_{n \le N} \eta_n - \sum_{j \le M_N} y_j \ll N^{\beta+\varepsilon}$ a.e.*

Begin by writing

$$\sum_{n \le N} \eta_n - \sum_{j \le M_N} y_j = \Bigg(\sum_{n \le P_{M_N}} \eta_n - \sum_{j \le M_N} (y_j + z_j) \Bigg) - \sum_{n=N+1}^{P_{M_N}} \eta_n + \sum_{j \le M_N} z_j.$$

In the next three lemmas, we estimate these three terms (following [37], Lemmas 7.3.2, 7.3.3 and 7.3.4). The result follows by combining these estimates.

LEMMA 2.12. $\sum_{n \le P_{M_N}} \eta_n - \sum_{j \le M_N} (y_j + z_j) \ll 1$ *a.e.*

PROOF. By Proposition 2.9, $\sum_{n \le \infty} |\eta_n - \eta_{\ell n}| < \infty$ a.e. Hence $|\sum_{n \le P_M} \eta_n - \sum_{j \le M} (y_j + z_j)| = |\sum_{n \le P_M} (\eta_n - \eta_{\ell n})| \le \sum_{n \le \infty} |\eta_n - \eta_{\ell n}| \ll 1$ a.e. □

LEMMA 2.13. *Let* $\beta = (\frac{1}{2} + \frac{1}{2}\alpha)/(1+Q)$. *For any* $\varepsilon > 0$, $\sum_{j \le M_N} z_j \ll N^{\beta+\varepsilon}$ *a.e.*

PROOF. We work componentwise, so without loss z_j is $\mathbb{R}$-valued. By Proposition 2.10, we have $|z_j|_p \ll j^{\alpha/2}$ and so $\sum_m^n Ez_j^2 \ll \sum_m^n j^\alpha \ll n^{1+\alpha} - m^{1+\alpha}$. By Lemmas 2.4 and 2.7 (with $\tilde{\tau} = \tau^\varepsilon$, where $\varepsilon = 1 - 2/p$) for all $i < j$,

$$|Ez_i z_j| \ll |z_i|_p |z_j|_p \tilde{\tau}^{j^\alpha} \le (i^\alpha \tilde{\tau}^{i^\alpha} j^\alpha \tilde{\tau}^{j^\alpha})^{1/2},$$

which is summable over $(i, j) \in \mathbb{N}^2$. We have shown that $E(\sum_{j=m}^n z_j)^2 \ll n^{1+\alpha} - m^{1+\alpha}$, for all $1 \le m \le n$. By Lemma 2.6, $\sum_{j \le M} z_j \ll M^\gamma$ a.e. for $\gamma > \frac{1}{2}(1+\alpha)$, and the result follows from Proposition 2.8. □

LEMMA 2.14. *Let* $\beta = (\frac{1}{p} + \frac{1}{2}Q)/(1+Q)$. *For any* $\varepsilon > 0$, $\sum_{n=N+1}^{P_{M_N}} \eta_n \ll N^{\beta+\varepsilon}$ *a.e.*

PROOF. Let $A_M = \max_{P_{M-1}+1 \le N \le P_M} |\sum_{n=N+1}^{P_M} \eta_n|$. By (2.2) and stationarity, $|A_M|_p \ll (P_M - P_{M-1})^{1/2} \ll M^{Q/2}$. Hence

$$P(A_M > M^\gamma) = P(A_M^p > M^{p\gamma}) \ll M^{-p(\gamma - Q/2)},$$

which is summable provided $\gamma > \frac{1}{p} + \frac{1}{2}Q$. By Borel–Cantelli, $A_M \ll M^\gamma$ a.e. and the result follows from Proposition 2.8. □

2.5. *Proof of Theorem* 2.1. Our proof follows the argument of Kuelbs and Philipp [27] which extends Berkes and Philipp [4]. Let $X_j = [j^Q]^{-1/2} y_j$. Note that $\mathcal{L}_1^j$ is an increasing sequence of σ-fields such that X_j is $\mathcal{L}_1^j$-measurable.

Let $a, b \in (0, \frac{1}{2}]$ be as in hypothesis (2.4).

PROPOSITION 2.15. *Let* $\gamma \in (0, bQ)$. *There exists* $\varepsilon > 0$ *such that* $E|E(\exp(i\langle u, X_j\rangle)|\mathcal{L}_1^{j-1}) - \exp(-\frac{1}{2}\langle u, \Sigma u\rangle)| \le C'(j^{-aQ} + j^{\gamma - Q/2})$ *for all* $u \in \mathbb{R}^d$ *satisfying* $|u| \le \varepsilon j^\gamma$.

PROOF. Let $f_N(u) = E\exp(i\langle u, \sum_{n \le N} \eta_n/\sqrt{N}\rangle)$, $g(u) = \exp(-\frac{1}{2}\langle u, \Sigma u\rangle)$, and write

$$\begin{aligned}
&E\{\exp(i\langle u, X_j\rangle)|\mathcal{L}_1^{j-1}\} - g(u) \\
&\quad = \big(E\{\exp(i\langle u, X_j\rangle)|\mathcal{L}_1^{j-1}\} - E\exp(i\langle u, X_j\rangle)\big) \\
&\qquad + \left(E\exp(i\langle u, [j^Q]^{-1/2} y_j\rangle) - E\exp\left(i\left\langle u, [j^Q]^{-1/2} \sum_{n \le [j^Q]} \eta_n \right\rangle\right)\right) \\
&\qquad + \big(f_{[j^Q]}(u) - g(u)\big) \\
&\quad = I + II + III.
\end{aligned}$$

Using Lemmas 2.5 and 2.7, $E|I| \ll \tau^{j^\alpha}$. Also, *III* is estimated by (2.4) so it remains to estimate *II*. Write $y_j = \sum^* \eta_{\ell n}$, where $\sum^*$ has length $[j^Q]$. By stationarity and Proposition 2.9,

$$\begin{aligned}
|II| &= \left| E\left(\exp\left(i\left\langle u, [j^Q]^{-1/2}\sum^* \eta_{\ell n}\right\rangle\right) - \exp\left(i\left\langle u, [j^Q]^{-1/2}\sum^* \eta_n\right\rangle\right)\right)\right| \\
&\le \left| \exp\left(i\left\langle u, [j^Q]^{-1/2}\sum^* (\eta_{\ell n} - \eta_n)\right\rangle\right) - 1\right|_1 \\
&\le \left| \left\langle u, [j^Q]^{-1/2}\sum^* (\eta_{\ell n} - \eta_n)\right\rangle\right|_1 \\
&\le \varepsilon j^\gamma [j^Q]^{-1/2}\left|\sum_{n\ge 1}(\eta_{\ell n} - \eta_n)\right|_1 \ll j^{\gamma - Q/2}
\end{aligned}$$

as required. □

PROPOSITION 2.16. *Let G be the distribution function of $N(0, \Sigma)$. Then $G\{u : |u| > T\} \le e^{-DT^2}$.*

PROOF. This is a straightforward calculation; see, e.g., [4], page 43. □

Let $\lambda_j = C'(j^{-aQ} + j^{\gamma - Q/2})$, $T_j = \varepsilon j^\gamma$, where $\gamma \in (0, bQ)$ is chosen below. By Propositions 2.15 and 2.16, we have

$$E|E\{\exp(i\langle u, X_j\rangle)|\mathcal{L}_1^{j-1}\} - g(u)| \le \lambda_j \qquad \text{for all } |u| \le T_j,$$
$$G\{u : |u| > \tfrac{1}{4}T_j\} \le \delta_j,$$

where $\delta_j = e^{-D' j^{2\gamma}}$. These are the hypotheses of [4], Theorem 1. Defining

$$\alpha_j = 16 d T_j^{-1} \log T_j + 4\lambda_j^{1/2} T_j^d + \delta_j$$

as in [4], we have

$$\alpha_j \ll j^{-\gamma}\log j + j^{d\gamma - aQ/2} + j^{(d+1/2)\gamma - Q/4},$$

which is summable provided $1 < \gamma < \frac{aQ/2-1}{d}$ and $1 < \gamma < \frac{Q/4-1}{d+1/2}$. We take γ slightly larger than 1 (which can be done provided $Q > \frac{1}{b}$) and Q slightly larger than $(2d+2)/a$ and $4d+6$ so that $\alpha_j \ll j^{-(1+\varepsilon)}$.

Applying [4], Theorem 1, we conclude that (passing to a richer probability space) there is a sequence of i.i.d. random variables Y_j with distribution $N(0, \Sigma)$ such that

$$|X_j - Y_j| \ll j^{-(1+\varepsilon)} \qquad \text{a.e.}$$

Let $W(t)$ be a Brownian motion with covariance Σ and define $Y_j^* = [j^Q]^{-1/2} \times (W(h_j) - W(h_{j-1}))$, where $h_j = \sum_{n=1}^{j} [n^Q] \sim j^{1+Q}$. Then $\{Y_j\} =_d \{Y_j^*\}$ and without loss (after passing to a richer probability space), $Y_j = Y_j^*$. We have

$$\sum_{j\le M} y_j = \sum_{j\le M} [j^Q]^{1/2} X_j = \sum_{j\le M} [j^Q]^{1/2}(X_j - Y_j) + \sum_{j\le M} W(h_j) - W(h_{j-1})$$
$$= \sum_{j\le M} [j^Q]^{1/2}(X_j - Y_j) + W(h_M).$$

Now

$$\sum_{j\le M} [j^Q]^{1/2}(X_j - Y_j) \ll \sum_{j\le M} j^{Q/2}\alpha_j \ll \sum_{j\le M} j^{Q/2-1} \ll M^{Q/2}$$
$$\ll N^{Q/(2(1+Q))}.$$

If $h_M > N$, then $h_M - N \le P_M - P_{M-1} \ll M^Q$, whereas if $h_M < N$ then $N - h_M < P_M - h_M = \sum_{j\le M}[j^\alpha] \ll M^{1+\alpha}$. Since $Q > 1 + \alpha$, we obtain $h_M - N \ll N^{Q/(1+Q)}$, and so $W(h_M) = W(N) + O(N^{Q/(2(1+Q))+\varepsilon})$. Combining these estimates with Theorem 2.11 we obtain

$$\sum_{n\le N} \eta_n = \sum_{n\le N} \eta_n - \sum_{j\le M} y_j + \sum_{j\le M} y_j = W(N) + O\big(N^{(1/p+Q/2)/(1+Q)+\varepsilon}\big).$$

Taking Q slightly larger than $B = \max\{\frac{2d+2}{a}, \frac{1}{b}, 4d+6\}$ yields the required result.

2.6. *Proof of Theorem 2.2.* Starting with η_n, $n = 1, 2, \dots$, we write $\eta_{\ell n} = E(\eta_n | \mathcal{F}_n^{n+\ell})$ and construct blocks Y_1, Z_1, Y_2, $Z_2, \dots$ as before, but note that the blocks Y_j, Z_j have nothing to do with the blocks y_j, z_j. In fact, we define Y_j to be blocks of fixed length $[N^c]$ and Z_j to be blocks of fixed length $[N^\delta]$, where $0 < \delta < c < 1$ are specified later. Also, let $\ell = [N^\delta]/2$ and let $M = [N/(N^c + N^\delta)] = N^{1-c} + O(N^{1-2c+\delta})$.

By analogy with Propositions 2.9 and 2.10, we have

$$\sum_{n\ge 1} |\eta_n - \eta_{\ell,n}|_p < \infty, \qquad |Y_j|_p \ll N^{c/2}, \qquad |Z_j|_p \ll N^{\delta/2}. \tag{2.6}$$

Given $u \in \mathbb{R}^d$, define $\sigma_u^2 = \langle \Sigma u, u\rangle$.

PROPOSITION 2.17. *Let $b \in (0, \min\{\frac{c}{4}, (\frac{1}{2} - \frac{1}{p})(1-c)\}]$. There exists $\varepsilon > 0$ such that*

$$\left| \prod_{j\le M} E\big(e^{i\langle u, Y_j/\sqrt{N}\rangle}\big) - e^{-\sigma_u^2/2} \right| \ll e^{-\sigma_u^2/2} \left\{ \frac{|u|^2}{N^{c-\delta}} + \frac{|u|^2}{N^{c/2}} + \frac{|u|^p}{N^{(p/2-1)(1-c))}} \right\}$$

for all $|u| \le \varepsilon N^b$.

PROOF. By (2.5), $|\langle u, \sum_{n\le N}\eta_n\rangle|_2 = N^{1/2}\sigma_u + O(|u|)$. We can write $Y_j = \sum^* \eta_{\ell n}$ where $\sum^*$ is a sum of length $[N^c]$. By (2.6) and stationarity,

$$|\langle u, Y_j\rangle|_2 = \left|\left\langle u, \sum{}^*\eta_n\right\rangle\right|_2 + O(|u|) = \left|\left\langle u, \sum_{n\le[N^c]}\eta_n\right\rangle\right|_2 + O(|u|)$$
$$= N^{c/2}\sigma_u + O(|u|).$$

Hence

$$E(\langle u, Y_j\rangle^2) = N^c\sigma_u^2 + O(N^{c/2}|u|^2).$$

Next, write $E(e^{i\langle u, Y_j/\sqrt{N}\rangle}) = 1 - r$, where

$$r = \tfrac{1}{2}E(\langle u, Y_j\rangle^2)/N + O\big(E(\langle u, Y_j\rangle^p)/N^{p/2}\big)$$
$$= \tfrac{1}{2}\sigma_u^2/N^{1-c} + O(|u|^2/N^{1-c/2}) + O\big(|u|^p/N^{p(1-c)/2}\big)$$

and $r^2 = O(|u|^4/N^{2(1-c)})$. Here, we have used (2.6) and the fact that r is small since $|u| \le \varepsilon N^{(1-c)/2}$. Hence

$$\log E\big(e^{i\langle u, Y_j/\sqrt{N}\rangle}\big) = -r + O(r^2)$$
$$= -\tfrac{1}{2}\sigma_u^2/N^{1-c} + O(|u|^2/N^{1-c/2}) + O\big(|u|^p/N^{p(1-c)/2}\big)$$

and

$$\sum_{j\le M}\log E\big(e^{i\langle u, Y_j/\sqrt{N}\rangle}\big)$$
$$= -\tfrac{1}{2}\sigma_u^2 + O(|u|^2/N^{c-\delta}) + O(|u|^2/N^{c/2}) + O\big(|u|^p/N^{(p/2-1)(1-c)}\big).$$

The last three terms on the right-hand side are small by the constraints on u, so exponentiation yields the result. □

PROPOSITION 2.18. *For all* $u \in \mathbb{R}^d$,

$$\big|E\big(e^{i\langle u, \sum_{n\le N}\eta_n/\sqrt{N}\rangle}\big) - E\big(e^{i\langle u, \sum_{j\le M}Y_j/\sqrt{N}\rangle}\big)\big| \ll |u|(1/N^{1/2-c/2} + 1/N^{c-1/2-\delta/2}).$$

PROOF. Write

$$\sum_{n\le N}\eta_n = \sum_{n\le M([N^c]+[N^\delta])}\eta_n + \sum{}^*\eta_n,$$

where $\sum^*$ has length smaller than $N^c + N^\delta$ and so $|\sum^*\eta_n|_1 \ll |Y_{N+1}|_1 + |Z_{N+1}|_1 \ll N^{c/2} + N^{\delta/2} \ll N^{c/2}$ by (2.6). Further,

$$\sum_{n\le M([N^c]+[N^\delta])}\eta_{\ell,n} = \sum_{j\le M}(Y_j + Z_j).$$

By (2.6), $|Z_j| \ll N^{\delta/2}$ and so $|\sum_{j\le M} Z_j|_1 \ll N^{1-c+\delta/2}$. Combining all of this with (2.6),

$$\left|\sum_{n\le N}\eta_n - \sum_{j\le M} Y_j\right|_1 \ll N^{c/2} + N^{1-c+\delta/2}$$

and the result follows from the estimate $|e^{ix} - e^{iy}| \le |x-y|$. □

PROPOSITION 2.19. *For all $u \in \mathbb{R}^d$,*

$$\left|E\left(\prod_{j\le M} e^{i\langle u, Y_j/\sqrt{N}\rangle}\right) - \prod_{j\le M} E\left(e^{i\langle u, Y_j/\sqrt{N}\rangle}\right)\right| \ll \tau^{N^\delta}$$

for a (modified) $\tau \in (0,1)$.

PROOF. For each j, we have that Y_j is measurable with respect to $\mathcal{F}_1^{j[N^c]+(j-1)[N^\delta]+[N^\delta/2]}$ whereas Y_{j+1} is measurable with respect to $\mathcal{F}_{j[N^c]+j[N^\delta]+1}^{\infty}$. It follows from (2.1) and Lemma 2.4 (with $p=1$, $r=s=\infty$) that $|E(\prod_{j\le q+1} e^{i\langle u, Y_j/\sqrt{N}\rangle}) - E(\prod_{j\le q} e^{i\langle u, Y_j/\sqrt{N}\rangle})E(e^{i\langle u, Y_{q+1}/\sqrt{N}\rangle})| \le 10\tau^{[N^\delta]-[N^\delta/2]+1} \ll \tau^{N^\delta/2}$. Inductively, we obtain

$$\left|E\left(\prod_{j\le M} e^{i\langle u, Y_j/\sqrt{N}\rangle}\right) - \prod_{j\le M} E\left(e^{i\langle u, Y_j/\sqrt{N}\rangle}\right)\right| \ll M\tau^{N^\delta/2} \ll N^{1-c}\tau^{N^\delta/2}$$

as required. □

COROLLARY 2.20. *Let $b \in (0, \min\{\frac{c}{4}, (\frac{1}{2}-\frac{1}{p})(1-c)\}]$. There exists $\varepsilon > 0$ such that*

$$|E\left(e^{i\langle u, \sum_{n\le N}\eta_n/\sqrt{N}\rangle}\right) - e^{-\sigma_u^2/2}| \ll e^{-\sigma_u^2/2}\left\{\frac{|u|^2}{N^{c-\delta}} + \frac{|u|^2}{N^{c/2}} + \frac{|u|^p}{N^{(p/2-1)(1-c)}}\right\} + |u|\left\{\frac{1}{N^{1/2-c/2}} + \frac{1}{N^{c-1/2-\delta/2}}\right\}$$

for all $|u| \le \varepsilon N^b$.

PROOF OF THEOREM 2.2. If $p \ge 3 - \frac{1}{2d+3}$, we take $c = \frac{2}{3}$ and $b = \frac{1}{12d+18}$. Then b satisfies the constraints $b \le \frac{c}{4}$ and $b \le (\frac{1}{2}-\frac{1}{p})(1-c)$ so that Corollary 2.20 applies. It follows that $|E(e^{i\langle u, \sum_{n\le N}\eta_n/\sqrt{N}\rangle}) - e^{-\sigma_u^2/2}| = O(N^b/N^{c-1/2-\delta/2}) = O(N^{-a'+\delta'})$, where $a' = \frac{d+1}{6d+9}$ and $\delta' = \delta/2$ is arbitrarily small. Hence hypothesis (2.4) holds with $b = \frac{1}{12d+18}$ for all $a < \frac{d+1}{6d+9}$. Now apply Theorem 2.1.

If $p \le 3 - \frac{1}{2d+3}$, we take $c = \frac{2pd+3p-2d-4}{2pd+3p-2}$ and $b = \frac{p-2}{4pd+6p-4}$. Again Corollary 2.20 applies and we obtain $|E(e^{i\langle u, \sum_{n\le N}\eta_n/\sqrt{N}\rangle}) - e^{-\sigma_u^2/2}| = O(1/$

$N^{(1/2-1/p)(1-c)}) = O(N^{-a})$, where $a = \frac{1}{p}(p-2)(d+1)$. Hence hypothesis (2.4) holds with these choices of a and b and the result follows from Theorem 2.1. $\square$

3. ASIP for Gibbs–Markov maps. In this section we prove the ASIP for weighted Lipschitz $\mathbb{R}^d$-valued observables of Gibbs–Markov maps [1]. Roughly speaking, these are uniformly expanding maps with countably many inverse branches and good distortion properties. We derive Theorem 1.3 as a consequence.

3.1. *Gibbs–Markov maps.* Let (Λ, m) be a Lebesgue space with a countable measurable partition α. Without loss, we suppose that all partition elements $a \in \alpha$ have $m(a) > 0$. Recall that a measure-preserving transformation $F : \Lambda \to \Lambda$ is a *Markov map* if Fa is a union of elements of α and $F|_a$ is injective for all $a \in \alpha$. Define α' to be the coarsest partition of Λ such that Fa is a union of atoms in α' for all $a \in \alpha$. (So α' is a coarser partition than α.) If $a_0, \dots, a_{n-1} \in \alpha$, we define the n-cylinder $[a_0, \dots, a_{n-1}] = \bigcap_{i=0}^{n-1} F^{-i} a_i$. It is assumed that F and α separate points in Λ (if $x, y \in \Lambda$ and $x \neq y$, then for n large enough there exist distinct n-cylinders that contain x and y).

Let $0 < \beta < 1$. We define a metric d_β on Λ by $d_\beta(x, y) = \beta^{s(x,y)}$ where $s(x, y)$ is the greatest integer $n \geq 0$ such that x, y lie in the same n-cylinder. Define $g = JF^{-1} = \frac{dm}{d(m \circ F)}$ and set $g_k = gg \circ F \cdots g \circ F^{k-1}$.

A Markov map F is *topologically mixing* if for all $a, b \in \alpha$ there exists $N \geq 1$ such that $F^n a \cap b \neq \varnothing$ for all $n \geq N$. A Markov map F is *Gibbs–Markov* if:

(i) *Big images property.* There exists $c > 0$ such that $m(Fa) \geq c$ for all $a \in \alpha$.
(ii) *Distortion.* $\log g|_a$ is Lipschitz with respect to d_β for all $a \in \alpha'$.

Let α_0^{k-1} denote the partition of Λ into length k cylinders $a = [a_0, \dots, a_{k-1}]$. The following result of [2] is stated explicitly in [29], Lemma 2.4(b).

LEMMA 3.1. *Let F be a topologically mixing Gibbs–Markov map. Then $|m(a \cap F^{-(N+k)}b) - m(a)m(b)| \leq C\tau^N m(a)m(b)^{1/2}$ for all $a \in \alpha_0^{k-1}$ and all measurable b.*

3.2. *Weighted Lipschitz observables.* Let $p \in [1, \infty)$. We fix a sequence of weights $R(a) > 0$ satisfying $|R|_p = (\sum_{a \in \alpha} m(a) R(a)^p)^{1/p} < \infty$. Given $\Phi : \Lambda \to \mathbb{R}$ continuous, define $|\Phi|_\beta$ to be the Lipschitz constant of Φ with respect to the metric d_β. Let $\|\Phi\|_\infty = \sup_{a \in \alpha} |\Phi 1_a|_\infty / R(a)$, $\|\Phi\|_\beta = \sup_{a \in \alpha} |\Phi 1_a|_\beta / R(a)$. Let $\mathcal{B}$ consist of the space of weighted Lipschitz functions with $\|\Phi\| = \|\Phi\|_\infty + \|\Phi\|_\beta < \infty$. Note in particular that $R \in \mathcal{B}$ and $\|R\| = 1$. We have the embeddings $\mathrm{Lip} \subset \mathcal{B} \subset L^p \subset L^1$, where Lip is the space of (globally) Lipschitz functions.

LEMMA 3.2. *Let $\Phi \in \mathcal{B}$ with $\int_\Lambda \Phi = 0$. Then $|\Phi - E(\Phi|\alpha_0^{k-1})|_p \leq \|\Phi\|_\beta \times |R|_p \beta^k$ for all $k \geq 1$.*

PROOF (cf. [29], Lemma 2.4(a)). Note that $E(\Phi|\alpha_0^{k-1})$ is constant on partition elements $a \in \alpha_0^{k-1}$ with value $\frac{1}{m(a)}\int_a \Phi\,dm$, and that $|\Phi 1_a - \frac{1}{m(a)}\int_a \Phi\,dm|_\infty \le |\Phi 1_a|_\beta \operatorname{diam}_\beta(a) \le \|\Phi\|_\beta R(a)\beta^k$. Hence, $|\Phi - E(\Phi|\alpha_0^{k-1})|_p^p \le (\|\Phi\|_\beta \beta^k)^p \times \sum_{a\in\alpha_0^{k-1}} R(a)^p m(a) = (\|\Phi\|_\beta \beta^k |R|_p)^p$. □

3.3. *A maximal inequality.* Given a measure-preserving transformation $F: \Lambda \to \Lambda$ of a probability space (Λ, m), the transfer (Perron–Frobenius) operator $L: L^1 \to L^1$ is defined by $\int_\Lambda L\Phi\Psi\,dm = \int_\Lambda \Phi\Psi\circ F\,dm$ for all $\Phi \in L^1$, $\Psi \in L^\infty$. This restricts to an operator on L^p, $1 \le p \le \infty$.

LEMMA 3.3. *Let* $\Phi \in L^p(\Lambda)$, $2 \le p < \infty$, *with* $L\Phi = 0$. *Then* $|\max_{0\le\ell\le N-1} |\sum_{n=\ell}^N \Phi\circ F^n||_p \le CN^{1/2}$.

PROOF. Note that $L = E(\cdot|F^{-1}\mathcal{M})$, where $\mathcal{M}$ is the underlying σ-algebra. By hypothesis the sequence $\{\Phi\circ F^n; n \ge 0\}$ is a reverse martingale difference sequence. Passing to the natural extension we obtain an L^p martingale difference sequence $\{w_n; n \in \mathbb{Z}\}$ such that $\Phi\circ F^n = w_{-n}$. By Burkholder's inequality [7] (which follows from [7], (1.4) and (3.3), and is stated explicitly in [36], (1)), we have $|\max_{1\le k\le N} |\sum_{n=0}^k w_n||_p \le CN^{1/2}$. Setting $\ell = N-k$ and using stationarity,

$$\max_{0\le\ell\le N-1}\left|\sum_{\ell}^{N}\Phi\circ F^n\right| =_d \max_{0\le\ell\le N-1}\left|\sum_{-N+\ell}^{0}\Phi\circ F^n\right| = \max_{1\le k\le N}\left|\sum_{0}^{k} w_n\right|,$$

proving the result. □

3.4. *Quasicompactness and the central limit theorem.* Let $F: \Lambda \to \Lambda$ be a topologically mixing Gibbs–Markov map with transfer operator $L: L^1 \to L^1$. It is well known [1, 29] that L restricts to a bounded operator on weighted Lipschitz observables $\Phi \in \mathcal{B}$ and $L(\mathcal{B}) \subset \text{Lip}$. Moreover $L: \mathcal{B} \to \mathcal{B}$ is *quasicompact*: $L1 = 1$ and the spectral radius of L restricted to $\mathcal{B}_0 = \{\Phi \in \mathcal{B} : \int_\Lambda \Phi\,dm = 0\}$ is strictly less than 1.

We define $\mathcal{B}^d$ to the be the space of $\mathbb{R}^d$ weighted Lipschitz observables, so $\Phi = (\Phi_1, \dots, \Phi_d) \in \mathcal{B}^d$ if and only if $\Phi_i \in \mathcal{B}$ for $i = 1, \dots, d$. Similarly, we define $\mathcal{B}_0^d$. We suppress the superscript for spaces such as L^p and Lip relying on the context.

PROPOSITION 3.4. *Suppose that* $\Phi \in \mathcal{B}_0^d$. *Then there exists* $\Psi \in \mathcal{B}_0^d$ *and* $\chi \in L^\infty$ *such that* $\Phi = \Psi + \chi\circ F - \chi$ *and* $L\Psi = 0$.

PROOF (cf. [29], proof of Corollary 2.3(c)). Define $\chi = \sum_{j=1}^\infty L^j\Phi$. This converges in $\mathcal{B}_0^d$ since the spectral radius of L is less than 1. Since $L(\mathcal{B}^d) \subset \text{Lip}$, we have $\chi \in L^\infty$. By construction, $L\Psi = 0$. □

Suppose that $p \geq 2$. Let $\Phi \in \mathcal{B}_0^d \subset L^2$ and assume that $L\Phi = 0$. Let $S_N = \sum_{n \leq N} \Phi \circ F^n$ and form the $d \times d$ matrix $S_N S_N^T$. We define the covariance matrix $\Sigma = \frac{1}{N} \int_\Lambda S_N S_N^T \, dm = \int_\Lambda \Phi \Phi^T \, dm$.

LEMMA 3.5. *Suppose that $p > 3$. Let $a = (p-3)/2$ for $p < 3 \leq 4$ and $a = \frac{1}{2}$ for $p \geq 4$. There exists $\varepsilon > 0$ such that*

$$\int_\Lambda \exp(i\langle u, S_N\rangle N^{-1/2}) \, dm - \exp\bigl(-\tfrac{1}{2}\langle u, \Sigma u\rangle\bigr) = O(N^{-a})$$

uniformly for $u \in \mathbb{R}^d$ satisfying $|u| \leq \varepsilon N^{1/2}$.

PROOF. We follow a standard argument establishing the central limit theorem with error term for systems with quasicompact transfer operator (see [35], Theorem 4.13, and references therein). Let S^{d-1} denote the unit sphere in $\mathbb{R}^d$. Given $u \in \mathbb{R}^d$, write $u = tv$ where $t \geq 0$ and $v \in S^{d-1}$. Define the twisted transfer operator $L_u : \mathcal{B} \to \mathcal{B}$ by $L_u \Psi = L(e^{i\langle u, \Phi\rangle} \Psi)$. Recall that 1 is an isolated eigenvalue for L. For u small, the spectral radius of L_u is $\exp P(u)$ where P is continuous and $P(0) = 0$.

Now $\int_\Lambda \exp(i\langle u, S_N\rangle N^{-1/2}) \, dm = \int_\Lambda (L_{uN^{-1/2}})^N 1 \, dm$, and since the leading eigenvalue of L_u is isolated, there exists $\gamma \in (0, 1)$ such that

$$\int_\Lambda \exp(i\langle u, S_N\rangle N^{-1/2}) \, dm - \exp(N P(uN^{-1/2})) \ll \gamma^N,$$

uniformly for $|u| \leq \varepsilon N^{1/2}$. Hence it remains to estimate $\exp(N P(uN^{-1/2})) - \exp(-\frac{1}{2}\langle u, \Sigma u\rangle)$.

We claim that P is $p-1$ times differentiable. Suppose for the moment that the claim is correct. If $p > 4$, then it follows from the argument in [35], page 66, that

$$P(u) = -\tfrac{1}{2}\langle u, \Sigma u\rangle - i P_3(v) t^3 + P_4(v, t) t^4,$$

where $P_3(v) \in \mathbb{R}$, $P_4(v, t) \in \mathbb{C}$ are bounded, and hence there exists $\varepsilon > 0$ such that [35], page 67,

$$\exp(N P(uN^{-1/2})) - \exp\bigl(-\tfrac{1}{2}\langle u, \Sigma u\rangle\bigr)\bigl(1 - i P_3(v) t^3 N^{-1/2}\bigr) = O(N^{-1/2}),$$

uniformly for $|u| \leq \varepsilon N^{1/2}$. [In fact, we obtain $O(N^{-1})$ for $p > 5$ and $O(N^{-(p-3)/2})$ for $4 < p < 5$.]

Still assuming the claim, for $p > 3$,

$$P(u) = -\tfrac{1}{2}\langle u, \Sigma u\rangle + E(v, t) t^{p-1},$$

where $E(v, t) \in \mathbb{C}$ is bounded, and hence

$$\exp(N P(uN^{-1/2})) - \exp\bigl(-\tfrac{1}{2}\langle u, \Sigma u\rangle\bigr) = O\bigl(N^{-(p-3)/2}\bigr),$$

uniformly for $|u| \leq \varepsilon N^{1/2}$.

We verify the claim by showing that $u \mapsto P_u$ is C^{p-1}. Write $p > 1 + k + \alpha$ where $k \in \mathbb{N}$ and $\alpha \in [0, 1)$. We show that $u \mapsto L_u$ is $C^{k+\alpha}$, when viewed as a map from $(-\delta, \delta)$ to bounded linear transformations on Lip.

For ease of notation, we consider the case $d = 1$. Recall that $L_u\Psi = L(e^{iu\Phi)}\Psi)$. Here, $\Phi \in \mathcal{B}$ is weighted Lipschitz with respect to the weight $R \in L^p$, and $\Psi \in \text{Lip}$ is globally Lipschitz. It is easily verified that $e^{iu\Phi}\Psi \in \mathcal{B}$ and $\|e^{iu\Phi}\Psi\|_{\mathcal{B}} \le (1 + \|\Phi\|_{\mathcal{B}})\|\Psi\|_{\text{Lip}}$. Moreover, $L : \mathcal{B} \to \text{Lip}$ is a bounded operator with norm C say, so $\|L_u\Psi\|_{\text{Lip}} \le C(1 + \|\Phi\|_{\mathcal{B}})\|\Psi\|_{\text{Lip}}$.

The fact that $L : \mathcal{B} \to \text{Lip}$ is bounded uses only that $R \in L^1$. Hence a similar argument shows that $\Psi \mapsto L_u(\Phi^{p-1}\Psi)$ is bounded (since $e^{iu\Phi}\Phi^{p-1}\Psi$ is weighted Lipschitz with respect to R^p). It follows that we can form the Taylor expansion

$$\begin{aligned} L_{u+h}\Psi &= L_u(e^{ih\Phi}\Psi) \\ &= L_u\Psi + ihL_u(\Phi\Psi) - (h^2/2!)L_u(\Phi^2\Psi) + \cdots \\ &\quad + \big((ih)^k/k!\big)L_u(\Phi^k\Psi) + o(h^k), \end{aligned}$$

establishing C^k differentiability. Similarly, the kth derivative $L_u(\Phi^k\cdot)$ is C^α, completing the verification of the claim. □

3.5. *Statement and proof of ASIP for Gibbs–Markov maps.*

THEOREM 3.6. *Suppose that $F : \Lambda \to \Lambda$ is a topologically mixing Gibbs–Markov map. Define the Banach space $\mathcal{B}^d$ corresponding to weights $R \in L^p$ where $p > 2$. Suppose that $\Phi : \Lambda \to \mathbb{R}^d$ is a mean zero observable in $\mathcal{B}^d$ with partial sums $S_N = \sum_{n=1}^N \Phi \circ F^n$. Then the conclusion of Theorem 1.6 is valid with error term $O(N^{\beta+\varepsilon})$ as in Remark 1.7.*

PROOF. We give the proof for $d \ge 2$. The improved estimate for $d = 1$ follows from Theorem A.5.

By Proposition 3.4, $S_N = \sum_{n=1}^N \Psi \circ F^n + O(1)$ a.e., where $L\Psi = 0$. Hence without loss we may suppose from the outset that $L\Phi = 0$.

Define $\eta_n = \Phi \circ F^n$ and $\xi_n = a_n$. Then $\eta_n = \Phi(\xi_n, \xi_{n+1}, \ldots)$. The assumptions on Φ imply that η_n lies in L^p and $E\eta_n = 0$. Hypotheses (2.1)–(2.3) follow from Lemmas 3.1, 3.3 and 3.2, respectively.

If $p > 3$, then hypothesis (2.4) holds with $b = \frac{1}{2}$ and a given by Lemma 3.2. Applying Theorem 2.1, we compute that $B = 4d + 6$ for $p \ge 4 - \frac{1}{2d+3}$ and $B = \frac{4d+4}{p-3}$ for $3 < p \le 4 - \frac{1}{2d+3}$. In particular, we obtain the required value of β for $p \ge \frac{20d+29}{6d+9}$.

Hypothesis (2.5) is satisfied for general $p > 2$, indeed $E(\sum_{n\le N} \eta_n)^2 = N\Sigma + O(1)$. (See, e.g., [29] when $d = 1$ and [20] for general d.) Applying Theorem 2.2 yields the required value of β for $2 < p \le \frac{20d+29}{6d+9}$. □

PROOF OF THEOREM 1.3. This reduces by standard techniques to a two-sided and then one-sided subshift of finite type. The latter is a special case of a Gibbs–Markov map with finite alphabet, hence $R \in L^\infty$. Theorem 1.3 follows from Theorem 3.6 with $p = \infty$. □

4. Applications to nonuniformly hyperbolic systems. In this section, we prove the vector-valued ASIP for large classes of nonuniformly hyperbolic systems. In Section 4.1, we consider nonuniformly expanding systems. In Section 4.2, we consider nonuniformly hyperbolic systems, proving Theorems 1.6 and 1.8. Some open problems are described in Section 4.3.

4.1. *Nonuniformly expanding systems.* Let (M, d) be a locally compact separable bounded metric space with Borel probability measure η and let $f : M \to M$ be a nonsingular transformation for which η is ergodic. Let $\Lambda \subset M$ be a measurable subset with $\eta(\Lambda) > 0$. We suppose that there is an at most countable measurable partition $\{\Lambda_j\}$ with $\eta(\Lambda_j) > 0$, and that there exist integers $R_j \geq 1$, and constants $\lambda > 1$; $C > 0$ and $\gamma \in (0, 1)$ such that for all j,

(1) $f^{R_j} : \Lambda_j \to \Lambda$ is a (measure-theoretic) bijection.
(2) $d(f^{R_j}x, f^{R_j}y) \geq \lambda d(x, y)$ for all $x, y \in \Lambda_j$.
(3) $d(f^k x, f^k y) \leq C d(f^{R_j}x, f^{R_j}y)$ for all $x, y \in \Lambda_j$, $k < R_j$.
(4) $g_j = \frac{d(\eta|_{\Lambda_j} \circ (f^{R_j})^{-1})}{d\eta|_\Lambda}$ satisfies $|\log g_j(x) - \log g_j(y)| \leq C d(x, y)^\gamma$ for almost all $x, y \in \Lambda$.
(5) $\sum_j R_j \eta(\Lambda_j) < \infty$.

A dynamical system f satisfying (1)–(5) is called *nonuniformly expanding*.

Define the *return time function* $R : \Lambda \to \mathbb{Z}^+$ by $R|_{\Lambda_j} \equiv R_j$ and the *induced map* $F : \Lambda \to \Lambda$ by $Fy = f^{R(y)}(y)$. It is well known that there is a unique invariant probability measure m on M that is equivalent to η.

THEOREM 4.1. *Let $f : M \to M$ be a nonuniformly expanding map satisfying* (1)–(5) *above. Assume moreover that $R \in L^p(\Lambda)$, $p > 2$. Let $\phi : M \to \mathbb{R}^d$ be a mean zero Hölder observable with partial sums $S_N = \sum_{n=1}^N \phi \circ f^n$. Then the conclusion of Theorem 1.6 is valid.*

PROOF. This is identical to the proof of [29], Theorem 2.9, so we just sketch the main steps. The induced map $F : \Lambda \to \Lambda$ is a topologically mixing Gibbs–Markov map with respect to the partition $\alpha = \{\Lambda_j\}$. The induced observable $\Phi : \Lambda \to \mathbb{R}^d$ given by $\Phi(y) = \sum_{\ell=0}^{R(y)-1} \phi(f^\ell y)$ is weighted Lipschitz and satisfies the ASIP by Theorem 3.6.

If $F : \Lambda \to \Lambda$ were the first return map, then the result would follow immediately from [31], Theorem 4.2 (see also [29], Theorem B.1). The general result is proved by passing to a Young tower [45] which is a Markov extension of $f : M \to M$ for which F is the first return map. □

REMARK 4.2. (a) The regularity assumption on ϕ in Theorem 4.1 can be replaced by the more general assumption that the induced observable Φ is weighted Lipschitz (with respect to the metric defined on the Gibbs–Markov system Λ).

(b) A similar result holds for nonuniformly expanding semiflows ([29], Corollary 2.12).

Naturally, Theorem 4.1 includes uniformly expanding and piecewise expanding maps where the partition is finite (with $p = \infty$). Further examples of nonuniformly expanding maps to which Theorem 4.1 applies include Alves–Viana maps, Liverani–Saussol–Vaienti (Pomeau–Manneville maps), multimodal maps, and circle maps with a neutral fixed point; see [29], Section 4.

4.2. *Nonuniformly hyperbolic systems.* As was the case in [29], the results in this paper apply to dynamical systems that are *nonuniformly hyperbolic in the sense of Young* [44] with return time function $R \in L^p$, $p > 2$.

Let $f : M \to M$ be a diffeomorphism (possibly with singularities) defined on a Riemannian manifold (M, d). We assume from the start that f preserves a "nice" probability measure m (one of the conclusions in Young [44] is that m is an SRB measure).

Fix a subset $\Lambda \subset M$ and a family of subsets of M called "stable disks" $\{W^s\}$ that are disjoint and cover Λ. The stable disk containing x is labeled $W^s(x)$.

(A1) There is a partition $\{\Lambda_j\}$ of Λ and integers $R_j \geq 1$ such that $f^{R_j}(W^s(x)) \subset W^s(f^{R_j}x)$ for all $x \in \Lambda_j$.

Define the return time function $R : \Lambda \to \mathbb{Z}^+$ by $R|_{\Lambda_j} = R_j$ and the induced map $F : \Lambda \to \Lambda$ by $F(x) = f^{R(x)}(x)$. Form the discrete suspension map $\hat{f} : \Delta \to \Delta$, where $\hat{f}(x, \ell) = (x, \ell + 1)$ for $\ell < R(x) - 1$ and $\hat{f}(x, R(x) - 1) = (Fx, 0)$. Define a separation time $s : \Lambda \times \Lambda \to \mathbb{N}$ by defining $s(x, x')$ to be the greatest integer $n \geq 0$ such that $F^k x$, $F^k x'$ lie in the same partition element of Λ for $k = 0, \ldots, n$. [If x, x' do not lie in the same partition element, then we take $s(x, x') = 0$.] For general points $p = (x, \ell)$, $p' = (x', \ell') \in \Delta$, define $s(p, q) = s(x, x')$ if $\ell = \ell'$ and $s(p, q) = 0$ otherwise. This defines a separation time $s : \Delta \times \Delta \to \mathbb{N}$. The projection $\pi : \Delta \to M$, $\pi(x, \ell) = f^\ell x$, satisfies $\pi f = \hat{f}\pi$.

(A2) There is a distinguished "unstable leaf" $W^u \subset \Lambda$ such that each stable disk intersects W^u in precisely one point, and there exist constants $C \geq 1$, $\alpha \in (0, 1)$ such that:

(i) $d(f^n x, f^n y) \leq C\alpha^n$ for all $y \in W^s(x)$, all $n \geq 0$, and
(ii) $d(f^n x, f^n y) \leq C\alpha^{s(x,y)}$ for all $x, y \in W^u$ and all $0 \leq n < R$.

Let $\bar{\Lambda} = \Lambda/\sim$ where $x \sim x'$ if $x \in W^s(x')$ and define the partition $\{\bar{\Lambda}_j\}$ of $\bar{\Lambda}$. We obtain a well-defined return time function $R : \bar{\Lambda} \to \mathbb{Z}^+$ and induced

map $\bar{F}:\bar{\Lambda}\to\bar{\Lambda}$. Let $\bar{f}:\bar{\Delta}\to\bar{\Delta}$ denote the quotient of $\hat{f}:\Delta\to\Delta$ where (x,ℓ) is identified with (x',ℓ') if $\ell=\ell'$ and $x'\in W^s(x)$. Let $\bar{\pi}:\Delta\to\bar{\Delta}$ denote the natural projection. The separation time on Δ drops down to a separation time on $\bar{\Delta}$.

(A3) The map $\bar{F}:\bar{\Lambda}\to\bar{\Lambda}$ and partition $\{\bar{\Lambda}_j\}$ separate points in $\bar{\Lambda}$. [It follows that $d_\theta(p,q)=\theta^{s(p,q)}$ defines a metric on $\bar{\Delta}$ for each $\theta\in(0,1)$.]

(A4) There exist invariant probability measures $\hat{m}$ on Δ and $\bar{m}$ on $\bar{\Delta}$ such that:

(i) $\pi:\Delta\to M$ and $\bar{\pi}:\Delta\to\bar{\Delta}$ are measure-preserving; and

(ii) $\bar{f}:\bar{\Delta}\to\bar{\Delta}$ is nonuniformly expanding in the sense of Section 4.1 with induced map $\bar{F}:\bar{\Lambda}\to\bar{\Lambda}$. [Conditions (2) and (3) are automatic.]

PROOF OF THEOREM 1.6. This reduces, as in the proof of [29], Theorem 3.4, to the ASIP for the nonuniformly expanding map $\bar{f}:\bar{\Delta}\to\bar{\Delta}$ and hence follows from Theorem 4.1. □

REMARK 4.3. Again, the regularity assumption on ϕ can be relaxed, and the result extends to nonuniformly hyperbolic flows.

Large classes of billiard maps and Lorentz flows, surveyed in [10] satisfy the vector-valued ASIP. These include dispersing billiards (with finite or infinite horizons) and the corresponding Lorentz flows (assuming finite horizons).

PROOF OF THEOREM 1.8. By periodicity, we can consider the quotient flow on the compact manifold $M=(\mathbb{T}^2-\Omega)\times S^1$. The Poincaré map $f:X\to X$ on the compact cross-section $X=\partial\Omega\times(\frac{-\pi}{2},\frac{\pi}{2})$ is called the billiard map or collision map. Young [45] showed that f is nonuniformly hyperbolic in the sense of Young with $R\in L^p$ for all $p>2$. By Theorem 1.6, the vector-valued ASIP holds for f with $p=\infty$. The collision time is uniformly bounded and piecewise Hölder, so it follows from [31] that the vector-valued ASIP holds for the Lorentz flow on M. Now take as an $\mathbb{R}^2$-valued observable the velocity coordinate $v:M\to S^1$. This is piecewise Hölder, and the lifted position in $\mathbb{R}^2$ is given by $q(T)=\int_0^T v\circ f_t\,dt$. Finally, nonsingularity of the covariance matrix was proved in [5]. □

Chernov and Zhang [11] consider three classes of billiards with slow mixing rates. The first and third classes are not covered by our results since it is shown only that $R\in L^{2-\varepsilon}$. The second class of *Bunimovich-type* billiards treated in [11] satisfies $R\in L^{3-\varepsilon}$. The vector-valued ASIP for such billiards (and the corresponding flows) is hence a consequence of Theorem 1.6 with error $\beta=\frac{6d+10}{12d+21}$.

As in [29], Theorem 1.6 also applies to Lozi maps and certain piecewise hyperbolic maps, Hénon-like maps and partially hyperbolic diffeomorphisms with mostly contracting direction.

A further important class of dynamical systems is *singular hyperbolic flows* [32]. Theorem 1.6 does not apply directly to such systems, but it establishes the vector-valued ASIP (with $p = \infty$) when combined with the techniques in Holland and Melbourne [24].

4.3. *Open problems.* Given a (non)uniformly hyperbolic flow f_t, the time-one map f_1 is only *partially hyperbolic*. For such maps the Gibbs–Markov/suspension formalism breaks down so the results in [29] and in this paper do not apply. By different methods, Melbourne and Török [30] proved that the scalar ASIP is typically valid for the time-one map of an Axiom A flow. They used rapid mixing properties to reduce to a reverse martingale difference sequence. Following [13, 20], this leads to the ASIP in reverse time and hence forward time (since the class of such flows is closed under time reversal). Similarly, the scalar ASIP for the time-one map of the planar periodic Lorentz gas with finite horizons is typically valid (since the flow is typically rapid mixing [28] and the class of flows is closed under time reversal).

PROBLEM 1. Prove that the vector-valued ASIP holds (at least typically) for time-one maps of Axiom A flows and/or planar periodic Lorentz gas with finite horizons.

Generally speaking, the hypotheses for a nonuniformly hyperbolic system are not time-symmetric so [13, 20, 30] fails.

PROBLEM 2. Obtain results on the scalar ASIP for time-one maps of nonuniformly hyperbolic flows and/or singular hyperbolic flows.

REMARK 4.4. (a) The Banach space-valued ASIP of [3] applies to Problem 1, with the caveats mentioned in Remark 1.11(a). In particular, the d-dimensional functional LIL is typically valid. These results do not apply to Problem 2.

(b) The (vector-valued) functional central limit theorem is typically valid in Problems 1 and 2 (combining the arguments in [20], Section 3.3, and [30]).

APPENDIX: SCALAR ASIP WITH ERROR TERM

In this appendix, we prove a scalar ASIP using martingale approximation and the method of Strassen [41]. This is precisely the result [37], Theorem 7.1, used in [29], but our purpose here is to obtain a better error term under assumptions appropriate for dynamical systems. This improves Theorem 2.1 when $d = 1$.

We assume hypotheses (2.1)–(2.3) and (2.5) (with $d = 1$) normalizing so that $\Sigma = 1$. Define $\{y_j\}$ as in Section 2.3. In particular, Theorem 2.11 is unchanged. Fix $Q > 1$, $\alpha > 0$. Eventually, $\alpha > 0$ is chosen arbitrarily small.

Law of large numbers for y_j^2.

LEMMA A.1. *Let $\gamma = \frac{1}{2}Q/(1+Q)$. Then $\sum_{j\le M_N} Ey_j^2 = N + O(N^{1/2+\gamma})$.*

PROOF (cf. [37], Lemma 7.3.5). By hypothesis (2.5), $E(\sum_{n\le N}\eta_n)^2 = a_N^2$ where $a_N^2 = N(1+O(N^{-1/2}))$. Write

$$\sum_{n\le N}\eta_n = \sum_{n\le P_M}\eta_n - \sum_{n=N+1}^{P_M}\eta_n = \sum_{n\le P_M}(\eta_n-\eta_{\ell n}) + \sum_{j\le M} y_j + \sum_{j\le M} z_j - \sum_{n=N+1}^{P_M}\eta_n.$$

Then

$$\begin{aligned}\Big\|\sum_{j\le M} y_j\Big\|_2 - a_N &= \Big\|\sum_{j\le M} y_j\Big\|_2 - \Big\|\sum_{n\le N}\eta_n\Big\|_2\\ &\le \Big\|\sum_{j\le M} z_j\Big\|_2 + \Big\|\sum_{n\le P_M}(\eta_n-\eta_{\ell n})\Big\|_2 + \Big\|\sum_{n=N+1}^{P_M}\eta_n\Big\|_2.\end{aligned}$$

By Proposition 2.9, $\|\sum_{n\le N}(\eta_n-\eta_{\ell n})\|_2 \ll 1$. By the proof of Lemma 2.13, $\|\sum_{j\le M} z_j\|_2^2 \ll M^{1+2\alpha}$ and so $\|\sum_{j\le M} z_j\|_2 \ll N^{(\frac{1}{2}+\alpha)/(1+Q)}$. By stationarity and (2.5), $\|\sum_{n=N+1}^{P_M}\eta_n\|_2^2 = \|\sum_{n\le P_M-N}\eta_n\|_2^2 \ll P_M - N \ll N^{Q/(1+Q)}$. Hence $\|\sum_{j\le M} y_j\|_2 = a_N + O(N^\gamma)$ and $E(\sum_{j\le M} y_j)^2 = N + O(N^{1/2+\gamma})$. Also, as in the proof of Lemma 2.13, $\sum_{i\ne j} Ey_iy_j \ll 1$. □

COROLLARY A.2. *Let $\beta = (\frac{1}{4}+\frac{1}{2}Q)/(1+Q)$. Then $\sum_{j\le M_N} Ey_j^2 = N + O(N^{2\beta})$.*

LEMMA A.3. *Let $\beta = (\frac{3}{4}-\frac{p}{8}+\frac{1}{2}Q)/(1+Q)$ for $2<p\le 4$ [and $\beta = (\frac{1}{4}+\frac{1}{2}Q)/(1+Q)$ for $p>4$]. Then for any $\varepsilon>0$, $\sum_{j\le M_N} y_j^2 - Ey_j^2 \ll N^{2\beta+\varepsilon}$ a.e.*

PROOF. The value of ε below may change from line to line. Define

$$w_j = \begin{cases} y_j^2 - Ey_j^2, & |y_j^2 - Ey_j^2| \le j^{1+Q+\varepsilon},\\ 0, & \text{otherwise.}\end{cases}$$

Note that $P(w_j \ne y_j^2 - Ey_j^2) = P(|y_j^2 - Ey_j^2| > j^{1+Q+\varepsilon}) \le 2\|y_j\|_2^2/j^{1+Q+\varepsilon} \ll j^{-(1+\varepsilon)}$, which is summable, so by Borel–Cantelli w_j fails to coincide with $y_j^2 - Ey_j^2$ only finitely often. Hence it suffices to estimate $\sum_{j\le M} w_j$. We do this by estimating $\sum_{j\le M}\tilde w_j$ and $\sum_{j\le M} Ew_j$ where $\tilde w_j = w_j - Ew_j$.

Again $\sum_{i\ne j} E\tilde w_i\tilde w_j \ll 1$. Also, $E\tilde w_j^2 \le |\tilde w_j^{p/2}|_1\|\tilde w_j^{2-p/2}\|_\infty \ll \|y_j\|_p^p \times \|w_j\|_\infty^{2-p/2} \ll j^{R-1}$, where $R = 3 - \frac{p}{2} + 2Q + \varepsilon$. Hence $E(\sum_{j=m}^n \tilde w_j)^2 \ll n^R - m^R$. By Lemma 2.6, $\sum_{j\le M}\tilde w_j \ll M^{R/2} \le N^{R/(2(1+Q))}$ for any $\varepsilon>0$.

Let $A = \{|y_j^2 - Ey_j^2| > j^{1+Q+\varepsilon}\}$. Then

$$Ew_j = -E\{(y_j^2 - Ey_j^2)I_A\} \ll \|y_j^2 - Ey_j^2\|_{p/2}\|1_A\|_{p/(p-2)}$$
$$\ll \|y_j\|_p^2\|1_A\|_{p/(p-2)} \ll j^{Q-(1+\varepsilon)(p-2)/p} = j^{S-1},$$

where $S = \frac{2}{p} - \varepsilon + Q$. Hence $\sum_{j\le M} Ew_j \ll M^S$. Thus, it suffices that $2\beta = \max\{\frac{1}{2}R/(1+Q), S/(1+Q)\} = \frac{1}{2}R/(1+Q)$. □

Martingale approximation.

LEMMA A.4. *Set $\mathcal{L}_j = \mathcal{L}_1^j = \sigma\{y_1, \dots, y_j\}$. There is a martingale difference sequence $\{Y_j, \mathcal{L}_j\}$ such that $y_j = Y_j + u_j - u_{j+1}$, where $\|u_j\|_q \ll \tilde{\tau}^{j^\alpha}$ for all $2 < q < p$.*

PROOF (cf. [37], Lemma 7.4.1). Define $u_j = \sum_{k=0}^\infty E(y_{j+k}|\mathcal{L}_{j-1})$. We estimate $\|E(y_{j+k}|\mathcal{L}_{j-1})\|_q$, which we write for convenience as $\|E(y|\mathcal{L})\|_q$. Note that

$$E|E(y|\mathcal{L})|^q = E\{E(y|\mathcal{L})E(y|\mathcal{L})|E(y|\mathcal{L})|^{q-2}\}$$
$$= E\{E\{yE(y|\mathcal{L})|E(y|\mathcal{L})|^{q-2}|\mathcal{L}\}\}$$
$$= E\{yE(y|\mathcal{L})|E(y|\mathcal{L})|^{q-2}\}.$$

Write $\frac{1}{q} + \frac{1}{s} = 1$. Then $\frac{1}{p} + \frac{1}{s} < 1$, so by Lemma 2.4,

$$E|E(y|\mathcal{L})|^q \le \|y\|_p\|E(y|\mathcal{L})^{q-1}\|_s\tilde{\tau}^{(j+k)^\alpha}.$$

Note that $\|E(y|\mathcal{L})^{q-1}\|_s = (E|E(y|\mathcal{L})|^q)^{1-1/q}$, and so dividing both sides by this yields $\|E(y|\mathcal{L})\|_q \le \|y\|_p\tilde{\tau}^{(j+k)^\alpha}$. Since $\sum_{k=0}^\infty \tilde{\tau}^{(j+k)^\alpha} \ll j\tilde{\tau}^{j^\alpha}$, the estimate for $\|u_j\|_q$ follows (increasing $\tilde{\tau}$ slightly).

At the same time, it follows immediately that $\sum_{k=0}^\infty |E(y_{j+k}|\mathcal{L}_j)|_1 < \infty$, which guarantees (see, e.g., [37], Lemma 2.1) that Y_j is a martingale difference sequence. □

COROLLARY A.5. *$\sum_{j\le M_N}(y_j - Y_j) \ll 1$ a.e. and $\sum_{j\le M_N}(y_j^2 - Y_j^2) \ll N^{1/2}$ a.e.*

PROOF. We have $\sum_{j\le M}(y_j - Y_j) = u_1 - u_{M+1}$, and hence certainly $|\sum_{j\le M}(y_j - Y_j)| \le \sum_{j\ge 1}|u_j|$. By Lemma A.4, $\sum_{j\ge 1}|u_j|_1 < \infty$ so that $\sum_{j\ge 1}|u_j| < \infty$ a.e. proving the first statement.

Set $v_j = u_j - u_{j+1}$. Then $Y_j^2 - y_j^2 = v_j^2 - 2y_jv_j$. Now $E\sum_{j=1}^\infty v_j^2 = \sum_{j=1}^\infty Ev_j^2 \ll \sum_{j=1}^\infty \tilde{\tau}^{2j^\alpha} < \infty$ by Lemma A.4. Hence $\sum_{j\le M} v_j^2 \le \sum_{j=1}^\infty v_j^2 < \infty$ a.e. Finally, $\sum_{j\le M} y_jv_j \le (\sum_{j\le M} y_j^2)^{1/2}(\sum_{j\le M} v_j^2)^{1/2} \ll (\sum_{j\le M} y_j^2)^{1/2} \ll N^{1/2}$ by Lemmas A.1 and A.3. □

LEMMA A.6. *Let X_k be a martingale difference sequence and $m \in (1,2]$. Suppose that $b_1 < b_2 < \cdots \to \infty$. If $\sum_{k\le n} b_k^{-m} E|X_k|^m < \infty$, then $\sum_{k\le n} X_k = o(b_n)$ a.e.*

PROOF. For $m = 2$, this is explicit in [19], page 238. For $m \in (1,2)$ it follows from a standard martingale result (Chow [12]), combined with Kronecker's lemma; this is implicit in the proof of [37], Lemma 7.4.4. □

LEMMA A.7. *Let $\beta = (\frac{1}{p} + \frac{1}{2}Q)/(1+Q)$ for $2 < p \le 4$ [and $\beta = (\frac{1}{4} + \frac{1}{2}Q)/(1+Q)$ for $p > 4$]. Then for any $\varepsilon > 0$, $\sum_{j\le M_N}(E(Y_j^2|\mathcal{L}_{j-1}) - Y_j^2) \ll N^{2\beta+\varepsilon}$ a.e.*

PROOF. Define $R_j = E(Y_j^2|\mathcal{L}_{j-1}) - Y_j^2$. Suppose that $\gamma > 2/p + Q$ and choose $q < p$ so that $\gamma > 2/q + Q$. Then

$$(j^\gamma)^{-q/2} E|R_j|^{q/2} \ll j^{-\gamma q/2} E|Y_j|^q \ll j^{-(\gamma - Q)q/2},$$

hence $\sum_{j=1}^\infty (j^\gamma)^{-q/2} E|R_j|^{q/2} < \infty$. Note that $\frac{q}{2} \in (1,2]$ and R_j is a martingale difference sequence, so it follows from Lemma A.6 that $\sum_{j\le M} R_j \ll M^\gamma$ and the result follows from Proposition 2.8. □

We now apply Strassen's martingale version of the Skorokhod embedding [41]. There exist nonnegative random variables T_j such that the sequences $\{\sum_{j\le M} Y_j, M \ge 1\}$ and $\{W(\sum_{j\le M} T_j), M \ge 1\}$ are equal in distribution.

PROPOSITION A.8. *For β as in Corollary A.2, Lemmas A.3 and A.7 and any $\varepsilon > 0$, $\sum_{j\le M_N} T_j - N \ll N^{2\beta+\varepsilon}$ a.e.*

PROOF. Let $\mathcal{A}_M = \sigma\{W(t) : 0 \le t \le \sum_{j\le M} T_j\}$, so $\mathcal{L}_M \subset \mathcal{A}_M$. Each T_j is $\mathcal{A}_j$-measurable, $E(T_j|\mathcal{A}_{j-1}) = E(Y_j^2|\mathcal{L}_{j-1})$ a.e., and $ET_j^p \ll E|Y_j|^{2p}$. In particular, the argument in Lemma A.7 implies that

$$\sum_{j\le M} \big(T_j - E(T_j|\mathcal{A}_{j-1})\big) \ll N^{2\beta} \qquad \text{a.e.} \tag{A.1}$$

Now write

$$\sum T_j - N = \sum \big(T_j - E(T_j|A_{j-1})\big) + \sum \big(E(Y_j^2|\mathcal{L}_{j-1}) - Y_j^2\big) + \sum Y_j^2 - N.$$

The result follows from (A.1), Corollaries A.2 and A.5 and Lemmas A.3 and A.7. □

THEOREM A.5. *Let $\beta = \frac{1}{2p} + \frac{1}{4}$ for $2 < p \le 4$ and $\beta = \frac{3}{8}$ for $p > 4$. For any $\varepsilon > 0$, $\sum_{n\le N} \eta_n = W(N) + O(N^{\beta+\varepsilon})$ a.e.*

PROOF. By Theorem 2.11 and Corollary A.5, it suffices to prove that $\sum_{j\le M} Y_j = W(N) + O(N^{\beta+2\varepsilon})$. Equivalently, $W(\sum_{j\le M} T_j) = W(N) + O(N^{\beta+2\varepsilon})$. Hence the result follows from Proposition A.8. □

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