

Superposition rules and stochastic Lie–Scheffers systems

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Abstract. This paper proves a version for stochastic differential equations of the Lie–Scheffers theorem. This result characterizes the existence of nonlinear superposition rules for the general solution of those equations in terms of the involution properties of the distribution generated by the vector fields that define it. When stated in the particular case of standard deterministic systems, our main theorem improves various aspects of the classical Lie–Scheffers result. We show that the stochastic analog of the classical Lie–Scheffers systems can be reduced to the study of Lie group valued stochastic Lie–Scheffers systems; those systems, as well as those taking values in homogeneous spaces are studied in detail. The developments of the paper are illustrated with several examples.

Résumé. Ce papier contient une généralisation du Théorème de Lie–Scheffers aux équations différentielles stochastiques. Ce résultat caractérise l'existence de règles de superposition non linéaires pour la solution générale de ces équations, en termes des propriétés d'involution de la distribution engendrée par les champs vecteurs qui les définissent. Dans le cas particulier des systèmes déterministes, notre théorème principal améliore certains aspects du théorème de Lie–Scheffers traditionnel. Nous montrons que l'analogie stochastique des systèmes de Lie–Scheffers classiques peuvent être réduits à l'étude des systèmes de Lie–Scheffers stochastiques à valeurs dans un groupe de Lie; ces systèmes, ainsi que ceux qui prennent des valeurs dans des espaces homogènes sont étudiés en détail. Les développements de ce papier sont illustrés avec plusieurs exemples.

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1. Introduction

A differential equation is said to have a *superposition rule* (a more explicit definition is provided in the next section) whenever any of its solutions can be written as a given (in general nonlinear) function of the initial condition and of a fixed set of particular solutions. The first characterization of the existence of superposition rules was given by the Norwegian mathematician Sophus Lie in a remarkable piece of work [20] where he established a link between the existence of superposition rules and what we nowadays call the Lie algebraic properties of the vector fields that define a time-dependent differential equation. This result is referred to as the *Lie–Scheffers theorem* and systems that satisfy its hypotheses as *Lie–Scheffers systems*.

Lie–Scheffers systems have been the subject of much attention due to their widespread occurrence in physics and mathematics. The reader is encouraged to check with [3,4], and references therein, for various presentations of the classical Lie–Scheffers theorem, an excellent collection of examples of applications of this theorem, and for historical remarks.

The main goal of this paper is the extension of the Lie–Scheffers theorem to stochastic differential equations. This generalization is stated in Theorem 3.1. It is worth emphasizing that the main result of the paper, Theorem 3.1,

cannot be seen just as a mere transcription of the deterministic Lie–Scheffers theorem into the context of Stratonovich stochastic integration by using the so-called Malliavin’s Transfer Principle [22]. This Principle states that whatever is true for standard differential equations also holds for Stratonovich stochastic differential equations; as we will see later on, there are purely stochastic conditions that appear in the statement of the theorem.

Additionally, in proving Theorem 3.1 we have carefully spelled out the regularity conditions needed for the result to be valid; those conditions are only vaguely evoked in the classical references or in the cited papers that study the deterministic case. More importantly, a careful construction of the proof has lead us to realize that the hypotheses under which we can guarantee the existence of superposition rules can be weakened: the Lie algebra condition in the classical theorem can be replaced by an involutivity hypothesis that is, in general, less restrictive.

The contents of the paper are structured as follows. Section 2 explains in detail the notion of superposition rule and includes a proposition that translates this concept into geometric terms. Section 3 contains the main theorem that we have already described.

Section 4 is dedicated to the study of Lie–Scheffers systems on Lie groups and homogeneous spaces; this case is particularly relevant since, as we show in the first result of that section (Proposition 4.1), classical Lie–Scheffers systems (roughly speaking, those generated by vector fields that close a Lie algebra) can be locally reduced to this case via a theorem due to Palais. In that section we also show, as an example, how Lévy stochastic processes can be seen as Lie group valued Lie–Scheffers systems. The section concludes with a brief presentation of the classical Wei–Norman method for solving Lie–Scheffers systems, adapted to the stochastic context.

Section 5 contains a discussion on how the existence of a superposition rule for a stochastic differential equation makes available a remarkable feature that has deserved certain attention in the context of standard stochastic differential equations, namely, the fact that the stochastic flow can be written as a fixed deterministic function of the Brownian forcing of the equation in question. Indeed, a well-known theorem by Ben Arous [2], that we state in the paper and whose proof is based on the use of stochastic Taylor expansions, shows that this property of the flow is available under exactly the same hypotheses as the classical Lie–Scheffers theorem. Our main theorem allows, admittedly only to a certain extent, the generalization of this statement to any stochastic differential equation that satisfies its hypotheses; more specifically, any SDE generated by vector fields that span an involutive distribution has a superposition rule and hence its flow can be written as a fixed deterministic function of the initial conditions and of a set of solutions that contain the stochastic behavior of the resulting map.

The paper concludes with a section that contains a number of examples that illustrate the developments of the paper.

2. Superposition rules for stochastic differential equations

Let (Ω, \mathcal{F}, P) be a probability space. We start by considering the stochastic differential equation

$$\delta\Gamma = S(X, \Gamma) \delta X, \tag{2.1}$$

where $X : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^l$ is a given \mathbb{R}^l -valued semimartingale and $S(x, z) : T_x \mathbb{R}^l \rightarrow T_z \mathbb{R}^n$ is a Stratonovich operator from \mathbb{R}^l to \mathbb{R}^n . Sometimes we will choose a basis in $T^* \mathbb{R}^l$ and will write down the Stratonovich operator $S(x, z)$ in terms of its components $(S_1(x, z), \dots, S_l(x, z))$ with respect to that basis.

Definition 2.1. A superposition rule of the stochastic differential equation (2.1) is a pair $(\Phi, \{\Gamma_1, \dots, \Gamma_m\})$, where $\Phi : \mathbb{R}^{n(m+1)} \rightarrow \mathbb{R}^n$ is a (not necessarily smooth) function and $\{\Gamma_i : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^n \mid i = 1, \dots, m\}$ is a set of particular solutions of (2.1) such that any solution Γ of (2.1) can be written, at least up to a sufficiently small stopping time τ , as

$$\Gamma = \Phi(z^1, \dots, z^n; \Gamma_1, \dots, \Gamma_m) =: \Phi(z; \Gamma_1, \dots, \Gamma_m),$$

where $z = (z^1, \dots, z^n)$ a set of n arbitrary constants associated with the initial condition of the solution Γ , that is, $\Gamma(0, \omega) = (z^1, \dots, z^n)$, for all $\omega \in \Omega$. We extend to the stochastic context the terminology used for standard differential equations and we will call Lie–Scheffers systems the stochastic differential equations that admit a superposition rule.

Remark 2.2. As we will see in examples later on in the paper, superposition rules exist only locally. That is why we can, without loss of generality, restrict our attention to stochastic differential equations on Euclidean spaces. Observe also that we are requiring that Φ does not depend on time, the probability space, or the noise X . This prevents us from using certain regularization techniques at the time of testing the existence of superposition rules. For example, when dealing with a deterministic differential equation, the standard transformation of a time-dependent system $\dot{\gamma} = f(t, \gamma)$ on \mathbb{R}^n , $f: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ into the autonomous one

$$\dot{\gamma} = f(t, \gamma) \quad \text{and} \quad \dot{i} = 1$$

on \mathbb{R}^{n+1} obtained by adding an extra trivial differential equation for the time is not allowed; indeed, if we find a superposition rule for the transformed autonomous system, that rule does not yield a superposition rule for the original system that satisfies the requirements of our definition, precisely due to the explicit dependence on time that appears in the superposition function.

In order to study the implications of the presence of a superposition rule we take a more geometric approach. Let Ψ be the function defined by

$$\begin{aligned} \Psi: \mathbb{R}^{n(m+2)} &\longrightarrow \mathbb{R}^n, \\ (z, q_0, q_1, \dots, q_m) &\longmapsto q_0 - \Phi(z; q_1, \dots, q_m). \end{aligned} \tag{2.2}$$

Notice that for any $z \in \mathbb{R}^n$, the function $\Psi_z := \Psi(z, \cdot): \mathbb{R}^{n(m+1)} \rightarrow \mathbb{R}^n$ is constant on a $(m+1)$ -tuple $(\Gamma, \Gamma_1, \dots, \Gamma_m)$ of solutions of the system (2.1), at least up to a given stopping time τ , provided that $\Gamma_{t=0} = z \in \mathbb{R}^n$ a.s. From now on we assume that all the solutions Γ that we are dealing with are constant a.s. at $t = 0$. Additionally, if the function Φ is smooth then the map $\Psi_z: \mathbb{R}^{n(m+1)} \rightarrow \mathbb{R}^n$ is a submersion for any fixed $z \in \mathbb{R}^n$, because

$$\text{rank} \left(\frac{\partial \Psi_z^j}{\partial q_0^i} \right)_{j,i=1, \dots, n} = \text{rank}(I_n) = n, \tag{2.3}$$

where I_n is the identity matrix of dimension n . Consequently, for any $z \in \mathbb{R}^n$, the level set $\Psi_z^{-1}(0) \subset \mathbb{R}^{n(m+1)}$ is a closed embedded submanifold of $\mathbb{R}^{n(m+1)}$ of dimension nm . That is, the function Ψ defines a family \mathcal{G} of regular nm -dimensional submanifolds \mathcal{G}_z via the zero level sets $\Psi_z^{-1}(0) = \{p \in \mathbb{R}^{n(m+1)} \mid \Psi(z, p) = 0\} =: \mathcal{G}_z$ of Ψ_z , for any $z \in \mathbb{R}^n$. The submanifolds \mathcal{G}_z are globally diffeomorphic to \mathbb{R}^{nm} via the restriction $\pi_m|_{\mathcal{G}_z}$ to \mathcal{G}_z of the projection $\pi_m: \mathbb{R}^{n(m+1)} = \mathbb{R}^n \times \overset{m+1}{\dots} \times \mathbb{R}^n \rightarrow \mathbb{R}^{nm} = \mathbb{R}^n \times \overset{m}{\dots} \times \mathbb{R}^n$ onto the last m \mathbb{R}^n factors. This is easy to see by verifying that the inverse $\mathcal{E}_z: \mathbb{R}^{nm} \rightarrow \mathcal{G}_z$ of $\pi_m|_{\mathcal{G}_z}$ is given by $\mathcal{E}_z(q_1, \dots, q_m) = (\Phi(z; q_1, \dots, q_m), q_1, \dots, q_m)$, which is obviously a diffeomorphism. In order to study the significance of the family of submanifolds \mathcal{G} we start by introducing the following definition.

Definition 2.3. Let $Y: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a vector field. The vector field

$$\begin{aligned} \tilde{Y}: \mathbb{R}^{n(m+1)} &\longrightarrow \mathbb{R}^{n(m+1)}, \\ (q_0, \dots, q_m) &\longmapsto (Y(q_0), \dots, Y(q_m)) \end{aligned}$$

is called the diagonal extension of Y .

It can be easily checked that the set of diagonal extensions of vector fields in $\mathfrak{X}(\mathbb{R}^n)$ are a subalgebra of $\mathfrak{X}(\mathbb{R}^{n(m+1)})$; more explicitly, for any $Y_1, Y_2, Y_3 \in \mathfrak{X}(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$,

$$[\tilde{Y}_1, \tilde{Y}_2 + \lambda \tilde{Y}_3] = [\tilde{Y}_1, \widetilde{Y_2 + \lambda Y_3}]. \tag{2.4}$$

The following proposition states that, roughly speaking, the family of submanifolds \mathcal{G} completely characterizes the superposition rule.

Proposition 2.4. *Suppose that the stochastic differential equation (2.1) admits a smooth superposition rule $(\Phi, \{\Gamma_1, \dots, \Gamma_m\})$. Suppose that $(\Gamma_1, \dots, \Gamma_m)_{t=0} = (p_1, \dots, p_m) \in \mathbb{R}^{mn}$ a.s. Then, there exists a family \mathcal{G} of closed embedded nm -dimensional submanifolds of $\mathbb{R}^{n(m+1)}$ such that for any $z \in \mathbb{R}^n$ there exists $\mathcal{G}_z \in \mathcal{G}$ such that $(\Gamma^z, \Gamma_1, \dots, \Gamma_m) \subset \mathcal{G}_z$, with Γ^z the solution of (2.1) such that $(\Gamma^z)_{t=0} = z$. Moreover, for any $\mathcal{G}_z \in \mathcal{G}$ the map $\pi_m|_{\mathcal{G}_z} : \mathcal{G}_z \rightarrow \mathbb{R}^{nm}$ is a diffeomorphism.*

Conversely, let \mathcal{G} be a family of (not necessarily embedded) submanifolds of $\mathbb{R}^{n(m+1)}$ diffeomorphic to \mathbb{R}^{nm} via π_m and $\{\Gamma_1, \dots, \Gamma_m\}$ a set of distinct solutions of (2.1) such that $(\Gamma_1, \dots, \Gamma_m)_{t=0} = (p_1, \dots, p_m) \in \mathbb{R}^{mn}$ a.s. Then, if for any point $z \in \mathbb{R}^n$ there is an element \mathcal{G}_z that contains the point (z, p_1, \dots, p_m) and the diagonal extensions $(\tilde{S}_1(X, \cdot), \dots, \tilde{S}_l(X, \cdot))$ of the vector fields $(S_1(X, \cdot), \dots, S_l(X, \cdot))$ that define (2.1) are tangent to \mathcal{G}_z when evaluated at $(\Gamma^z, \Gamma_1, \dots, \Gamma_m)$, then (2.1) admits a (possibly nonsmooth) superposition rule.

Proof. In view of the remarks preceding Definition 2.3 we just need to prove that having a family \mathcal{G} that satisfies the hypotheses in the statement allows us to recover the superposition rule.

Let $\{\Gamma_1, \dots, \Gamma_m\}$ be the set of fixed distinct solutions of (2.1). Denote $p_i = (\Gamma_i)_{t=0}$ the (necessarily different) constant initial conditions of $\Gamma_i, i = 1, \dots, m$. Let $z = (z^1, \dots, z^n) \in \mathbb{R}^n$ be a point and let \mathcal{G}_z be the submanifold in \mathcal{G} such that $(z, p_1, \dots, p_m) \in \mathcal{G}_z$; by hypothesis, this manifold is diffeomorphic to \mathbb{R}^{nm} via the map $\varphi_z = \pi_m|_{\mathcal{G}_z}$, where $\pi_m : \mathbb{R}^{n(m+1)} \rightarrow \mathbb{R}^{nm}$ is the projection onto the last nm factors. In other words, the last nm coordinates of a point in $\mathbb{R}^{n(m+1)}$ serve as global coordinates of \mathcal{G}_z . Introduce the projection

$$\begin{aligned} \pi_{\mathbb{R}^n}^0 : \mathbb{R}^{n(m+1)} &\longrightarrow \mathbb{R}^n, \\ (q_0, \dots, q_m) &\longmapsto q_0. \end{aligned} \quad (2.5)$$

We now define

$$(\Gamma_0)_t(\omega) := \pi_{\mathbb{R}^n}^0 \circ \varphi_z^{-1}((\Gamma_1)_t(\omega), \dots, (\Gamma_m)_t(\omega)). \quad (2.6)$$

It is immediate to see that $(\Gamma_0)_{t=0} = z$ and that Γ_0 is a semimartingale because, by construction, it is a composition of smooth functions with semimartingales. Let now Γ^z be the unique solution of (2.1) with a.s. initial condition $z \in \mathbb{R}^n$. We will proceed by proving that Γ_0 defined in (2.6) equals Γ^z and we will therefore have a superposition rule Φ given by the map $\Phi(z; \Gamma_1, \dots, \Gamma_m) := \pi_{\mathbb{R}^n}^0 \circ \varphi_z^{-1}(\Gamma_1, \dots, \Gamma_m)$. Notice that unless additional hypotheses are assumed on the family \mathcal{G} , there is no guarantee on the smoothness of Φ on the z variable.

In order to prove that Γ_0 equals Γ^z , denote by $(q^k; k = 1, \dots, n)$ the coordinates on \mathbb{R}^n and by $(q_a^k; k = 1, \dots, n; a = 0, \dots, m)$ the coordinates on $\mathbb{R}^{n(m+1)}$. Let $F_k^a : \mathbb{R}^{nm} \rightarrow \mathbb{R}^n$ and $X_k^a : \mathbb{R}^{nm} \rightarrow \mathbb{R}^{n(m+1)}$ be the maps defined as

$$\begin{aligned} F_k^a(q_1, \dots, q_m) &= T_{(q_1, \dots, q_m)}(\pi_{\mathbb{R}^n}^0 \circ \varphi_z^{-1} \circ \pi_m) \left(\frac{\partial}{\partial q_a^k} \right), \\ X_k^a(\varphi_z^{-1}(q_1, \dots, q_m)) &= T_{(q_1, \dots, q_m)}(\varphi_z^{-1} \circ \pi_m) \left(\frac{\partial}{\partial q_a^k} \right) = (F_k^a(q_1, \dots, q_m), 0, \overbrace{a-1}^{n \text{ entries}}, (0, k-1, 1, \dots, 0), \overbrace{m-a}^{n \text{ entries}}, 0), \end{aligned}$$

where $a = 1, \dots, m, k = 1, \dots, n$. Observe that, by construction, the nm vector fields X_k^a are linearly independent and span $T_q \mathcal{G}_z$ at any $q \in \mathcal{G}_z$, since φ_z^{-1} is a diffeomorphism from \mathbb{R}^{nm} to \mathcal{G}_z .

Now, we notice that for any $j = 1, \dots, l$, the vectors

$$\tilde{S}_j(X; \Gamma^z, \Gamma_1, \dots, \Gamma_m) = (S_j(X, \Gamma^z), S_j(X, \Gamma_1), \dots, S_j(X, \Gamma_m)) \quad (2.7)$$

are by hypothesis tangent to \mathcal{G}_z . Additionally, due to (2.6) and the Stratonovich differentiation rules we can write

$$\delta \Gamma_0 = \sum_{a=1}^m \sum_{k=1}^n F_k^a(\Gamma_1, \dots, \Gamma_m) \delta \Gamma_a^k = \sum_{a=1}^m \sum_{k=1}^n \sum_{j=1}^l F_k^a(\Gamma_1, \dots, \Gamma_m) S_j^k(X, \Gamma_a) \delta X^j. \quad (2.8)$$

Moreover,

$$\left(\sum_{a=1}^m \sum_{k=1}^n F_k^a(\Gamma_1, \dots, \Gamma_m) S_j^k(X, \Gamma_a), S_j(X, \Gamma_1), \dots, S_j(X, \Gamma_m) \right) \in \mathbb{R}^{n(m+1)} \tag{2.9}$$

belongs also to $T\mathcal{G}_z$ for any $j = 1, \dots, l$, since (2.9) can be written as a linear combination of the nm linearly independent vector fields X_k^a . Indeed,

$$\left(\sum_{a=1}^m \sum_{k=1}^n F_k^a(\Gamma_1, \dots, \Gamma_m) S_j^k(X, \Gamma_a), S_j(X, \Gamma_1), \dots, S_j(X, \Gamma_m) \right) = \sum_{a=1}^m \sum_{k=1}^n S_j^k(X, \Gamma_a) X_k^a(\Gamma_1, \dots, \Gamma_m).$$

Subtracting (2.9) from (2.7), we see that for any $j = 1, \dots, l$,

$$W_j := \left(S_j(X, \Gamma^z) - \sum_{a=1}^m \sum_{k=1}^n F_k^a(\Gamma_1, \dots, \Gamma_m) S_j^k(X, \Gamma_a), 0, \dots, 0 \right) \in T\mathcal{G}_z.$$

Any of these vector fields, if different from zero, is obviously linearly independent from all the X_k^a , $a = 1, \dots, m$, $k = 1, \dots, n$. If that is the case we could therefore conclude that $\dim(\mathcal{G}_z)$ is strictly bigger than nm , which is obviously a contradiction. Therefore, $W_j = 0$ necessarily, and hence

$$S_j(X, \Gamma^z) = \sum_{a=1}^m \sum_{k=1}^n F_k^a(\Gamma_1, \dots, \Gamma_m) S_j^k(X, \Gamma_a),$$

which guarantees that Γ_0 is a solution of (2.1) because by (2.8)

$$\delta \Gamma_0 = \sum_{j=1}^l S_j(X, \Gamma^z) \delta X^j = \delta \Gamma^z. \tag{2.10}$$

□

Remark 2.5. In the previous proposition we saw how the tangency of the diagonal extensions of the vector fields that define the SDE to the submanifolds in \mathcal{G} is a sufficient condition to ensure the existence of a superposition rule. Is it necessary? Suppose that we have a smooth superposition rule $(\Phi, \Gamma_1, \dots, \Gamma_m)$ and let Ψ be the associated map introduced in (2.2). As we have that $\Psi_z(\Gamma^z, \Gamma_1, \dots, \Gamma_m) = 0$, the Stratonovich differentiation rules yield

$$0 = \sum_{i=1}^n \sum_{a=0}^m \frac{\partial \Psi_z}{\partial q_a^i}(\Gamma^z, \Gamma_1, \dots, \Gamma_m) \delta \Gamma_a^i = \sum_{j=1}^l \sum_{i=1}^n \sum_{a=0}^m \frac{\partial \Psi_z}{\partial q_a^i}(\Gamma^z, \Gamma_1, \dots, \Gamma_m) S_j^i(X, \Gamma_a) \delta X^j. \tag{2.10}$$

A sufficient condition for this identity to hold is that, for any $j \in \{1, \dots, l\}$,

$$\sum_{i=1}^n \sum_{a=0}^m \frac{\partial \Psi_z}{\partial q_a^i}(\Gamma^z, \Gamma_1, \dots, \Gamma_m) S_j^i(X, \Gamma_a) = 0 \tag{2.11}$$

or, equivalently, that the diagonal extensions $\tilde{S}_j(X, \Gamma^z, \Gamma_1, \dots, \Gamma_m)$ are tangent to the elements of the family of submanifolds \mathcal{G} given by the zero fibers of the maps Ψ_z . Additionally, one can find situations in which (2.10) implies (2.11): for instance if $j = 1$ and (like in the case of the Brownian motion) the quadratic variation $[X, X]$ is a strictly increasing process, a straightforward application of the Doob–Meyer decomposition and the Itô isometry make in this case (2.10) and (2.11) equivalent.

Remark 2.6. If we add to the hypotheses of Proposition 2.4 that for any $z \in \mathbb{R}^n$ and for any $(p_1, \dots, p_m) \in \mathbb{R}^{nm}$ there exist a submanifold \mathcal{G}_z in \mathcal{G} such that $(z, p_1, \dots, p_m) \in \mathcal{G}_z$ (for instance when \mathcal{G} is a foliation of $\mathbb{R}^{n(m+1)}$ whose leaves are diffeomorphic to \mathbb{R}^{nm} via π_m) then the superposition function that we constructed in the proof of that result has the

following extremely convenient property: the superposition function is the same for any fundamental sets of solutions $\{\Gamma_1, \dots, \Gamma_m\}$ that we may want to choose. In other words, once Φ is known, we can take m arbitrary independent solutions of (2.1) to write down any solution. This situation frequently occurs in mechanics; see for instance, the study of the classical Riccati equation in [5].

3. The stochastic Lie–Scheffers theorem

The main goal of this section is proving a theorem that characterizes the existence of a superposition rule for a stochastic differential equation in terms of the integrability properties of the distribution spanned by the vector fields that define it. This can be translated into a Lie algebraic requirement, which allows us to recover the classical Lie–Scheffers theorem in the stochastic context (Corollary 3.5).

In order to have at hand the necessary concepts to state the main theorem, we start by briefly recalling some standard results on generalized distributions due to Stefan [24,25] and Sussman [26]. Let M be a smooth manifold, $\mathcal{D} \subset \mathfrak{X}(M)$ be a family of smooth vector fields, and D the smooth generalized distribution spanned by \mathcal{D} . Let $G_{\mathcal{D}}$ be the pseudogroup of transformations generated by the flows of the vector fields in \mathcal{D} and constructed as follows: let $k \in \mathbb{N}^*$ be a positive natural number, \mathcal{X} an ordered family $\mathcal{X} = (X_1, \dots, X_k)$ of k elements of \mathcal{D} , and T a k -tuple $T = (t_1, \dots, t_k) \in \mathbb{R}^k$ such that F_t^i denotes the (locally defined) flow of X_i , $i \in \{1, \dots, k\}$, t_i ; the elements \mathcal{F}_T of $G_{\mathcal{D}}$ are the locally defined diffeomorphisms of the form $\mathcal{F}_T = F_{t_1}^1 \circ F_{t_2}^2 \circ \dots \circ F_{t_k}^k$. Two points x and y in M are said to be $G_{\mathcal{D}}$ -equivalent, if there exists a diffeomorphism $\mathcal{F}_T \in G_{\mathcal{D}}$ such that $\mathcal{F}_T(x) = y$. The relation $G_{\mathcal{D}}$ -equivalent is an equivalence relation whose equivalence classes are called the $G_{\mathcal{D}}$ -orbits, that are sometimes referred to as the *accessible sets* associated to the family \mathcal{D} .

Given the family \mathcal{D} and the associated pseudogroup $G_{\mathcal{D}}$ we can define another family \mathcal{D}' of vector fields as

$$\mathcal{D}' := \{T\mathcal{F}_T \cdot X \mid X \in \mathcal{D}, \mathcal{F}_T \in G_{\mathcal{D}}\},$$

that clearly extends \mathcal{D} , that is, $\mathcal{D} \subset \mathcal{D}'$. The distribution D' spanned by the elements of \mathcal{D}' is by construction $G_{\mathcal{D}}$ -invariant. That is, for each $\mathcal{F}_T \in G_{\mathcal{D}}$ and for each $z \in M$ in the domain of \mathcal{F}_T ,

$$T_z \mathcal{F}_T(D'(z)) = D'(\mathcal{F}_T(z)). \tag{3.1}$$

Moreover, since $(\mathcal{D}')' = \mathcal{D}'$ by construction, the Stefan–Sussmann theorem guarantees that it is *completely integrable* in the sense that for every point $z \in M$, there exists an integral manifold of D' everywhere of maximal dimension which contains z . The maximal integral manifolds of a completely integrable generalized distribution on M form a *generalized foliation* of M (see for instance [8]). A leaf of a generalized foliation is *regular* if it has a neighborhood where the singular foliation induces a regular foliation by restriction. A point is regular if it belongs to a regular leaf. Regular points are open and dense in M ([8], Theorem 2.2). We will refer to D' (respectively \mathcal{D}') as the *Stefan–Sussmann extension* of D (respectively \mathcal{D}). The Stefan–Sussmann theorem also establishes an equivalence between the $G_{\mathcal{D}}$ -invariance of D ($D' = D$) and its complete integrability; additionally, if D is a completely integrable distribution, then its integral manifolds are the $G_{\mathcal{D}}$ -orbits. When the distribution D has constant dimension, the Stefan–Sussmann theorem reduces to the celebrated and especially convenient Frobenius theorem which states that D is integrable if and only if D is involutive. Recall that D is *involutive* if $[X, Y]$ takes values in D whenever X and Y are vector fields with values in D .

In the sequel, we will use the following notation in order to be able to handle diagonal extensions of different dimensions. Given $l \in \mathbb{N}$ and $X \in \mathfrak{X}(\mathbb{R}^n)$, we will denote by $\tilde{X}^l \in \mathfrak{X}(\mathbb{R}^{ln})$ the diagonal extension of X to \mathbb{R}^{ln} . For the sake of consistency with the previous section \tilde{X} means \tilde{X}^{m+1} .

Theorem 3.1 (Lie–Scheffers’ theorem for SDE). *Let*

$$\delta\Gamma = S(X, \Gamma) \delta X \tag{3.2}$$

be a stochastic differential equation on \mathbb{R}^n , where $X: \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^l$ is a given \mathbb{R}^l -valued semimartingale and $S(x, z): T_x \mathbb{R}^l \rightarrow T_z \mathbb{R}^n$ is a Stratonovich operator from \mathbb{R}^l to \mathbb{R}^n . Let V be an arbitrary open neighborhood of \mathbb{R}^n . Then,

(i) If the X -dependent vector fields $\{S_1(X, \cdot), \dots, S_l(X, \cdot)\}$ can be expressed on V as

$$S_j(X, z) = \sum_{i=1}^r b_j^i(X) Y_i(z) \in T_z \mathbb{R}^n, \quad b_j^i \in C^\infty(\mathbb{R}^l), \quad z \in V, \tag{3.3}$$

and the distribution D spanned by the vector fields $\mathcal{D} = \{Y_1, \dots, Y_r\} \subset \mathfrak{X}(V)$ is involutive, then (3.2) admits a local superposition rule.

(ii) Conversely, suppose that (3.2) admits a superposition rule $(\Phi, \{\Gamma_1, \dots, \Gamma_m\})$ and that the diagonal extensions $\{\tilde{S}_1(X, \cdot), \dots, \tilde{S}_l(X, \cdot)\}$ to $\mathbb{R}^{n(m+1)}$ are tangent to the family \mathcal{G} of nm -dimensional submanifolds of $\mathbb{R}^{n(m+1)}$ associated to this superposition rule (see Proposition 2.4). Let $\tilde{D}(q) := \text{span}\{\tilde{S}_j(X_t, q) \mid j \in \{1, \dots, l\}, t \in \mathbb{R}_+\}$, $q \in \mathbb{R}^{n(m+1)}$, \tilde{D}' the Stefan–Sussmann extension of \tilde{D} , and \mathcal{G}_0 its associated generalized foliation. Let $z \in \mathbb{R}^n$, $p_i = (\Gamma_i)_{t=0}$, and suppose that $p = (z, p_1, \dots, p_m) \in \mathbb{R}^{n(m+1)}$ belongs to a regular leaf $(\mathcal{G}_0)_z$ of \mathcal{G}_0 . Then, there exists an open neighborhood V of z , a family of vector fields $\{Y_1, \dots, Y_r\} \subset \mathfrak{X}(V)$, and a family of functions $\{b_j^i\}_{j=1, \dots, l}^{i=1, \dots, r} \subset C^\infty(\mathbb{R}^l)$ such that

$$S_j(X, v) = \sum_{i=1}^r b_j^i(X) Y_i(v) \tag{3.4}$$

for any $v \in V$. Moreover, the vector fields $\{Y_1, \dots, Y_r\}$ form a real Lie algebra.

Proof. (i) Given $l \in \mathbb{N}$, we define $V^l := V \times \dots \times V$ and $d_l := \max_{q \in V^l} \{\dim(\text{span}\{\tilde{Y}_1^l(q), \dots, \tilde{Y}_r^l(q)\})\}$. Notice that for any $l \in \mathbb{N}$ one has $d_l \leq d_{l+1}$ and $d_l \leq r$. Let $m \in \mathbb{N}$ be the smallest number for which $d_m = d_{m+1}$ and let $q_0 \in V^{m+1}$ be such that

$$\dim(\text{span}\{\tilde{Y}_1^{m+1}(q_0), \dots, \tilde{Y}_r^{m+1}(q_0)\}) = d_{m+1}. \tag{3.5}$$

The maximality of the dimension of $\text{span}\{\tilde{Y}_1^{m+1}, \dots, \tilde{Y}_r^{m+1}\}$ at q_0 implies that there exists a neighborhood U of q_0 in V^{m+1} for which $\dim(\text{span}\{\tilde{Y}_1^{m+1}(q), \dots, \tilde{Y}_r^{m+1}(q)\}) = d_{m+1}$, for all $q \in U$. Indeed, the expression (3.5) is equivalent to saying that the $r \times n(m+1)$ matrix $M(q)$ with entries $M_{ij}(q) := (\tilde{Y}_i^{m+1}(q))^j$ has rank d_m when evaluated at q_0 which, in turn, amounts to the existence of a non-vanishing minor $M_{d_{m+1}}(q_0)$ of $M(q_0)$ of order d_{m+1} . Since the minor $M_{d_{m+1}}(q)$ depends smoothly on q and $M_{d_{m+1}}(q_0) \neq 0$, there exists an open neighborhood U of q_0 in V^{m+1} for which $M_{d_{m+1}}(q) \neq 0$, for any $q \in U$. This implies that $\dim(\text{span}\{\tilde{Y}_1^{m+1}(q), \dots, \tilde{Y}_r^{m+1}(q)\}) \geq d_{m+1}$, for all $q \in U$. However, the maximality used in the definition of d_{l+1} implies that the previous inequality is necessarily an equality.

Consequently, we have found an open set $U \subset V^{m+1}$ in which the distribution D spanned by the family $\{\tilde{Y}_1^{m+1}, \dots, \tilde{Y}_r^{m+1}\}$ has constant rank. Moreover, (2.4) and the hypothesis on $\{Y_1, \dots, Y_r\}$ being in involution imply by the classical Frobenius theorem that D is integrable. Let \mathcal{G}_0 be the family of maximal integrable leaves of D that form a foliation of U^{m+1} . Now, shrinking U if necessary and using foliation coordinates for \mathcal{G}_0 , we extend the distribution D to another integrable distribution $\bar{D} \supset D$ of rank nm whose integrable leaves \mathcal{G} contain those of \mathcal{G}_0 , and for which the restrictions of $\pi_m : \mathbb{R}^{n(m+1)} \rightarrow \mathbb{R}^{nm}$ to the leaves in \mathcal{G} are diffeomorphisms onto their images.

Let now $\{p_1, \dots, p_m\}$ be a set of m distinct points in V such that $(p_1, \dots, p_m) \in \pi_m(U)$ and $\{\Gamma_1, \dots, \Gamma_m\}$ the solutions of (3.2) such that $(\Gamma_1, \dots, \Gamma_m)_{t=0} = (p_1, \dots, p_m)$ a.s. Let $\Gamma := (\Gamma_1, \dots, \Gamma_m)$ and τ the stopping time defined as $\tau := \inf\{t > 0 \mid \Gamma_t \notin \pi_m(U)\}$. Since the vector fields

$$\tilde{S}_j^{m+1}(X, \Gamma) = \sum_{i=1}^r b_j^i(X) \tilde{Y}_i^{m+1}(\Gamma)$$

are tangent to the integral leaves of \mathcal{G}_0 and hence to those of \mathcal{G} , at least up to time τ , Proposition 2.4 guarantees the existence of a local superposition rule.

(ii) We start the proof by providing a lemma that will be needed in our argument.

Lemma 3.2. Let $\{Y_1, \dots, Y_r\} \subset \mathfrak{X}(\mathbb{R}^n)$ with $r \leq mn$ and let $\{\tilde{Y}_1, \dots, \tilde{Y}_r\}$ be the corresponding diagonal extensions to $\mathbb{R}^{n(m+1)}$. Suppose that $\{T_q \pi_m(\tilde{Y}_1(q)), \dots, T_q \pi_m(\tilde{Y}_r(q))\}$ are linearly independent for any q in a neighborhood

$U \subseteq \mathbb{R}^{n(m+1)}$. If the sum $\sum_{i=1}^r b^i \tilde{Y}_i$ with $b^i \in C^\infty(U)$, $i = 1, \dots, r$, is again a diagonal extension then the functions b^i are necessarily the pull-back by π_m of a family of functions in $C^\infty(\pi_m(U))$. More specifically, if $(q_a^j; j = 1, \dots, n; a = 0, \dots, m)$ are coordinates for $\mathbb{R}^{n(m+1)}$, then the functions $\{b^i\}_{i=1, \dots, r}$ do not depend on $(q_a^j; j = 1, \dots, n)$.

Proof. Using the coordinates $(q^j; j = 1, \dots, n)$ for \mathbb{R}^n , there exists a family of functions $A_i^j \in C^\infty(\mathbb{R}^n)$, $i \in \{1, \dots, r\}$, $j \in \{1, \dots, n\}$, such that the vector fields $\{Y_1, \dots, Y_r\} \subset \mathfrak{X}(\mathbb{R}^n)$ can be written as

$$Y_i(q) = \sum_{j=1}^n A_i^j(q) \frac{\partial}{\partial q^j}$$

which implies that the diagonal extensions have the expression

$$\tilde{Y}_i(q_0, \dots, q_m) = \sum_{a=0}^m \sum_{j=1}^n A_i^j(q_a) \frac{\partial}{\partial q_a^j}.$$

Then, if we assume that

$$\sum_{i=1}^r b^i(q_0, \dots, q_m) \tilde{Y}_i(q_0, \dots, q_m) = \sum_{i=1}^r \sum_{a=0}^m \sum_{j=1}^n b^i(q_0, \dots, q_m) A_i^j(q_a) \frac{\partial}{\partial q_a^j}$$

is a diagonal extension on U , then there exist some functions $\{B^i\}_{i=1, \dots, r} \subset C^\infty(\mathbb{R}^n)$ such that

$$\sum_{i=1}^r b^i(q_0, \dots, q_m) A_i^j(q_a) \Big|_U = B^j(q_a)|_U, \quad a = 0, \dots, m, \quad j = 1, \dots, n.$$

That is, the r functions $b^i(q_0, \dots, q_m)$ solve the following subsystem of linear equations

$$\begin{pmatrix} \mathcal{A}(q_0) \\ \mathcal{A}(q_1) \\ \vdots \\ \mathcal{A}(q_m) \end{pmatrix} \begin{pmatrix} b^1(q_0, \dots, q_m) \\ \vdots \\ b^r(q_0, \dots, q_m) \end{pmatrix} = \begin{pmatrix} \mathcal{B}(q_0) \\ \mathcal{B}(q_1) \\ \vdots \\ \mathcal{B}(q_m) \end{pmatrix}, \quad (3.6)$$

where \mathcal{A} and \mathcal{B} are the $n(m+1) \times r$ and $n(m+1) \times 1$ matrices, respectively, defined as $\mathcal{A}(q_a)_{ij} = A_j^i(q_a)$ and $\mathcal{B}(q_a)_i = B^i(q_a)$, $a = 0, \dots, m$. Now, the hypothesis on the linear independence of $\{T\pi_m(\tilde{Y}_1), \dots, T\pi_m(\tilde{Y}_r)\}$ implies that the rank of the matrix $(\mathcal{A}(q_1), \dots, \mathcal{A}(q_m))$ is $r \leq nm$ and hence (3.6) has a unique solution which coincides with the unique solution of the system

$$\begin{pmatrix} \mathcal{A}(q_1) \\ \vdots \\ \mathcal{A}(q_m) \end{pmatrix} \begin{pmatrix} b^1(q_0, \dots, q_m) \\ \vdots \\ b^r(q_0, \dots, q_m) \end{pmatrix} = \begin{pmatrix} \mathcal{B}(q_1) \\ \vdots \\ \mathcal{B}(q_m) \end{pmatrix}. \quad (3.7)$$

Since there is no dependence on the coordinates q_0 in the augmented matrix associated to the system (3.7), its solution (b^1, \dots, b^r) does not therefore depend on q_0 , as required. \square

Suppose now that the stochastic differential equation (3.2) admits a superposition rule and that we are in the hypotheses of the theorem. We start by emphasizing that since the vector fields $\{\tilde{S}_1(X, \cdot), \dots, \tilde{S}_l(X, \cdot)\}$ are, by hypothesis, tangent to the elements of the family \mathcal{G} then their flows leave invariant those submanifolds and hence, the Stefan–Sussmann extension \tilde{D}' of \tilde{D} is also tangent to the elements of \mathcal{G} . This argument guarantees that, given the regular leaf $(\mathcal{G}_0)_z$ of \mathcal{G}_0 , then there exists an element \mathcal{G}_z in \mathcal{G} that contains it.

Now since $p = (z, p_1, \dots, p_m) \in \mathbb{R}^{n(m+1)}$ belongs to a regular leaf $(\mathcal{G}_0)_z$ of \mathcal{G}_0 , then there is an open neighborhood U of p where we can choose (taking regular foliation coordinates) a family of linearly independent vector

fields $\{\tilde{Y}_1, \dots, \tilde{Y}_r\} \subset \mathfrak{X}(\mathbb{R}^{n(m+1)})$ that span the tangent spaces to the leaves of $\mathcal{G}_0 \cap U$. The vector fields $\{\tilde{Y}_1, \dots, \tilde{Y}_r\}$ can be chosen as the diagonal extensions of r vector fields $\{Y_1, \dots, Y_r\} \subset \mathfrak{X}(\mathbb{R}^n)$, since the Stefan–Sussmann extension $\tilde{D}' = \text{span}\{T\tilde{\mathcal{F}}_T \cdot \tilde{S}_i(X, \cdot) \mid i \in \{1, \dots, l\}, \tilde{\mathcal{F}}_T \in G_{\mathcal{D}}\}$ of \tilde{D} is made of diagonal extensions. Indeed, in order to see that \tilde{D}' is spanned by diagonal extensions, it suffices to notice that the flow \tilde{F}_t of the diagonal extension $\tilde{Y} \in \mathfrak{X}(\mathbb{R}^{n(m+1)})$ of a vector field $Y \in \mathfrak{X}(\mathbb{R}^n)$ is $\tilde{F}_t(q_0, \dots, q_m) = (F_t(q_0), \dots, F_t(q_m))$, with F_t the flow of Y ; hence

$$\begin{aligned} T_q \tilde{F}_t(\tilde{Y}(q)) &= (T_{q_0} F_t \times \dots \times T_{q_m} F_t)(\tilde{Y}(q)) \\ &= (T_{q_0} F_t(Y(q_0)), \dots, T_{q_m} F_t(Y(q_m))) = (\widetilde{TF_t(Y)})(q) \end{aligned}$$

is again a diagonal extension. Given that by (2.4) diagonal extensions form an algebra, the statement follows.

Moreover, since the distribution $\tilde{D}'|_U$ is regular and integrable then it is necessarily integrable in the sense of Frobenius, that is, there exist functions $\{c_{ij}^k\}_{i,j,k=1,\dots,r} \subset C^\infty(\mathbb{R}^{n(m+1)})$ such that

$$[\tilde{Y}_j, \tilde{Y}_i] = \sum_{k=1}^r c_{ji}^k \tilde{Y}_k. \tag{3.8}$$

Now, as $[\tilde{Y}_j, \tilde{Y}_i] = [\widetilde{Y_j}, \widetilde{Y_i}]$, we conclude that $\sum_{k=1}^r c_{ji}^k \tilde{Y}_k$ is a diagonal extension. Also, as the projection π_m is a local diffeomorphism when restricted to $U \cap \mathcal{G}_z$, the family of vectors $\{T\pi_m(\tilde{Y}_1), \dots, T\pi_m(\tilde{Y}_r)\}$ is necessarily linearly independent. In these circumstances Lemma 3.2 implies that the coefficients $\{c_{ij}^k\}_{i,j,k=1,\dots,r}$ do not depend on the first n coordinates q_0^j , $j = 1, \dots, n$. We now apply $\pi_{\mathbb{R}^n}^0$ (see (2.5)) on both sides of (3.8) and we obtain

$$[Y_j, Y_i](v) = \sum_{k=1}^r c_{ji}^k(q_1, \dots, q_m) Y_k(v), \tag{3.9}$$

where $v \in V := \pi_{\mathbb{R}^n}^0(U)$ and $(q_1, \dots, q_m) \in \mathbb{R}^{nm}$ is any arbitrary point such that $(v, q_1, \dots, q_m) \in U$. Since the left-hand side of (3.9) does not depend on (q_1, \dots, q_m) then the dependence of the coefficients $c_{ji}^k(q_1, \dots, q_m)$ on those coordinates is necessarily trivial which allows us to conclude that $\{Y_1, \dots, Y_r\}$ close a Lie algebra.

Finally, since the vector fields $\tilde{S}_j(X, \cdot)$ are tangent to \mathcal{G}_0 , $j = 1, \dots, l$, then there is a family of X -dependent functions $b_j^i(X, \cdot) \in C^\infty(U)$ such that

$$\tilde{S}_j(X, q) = \sum_{i=1}^r b_j^i(X, q) \tilde{Y}_i(q)$$

for any $q \in U$. As $\tilde{S}_j(X, \cdot)$ is also a diagonal extension, we can use again Lemma 3.2 in order to prove that the functions $\{b_j^i\}_{j=1,\dots,l}^{i=1,\dots,r}$ do not depend on q_0 . Consequently,

$$\tilde{S}_j(X, q) = \sum_{i=1}^r b_j^i(X, (q_1, \dots, q_m)) \tilde{Y}_i(p). \tag{3.10}$$

As we did in the previous paragraph, we apply $\pi_{\mathbb{R}^n}^0$ on both sides of (3.10)

$$S_j(X, v) = \sum_{i=1}^r b_j^i(X, (q_1, \dots, q_m)) Y_i(v)$$

for any $v \in V$. Again, we realize that since the left-hand side of this equation is independent of (q_1, \dots, q_m) , the dependence of the functions b_j^i on the coordinates (q_1, \dots, q_m) is necessarily trivial, which yields expression (3.4). \square

Remark 3.3. *Theorem 3.1 is a generalization for stochastic differential equations of the classical Lie–Scheffers theorem stated for time-dependent ordinary differential equations. That theorem claims that a differential equation $\dot{y} = Y(t, y)$ on \mathbb{R}^n given by a time-dependent vector field $Y(t, \cdot) \in \mathfrak{X}(\mathbb{R}^n)$, $t \in \mathbb{R}$, admits a superposition rule if and only if Y can be locally written in the form $Y(t, y) = \sum_{i=1}^r f^i(t)Y_i(y)$, where $\{f^i\}_{i=1,\dots,r} \subset C^\infty(\mathbb{R})$ and $\{Y_1, \dots, Y_r\} \subset \mathfrak{X}(\mathbb{R}^n)$ form a (real) Lie subalgebra of $(\mathfrak{X}(M), [\cdot, \cdot])$ (see [4] and [3]). In relation to the traditional presentation of the Lie–Scheffers theorem, our Theorem 3.1:*

(i) *Weakenes the hypotheses under which we can guarantee the existence of superposition rules. The involutivity of the vector fields $\{Y_1, \dots, Y_r\}$ is, in general, less restrictive than requiring that they form a Lie algebra over the reals. We know a posteriori by the second part of Theorem 3.1 that, around regular points, if there exists a superposition rule, the components $\{S_1, \dots, S_l\}$ of the Stratonovich operator can also be expressed in terms of a family of vector fields that close a Lie algebra.*

(ii) *Carefully spells out the regularity conditions under which we have a converse; those conditions are only vaguely evoked in the already cited deterministic papers.*

(iii) *It is worth noticing that, apart from the two points that we just explained, Theorem 3.1 cannot be seen as a mere transcription of the deterministic Lie–Scheffers theorem into the context of Stratonovich stochastic integration by using the so-called Malliavin’s Transfer Principle [22] due to the purely stochastic conditions that appear in the statement of the theorem. Those additional requirements have to do with the tangency of the diagonal extensions of the components of the Stratonovich operator to the family of submanifolds associated to the superposition rule (see also Remark 2.5).*

Remark 3.4. *An interesting research problem would be the formulation of a Lie–Scheffers theorem in the context of Rough Paths Theory [21]. Such result seems to us plausible and would yield Theorem 3.1 as a particular case.*

In the next corollary, we show for the sake of completeness how the classical statement of the Lie–Scheffers theorem (generalized to SDEs) can be easily obtained out of Theorem 3.1.

Corollary 3.5. *Using the notation in Theorem 3.1, suppose that the X -dependent family of vector fields $\{S_1(X, \cdot), \dots, S_l(X, \cdot)\}$ that define the stochastic differential equation (2.1) can be expressed as*

$$S_j(X, z) = \sum_{i=1}^r b_j^i(X)Y_i(z) \in T_z\mathbb{R}^n, \quad b_j^i \in C^\infty(\mathbb{R}^l), \quad z \in \mathbb{R}^n.$$

Let $\text{Lie}\{Y_1, \dots, Y_r\}$ be the real Lie subalgebra of $(\mathfrak{X}(\mathbb{R}^n), [\cdot, \cdot])$ generated by the family $\{Y_1, \dots, Y_r\} \subset \mathfrak{X}(\mathbb{R}^n)$. If $\text{Lie}\{Y_1, \dots, Y_r\}$ is finite dimensional then (2.1) has a superposition rule.

Proof. Let D and D_2 be the generalized distributions associated with the families of vector fields $\mathcal{D} = \{Y_1, \dots, Y_r\}$ and $\mathcal{D}_2 = \text{Lie}\{Y_1, \dots, Y_r\}$, respectively. Observe that if $D(z) \subsetneq D_2(z)$, $z \in \mathbb{R}^n$, then since $\text{Lie}\{Y_1, \dots, Y_r\}$ is finite dimensional, we can always complete the family $\{Y_1, \dots, Y_r\}$ with a finite number of vectors $\{Z_1, \dots, Z_s\} \subset \mathcal{D}$ such that $D(z) = D_2(z)$. We then write the X -dependent vector fields $\{S_1(X, \cdot), \dots, S_l(X, \cdot)\}$ as

$$S_j(X, z) = \sum_{i=1}^r b_j^i(X)Y_i(z) + \sum_{k=1}^s a_j^k(X)Z_k(z), \quad z \in \mathbb{R}^n,$$

with $a_j^k = 0$ for any $j = 1, \dots, l$ and any $k = 1, \dots, s$. Therefore, we may simply suppose that $D(z) = \text{span}\{\text{Lie}\{Y_1, \dots, Y_r\}(z)\}$, $z \in \mathbb{R}^n$ and since D_2 is trivially involutive, the corollary follows from Theorem 3.1(i). \square

4. Lie–Scheffers systems and stochastic differential equations on Lie groups and homogeneous spaces

The Lie–Scheffers systems that are defined by a set of vector fields that generate a finite dimensional Lie algebra, that is, those that satisfy the hypothesis of Corollary 3.5 or of Theorem 5.1 can be reformulated in the language of group actions. More specifically, as we see in the next proposition, such systems come down locally to studying the solutions of an equivalent Lie–Scheffers system on a Lie group.

Proposition 4.1. Consider a stochastic differential equation that satisfies the hypotheses of Corollary 3.5. Let $z \in M$ be a point such that there exists a neighborhood V of z in which the dimension of $\text{Lie}\{Y_1, \dots, Y_r\}$ is constant. Then, shrinking V if necessary, there exists a Lie group G such that $\dim(G) = \dim(\text{Lie}\{Y_1, \dots, Y_r\}|_V)$, a group action $\mathcal{E} : G \times V \rightarrow V$, and Lie algebra elements $\{\xi_1, \dots, \xi_r\} \subset \mathfrak{g}$ such that

$$Y_i(z) = \xi_i^M(z) := \left. \frac{d}{dt} \right|_{t=0} \mathcal{E}(\exp(t\xi_i), z), \quad z \in V. \tag{4.1}$$

Moreover, the solution starting at $z \in M$ of the restriction to V of the stochastic differential equation may be expressed as

$$\Gamma_t^z = \mathcal{E}(g_t, z), \tag{4.2}$$

where $g_t : \mathbb{R}_+ \times \Omega \rightarrow G$ is the semimartingale solution of the stochastic differential equation on G

$$\delta g_t = \sum_{i=1}^r \xi_i^G(g_t) \delta X_t^i \tag{4.3}$$

with initial condition $g_{t=0} = e$ a.s.

Proof. Since the statement of the proposition is local we can always assume that the vector fields $\{Y_1, \dots, Y_r\}$ are complete by multiplying them by a compactly supported bump function and by restricting ourselves to an open neighborhood V consistent with that construction. In that situation and if $\dim(\text{Lie}\{Y_1, \dots, Y_r\}|_V) < \infty$, Palais showed in [23] (see corollary in p. 97 and Theorem III in p. 95) that there exists a unique connected Lie group G contained in the group of diffeomorphisms of M and a left action $\mathcal{E} : G \times M \rightarrow M$ such that (4.1) holds and $T_e \mathcal{E}_z : \mathfrak{g} \rightarrow \text{Lie}\{Y_1, \dots, Y_r\}(z)$ is an isomorphism, for any $z \in V$.

Let now $g_t : \mathbb{R}_+ \times \Omega \rightarrow G$ be the solution semimartingale of the stochastic differential equation on G

$$\delta g_t = \sum_{i=1}^r \xi_i^G(g_t) \delta X_t^i, \tag{4.4}$$

where $\xi_i^G \in \mathfrak{X}(G)$ denotes the right-invariant infinitesimal generator associated to $\xi_i \in \mathfrak{g}$ via the left translations of G on G . Given that any two infinitesimal generators ξ^G and ξ^M , $\xi \in \mathfrak{g}$, are related by the formula $T_g \mathcal{E}_z(\xi^G) = \xi^M(\mathcal{E}(g, z))$, $g \in G$, $z \in V$, it is straightforward to verify that if g_t is a solution of (4.3) with initial condition $g_{t=0} = e$ a.s., then

$$\Gamma_t^z = \mathcal{E}(g_t, z),$$

is the solution of $\delta \Gamma_t = \sum_{i=1}^r Y_i(\Gamma_t) \delta X_t^i$ such that $\Gamma_0 = z$, a.s. □

Remark 4.2. Observe that (4.2) may be understood as a general reformulation of (5.2) (see also [2], Theorem 19). Processes of the type $\Gamma_t^z = \mathcal{E}(g_t, z)$ defined using a group action are sometimes called one point motions [19].

The proposition that we just proved shows that for Lie–Scheffers systems defined by vector fields that generate a finite dimensional Lie algebra \mathfrak{g} , it is the associated Lie–Scheffers system on the Lie group G (4.3) that really matters. This is the subject of the rest of this section.

Stochastic differential equations on Lie groups

Let now G be an arbitrary connected Lie group and \mathfrak{g} its Lie algebra. Let $\{\xi_1, \dots, \xi_l\}$ and $\{\varepsilon^1, \dots, \varepsilon^l\}$ be dual bases of \mathfrak{g} and \mathfrak{g}^* , respectively. Left (respectively, right) translations on G will be denoted by $L : G \times G \rightarrow G$ (respectively,

$R : G \times G \rightarrow G$). With the same notation that we have used so far, let

$$S(\mu, g) : T_\mu \mathfrak{g} \simeq \mathfrak{g} \longrightarrow T_g G, \tag{4.5}$$

$$\eta \longmapsto \sum_{i=1}^l \xi_i^G(g)(\varepsilon^i, \eta) = \eta^G(g)$$

be a Stratonovich operator from \mathfrak{g} to G , where η^G denotes the infinitesimal generator associated to the G -action on itself by left translations. Consider the stochastic differential equation associated to (4.5),

$$\delta g_t = \sum_{i=1}^l \xi_i^G(g_t) \delta X_t^i \tag{4.6}$$

for some driving noise (semimartingale) $X : \mathbb{R}_+ \times \Omega \rightarrow \mathfrak{g}$. Using the equivariance of the vector fields $\xi^G \in \mathfrak{X}(G)$ with respect to right translations, that is, $T_h R_g(\xi^G(h)) = \xi^G(R_g(h))$ for any $g, h \in G$, and $\xi \in \mathfrak{g}$, it is immediate to check that if Γ^e is the solution of (4.6) with initial condition $\Gamma_{t=0}^e = e$ a.s., then the solution Γ_t^g starting at $g \in G$ is given by

$$\Gamma_t^g = L_{\Gamma_t^e} g = R_g(\Gamma_t^e). \tag{4.7}$$

In other words, the stochastic differential equation (4.6) has a superposition rule in the sense of Definition 2.1 and the superposition function Φ is given by

$$\Phi : G \times G \longrightarrow G,$$

$$(h, g) \longmapsto L_h g = R_g h.$$

It is also worth noticing that (4.6) is *stochastically complete* ([9], Chapter VII, Section 6) since it is a left-invariant system. Therefore any solution of (4.6) is defined for all $(t, \omega) \in \mathbb{R}_+ \times \Omega$ and, consequently, so is any one point motion and, in particular, any solution of any Lie–Scheffers system on a manifold M which can be globally considered as induced by a group action $\mathcal{E} : G \times M \rightarrow M$.

Lévy processes and Lie–Scheffers systems

This is an important class of Lie group valued stochastic processes and, as we will now see, a class of examples of Lie–Scheffers systems. Recall that a continuous process $g : \mathbb{R}_+ \times \Omega \rightarrow G$ is called a right Lévy process if, for any $0 = t_0 < t_1 < t_2 < \dots < t_n$, the increments

$$g_{t_0}, g_{t_0} g_{t_1}^{-1}, g_{t_1} g_{t_2}^{-1}, \dots, g_{t_{n-1}} g_{t_n}^{-1} \tag{4.8}$$

are *independent* and *stationary*. This means that the random variables in (4.8) are mutually independent and that their distributions only depend on the differences $t_i - t_{i-1}$, $i \in \{1, \dots, n\}$. If $g_{t_0} \neq e$ a.s., we define $g_t^e = g_t g_{t_0}^{-1}$, which is a right Lévy process starting at the identity.

We are now going to see that continuous Lévy processes and Lie–Scheffers systems are closely related. First of all, recall that any right Lévy process on a locally compact topological group with a countable basis of open sets is a Markov process with a right-invariant Feller transition semigroup $\{P_t\}_{t \in \mathbb{R}_+}$ given by $P_t f(g) := E[f(g_t^e g)]$, $g \in G$, where $f : G \rightarrow \mathbb{R}$ is any measurable function. Conversely, any rightinvariant *continuous* Markov process is a right Lévy process ([19], Proposition 1.2). Moreover, if $g : \mathbb{R}_+ \times \Omega \rightarrow G$ is a right Lévy process, then there exists an l -dimensional Brownian motion $B : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^l$ with respect to the natural filtration $\{\mathcal{F}_t^e\}_{t \in \mathbb{R}_+}$ of the process g_t^e , $l = \dim(\mathfrak{g})$, with covariance matrix $(a_{ij})_{i,j=1,\dots,l}$ and some constants $\{c_i\}_{i=1,\dots,l}$ such that

$$f(g_t) = f(g_0) + \sum_{i=1}^l \int_0^t \xi_i^G[f](g_s) \delta B_s^i + \sum_{i=1}^l c_i \int_0^t \xi_i^G[f](g_s) ds$$

for any $f \in C^2(G)$ and where, as before, $\{\xi_1, \dots, \xi_l\}$ is a basis of \mathfrak{g} ([19], Theorem 1.2). This expression amounts to saying that the Lévy process $g: \mathbb{R}_+ \times \Omega \rightarrow G$ satisfies the stochastic differential equation

$$\delta g_t = \sum_{i=1}^l c_i \xi_i^G(g_s) \delta s + \sum_{i=1}^l \xi_i^G(g_s) \delta B_s^i,$$

and hence by Corollary 3.5 we can conclude that *any continuous right Lévy process is a solution of a right-invariant Lie–Scheffers system*. Additionally, it can be shown in this context (see [19], Theorem 1.2) that one point motions obtained out of a G -action $\mathcal{E}: G \times M \rightarrow M$ are Markov processes with Feller transition semigroup $\{P_t^M\}_{t \in \mathbb{R}_+}$

$$P_t^M f(z) = E[f(\mathcal{E}(g_t^e, z))], \quad z \in M, \quad f \in C(M).$$

Lie–Scheffers systems on homogeneous spaces

Let $H \subset G$ be a closed subgroup of G and consider the *homogeneous space* $G/H = \{gH \mid g \in G\}$ with the unique smooth structure that makes the projection $\pi_H: G \rightarrow G/H$ into a submersion. The group G acts on G/H via the map $\lambda: G \times G/H \rightarrow G/H$ on G/H defined by $(h, gH) \mapsto (hg)H$. It is immediate to check that the infinitesimal generators associated to the left G -actions on G and on G/H are π_H -related, that is,

$$T_g \pi_H(\xi^G(g)) = \xi^{G/H}(\pi_H(g))$$

for any $g \in G$, any $\xi \in \mathfrak{g}$, and where $\xi^{G/H}(gH) = \frac{d}{dt}|_{t=0} \lambda_{\exp(t\xi)}(gH)$. This straightforward observation has as an immediate consequence the next proposition:

Proposition 4.3. *Let $X: \mathbb{R}_+ \times \Omega \rightarrow \mathfrak{g}$ be a \mathfrak{g} -valued semimartingale, G a Lie group, and $H \subset G$ a closed subgroup. Let Γ be a solution of the Lie–Scheffers system defined by X and the Stratonovich operator (4.5) with initial condition $\Gamma_{t=0}$. Then, $\pi_H(\Gamma)$ is a solution of the Lie–Scheffers system on G/H*

$$\delta \bar{\Gamma} = \sum_{j=1}^l \xi_j^{G/H}(\bar{\Gamma}_t) \delta X_t^j \tag{4.9}$$

with initial condition $\pi_H(\Gamma_{t=0})$.

Observe that since the Stratonovich operator (4.5) is right-invariant by the action of G , and therefore H -invariant, and that since this action is free and proper, the previous proposition can be seen as a particular case of the *Reduction theorem* in [18]. The next theorem is a transcription of the *Reconstruction theorem* in [18] into the present context and describes how to construct solutions in the opposite direction, that is, it tells us *how to construct a solution Γ of the Lie–Scheffers system (4.6) out of the solutions of two other dimensionally smaller Lie–Scheffers systems*: first, a solution of the reduced system (4.9) and second, another solution of a new Lie–Scheffers system, now on H .

Theorem 4.4. *Let $X: \mathbb{R}_+ \times \Omega \rightarrow \mathfrak{g}$ be a \mathfrak{g} -valued semimartingale, G a Lie group, $H \subset G$ a closed subgroup, and S the Stratonovich operator defined in (4.5). Let $R: H \times G \rightarrow G$ be the (right) action of H on G by right translations and A an auxiliary principal connection on $\pi_H: G \rightarrow G/H$. Then, any solution Γ of the system (4.6) can be written in the form*

$$\Gamma_t = R_{h_t} g_t = g_t h_t.$$

In this statement, $g: \mathbb{R}_+ \times \Omega \rightarrow G$ is a G -valued semimartingale horizontal with respect to A , i.e., $\int \langle A, \delta g_t \rangle = 0 \in \mathfrak{g}$, $g_{t=0} = \Gamma_{t=0}$, and such that $\pi_H(g_t)$ is a solution of the reduced system (4.9). On the other hand, $h: \mathbb{R}_+ \times \Omega \rightarrow H$ is an H -valued semimartingale that satisfies the stochastic differential equation

$$\delta h_t = \tilde{R}(Y_t, h_t) \delta Y_t \tag{4.10}$$

with initial condition $h_{t=0} = e$, and associated to the Stratonovich operator

$$\begin{aligned} \tilde{R}(\xi, h) : T_\xi \mathfrak{h} &\longrightarrow T_h H, \\ \eta &\longmapsto T_e R_h(\eta) = \eta^H(h) \end{aligned} \tag{4.11}$$

and the stochastic component $Y : \mathbb{R}_+ \times \Omega \rightarrow \mathfrak{h}$ given by

$$Y = \sum_{i=1}^l \int A_{g_t}(\xi_i^G(g_t)) \delta X^i.$$

Proof. See [18], Theorem 3.2 and Proposition 3.4. □

4.1. The Wei–Norman method for solving stochastic Lie–Scheffers systems

The method that we are going to develop in this subsection is a generalization to stochastic systems of the one proposed by Wei and Norman in [27,28] in order to solve by quadratures time evolution equations of the form $\frac{dU_t}{dt} = H_t U_t$ that appear in quantum mechanics, where both U_t and H_t are bounded linear operators on a suitable Hilbert space. This method has already been adapted by Cariñena and Ramos [6] to the study of deterministic Lie–Scheffers systems on Lie groups and it is their approach that we will follow. As we will see later on, the power of this method and the ease of its implementation depends strongly on the algebraic structure of the Lie algebra \mathfrak{g} of the group G where the solutions of the stochastic differential equation take values.

Let $\Gamma : \mathbb{R} \times \Omega \rightarrow G$ be the solution of (4.6) such that $\Gamma_{t=0} = e \in G$ a.s.; we write it down in terms of second kind canonical coordinates with respect to a basis $\{\xi_1, \dots, \xi_l\}$ of the Lie algebra \mathfrak{g} . That is,

$$\Gamma_t = \exp(d_t^1 \xi_1) \cdots \exp(d_t^l \xi_l), \tag{4.12}$$

where $\{d_t^1, \dots, d_t^l\}$ is a family of real-valued semimartingales, $d^i : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$, such that $d_{t=0}^i = 0$ a.s. for any $i = 1, \dots, l$. Notice that the expression (4.12) is only valid up to the exit time of Γ from the neighborhood U_e of $e \in G$ where the second kind of canonical coordinates for G around the origin are valid. The key idea in this method is that if the functions d^i were differentiable then

$$\frac{d\Gamma_t}{dt} = T_e R_{\Gamma_t} \left(\sum_{i=1}^l \dot{d}_t^i \left(\prod_{j<i} \text{Ad}_{\exp(d_t^j \xi_j)} \right) \xi_i \right)$$

(see [6], Eqs (33) and (34)), where $\text{Ad}_g(\eta) \in \mathfrak{g}$ is the adjoint representation of G on \mathfrak{g} , $g \in G$, $\eta \in \mathfrak{g}$. In our setup we obviously cannot invoke the differentiability of the functions d^i ; however, applying the Stratonovich differentiation rules to (4.12) with d^i our real-valued semimartingales, $i = 1, \dots, l$, we have

$$\delta \Gamma_t = T_e R_{\Gamma_t} \left(\sum_{i=1}^l \delta d_t^i \left(\prod_{j<i} \text{Ad}_{\exp(d_t^j \xi_j)} \right) \xi_i \right).$$

This expression implies that for any right-invariant one-form $\mu^G \in \Omega(G)$, that is, $\mu^G(g) = T_g^* R_{g^{-1}}(\mu)$ for any $g \in G$ and a fixed $\mu \in \mathfrak{g}^*$,

$$\int \langle \mu^G, \delta \Gamma \rangle = \left\langle \mu, \sum_{i=1}^r \int \left(\prod_{j<i} \text{Ad}_{\exp(\sum_{j=1}^l d_t^j v_j)} \right) \xi_i \delta d_t^i \right\rangle. \tag{4.13}$$

At the same time, it is clear that $\int \langle \mu^G, \delta \Gamma \rangle = \langle \mu, X \rangle$ and hence (4.13) implies that

$$X = \sum_{i=1}^l \int \left(\prod_{j<i} \text{Ad}_{\exp(d_t^j \xi_j)} \right) \xi_i \delta d_t^i.$$

Using the identity $\text{Ad}_{\exp(\eta)} = e^{\text{ad}(\eta)} = \sum_{n \geq 0} \frac{1}{n!} \text{ad}(\eta) \circ \dots \circ \text{ad}(\eta)$, for any $\eta \in \mathfrak{g}$, and writing $X = \sum_{i=1}^l X^i \xi_i$, we get the relation

$$\sum_{i=1}^l X^i \xi_i = \sum_{i=1}^l \int \left(\prod_{j < i} e^{\text{ad}(d_t^j \xi_j)} \right) \xi_i \delta d_t^i. \tag{4.14}$$

The system of stochastic differential equations (4.14) can be solved for the semimartingales d_t^i , $i = 1, \dots, m$ by quadratures if the Lie algebra \mathfrak{g} is solvable (see [27,28]) and, in particular, for nilpotent Lie algebras. The solvable case was extensively studied in [17] where similar conclusions were presented using a different approach.

As a simple example consider the affine group in one dimension \mathcal{A}_1 , that is, the group of affine transformations of the real line. Any element of \mathcal{A}_1 can be expressed as a pair of real numbers (a_0, a_1) with $a_1 \neq 0$ defining the affine transformation $x \mapsto a_1 x + a_0$. The product $*$: $\mathcal{A}_1 \times \mathcal{A}_1 \rightarrow \mathcal{A}_1$ in \mathcal{A}_1 is

$$(a_0, a_1) * (b_0, b_1) = (a_0 + a_1 b_0, a_1 b_1).$$

If $\{\xi_0 = (1, 0), \xi_1 = (0, 1)\}$ is a basis of the Lie algebra \mathfrak{a}_1 of \mathcal{A}_1 , it is immediate to check that

$$[\xi_0, \xi_1] = \text{ad}_{\xi_0}(\xi_1) = -\xi_0. \tag{4.15}$$

Furthermore, the infinitesimal generators associated to the left action of \mathcal{A}_1 on itself are

$$\xi_0^{\mathcal{A}_1}(x, y) = \frac{\partial}{\partial x} \quad \text{and} \quad \xi_1^{\mathcal{A}_1}(x, y) = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}.$$

A typical Lie system on \mathcal{A}_1 would be, for instance, the following Stratonovich differential equation on the upper half-plane $H_+ = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$,

$$\delta \Gamma_x = dt + \Gamma_x \delta B_t, \quad \delta \Gamma_y = \Gamma_y \delta B_t$$

obtained as a particular case of (4.5) when $G = \mathcal{A}_1$ and $X = (t, B)$, where $B: \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ is a Brownian motion. More generally, let $X: \mathbb{R}_+ \times \Omega \rightarrow \mathfrak{a}_1$ be an \mathfrak{a}_1 -valued semimartingale and write $X = X^0 \xi_0 + X^1 \xi_1$, with X^0 and X^1 real semimartingales. Then, using (4.15), (4.14) reads in this particular case

$$X^0 \xi_0 + X^1 \xi_1 = \int \xi_0 \delta d_t^0 + \int (\xi_1 - d_t^0 \xi_0) \delta d_t^1 = \left(\int \delta d_t^0 - \int d_t^0 \delta d_t^1 \right) \xi_0 + \left(\int \delta d_t^1 \right) \xi_1.$$

Putting together the terms that go both with ξ_1 and ξ_0 , respectively, we obtain

$$d_t^1 = X_t^1, \quad d_t^0 = X_t^0 + \int_0^t d_s^0 \delta X_s^1$$

and hence

$$d_t^0 = e^{X_t^1} \left(\int_0^t \delta X_s^0 e^{-X_s^1} \right).$$

5. The flow of a stochastic Lie–Scheffers system

Theorem 3.1 claims, roughly speaking, that the stochastic system (2.1) admits a superposition rule $(\Phi, \{\Gamma_1, \dots, \Gamma_m\})$ if the components of the Stratonovich operator $S(x, z): T_x \mathbb{R}^l \rightarrow T_z \mathbb{R}^n$, $x \in \mathbb{R}^l$, $p \in \mathbb{R}^n$, that define it may be written as $S_j(X, z) = \sum_{i=1}^r b_j^i(X) Y_i(z)$, where $b_j^i \in C^\infty(\mathbb{R}^l)$ and $\{Y_1, \dots, Y_r\} \subseteq \mathfrak{X}(\mathbb{R}^n)$ span an involutive distribution. The converse of this statement is also true provided that, for a given initial condition $z \in \mathbb{R}^n$, the point $(z, (\Gamma_1, \dots, \Gamma_m)_{t=0})$ is a regular point of the foliation \mathcal{G}_0 generated by the diagonal extensions of $\{S_1(X, \cdot), \dots, S_m(X, \cdot)\}$. Notice that this

is a reasonable condition since the set of regular points of a generalized foliation is open and dense ([8], Theorem 2.2). Moreover, when this happens, the vector fields $\{Y_1, \dots, Y_r\}$ form a real Lie algebra.

The condition on the vector fields $\{Y_1, \dots, Y_r\}$ forming a real finite dimensional Lie algebra or, more generally, $\dim(\text{Lie}\{Y_1, \dots, Y_r\}) < \infty$, are particularly appealing since these are algebraic requirements that we may expect to be easily verified for stochastic differential equations of a certain type. Moreover, these conditions have consequences that go beyond Corollary 3.5. More specifically, we will show that if $\dim(\text{Lie}\{Y_1, \dots, Y_r\}) < \infty$, then the general solution of a stochastic differential equation *can be written by composing a deterministic function with a suitable noise*. In the following paragraphs we are going to give a precise meaning to this statement and will put it in the context of well-known results available in the literature.

Traditionally, stochastic differential equations on a manifold M have been presented as

$$\delta \Gamma_t = Y_0(\Gamma_t) dt + \sum_{i=1}^r Y_i(\Gamma_t) \delta B_t^i, \tag{5.1}$$

where $\{Y_0, \dots, Y_r\} \subseteq \mathfrak{X}(M)$ and $B: \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^r$ is an r -dimensional Brownian motion defined on a standard filtered probability space $(\Omega, \mathcal{F}_t, P)$. For the sake of having a more compact notation, we write $B_t^0 := t$. The flow of such a stochastic differential equation may be locally written, that is, up to a given stopping time τ , by means of a Taylor series expansion that comes out of Picard’s iterative method for solving stochastic differential equations. In order to be more explicit we introduce some notation. Let $J = \{j_1, \dots, j_n\}$, $j_i \in \{0, \dots, r\}$, $1 \leq i \leq n$, be a multi-index of size n . $\|J\|$ will denote the *degree* of J that, by definition, is the size of J plus the number of zeros in the n -tuple (j_1, \dots, j_n) . For any $J = \{j_1, \dots, j_n\}$, we consider the iterated Stratonovich multiple integral

$$B_t^J = \int_{0 < t_1 < \dots < t_n < t} \dots \int \delta B_{t_1}^{j_1} \dots \delta B_{t_n}^{j_n}.$$

In addition, Y_J will denote

$$Y_J := [Y_{j_1}, [Y_{j_2}, \dots, [Y_{j_{n-1}}, Y_{j_n}]]].$$

If $Y \in \mathfrak{X}(M)$ is a vector field on the manifold M , we will use the following notation for its flow: $\exp(sY)(z)$ denotes the solution at time s of the ordinary differential equation $\dot{\gamma} = Y(\gamma)$ with initial condition $\gamma(0) = z$. Then,

Theorem 5.1 ([2], Theorem 20). *With the notation introduced so far, if $\dim(\text{Lie}\{Y_0, \dots, Y_r\}) < \infty$ and $\text{span}\{\text{Lie}\{Y_0, \dots, Y_r\}\}$ has constant dimension on a neighborhood V of the point $z \in M$, then there exists a stopping time τ such that the solution of (5.1) with initial condition z can be expressed as*

$$\Gamma_t^z = \exp\left(\sum_{n=1}^{\infty} \sum_{\|J\|=n} \beta_J B_t^J\right)(z) \tag{5.2}$$

up to time τ . In this expression, $\beta_J := \sum_{\sigma \in S_n} \frac{(-1)^{e(\sigma)}}{n^2 \binom{n-1}{e(\sigma)}} Y_{\sigma(J)}$, S_n denotes the permutation group of n elements, and $e(\sigma)$ is the cardinality of the set $\{j \in \{1, \dots, n-1\} | \sigma(j) > \sigma(j+1)\}$.

If the finiteness condition on the dimensionality of the Lie algebra generated by the vector fields is not available but, nevertheless, $\{Y_0, \dots, Y_r\}$ are Lipschitz vector fields, then the solution of (5.1) starting at $z \in M$ can always be approximated by a process like (5.2): if ζ_t^N denotes the *finite* sum $\sum_{n=1}^N \sum_{\|J\|=n} \beta_J B_t^J$, then

$$\Gamma_t^z = \exp(\zeta_t^N)(z) + t^{N/2} R_N(t),$$

where the error term $R_N(t)$ is bounded in probability when t tends to 0 ([7], Theorem 2.1). The expression (5.2) also holds if instead of the hypotheses of Theorem 5.1 we require M to be an analytic manifold and $\{Y_0, \dots, Y_r\}$ a family of real analytic vector fields ([2], Theorem 10). An important consequence of Theorem 5.1 lies in the fact that

the general solution of the stochastic differential equation (5.1) may be written, at least locally and up to a suitable stopping time τ , as the composition of a deterministic and smooth function, namely, the flow exponential, with the diffusion that defines the stochastic differential equation (see [15] for a complementary reading). From this point of view, there is a strong resemblance between Theorem 5.1 and Theorem 3.1:

- First, by Corollary 3.5, all the systems that satisfy the hypotheses of Theorem 5.1 admit a superposition rule.
- Second, the superposition rule allows us to write any solution as a composition of the deterministic function and the set of solutions $\{\Gamma_1, \dots, \Gamma_m\}$ that are responsible for the stochastic behavior of the resulting flow.

We conclude by quoting two references that study the nilpotent case (that is, the Lie algebra $\text{Lie}\{Y_0, \dots, Y_r\}$ is nilpotent); this case has deserved special attention in the literature (see, e.g., [17]) because in that situation the Taylor series expansion of the flow in terms of iterated integrals in (5.2) becomes finite. We also recommend the excellent exposition in [1] for a complementary approach to the subject of Taylor series approximation of the general solution of (5.1); in this book it is shown that, for instance, the Carnot group of depth $N = \dim(\text{Lie}\{Y_0, \dots, Y_r\})$ can be used in the nilpotent case to integrate the Lie algebra action of $\text{Lie}\{Y_0, \dots, Y_r\}$ when one writes, as we did in the previous section, a Lie–Scheffers system as a stochastic differential equation on a Lie group that acts on the manifold in question.

6. Examples

6.1. Inhomogeneous linear systems

Let $A_k : \mathbb{R} \rightarrow M_n(\mathbb{R})$ be an $n \times n$ time-dependent real matrix and $B_k : \mathbb{R} \rightarrow \mathbb{R}^n$ be a time-dependent vector for any $k = 1, \dots, l$. Let $X : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^l$ be a semimartingale. An inhomogeneous linear system is a system of stochastic differential equations on \mathbb{R}^n that may be written as

$$\delta \Gamma_t = \sum_{k=1}^l (A_k(t)(\Gamma_t) - B_k(t)) \delta X_t^k. \tag{6.1}$$

Let (q^1, \dots, q^n) be coordinates for \mathbb{R}^n . It is an exercise to check that (6.1) can be equivalently written as

$$\delta \Gamma_t = \sum_{k=1}^l \sum_{i,j=1}^n (A_k)_i^j(t) Y_j^i(\Gamma_t) \delta X_t^k + \sum_{k=1}^l \sum_{i,j=1}^n (B_k)^j(t) Z_j(\Gamma_t) \delta X_t^k,$$

where the vector fields $Y_j^i, Z_j \in \mathfrak{X}(\mathbb{R}^n)$, $i, j, k = 1, \dots, n$, are given by

$$Y_j^i = q^i \frac{\partial}{\partial q^j}, \quad Z_j = \frac{\partial}{\partial q^j}.$$

Given that

$$[Y_j^i, Y_l^k] = \delta_j^k Y_l^i - \delta_l^j Y_j^k, \quad [Y_j^i, Z_k] = -\delta_k^i Z_j \quad \text{and} \quad [Z_i, Z_j] = 0$$

we see that the vectors $\{Y_j^i, Z_k \mid i, j, k = 1, \dots, n\} \subset \mathfrak{X}(\mathbb{R}^n)$ span a Lie algebra isomorphic to the $(n^2 + n)$ -dimensional Lie algebra of the group of affine transformations of \mathbb{R}^n . Therefore, the system (6.1) satisfies the hypotheses of Theorem 3.1 and hence it admits a superposition rule. In order to explicitly construct the superposition rule, let Γ^{e_j} be the solution of the homogeneous part of (6.1),

$$\delta \Gamma_t = \sum_{k=1}^l A_k(t)(\Gamma_t) \delta X_t^k$$

with initial solution $\Gamma_{t=0}^{e_j} = e_j \in \mathbb{R}^n$ a.s., where $e_j = (0, \overset{j-1}{\dots}, 0, 1, 0, \dots, 0)$ for any $j = 1, \dots, n$. Let $\bar{\Gamma}$ be a particular solution of (6.1) with initial condition $\bar{\Gamma}_{t=0} = 0 \in \mathbb{R}^n$ a.s. Then,

$$\Gamma_t = \sum_{j=1}^n z^j \Gamma_t^{e_j} + \bar{\Gamma}_t$$

is the general semimartingale solution of (6.1) starting at $z = (z^1, \dots, z^n) \in \mathbb{R}^n$.

6.2. The stochastic exponential of a Lie group

Let G be a Lie group and \mathfrak{g} be its Lie algebra. Let $\{\xi_1, \dots, \xi_l\}$ be a basis of \mathfrak{g} and $X : \mathbb{R}_+ \times \Omega \rightarrow \mathfrak{g}$ be a \mathfrak{g} -valued semimartingale. Observe that X can be written as $X = \sum_{i=1}^l a_t^i \xi_i$ for a family of real semimartingales $a^i : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$, $i = 1, \dots, l$. Following [13] and [12], we define the (left) *stochastic exponential* $\mathcal{E}(X) : \mathbb{R}_+ \times \Omega \rightarrow G$ of X as the unique solution of the Lie–Scheffers system on G given by

$$\delta \Gamma_t = \sum_{i=1}^l (\xi_i)^G(\Gamma_t) \delta a_t^i$$

with initial condition $\Gamma_{t=0} = e \in G$ a.s. Unlike the conventions used in Section 4, the vector fields $(\xi_i)^G \in \mathfrak{X}(G)$ here are not the right-invariant vector fields built from ξ_i , $i = 1, \dots, l$, but the left-invariant ones. That is,

$$(\xi_i)^G(g) = T_e L_g(\xi_i), \quad g \in G.$$

Except for the fact that $(\xi_i)^G \in \mathfrak{X}(G)$, $i = 1, \dots, l$, are now left-invariant, solving a Lie–Scheffers system on a Lie group such as those presented in Section 4 amounts to computing the stochastic exponential of a given \mathfrak{g} -valued semimartingale X .

The stochastic exponential establishes a bijection between \mathfrak{g} -valued local martingales and martingales on G with respect to certain connections. Recall that, given an affine connection $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ on a manifold M , a M -valued semimartingale $\Gamma : \mathbb{R}_+ \times \Omega \rightarrow M$ is said to be a ∇ -martingale (or a martingale with respect to ∇) provided that

$$f(\Gamma) - f(\Gamma_{t=0}) - \frac{1}{2} \int \text{Hess } f(d\Gamma, d\Gamma)$$

is a real local martingale for any $f \in C^\infty(M)$, where $\text{Hess } f : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow C^\infty(M)$ is the bilinear form defined as

$$\text{Hess } f(Y, Z) = Y[Z[f]] - \nabla_Z Y[f]$$

for any $Y, Z \in \mathfrak{X}(M)$ (see [11], Chapter IV). When $M = G$ is a Lie group, one can construct left-invariant connections ∇ by using bilinear skew-symmetric forms $\alpha : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ on the Lie algebra \mathfrak{g} via the definition

$$\nabla_{\xi} \eta^G := \alpha(\xi, \eta), \quad \xi, \eta \in \mathfrak{g}.$$

The curves $\exp(t\xi) \in G$, where $\xi \in \mathfrak{g}$ and $\exp : \mathfrak{g} \rightarrow G$ is the Lie algebraic exponential, coincide with the geodesics $c(t)$ with respect to these connections that start at $e \in G$ and that satisfy $\dot{c}(0) = \xi$. It can be shown ([12], Lemma 1.4) that the connections built from $\alpha = 0$ and $\alpha(\xi, \eta) = \frac{1}{2}[\xi, \eta]$ induce the same ∇ -martingales on G . Moreover, with respect to these two connections, the set of ∇ -martingales consists precisely of the processes of the form $\Gamma_0 \mathcal{E}(X)$ where X is a \mathfrak{g} -valued local martingale and Γ_0 a G -valued \mathcal{F}_0 -measurable random variable ([12], Proposition 1.9). This expression provides the bijection between \mathfrak{g} -valued local martingales and ∇ -martingales on G that we announced above.

6.3. Geometric Brownian motion

Let (\mathbb{R}_+, \cdot) be the Abelian Lie group of strictly positive real numbers endowed with the standard product. Its Lie algebra is simply \mathbb{R} and, for any $\xi \in \mathbb{R}$, the Lie algebra exponential coincides with the standard exponential, that is, $\exp \xi = e^\xi$; consequently, the infinitesimal generator (right- or left-invariant) is

$$\xi^{\mathbb{R}_+}(q) = \xi q \quad \text{for any } q \in \mathbb{R}.$$

Let $G = \mathbb{R}_+ \times \dots \times \mathbb{R}_+$ be the Lie group constructed as the direct product of n copies of (\mathbb{R}_+, \cdot) . Its product map $\cdot : G \times G \rightarrow G$ is obviously $(a_1, \dots, a_n) \cdot (b_1, \dots, b_n) = (a_1 b_1, \dots, a_n b_n)$, $a_i, b_i \in \mathbb{R}_+$ for any $i = 1, \dots, n$, and its Lie algebra is $\mathfrak{g} = T_1 \mathbb{R}_+ \times \dots \times T_1 \mathbb{R}_+ \simeq \mathbb{R} \times \dots \times \mathbb{R} = \mathbb{R}^n$. Let $\{\xi_i = (0, \dots, 1, \dots, 0) \mid i = 1, \dots, n\}$ be the canonical basis of $\mathfrak{g} = \mathbb{R}^n$, $\mu = (\mu^1, \dots, \mu^n)$, $\sigma = (\sigma^1, \dots, \sigma^n) \in \mathfrak{g}$ a couple of elements of \mathfrak{g} , $B : \mathbb{R}_+ \times \Omega \rightarrow \mathfrak{g}$ a n -dimensional Brownian motion on some filtered probability space $(\Omega, P, \{\mathcal{F}_t\}_{t \in \mathbb{R}_+})$, and consider the following Lie–Scheffers system on G

$$\delta \Gamma_t = \left(\mu - \frac{1}{2} \sigma^2 \right)^G (\Gamma_t) dt + \sum_{i=1}^n \sigma^i \xi_i^G (\Gamma_t) \delta B_t^i, \tag{6.2}$$

where $\sigma^2 = ((\sigma^1)^2, \dots, (\sigma^n)^2)$. Using coordinates (q^1, \dots, q^n) in G we can rewrite (6.2) as

$$\delta q_t^i = \left(\mu^i - \frac{1}{2} (\sigma^i)^2 \right) q_t^i dt + \sigma^i q_t^i \delta B_t^i, \quad i = 1, \dots, n,$$

which may be rewritten in terms of Itô integrals as

$$dq_t^i = \mu^i q_t^i dt + \sigma^i q_t^i dB_t^i, \quad i = 1, \dots, n. \tag{6.3}$$

The solutions of the n -dimensional system of stochastic differential equation (6.3) are usually referred to as the geometric Brownian motion which is well known for its use in the Black–Scholes theory of derivatives pricing as a model for the time evolution of the prices of n assets in a complete and arbitrage-free financial market.

The well-known solution of the differential equation (6.3) can be easily obtained by using the stochastic version of the Wei–Norman method that we introduced in Section 4.1. Indeed, let $q_t = \exp(a_t^1 \xi_1) \cdots \exp(a_t^n \xi_n)$ be the solution of (6.3) starting at $e = (1, \dots, 1) \in G$, where $a^i : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ are real semimartingales such that $a_{t=0}^i = 0$ a.s. for any $i = 1, \dots, n$. Since the Lie algebra \mathfrak{g} of G is Abelian, and (6.2) is written in Lie–Scheffers form

$$\delta \Gamma_t = \sum_{i=1}^n \xi_i^G (\Gamma_t) \delta X_t^i$$

by taking the noise semimartingale $X := ((\mu^1 - \frac{(\sigma^1)^2}{2})t + \sigma^1 B_t^1, \dots, (\mu^n - \frac{(\sigma^n)^2}{2})t + \sigma^n B_t^n)$, Eq. (4.14) in the Wei–Norman method reduces to

$$(\mu^1 - (\sigma^1)^2/2, \dots, \mu^n - (\sigma^n)^2/2)t + (\sigma^1 B_t^1, \dots, \sigma^n B_t^n) = \sum_{i=1}^n \xi_i a_t^i,$$

which implies that $a_t^i = (\mu^i - (\sigma^i)^2/2)t + \sigma^i B_t^i$ for any $i = 1, \dots, n$. Now, since the exponential map is given by

$$\exp : \mathfrak{g} \longrightarrow G = \mathbb{R}_+^n,$$

$$\xi = \sum_{i=1}^n \xi^i \xi_i \longmapsto (e^{\xi^1}, \dots, e^{\xi^n}),$$

where e^x is the standard exponential function, we recover the well-known result that the general solution q_t of (6.3) starting at $q_0 \in \mathbb{R}_+^n$ is

$$q_t = (q_0^1 e^{(\mu^1 - (\sigma^1)^2/2)t + \sigma^1 B_t^1}, \dots, q_0^n e^{(\mu^n - (\sigma^n)^2/2)t + \sigma^n B_t^n}).$$

6.4. *Brownian motion on reductive homogeneous spaces and symmetric spaces*

Let G be a Lie group and $H \subseteq G$ be a closed subgroup. We say that the homogeneous space $M = G/H$ is *reductive* if the Lie algebra \mathfrak{g} of G may be decomposed into as a direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ where \mathfrak{h} is the Lie algebra of H and \mathfrak{m} is a subspace invariant under the action of Ad_H . That is, $\text{Ad}_h(\mathfrak{m}) \subseteq \mathfrak{m}$ for any $h \in H$ and, consequently, $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$. Suppose now that the reductive homogeneous space M is Riemann manifold with Riemannian metric η and that the transitive action of G leaves the metric η -invariant. We want to define Brownian motions on (M, η) by reducing a suitable process defined on G . The notation and most of the results in this example, in addition to a comprehensive exposition on homogeneous spaces, can be found in [14] and [16]. The reader is encouraged to check with [10] to learn more about the geometry of homogeneous spaces in the stochastic context.

We start by recalling that an M -valued process Γ is a Brownian motion whenever

$$f(\Gamma) - f(\Gamma_0) - \frac{1}{2} \int \Delta(f)(\Gamma_s) ds$$

is a real valued local semimartingale for any $f \in C^\infty(M)$, where Δ denotes the Laplacian. As is widely known, the Laplacian is defined as the trace of the Hessian associated to the Riemannian connection ∇ of η . That is,

$$\Delta(f)(m) = \sum_{i=1}^r (\mathcal{L}_{Y_i} \circ \mathcal{L}_{Y_i} - \nabla_{Y_i} Y_i)(f)(m),$$

where $\{Y_1, \dots, Y_r\} \subset \mathfrak{X}(M)$ is family of vector field such that $\{Y_1(m), \dots, Y_r(m)\}$ is an orthonormal basis of $T_m M$, $m \in M$.

Let $o \in M$ denote the equivalent class of H in M . We have assumed that (M, η) is a Riemann manifold with a (left) G -invariant metric η . Since η is G -invariant and Φ is transitive, the only thing that really matters as far as the characterization of η is concerned is the symmetric bilinear form $\eta_o : T_o M \times T_o M \rightarrow T_o M$. It can be easily proved that there is a natural one-to-one correspondence between the G -invariant Riemannian metrics η on $M = G/H$ and the Ad_H -invariant positive definite symmetric bilinear forms B on $T_o M = \mathfrak{g}/\mathfrak{h}$ ([16], Chapter X, Proposition 3.1). The correspondence is given by

$$\eta(\xi_1^M, \xi_2^M) = B(T_e \pi(\xi_1), T_e \pi(\xi_2)),$$

where $\xi_1, \xi_2 \in \mathfrak{g}$, $\pi : G \rightarrow G/H$ is the canonical submersion, and $\xi^M \in \mathfrak{X}(M)$ denotes the infinitesimal generator associated with $\xi \in \mathfrak{g}$. In addition, if M is reductive then the bilinear form B may be regarded as defined on \mathfrak{m} , $B : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathbb{R}$, since $T_o M$ is naturally isomorphic to \mathfrak{m} , which is an Ad_H -invariant subspace of \mathfrak{g} . The Riemannian connection ∇ of the metric η associated to such a bilinear form B is given by

$$\nabla_{\xi_1^M} \xi_2^M = \frac{1}{2} [\xi_1^M, \xi_2^M] + (U(\xi_1, \xi_2))^M \tag{6.4}$$

([16], Chapter X, Theorem 3.3). In this expression ξ_1 and ξ_2 belong to \mathfrak{m} and $U : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m}$ is the bilinear mapping defined by

$$2B(U(\xi_1, \xi_2), \xi_3) = B(\xi_1, [\xi_3, \xi_2]_{\mathfrak{m}}) + B([\xi_3, \xi_1]_{\mathfrak{m}}, \xi_2),$$

where $[\cdot, \cdot]_{\mathfrak{m}}$ is such that $[\cdot, \cdot] = [\cdot, \cdot]_{\mathfrak{h}} + [\cdot, \cdot]_{\mathfrak{m}}$ with $[\cdot, \cdot]_{\mathfrak{h}} \in \mathfrak{h}$ and $[\cdot, \cdot]_{\mathfrak{m}} \in \mathfrak{m}$. A consequence of (6.4) is that the Laplacian Δ takes the expression $\Delta(f)(m) = \sum_{i=1}^r (\mathcal{L}_{\xi_i^M} \circ \mathcal{L}_{\xi_i^M} + U(\xi_i, \xi_i)^M)(f)(m)$, $m \in M = G/K$, where $\{\xi_1^M, \dots, \xi_r^M\}$ is an orthonormal basis of $T_m M$.

The most important examples of reductive homogeneous spaces are symmetric spaces. In that case, G is the connected component of the isometric group $I(M) \subseteq \text{Diff}(M)$ of the symmetric space (M, η) containing $e = \text{Id}$. In order to identify the symmetric space (M, η) with a reductive space, take $o \in M$ a fixed point and let s be a geodesic symmetry at o . Then the Lie group G acts on M transitively and, if H denotes the isotropy group of o , M is diffeomorphic to G/H ([14], Chapter IV, Theorem 3.3). Suppose that $\dim(G) < \infty$ and let $\sigma : G \rightarrow G$ be the involutive automorphism of G defined by $\sigma(g) = s \circ \Phi_g \circ s$ for any $g \in G$, where $\Phi : G \times M \rightarrow G$ denotes as usual the left action of G on M . It is a matter of fact that $T_e\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$ induces an involutive automorphism of \mathfrak{g} . That is, $T_e\sigma \circ T_e\sigma = \text{Id}$ but $T_e\sigma \neq \text{Id}$. Let \mathfrak{h} and \mathfrak{m} be the eigenspaces of \mathfrak{g} associated with the eigenvalues 1 and -1 of $T_e\sigma$, respectively, such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. It can be checked that \mathfrak{h} is a Lie subalgebra of \mathfrak{g} ,

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h},$$

and $\text{Ad}_H(\mathfrak{m}) \subseteq \mathfrak{m}$ ([16], Chapter XI, Propositions 2.1 and 2.2). Moreover, the symmetric space G/K has a unique affine connection ∇ -invariant under the action of G . This is actually the Riemannian connection ([16], Chapter XI, Theorem 3.3) so that (6.4) reads

$$\nabla_{\xi_1^M} \xi_2^M = 0$$

for any pair of left-invariant vector fields ξ_1^M and ξ_2^M .

Returning to the general case, let $\{\xi_1, \dots, \xi_r\}$ be a basis of \mathfrak{m} such that $\{T_e\pi(\xi_1), \dots, T_e\pi(\xi_r)\}$ is an orthonormal basis of $T_o(G/K)$ with respect to η_o and let $\{\xi_1^G, \dots, \xi_r^G\} \subset \mathfrak{X}(G)$ be now the corresponding family of *right-invariant* vector fields built from $\{\xi_1, \dots, \xi_r\}$. Observe that $\{\xi_1^M(m), \dots, \xi_r^M(m)\}$ is an orthonormal basis of $T_m(G/K)$ due to the transitivity of the action and to the G -invariance of the metric η . Consider now the Stratonovich stochastic differential equation

$$\delta g_t = \sum_{i=1}^r \xi_i^G(g_t) \delta B_t^i + \sum_{i=1}^r U(\xi_i, \xi_i)^G(g_t) dt, \tag{6.5}$$

where (B_t^1, \dots, B_t^r) is a \mathbb{R}^r -valued Brownian motion. The stochastic system (6.5) is by definition K -invariant with respect to the natural *right action* $R : K \times G \rightarrow G$, $R_k(g) = gk$ for any $g \in G$ and $k \in K$. In addition, it is straightforward to check that the projection $\pi : G \rightarrow G/K$ send any right-invariant vector field $\xi^G \in \mathfrak{X}(G)$, $\xi \in \mathfrak{g}$, to the infinitesimal generator $\xi^M \in \mathfrak{X}(M)$ of the G -action $\Phi : G \times M \rightarrow M$. Hence (6.5) projects to the stochastic system

$$\delta \Gamma_t = \sum_{i=1}^r \xi_i^M(\Gamma_t) \delta B_t^i + \sum_{i=1}^r U(\xi_i, \xi_i)^M(\Gamma_t) dt \tag{6.6}$$

on M by Proposition 4.3. It is evident that the solutions of (6.6) have as a generator the second order differential operator $\frac{1}{2} \sum_{i=1}^r (\mathcal{L}_{\xi_i^M} \circ \mathcal{L}_{\xi_i^M} + U(\xi_i, \xi_i)^M)$ and they are therefore Brownian motions.

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