

# Multiclass Hammersley–Aldous–Diaconis process and multiclass-customer queues

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Received 27 July 2007; revised 29 January 2008; accepted 1 February 2008

**Abstract.** In the Hammersley–Aldous–Diaconis process, infinitely many particles sit in  $\mathbb{R}$  and at most one particle is allowed at each position. A particle at  $x$ , whose nearest neighbor to the right is at  $y$ , jumps at rate  $y - x$  to a position uniformly distributed in the interval  $(x, y)$ . The basic coupling between trajectories with different initial configuration induces a process with different classes of particles. We show that the invariant measures for the two-class process can be obtained as follows. First, a stationary  $M/M/1$  queue is constructed as a function of two homogeneous Poisson processes, the arrivals with rate  $\lambda$  and the (attempted) services with rate  $\rho > \lambda$ . Then put first class particles at the instants of departures (effective services) and second class particles at the instants of unused services. The procedure is generalized for the  $n$ -class case by using  $n - 1$  queues in tandem with  $n - 1$  priority types of customers. A *multi-line* process is introduced; it consists of a coupling (different from Liggett's basic coupling), having as invariant measure the product of Poisson processes. The definition of the multi-line process involves the *dual points* of the space–time Poisson process used in the graphical construction of the reversed process. The coupled process is a transformation of the multi-line process and its invariant measure is the transformation described above of the product measure.

**Résumé.** Dans un processus de Hammersley nous considérons une infinité de particules sur la droite réelle; et il ne peut pas y avoir plus d'une particule sur chaque position. Une particule située en  $x$  et ayant pour plus proche voisine (sur sa droite) une particule située en  $y$ , saute avec un taux  $y - x$  à une position aléatoire choisie uniformément dans l'intervalle  $(x, y)$ . Le couplage basique entre des trajectoires ayant des configurations initiales différentes induit un processus avec des particules de classes différentes. Nous donnons une construction explicite de la mesure invariante pour le processus ayant  $n$  classes de particules. Pour démontrer que la mesure est invariante nous introduisons un autre processus appelé "multi-ligne". La mesure invariante pour ce processus est un produit de plusieurs processus de Poisson. La définition du processus multi-ligne met en jeu les "points duaux" (de l'espace-temps), qui apparaissent naturellement dans la construction graphique du processus renversé par rapport au temps.

MSC: 60K35; 60K25; 90B22

**Keywords:** Multi-class Hammersley–Aldous–Diaconis process; Multiclass queuing system; Invariant measures

## 1. The Hammersley–Aldous–Diaconis process

The state space  $\mathcal{X}$  is an appropriate subset of locally finite subsets of  $\mathbb{R}$ . Elements of  $\mathcal{X}$  are called configurations and elements of a configuration  $\eta \in \mathcal{X}$  are called *particles*. The *Hammersley–Aldous–Diaconis* (HAD) *process* [1, 15] can be informally described by saying that a particle sitting at  $r \in \mathbb{R}$  waits an exponentially distributed random time with rate equal to the distance to the nearest particle to its right, located at  $r' > r$  (say) to jump to a site uniformly distributed in the interval  $[r, r']$ . Alternatively, bells ring at space–time points at rate 1 and, when a bell rings at  $(r, t)$ , the nearest particle to the left of  $r$  at time  $t$  – jumps to  $r$ .

A Harris graphical construction of the process is the following: Let  $\omega \in \Omega$  be a homogeneous rate-1 Poisson process in the space–time space  $\mathbb{R} \times \mathbb{R}^+$  (or later in  $\mathbb{R}^2$ ), where  $\Omega$  is the set of locally finite subsets of  $\mathbb{R}^2$ . We

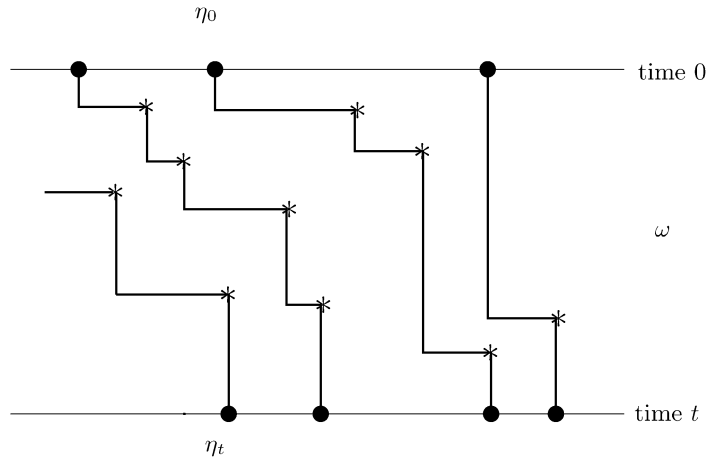


Fig. 1. Harris construction. Stars represent the space–time Poisson points in  $\omega$ . Balls represent particle positions at times 0 and  $t$ .

shall use *points* to refer to space–time elements of  $\omega$ . If  $\eta$  is the particle configuration at time  $t-$  and  $(r, t) \in \omega$ , then at time  $t$  the configuration jumps to  $\eta \setminus \{u\} \cup \{r\}$ , where  $u = u(\eta, r)$  is the nearest particle in  $\eta$  to the left of  $r$ . See Figure 1. This construction is well defined in a finite region as the points can be well ordered by time [1]. The construction in  $\mathbb{R}$  was performed by Aldous and Diaconis [1] and then by Seppäläinen [21] in the state space  $\mathcal{X} = \{\eta: \lim_{s \rightarrow \infty} |\eta \cup [0, s]|^2/s = \infty\}$  that we adopt. Here  $|\cdot|$  counts the number of elements of a finite set. Homogeneous Poisson processes in  $\mathbb{R}$  give mass 1 to  $\mathcal{X}$ .

For fixed initial configuration  $\eta$  and points  $\omega$ , the process  $(\eta_t, t \geq 0)$  is a deterministic function of  $\eta$  and  $\omega$  denoted  $\eta_t = \Phi(t, \eta, \omega)$ ,  $t \geq 0$ . In this case we say that the process is *governed by*  $\omega$  with initial configuration  $\eta$ . It satisfies

$$\eta_t = \Phi(t - s, \eta_s, \tau_s \omega), \quad (1)$$

for all  $0 \leq s < t$ , where  $\tau$  is the time translation operator defined by  $\tau_s \omega = \{(x, t - s), (x, t) \in \omega\}$ .

### Coupled and multiclass process

A joint construction of HAD processes  $((\eta_t^i, t \geq 0), i = 1, \dots, n)$  with initial configurations  $\eta^1, \dots, \eta^n$  governed by the same points  $\omega$  is called *coupled process*. It is defined by

$$\eta_t^i = \Phi(t, \eta^i, \omega). \quad (2)$$

We slightly abuse notation, writing  $\eta = (\eta^1, \dots, \eta^n)$  and  $\eta_t = \Phi(t, \eta, \omega) \in \mathcal{X}^n$ . In the coupled process when a point is at  $(x, t)$ , the closest particle to the left of  $x$  in each marginal jumps to  $x$  simultaneously.

Consider initial ordered particle configurations  $\eta^1 \subset \dots \subset \eta^n$  and run the coupled process. The order is maintained at later times. Define  $\mathcal{X}^{n\uparrow} = \{(\eta^1, \dots, \eta^n) \in \mathcal{X}: \eta^1 \subset \dots \subset \eta^n\}$ . Define  $R: \mathcal{X}^{n\uparrow} \rightarrow \mathcal{X}^n$  by

$$(R\eta)^k = \eta^k \setminus \eta^{k-1}. \quad (3)$$

The process  $\xi_t = R\eta_t$  is called the *multiclass process*. Particles in  $\xi_t^i$  are called *i-class particles*. The multiclass process is just a convention to describe a coupled process with ordered initial configurations. The map  $R$  is invertible; its inverse is given by  $(R^{-1}\xi)^k = \xi^1 \cup \dots \cup \xi^k$ . The process  $\xi_t$  governed by  $\omega$  with initial configuration  $\xi$  is defined by

$$\xi_t = \Upsilon(t, \xi, \omega) = R\Phi(t, R^{-1}\xi, \omega). \quad (4)$$

*Invariant measures*

The Poisson process with density  $\lambda$  is an invariant measure for the HAD process for all  $\lambda > 0$  [1]. Our main result is to construct invariant measures for the multiclass process. The resulting measure coincides with the law of the departure process of a stationary multiclass-customer queue system.

Let  $A$  and  $S$  be particle configurations in  $\mathcal{X}$ . Think of  $\mathbb{R}$  as time and construct a continuous time random walk  $(Z_r, r \in \mathbb{R})$  jumping one unit up at times in  $A$  and one unit down at times in  $S$ . We fix  $Z_0 = 0$  but since we are interested in the increments, the position at a given time is not important. The increments satisfy

$$Z_r - Z_s = |A \cap [r, s)| - |S \cap [r, s)|. \tag{5}$$

Assume that  $Z_r \rightarrow \mp\infty$  as  $r \rightarrow \pm\infty$ ; this implies  $Z_r$  visits each site a finite number of times. Let  $U(A, S)$  be the times in  $S$  that  $Z_r$  attains a new record down and  $D(A, S)$  its complement:

$$U(A, S) := \left\{ r \in S : Z_r < \inf_{r' < r} Z_{r'} \right\}, \tag{6}$$

$$D(A, S) := S \setminus U(A, S).$$

We interpret  $A$  as arrival times of customers to a one-server queue and  $S$  as the times at which service occurs. At these service times there are two possibilities: either there is a customer in the system, producing a *departure*, or there is no customer, in this case there is an *unused* service. We collect the departure times in  $D$  and the unused service times in  $U$ . In particular, suppose that  $A$  and  $S$  are stationary Poisson processes, independent from each other, with rates  $\lambda < \lambda'$ , respectively. Then  $Z_r$  satisfies the conditions above. Let  $Q_r = (Z_r - \inf_{r' \leq r} Z_{r'})$ ; then the process  $Q_r$  is stationary Markov, and satisfies

$$Q_r - Q_{r-} = \mathbf{1}\{r \in A\} - \mathbf{1}\{r \in D : Q_{r-} > 0\}. \tag{7}$$

The transition rates of this Markov process are  $\lambda$  from state  $i$  to state  $i + 1$  for all  $i \geq 0$ , and  $\lambda'$  from state  $i$  to state  $i - 1$  for all  $i \geq 1$ . Hence  $Q$  is the queue-length process of a stationary  $M/M/1$  queue, with arrival and service rates  $\lambda$  and  $\lambda'$ , respectively. The points  $D(A, S)$  are the departure times from this queue (the times when the queue-length decreases by 1). Figure 2 illustrates the original Poisson processes  $A$  and  $S$ , the random walk  $Z$  and the queue  $Q$ .

The operator  $D$  has the following properties:

- (i)  $D(\alpha^1, \alpha^2) \subset \alpha^2$ ;
- (ii) If  $\tilde{\alpha}^1 \subset \alpha^1$ , then  $D(\tilde{\alpha}^1, \alpha^2) \subset D(\alpha^1, \alpha^2)$ ;
- (iii) If  $\alpha^1$  and  $\alpha^2$  are independent one-dimensional Poisson processes of densities  $\rho^1$  and  $\rho^2$  with  $\rho^1 < \rho^2$ , then  $D(\alpha^1, \alpha^2)$  is also a one-dimensional Poisson process with density  $\rho^1$ .

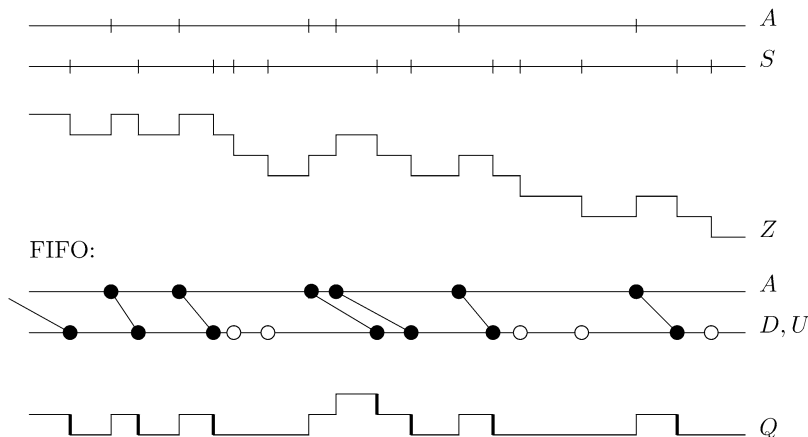


Fig. 2. Arrivals and services. Each arrival is linked to its departure time, running the queue according to the FIFO (first-in-first-out) schedule.

Part (iii) is Burke’s theorem for an  $M/M/1$  queue [3].

Let  $\nu^\lambda$  be the law of a Poisson process of rate  $\lambda$ . For  $\rho^1 < \dots < \rho^n$  define  $\nu := \nu^{\rho^1} \times \dots \times \nu^{\rho^n}$ . Let  $\alpha = (\alpha^1, \dots, \alpha^n) \in \mathcal{X}^n$  be a multi-line configuration with law  $\nu$ . Define a sequence of operators  $D^{(n)} : \mathcal{X}^n \mapsto \mathcal{X}$  as follows: Let  $D^{(1)}(\alpha^1) = \alpha^1$ , and then recursively for  $n \geq 2$ , let

$$D^{(n)}(\alpha^1, \alpha^2, \dots, \alpha^n) = D(D^{(n-1)}(\alpha^1, \dots, \alpha^{n-1}), \alpha^n). \tag{8}$$

The configuration  $D^{(n)}(\alpha^1, \alpha^2, \dots, \alpha^n)$  represents the departure process from a system of  $(n - 1)$  queues in tandem. The arrival process to the first queue is  $\alpha^1$ . The service process of the  $k$ th queue is  $\alpha^{k+1}$ , for  $k = 1, \dots, n - 1$ . Finally, for  $k = 2, \dots, n - 1$ , the arrival process to the  $k$ th queue is given by the departure process of the  $(k - 1)$ st queue. This is known as a system of  $\cdot/M/1$  queues in tandem.

Note  $D^{(2)}(\alpha^1, \alpha^2) = D(\alpha^1, \alpha^2)$ . By applying (i)–(iii) above repeatedly, we obtain that  $D^{(n)}(\alpha^1, \dots, \alpha^n) \subset D^{(n-1)}(\alpha^2, \dots, \alpha^n) \subset \dots \subset \alpha^n$  and if  $\alpha^1, \dots, \alpha^n$  are independent one-dimensional Poisson processes of densities  $\rho^1 < \dots < \rho^n$ , then  $D^{(n)}(\alpha^1, \dots, \alpha^n)$  also is a one-dimensional Poisson process with density  $\rho^1$ .

Define the configuration  $\eta = (\eta^1, \dots, \eta^n)$  by

$$\eta^k = D^{(n-k+1)}(\alpha^k, \alpha^{k+1}, \dots, \alpha^n). \tag{9}$$

By construction  $\eta^k \subset \eta^{k+1}$  for all  $k = 1, \dots, n - 1$  and for each  $k$ ,  $\eta^k$  has marginal distribution  $\nu^{\rho^k}$ .

Define the map  $C : \mathcal{X}^n \mapsto \mathcal{X}^{n\uparrow}$  by  $C\alpha = \eta$ . Define the measure  $\pi$  on  $\mathcal{X}^{n\uparrow}$  as the law of  $\eta$  (that is,  $\pi = C\nu$ ). Define the multiclass measure  $\mu$  as the law of  $\xi = R\eta$ , that is,  $\mu = RC\nu$ . Call  $M = RC$ . See Fig. 3.

**Theorem 1.** *If  $\alpha$  has law  $\nu$  with  $\rho^1 < \dots < \rho^n$ , then the law  $M\nu$  of  $\xi = M\alpha$  is invariant for the multiclass process  $\xi_t$  defined in (4). Equivalently, the law  $C\nu$  of  $C\alpha$  is invariant for the coupled process  $\eta_t$  defined in (2).*

To prove Theorem 1 we introduce later a new process  $\alpha_t = (\alpha_t^1, \dots, \alpha_t^n) \in \mathcal{X}^n$  called a *multi-line process*. We show that the product measure  $\nu$  is invariant for  $\alpha_t$  and that if  $\alpha$  is the initial configuration for the multi-line process  $\alpha_t$  then  $(C\alpha_t, t \geq 0)$  is the coupled process with initial configuration  $C\alpha$ . As a consequence,  $C\nu$  is invariant for the coupled process and  $M\nu$  is invariant for the multiclass process. Those results are proven in Propositions 4 and 5.

The right density of a configuration  $\eta \in \mathcal{X}$  is the limit as  $r$  goes to infinity of the number of  $\eta$ -particles in  $[0, r]$  divided by  $r$ , if this limit exists. Analogously, we define the left density and call simply the *density* of  $\eta$  the result when the left and right density are the same. The density of an ergodic measure  $\nu$  in  $\mathcal{X}$  is the unique value  $\lambda$  such that the set of configurations with density  $\lambda$  has  $\nu$  probability one.

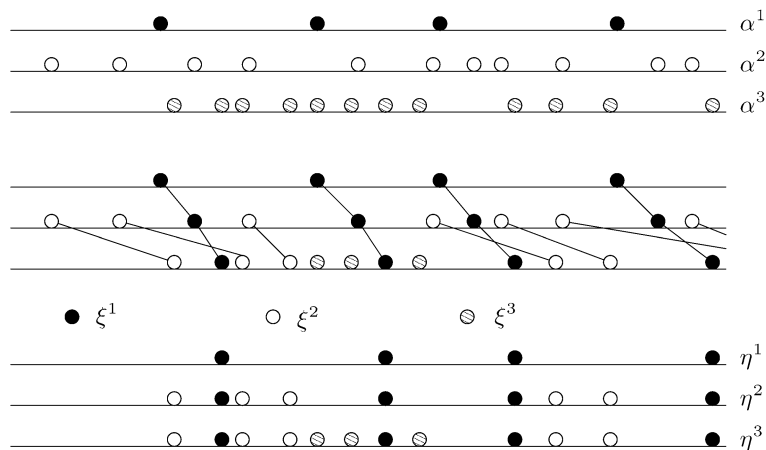


Fig. 3. Construction of a coupled configuration  $\eta$  and a multiclass configuration  $\xi$  from a multi-line configuration  $\alpha$ .  $\xi^k$  particles are filled differently according to its class. In the middle picture, arrivals have been linked with their departure times in each server, according to the FIFO schedule and the customer class.

### Uniqueness

The next result says that the measure  $C\nu$  is the unique invariant measure for the coupled process in the set of ergodic measures on  $\mathcal{X}^n$  with marginal densities  $\rho$ . Its domain of attraction includes the ergodic measures with densities  $\rho$ . Let  $\rho^1 < \dots < \rho^n$ ,  $\rho = (\rho^1, \dots, \rho^n)$  and  $\rho' = (\rho^1, \rho^2 - \rho^1, \dots, \rho^n - \rho^{n-1})$ . Let  $\mathcal{M}^n(\rho)$  be the set of ergodic measures on  $\mathcal{X}^n$  such that the  $k$ th marginal has density  $\rho^k$  for  $k = 1, \dots, n$ .

**Theorem 2.** *Assume the conditions of Theorem 1. Then,*

- (1) *The measure  $M\nu$  is the unique invariant measure for the multiclass process in  $\mathcal{M}^n(\rho')$ . The multiclass process starting with a measure in  $\mathcal{M}^n(\rho')$  converges weakly to  $M\nu$  as  $t \rightarrow \infty$ .*
- (2) *The measure  $C\nu$  is the unique invariant measure for the coupled process in  $\mathcal{M}^n(\rho)$ . The coupled process starting with a measure in  $\mathcal{M}^n(\rho)$  converges weakly to  $C\nu$  as  $t \rightarrow \infty$ .*

For all  $\lambda > 0$  and almost all realizations  $\omega$  of the points, there is a unique stationary realization of the HAD process at density  $\lambda$  governed by  $\omega$ . This is shown by the next theorem; notice that the notation  $\eta_t$  in this theorem is used for the HAD process on  $\mathcal{X}$ , that is, with  $n = 1$ .

**Theorem 3.** *For each  $\lambda > 0$  there exists an essentially unique function  $H_\lambda$  mapping elements  $\omega$  of  $\Omega$  to HAD trajectories  $(\eta_t, t \in \mathbb{R})$  such that:*

- (i) *The induced law of  $(\eta_t, t \in \mathbb{R}) = H_\lambda(\omega)$  is stationary in time.*
- (ii) *The marginal law of  $\eta_t$  for each  $t$  is space-ergodic with particle density  $\lambda$ .*
- (iii) *With probability 1,  $(\eta_t, t \in \mathbb{R})$  is a HAD evolution governed by  $\omega$ .*

(Here “essentially unique” means that if  $H'_\lambda$  is another function satisfying the three conditions, then  $H_\lambda(\omega) = H'_\lambda(\omega)$  with probability 1.) Then in fact the marginal law of  $\eta_t$  for each  $t$  is  $\nu^\lambda$ .

To stress the dependence on  $\lambda$  call  $\eta_t^\lambda$  the process constructed in Theorem 3. The construction implies  $\eta_t^\lambda \subset \eta_t^{\lambda'}$  if  $\lambda < \lambda'$ . The union of  $\eta_t^\lambda$  in  $\lambda$  is a countable dense set of  $\mathbb{R}$ . It consists on the space coordinates of the points in  $\omega$  with time coordinate less than  $t$ .

Theorems 2 and 3 are proven in Section 4, based on Proposition 10 which considers the coupled process  $(\eta_t^1, \eta_t^2)$  starting with two independent configurations with the same density  $\lambda$ . If one of the configurations is a Poisson process and the other comes from an ergodic distribution, then, with probability one, the density  $c(t)$  of positions where the two configurations differ at time  $t$  is deterministic and converges to zero as  $t$  grows. The proof of the proposition follows an argument of Ekhaus and Gray [7] as implemented by Mountford and Prabhakar [19].

### Dual points

The stationary realization  $(\eta_t^\lambda, t \in \mathbb{R})$  of the HAD process governed by  $\omega$  and density  $\lambda$  of Theorem 3 induces a point configuration on  $\mathbb{R} \times \mathbb{R}$  consisting on the space–time positions of the particles just before jumps; see Figure 4. Cator and Groeneboom [4] call them *dual points* and prove that they have the same Poisson law as  $\omega$  for all  $\lambda > 0$ . The proof is based on two facts: (a) the reverse HAD process with respect to  $\nu^\lambda$  is also a HAD process, with jumps to the left instead of to the right and (b) a trajectory of the process uniquely determines the points governing it.

### Multi-line process

The dual points of a stationary realization of the HAD process and a new density produce a new stationary realization of the HAD process. Given  $n$  densities and repeating the procedure  $n - 1$  times, we construct  $n$  stationary realizations of the HAD process at the given densities coupled in such a way that the  $k$ th realization is governed by the dual points of the  $(k + 1)$ th realization. The resulting process, called the *multi-line process*, is constructed as follows:

Fix densities  $\rho^1, \dots, \rho^n$  and a realization of the points  $\omega$ . Call  $\alpha_t^n = (\alpha_t^n, t \in \mathbb{R})$  the stationary realization of the HAD process at density  $\rho^n$  governed by  $\omega$  (which is well defined by Theorem 3). For  $k = n - 1, \dots, 1$  set  $\omega^k = \text{dual}$

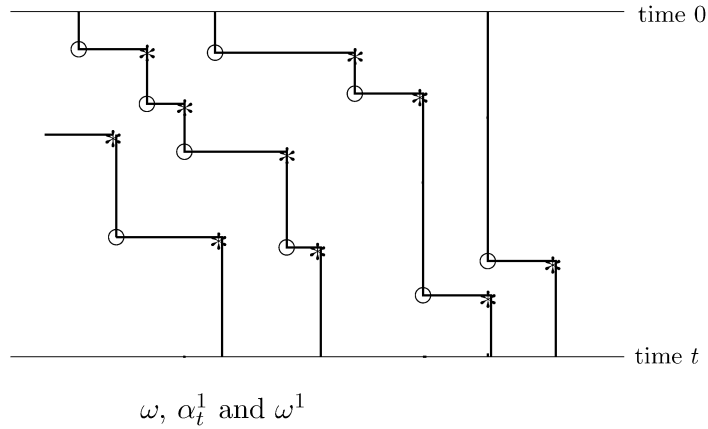


Fig. 4. The dual points are represented by circles.

points generated by  $\rho^k$  and  $\omega^{k+1}$  and  $\alpha_t^{k-1}$  = the stationary realization of the HAD process at density  $\rho^{k-1}$  governed by  $\omega^k$ . Define  $(\alpha_t, t \in \mathbb{R})$  as the *multi-line (stationary) process* given by  $\alpha_t = (\alpha_t^1, \dots, \alpha_t^n)$ . The generator of the multi-line process is given later in (16). The multi-line process can be constructed for times in  $[0, \infty)$  for any initial distribution; we do so in Section 2. In this case the  $n - 1$  first marginals do not follow the HAD dynamics unless the initial distribution is a product of Poisson processes. We prove two results in Section 2:

**Proposition 4.** *Let  $\rho = (\rho^1, \dots, \rho^k)$  be positive densities. The product of Poisson processes  $\nu^{\rho^1} \times \dots \times \nu^{\rho^k}$  is the unique invariant measure for the multi-line process with marginal densities  $\rho$ .*

**Proposition 5.** *Take a multi-line configuration  $\alpha$  such that the multiclass configuration  $M\alpha$  is well defined. Let  $\alpha_t$  be the multi-line process with initial configuration  $\alpha$  governed by  $\omega$ . Then  $(M\alpha_t, t \geq 0)$  is the multiclass process with initial distribution  $M\alpha$  and  $(C\alpha_t, t \geq 0)$  is the coupled process with initial distribution  $C\alpha$ , both governed by  $\omega$ .*

The proof of Theorem 1 is a consequence of Proposition 4 (without the uniqueness part) and Proposition 5.

Prähofer and Spohn [20] and Patrik Ferrari [13] introduce a *multilayer dynamics* based on the dual points and related to our multi-line process by a rotation of  $45^\circ$  (roughly speaking). See for example the pictures in [13], particularly Figure 2. In the context of these multilayer dynamics, all the densities  $\rho_k$  introduced would be the same (and equal to 1, say).

*Multiclass Burke’s theorem*

The invariant measure for the multiclass HAD is also a fixed point for a multiclass  $\cdot/M/1$  queue.

To make this more precise we need to introduce more notation in the construction of  $\eta$  as a function  $C$  of  $\alpha$ . As in [11] we label multiclass configurations  $\xi^{k,i}$  as functions of the multi-line configuration  $\alpha$ . Let  $\xi^{1,1} = \alpha^1$  and define

$$\begin{aligned} \xi^{k,i} &:= D(\xi^{k-1,i}, \alpha^k \setminus (\xi^{k,1} \cup \dots \cup \xi^{k,i-1})), \\ \xi^{k,k} &:= U(\alpha^{k-1}, \alpha^k) \end{aligned} \tag{10}$$

for  $k = 1, \dots, n$  and  $i = 1, \dots, k$ . Interpret these configurations as the service times of customers of  $k$  classes in the  $k$ th system; at times in  $\xi^{k-1,i}$  customers of class  $i$  arrive to the  $k$ th system and are served at times in  $\xi^{k,i}$ . The service schedule respects the classes: first-class customers are served first, second-class customers are served when there are no first-class customers waiting, etc. End of service occurs at times in  $\alpha^k$ . The unused service times of the  $k$ th system are in  $\xi^{k,k}$ . One can think that the  $k$ th system has infinitely many customers of class  $k$  waiting, so that they are served at the service times unused by the lower class clients. With this interpretation  $\xi^{1,1} = \alpha^1$  are the service times of a system with infinitely many customers in queue at all times. Assuming FIFO schedule (first-in-first-out) for customers of the

same class we can link each time in  $\alpha^k$  with the corresponding service time in  $\alpha^{k+1}$ . This is the meaning of the links in Figure 3. In each line  $k$ , the particles linked to some particle in the first line will be first-class particles; particles linked to particles in the second line but not to particles in the first line are second-class particles and so on.

Let  $M: \tilde{\mathcal{X}}^n \rightarrow \mathcal{X}^n$  be the map between multi-line configuration  $\alpha$  and the multiclass configuration  $\xi = (\xi^{n,1}, \dots, \xi^{n,n})$  using (6) and (10):

$$M\alpha := (\xi^{n,1}, \dots, \xi^{n,n}), \tag{11}$$

where  $\tilde{\mathcal{X}}^n \subset \mathcal{X}^n$  is the subspace of  $\mathcal{X}$  where  $M$  is well defined. This coincides with the map  $M = RC$  defined just after (9).

The  $n$ th multiclass stationary queuing system  $(Q_r^n, r \in \mathbb{R})$  in  $\{0, 1, \dots\}^{\{1, \dots, n-1\}}$  is defined as a deterministic function of the arrivals  $(\xi^{n-1,1}, \dots, \xi^{n-1,n-1})$  and the services  $\alpha^n$  by

$$Q_r^n(j) - Q_{r-}^n(j) = \mathbf{1}\{r \in \xi^{n-1,j}\} - \mathbf{1}\{r \in \alpha^n: Q_{r-}^n(i) = 0 \text{ for } i < j; Q_{r-}^n(j) > 0\}, \tag{12}$$

where  $Q_r^n(j)$  is the number of customers of class  $j$  in the  $n$ th system at time  $r$ . The existence of a process satisfying (12) can be proved by induction on  $n$  by observing that at times  $r$  in  $\xi^{n,n}$  the queue is empty:  $Q_r^n(j) = 0$  for all  $j < n$ . The process  $Q_r^n$  is not Markov unless the arrival process is a product of homogeneous Poisson and the service process is also Poisson. The departure times of class  $j$  in system  $n$  are given [equivalently to (10)] by

$$\xi^{n,j} = \{r \in \alpha^n: Q_{r-}^n(i) = 0 \text{ for } i < j \text{ and } Q_{r-}^n(j) > 0\} \tag{13}$$

for  $j = 1, \dots, n - 1$ . As before  $\xi^{n,n}$  is just the set of unused service times in the  $n$ th system.

Burke's theorem [3] says that the departures of a stationary  $M/M/1$  queue have the same Poisson law as the arrivals. It applies to our case for  $n = 2$ :  $Q_r^2(1)$  is an  $M/M/1$  queue with arrivals  $\xi^{1,1}$ , Poisson of rate  $\rho^1$  and departures  $\xi^{2,1}$ . The extension to the multiclass system says that the departures of the  $(n - 1)$  classes of the  $n$ th system  $(\xi^{n,1}, \dots, \xi^{n,n-1})$  have the same law as the arrivals  $(\xi^{n-1,1}, \dots, \xi^{n-1,n-1})$  to the same system. This is one of the consequences of the multiclass Burke's theorem as follows:

**Theorem 6 (Multiclass Burke).** Fix  $k \geq 1$  and for  $n \geq k$  let  $\alpha \in \mathcal{X}^n$  have law  $\nu$  with  $\rho^1 < \dots < \rho^n$ . Then the law of  $(\xi^{n,1}, \dots, \xi^{n,k})$  (= first  $k$  coordinates of  $M\alpha$ ) is independent of  $n$ .

**Proof.** Let  $\eta_t^{(n)}$  be the coupled process with initial invariant distribution  $C\nu^{(n)}$ , where  $\nu^{(n)}$  is the product measure in  $\mathcal{X}^n$  with marginals  $\nu^{\rho^k}$ ,  $\rho^1 < \dots < \rho^k$ . Since the evolution of the first  $k$  coordinates is Markovian, the marginal law of the first  $k$  coordinates of  $\eta_t^{(n)}$  under  $C\nu^{(n)}$  is invariant for the first  $k$  coordinates of the process  $\eta_t^{(n)}$ . By uniqueness of the invariant measure of Theorem 2, the law of these marginals must coincide with  $C\nu^{(k)}$ , the invariant measure for  $\eta_t^{(k)}$ , the process with  $k$  lines and the same densities. Since  $(\xi^{n,1}, \dots, \xi^{n,k}) = \text{first } k \text{ coordinates of } R\eta^{(n)}$ , its law coincides with the law of  $R\eta^{(k)} = M\alpha^{(k)}$ , where  $\alpha^{(k)}$  is the multi-line configuration with law  $\nu^{(k)}$ .  $\square$

*The process in a cycle*

Similar results can be proven for the multiclass process in a cycle  $\mathbb{R}_N$ . The state space  $\mathcal{X}_N$  is the set of finite configurations contained in  $[0, N]$ . The points  $\omega$  are restricted to  $\mathbb{R}_N \times \mathbb{R}^+$ . When the configuration at time  $t-$  is  $\eta_{t-} = \eta$  and there is a point at  $(x, t)$ , if there are no  $\eta$  particles to the left of  $x$ , then the rightmost particle of  $\eta$  jumps to  $x$ . The coupled process is defined as in (2). Take two finite particle configurations  $A_N$  and  $S_N$  in  $\mathbb{R}_N$  with  $|S_N| > |A_N|$ . Extend these configurations periodically to two infinite configurations  $A$  and  $S$ . Construct  $(Z_t, t \in \mathbb{R})$ , a periodic process satisfying (5). The resulting periodic configurations of departures  $D(A, S)$  and unused services  $U(A, S)$  are periodic and induce configurations  $D_N$  and  $U_N$  in  $\mathbb{R}_N$ . We construct a (unique) queue  $Q_r$ ,  $r \in \mathbb{R}_N$  such that it satisfies (7), which has value  $Q_r = 0$  for  $r \in U_N$ ; actually  $Q_r$  so constructed is the minimal process satisfying (7).

Given a multi-line configuration  $\alpha$  in  $\mathcal{X}_N^n$  we construct a multiclass configuration  $\xi = M_N\alpha \in \mathcal{X}_N^n$ , where  $M_N$  is defined as the map  $M$  substituting  $D$  and  $U$  by  $D_N$  and  $U_N$ . The analogues to Theorems 1, 2 and 6 hold for this process. Letting  $\underline{\ell} = (\ell^1, \dots, \ell^n)$ , the set  $\mathcal{X}^n(\rho)$  of Theorem 2 must be substituted by  $\mathcal{X}_N^n(\underline{\ell})$ , the set of configurations with exactly  $\ell^k$  particles in the  $k$ th cycle. The proof of the analogue to Theorem 2 is easy in this case.



*Regeneration properties of the invariant measure*

In the two-class invariant measure, the second-class particles are regeneration events. More precisely, let  $\mu$  on  $\mathcal{X}^2$  be a translation invariant measure with marginal densities  $\rho^1, \rho^2$ . The Palm measure “conditioned to have a second class particle at the origin” is the measure  $\hat{\mu}$  defined by  $\hat{\mu}f = (1/|I|\rho^2) \int d\mu(\xi) \sum_{r \in \xi^2 \cap I} f(\theta_r \xi)$ , for any measurable bounded set  $I \subset \mathbb{R}$ , where  $|I|$  is the Lebesgue measure of  $I$  and  $\theta_r$  is translation by  $r$ . The multiclass invariant measure  $M\nu$  of Theorem 1 conditioned to have a second-class particle at the origin satisfies that: (a) the configuration to the left of the origin is independent of the configuration to its right; (b) both the distribution of first-class particles to the right of the origin and the positions of the first plus second-class particles to the left of the origin are Poisson processes (or product measures in the discrete-space case). These properties were proven by [5] for the two-class invariant measure for the totally asymmetric simple exclusion process (TASEP); alternative probabilistic proofs can be found in [9]. Angel [2] shows that these properties follow easily from the two-class representation  $\xi = M\alpha$  that he called “collapsing procedure”. In Section 5 we show that as in the discrete case [11], there are *regeneration strings* for the multiclass invariant measure  $M\nu$ .

*Microscopic shocks*

The process with two classes of particles is a crucial tool to define shock measures [8,10]. Taking a configuration  $(\xi^1, \xi^2)$  distributed according to the invariant measure  $M\nu$  for the two-class process conditioned to have a  $\xi^2$  particle at the origin and considering the configuration consisting of all particles (of any class) to the left of the origin and the first-class particles to the right of it, the distribution of the resulting configuration is invariant for the process as seen from an isolated second-class particle. We discuss this item in Section 6.

*Multiclass HAD process in  $\mathbb{Z}$  and other discrete processes*

The approach has been implemented for a discrete-space multiclass HAD process and other discrete-space processes on  $\mathbb{Z}$ . In fact, this research started with the extension of Angel’s [2] result to the multiclass process for the discrete HAD and TASEP we performed in [11] and [12]. There are two main points in this paper: one is to show that the approach works also in continuous space (the HAD process is the most natural model to study TASEP like questions in  $\mathbb{R}$ ); the other is to show some results that are only announced in [12]. In particular, we show the uniqueness of the multiclass invariant measure with given marginal densities. The use of dual points is more intuitive in the continuous HAD than in the discrete processes. Indeed, the dual points in this case are identified by the trajectory of the process, while in the discrete cases it is necessary to appeal auxiliary spin–flip processes to identify the dual points.

The existence of an invariant measure for the two-class TASEP is first shown by Liggett [16,17] and then computed by Derrida, Janowsky and Lebowitz [5], see also [9,22]. The description of the two-class invariant measures by Angel [2] and Duchi and Schaeffer [6] was the starting point of our multiclass version [11,12].

**2. The multi-line process**

The multi-line process has configurations  $\alpha_t = (\alpha_t^1, \dots, \alpha_t^n)$  in  $\mathcal{X}^n$ . Given a multi-line configuration  $\alpha$  and a position  $x \in \mathbb{R}$  let  $x^n$  be the position of the closest particle of  $\alpha^n$  to the left of  $x$  and inductively for lines  $k = n - 1, \dots, 1$  let  $x^k$  be the position of the closest  $\alpha^k$  particle to the left of  $x^{k+1}$ . When the configuration at time  $t$  is  $\alpha$  and there is an  $\omega$  point in  $(x, t)$ , the  $\alpha^n$ -particle at  $x^n$  jumps to  $x$  at time  $t$  and simultaneously the  $\alpha^k$  particle located at  $x^k$  jumps to  $x^{k+1}$  for  $k = n - 1, \dots, 1$ . The function  $J : \mathcal{X}^n \times \mathbb{R} \rightarrow \mathcal{X}^n$  defined by

$$(J(\alpha, x))^k = \alpha^k \cup \{x^{k+1}\} \setminus \{x^k\} \tag{14}$$

maps the multi-line configuration  $\alpha$  before the jumps produced by  $x$  to the configuration after the jumps; see upper part of Figure 5. The configuration  $\alpha_t$  of the multi-line process at time  $t$  is a function of the initial configuration  $\alpha$  and the points  $\omega$  called  $\Psi$ :

$$\alpha_t := \Psi(t, \alpha, \omega). \tag{15}$$



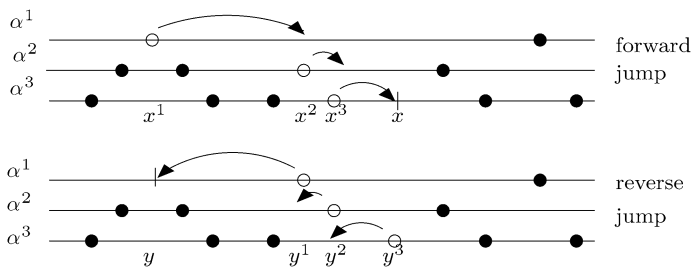


Fig. 5. Effect of a point at  $x$  for the multi-line process and the reverse jump (due to a point at  $y$  in the reverse).

The generator of the process is given by

$$\mathcal{L}f(\alpha) = \int_{\mathbb{R}} dx [f(J(\alpha, x)) - f(\alpha)]. \tag{16}$$

This definition is equivalent to the one given in the introduction. For  $t$  such that  $(x, t)$  is in  $\omega$  denote  $x^{n+1}(t) = x$  and  $x^k(t)$  the position at time  $t -$  of the particle in the  $k$ th line jumping due to the point in  $(x, t)$ . Then  $\omega^k = \{(x^k(t), t) : (x, t) \in \omega\}$  are the dual points of  $\omega^{k+1}$  which in turn govern the process  $\alpha_t^k$ .

We define another multi-line HAD process  $\alpha_t^*$  in  $\mathcal{X}^n$  and show that it is the reverse of  $\alpha_t$  with respect to the product of Poisson processes  $\nu$ . For  $\alpha_t^*$  the points govern the first line but produce jumps to the left: when a point appears at  $y$ , it calls the closest  $\alpha^1$  particle to the right of  $y$ , located at  $y^1$ . Simultaneously the closest  $\alpha^2$  particle to the right of  $y^1$ , located at  $y^2$  jumps to  $y^1$ , and so on. Calling  $y_0 = y$ , the positions  $(y_0, \dots, y_n)$  are defined as a function of  $\alpha$  and  $y$ . The multi-line configuration obtained after the jumps produced by a point at  $y$  is called  $J^*(\alpha, y)$ ; its  $k$ th line is given by

$$(J^*(\alpha, y))^k = \alpha^k \cup \{y^{k-1}\} \setminus \{y^k\}. \tag{17}$$

The operators  $J$  and  $J^*$  are the inverse of each other in the following (equivalent) senses:

$$J^*(J(\alpha, x), x^1) = \alpha; \quad J(J^*(\alpha, y), y^n) = \alpha \tag{18}$$

(in Figure 5,  $x^1 = y$  and  $y^n = x$ ). The generator of  $\alpha_t^*$  is given by

$$\mathcal{L}^* f(\alpha) = \int_{\mathbb{R}} dy [f(J^*(\alpha, y)) - f(\alpha)]. \tag{19}$$

**Proposition 7.** For any choice of positive densities  $\rho = (\rho^1, \dots, \rho^n)$ , the multi-line process with generator  $\mathcal{L}^*$  is the reverse of the multi-line process with generator  $\mathcal{L}$  with respect to the product measure  $\nu$  on  $\mathcal{X}^n$  with densities  $\rho$ . As a consequence,  $\nu$  is invariant for both  $\mathcal{L}$  and  $\mathcal{L}^*$ .

**Proof.** The proof is based on the observations: (a) the configurations  $\alpha$  and  $J(\alpha, r)$  and the configuration after a jump produced by a Poisson point at  $r$  have the same probability weight; (b) the rate to jump from  $\alpha$  to  $J(\alpha, r)$  is the same as the rate in the reverse process to jump from  $\alpha' = J(\alpha, r)$  to  $J^*(\alpha', y) = \alpha$ ; (c) the rate of exiting any configuration  $\alpha$  is the same for the direct and the reverse process. We formalize this with a generator computation.

It suffices to show that for bounded functions  $f$  and  $g$  depending on finite regions in  $\mathbb{R}^n$ ,  $\nu(g\mathcal{L}f) = \nu(f\mathcal{L}^*g)$ ; that is,

$$\int \nu(d\alpha) \int_{\mathbb{R}} g(\alpha) dx [f(J(\alpha, x)) - f(\alpha)] = \int \nu(d\alpha) \int_{\mathbb{R}} f(\alpha) dx [g(J^*(\alpha, x)) - g(\alpha)].$$

Since the term subtracting in both terms is the same, it suffices to prove that

$$\int \nu(d\alpha) \int_{\mathbb{R}} g(\alpha) f(J(\alpha, x)) dx = \int \nu(d\alpha) \int_{\mathbb{R}} g(\alpha) f(J^*(\alpha, x)) dx. \tag{20}$$

By conditioning on the positions of the jumping particles  $(x^1, \dots, x^n)$  the left term in (20) can be written as

$$\int_{\mathbb{R}} dx \int_{x^1 < \dots < x^n < x} \prod_{k=1}^n (\rho^k e^{-\rho^k (x^{k+1} - x^k)}) dx^1 \dots dx^n \int \nu_x(d\alpha) g(\alpha) f(J^*(\alpha, x)), \quad (21)$$

where  $\nu_x$  is the measure  $\nu$  conditioned to have at each line  $k$  a particle at position  $x^k$  and no particle in  $(x^k, x^{k+1})$ . Change variables: call  $\alpha' = J(\alpha, x)$  so that, by (18)  $\alpha = J^*(\alpha', y)$  and  $y^k = x^{k+1}$ ,  $k = 0, \dots, n$  to get that (21) equals

$$\int_{\mathbb{R}} dy \int_{y < y^1 < \dots < y^n} \prod_{k=1}^n (\rho^k e^{-\rho^k (y^k - y^{k-1})}) dy^1 \dots dy^n \int \nu^y(d\alpha') g(J^*(\alpha', y)) f(\alpha'), \quad (22)$$

where  $\nu^y$  is the measure  $\nu$  conditioned to have at each line  $k$  a particle at position  $y^k$  and no particles in  $(y^{k-1}, y^k)$ . Expression (22) is just the right-hand side of (20).  $\square$

The set of *dual points* of the HAD process starting with a Poisson process of rate  $\lambda$  is a Poisson process on  $\mathbb{R} \times \mathbb{R}^+$ . This is proven in Theorem 3.1 in [4] under the title ‘‘Burke’s theorem for Hammersley processes’’. We state and prove this fact in our context.

**Proposition 8.** *If  $\eta \in \mathcal{X}$  has law  $\nu^\lambda$  and  $\omega$  is a rate-1 homogeneous Poisson process in  $\mathbb{R} \times \mathbb{R}^+$ , then the dual points of the trajectory  $(\eta_t, t \geq 0)$  form a rate-1 homogeneous Poisson process in  $\mathbb{R} \times \mathbb{R}^+$ . Furthermore for  $t > 0$ ,  $\eta_T$  is independent of the dual points contained in  $\mathbb{R} \times [0, T]$ .*

**Proof.** By Proposition 7 (with  $n = 1$ ), the reverse HAD process with respect to  $\nu^\lambda$  is just a HAD process with the drift to the left. Assume we have constructed the process  $(\eta_t, 0 \leq t \leq T)$  with initial law  $\nu^\lambda$  and Poisson points  $\omega$  on  $\mathbb{R} \times [0, T]$ . The reverse process  $\eta_s^* = \eta_{T-s-}$ ,  $0 \leq s \leq T$ , is governed by the dual points in  $\mathbb{R} \times [0, T]$ . Since the reverse process started in equilibrium  $\nu^\rho$ , the points governing the reverse process must be also Poisson in  $\mathbb{R} \times [0, T]$ .

The dual points in  $\mathbb{R} \times [0, T]$  are the points governing the future of the reverse process  $\eta_t^* := \eta_{T-t-}$  which is in equilibrium. Hence it is a Poisson process independent of the initial configuration  $\eta_0^* = \eta_T$ .  $\square$

In the proof of Theorem 3 we use Proposition 8 to construct a stationary trajectory  $(\eta_t, t \in \mathbb{R})$  of the HAD process governed by points  $\omega$  with marginal law  $\nu^\lambda$ . The dual points of this trajectory are, as  $\omega$ , a homogeneous Poisson process in  $\mathbb{R}^2$ .

### 3. Multi-line and multiclass processes

In this section we prove that the projection of the multi-line process on the multiclass space using the map  $M$  is just the multiclass process. The analogous statement is true for the coupled process. Recall from (15) that given an initial multi-line configuration  $\alpha$  and a homogeneous rate-1 Poisson process  $\omega$  in  $\mathbb{R} \times \mathbb{R}^2$  the process  $(\alpha_t, t \geq 0)$ , where  $\alpha_t = \Psi(t, \alpha, \omega)$  is a realization of the multi-line process. On the other hand, given an initial configuration  $\xi$  and the same  $\omega$ , the process  $(\xi_t, t \geq 0)$  with  $\xi_t = \Phi(t, \xi, \omega)$  is the multiclass process.

**Proposition 9.** *Assume  $C\alpha$  is well defined. Then almost surely:*

$$C(\Psi(t, \alpha, \omega)) = \Phi(t, C\alpha, \omega), \quad (23)$$

$$M(\Psi(t, \alpha, \omega)) = \Upsilon(t, M\alpha, \omega). \quad (24)$$

*In particular, the law of  $C\alpha_t$  is the same as the law of  $\eta_t$ .*

**Proof.** Expression (24) follows then from (23), the identity  $M = RC$  and (4). Fix  $x \in \mathbb{R}$ ; it suffices to show

$$H(C\alpha, x) = CJ(\alpha, x), \quad (25)$$

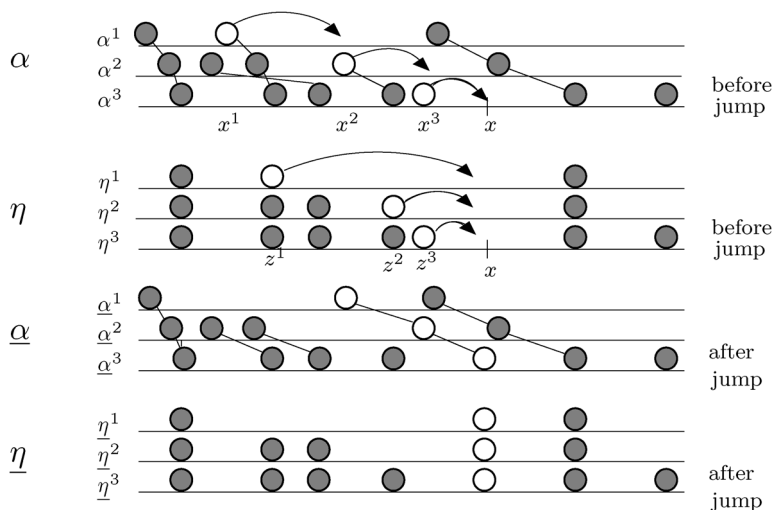


Fig. 6. Effect of a point at  $x$  for the multi-line process and the coupled process.  $\underline{\alpha} = J(\alpha, x)$  and  $\underline{\eta} = H(\eta, x)$ . Taking  $\eta = C\alpha$  we see that  $\underline{\eta} = C\underline{\alpha}$ .

where  $H(\eta, x)$  is the coupled Hammersley configuration obtained after a point at  $(x, t)$  (for some  $t$ ) if  $\eta_{t-} = \eta$  and  $J(\alpha, x)$  is the resulting multi-line configuration after a point at  $(x, t)$  if  $\alpha_{t-} = \alpha$ : its  $k$ th coordinates satisfy

$$\begin{aligned} (H(\eta, x))^k &= \eta \setminus \{z^k\} \cup \{x\}, \\ (J(\alpha, x))^k &= \alpha^k \setminus \{x^k\} \cup \{x^{k+1}\}, \end{aligned}$$

where  $z^k$  is the position of the closest  $\eta^k$  particle to the left of  $x$  and  $x^k$  is the position of the particle jumping on line  $k$  in the multi-line process; here  $x^{n+1} = x$ .

Let  $\eta = C\alpha$  be the coupled configuration obtained from  $\alpha$  and denote  $\underline{\alpha} = J(\alpha, x)$  and  $\underline{\eta} = H(\eta, x)$ . With this notation (25) is  $C\underline{\alpha} = \underline{\eta}$ ; see Figure 6. We first prove (25) for  $n = 2$  lines and then use this case to proceed by induction.

*The case  $n = 2$ .* The second line of both processes is governed by the point at  $x$  with the same Hammersley rule. This implies they coincide in the second line:  $(C\underline{\alpha})^2 = (\underline{\eta})^2$ . There is an  $\underline{\alpha}^2$  service time at  $x$  and an  $\underline{\alpha}^1$  particle at  $x^2$ . This particle is served in line 2 at time  $x$  unless there is another  $\underline{\alpha}^1$  particle to the left of  $x^2$  served at that time. In any case, an  $\underline{\alpha}^1$  particle is served at  $x$  in line 2. This means  $(C\underline{\alpha})^1$  has a particle at  $x$ . To see that  $(C\underline{\alpha})^1$  has no particle at  $z^1$  and that otherwise it is equal to  $\eta$  we consider two cases depending on whether  $x^2$  serves an  $\underline{\alpha}^1$  particle or not.

(1) The  $\underline{\alpha}^2$  time at  $x^2$  serves an  $\underline{\alpha}^1$  particle. Hence, there is an  $\eta^1$  particle at  $x^2$ . Since there are no  $\alpha^2$  services between  $x^2$  and  $x$  (by definition of  $x^2$ ), there cannot be  $\eta^1$  particles either and  $\underline{\eta}^1 = \eta^1 \setminus \{x^2\} \cup \{x\}$ . Let's check that also  $C\underline{\alpha}$  is of this form. Let  $y$  be the position of the  $\alpha^1$  particle served at  $x^2$  in line 2. Clearly  $y \leq x^1$ . The  $\alpha^1$  particles to the left of  $y$  are served in line 2 before  $x^2$  and hence its service time is not affected if  $x^1$  and  $x^2$  are translated to the right. This means  $C\underline{\alpha}$  coincides with  $\underline{\eta}$  at  $x$  and to the left of  $x$ . The  $\alpha^1$  particles to the right of  $y$  are served after  $x$ . The displacement of the service time from  $x^2$  to  $x$  just changes the service time of the particle at  $y$ , leaving the other service times unchanged. The displacement of the  $\alpha^1$  particle from  $x^1$  to  $x^2$  does not change the order or arrival of the particles or their service times at  $\alpha^2$ , as those times are after  $x$ . This implies  $C\underline{\alpha} = \underline{\eta}$  in this case.

(2) The  $\alpha^2$  time at  $x^2$  does not serve an  $\alpha^1$  particle. The  $\alpha^1$  particle at  $x^1$  is served at some time  $z$  before  $x^2$ . There are no  $\alpha^1$  particles between  $x^1$  and  $x^2$ . The translation of the  $\alpha^1$  particle from  $x^1$  to  $x^2$  and the  $\alpha^2$  particle from  $x^2$  to  $x$  does not change the service times of the  $\alpha^1$  particles to the left of  $x^1$ , which are served before  $x^2$ . By the FIFO schedule, particles that arrive after  $x^2$  are served after particles that arrive before. Hence, the translations do not change the service times of the particles that arrive after  $x^2$ .

*The induction step.* From the definition of  $C$ , we have

$$\eta_t^k = D^{(n-k+1)}(\alpha_t^k, \dots, \alpha_t^n). \tag{26}$$

From the observation just before (9),  $\eta_t^k$  has distribution  $\nu^{\rho^k}$ . So we need to show that the RHS of (26) is a HAD trajectory governed by  $\omega$ . Since  $(\alpha^k, \dots, \alpha^n)$  is itself a multi-line process (with  $n - k + 1$  lines) governed by  $\omega$ , it is enough to show that, for any  $n$ ,  $D^{(n)}(\alpha_t^1, \dots, \alpha_t^n)$  is a HAD trajectory governed by  $\omega$ . But from the definitions of  $D^{(n)}$  and of the multi-line process,

$$D^{(n)}(\alpha_t^1, \dots, \alpha_t^n) = D^{(2)}(D^{(n-1)}(\alpha_t^1, \dots, \alpha_t^{n-1}), \alpha_t^n).$$

This and the fact that  $(\alpha_t^1, \dots, \alpha_t^{n-1})$  is an  $(n - 1)$ -line multi-line process governed by  $\omega^{n-1}$  concludes the induction step.  $\square$

#### 4. Uniqueness of the invariant measure

In this section we prove Theorems 2 and 3 using the following proposition. It says that the marginals at time  $t$  of the coupled Hammersley process with initial independent marginals will coincide as  $t \rightarrow \infty$  if the first initial marginal is a Poisson process and the second one is an ergodic distribution with the same density as the first.

**Proposition 10.** *Let  $\lambda > 0$  and  $\eta, \tilde{\eta} \in \mathcal{X}^2$  be independent particle configurations with law  $\nu^\lambda$ , the Poisson process and  $\tilde{\nu}^\lambda$ , an ergodic process in  $\mathcal{X}$  with density  $\lambda$ , respectively. Let  $(\eta_t, \tilde{\eta}_t)$  be the coupled process with initial configuration  $(\eta, \tilde{\eta})$ . Then with probability one, both  $\eta_t \setminus \tilde{\eta}_t$  and  $\tilde{\eta}_t \setminus \eta_t$  have a deterministic density denoted  $c(t) = c(t; \tilde{\nu}^\lambda)$  which is decreasing and converges to 0 as  $t \rightarrow \infty$ .*

An immediate consequence of the proposition and the translation invariance of the law of the coupled process at each given time is that the probability of the event  $\{A \cap \eta_t = A \cap \tilde{\eta}_t\}$  converges to 1 as  $t \rightarrow \infty$  for any bounded interval  $A \subset \mathbb{R}$ .

Before proving the proposition we show how it implies Theorems 2 and 3.

**Proof of Theorem 2.** Assume  $\eta$  and  $\tilde{\eta}$  in  $\mathcal{X}^n$  are independent with laws  $\mu$  and  $\tilde{\mu}$ , respectively and let  $(\eta_t, \tilde{\eta}_t)$  be a  $2n$ -coordinates coupled process starting with configuration  $(\eta, \tilde{\eta})$ . Assume  $\mu$  and  $\tilde{\mu}$  are invariant for the coupled process and that the marginal law of both  $\eta^k$  and  $\tilde{\eta}^k$  is a Poisson process with rate  $\rho^k$  for all  $k$ . Then, for bounded  $f$  depending on a bounded interval  $A$ :

$$|\mu f - \tilde{\mu} f| = \left| \int \mu(d\eta) f(\eta) - \int \tilde{\mu}(d\tilde{\eta}) f(\tilde{\eta}) \right| \quad (27)$$

$$= \left| \int \int \mu(d\eta) \tilde{\mu}(d\tilde{\eta}) \mathbb{E}(f(\eta_t) - f(\tilde{\eta}_t)) \right| \quad (28)$$

$$\leq \int \int \mu(d\eta) \tilde{\mu}(d\tilde{\eta}) \|f\|_\infty \mathbb{P}(A \cap \eta_t \neq A \cap \tilde{\eta}_t) \quad (29)$$

which tends to zero by Proposition 10 applied to each coordinate. This implies  $\mu = \tilde{\mu}$ .  $\square$

**Proof of Theorem 3.** First construct a double infinite realization and the corresponding points. Fix particles  $\eta$  with law Poisson of rate  $\lambda$  and Poisson points  $\omega^- \cup \omega^+$ , the subsets of points with negative and positive time coordinates, respectively. Run the process  $\eta_t = \Phi(t, \eta, \omega^+)$  for  $t \geq 0$  using the points  $\omega^+$ . Run the reverse process backwards  $\eta_{-t}^* = \Phi^*(-t, \eta, \omega^-)$  starting from the same configuration using the points  $\omega^-$ . Here  $\Phi^*(-t, \eta, \omega^-) = \Phi(t, \eta, \text{TR}(\omega^-))$ , where  $\text{TR}(\omega) = \{(x, -t), (x, t) \in \omega\}$  are the points of  $\omega$  reflected with respect to the line  $\{t = 0\}$ . Let the dual points  $D^-(\omega^-, \eta)$  be the positions of the particles of  $\eta_t^*$  just before jumps. By Proposition 8,  $D^-(\omega^-, \eta)$  is a Poisson process of points. Let  $\omega = D^-(\omega^-, \eta) \cup \omega^+$  be the configuration consisting on the dual points of the reverse process for negative times and the original points for positive times. The points  $\omega$  are Poisson and govern a stationary process having configuration  $\eta$  at time zero. This constructs simultaneously Poisson points  $\omega$  and a stationary trajectory governed by  $\omega$ .

For uniqueness we need to show that if  $(\tilde{\eta}_s) = (\tilde{\eta}_s, -\infty < s < \infty)$  is a stationary evolution governed by  $\omega$  such that the time marginal  $\tilde{\eta}_s$  is ergodic with density  $\lambda$  then  $\tilde{\eta}_s = \eta_s$  for all  $s$ . By Theorem 2 the marginal law of  $\tilde{\eta}_t$  is Poisson of parameter  $\lambda$  for all  $t$ . Hence both  $(\tilde{\eta}_s)$  and  $(\eta_s)$  are space–time ergodic processes but the joint process  $((\eta_s), (\tilde{\eta}_s))$  is not necessarily ergodic. Fix a positive  $t$  and introduce an auxiliary process  $(\eta'_s, s \geq -t)$  governed by  $\omega$ . The initial configuration  $\eta'_{-t}$  is Poisson of rate  $\lambda$  and independent of the configurations  $(\eta_{-t}, \tilde{\eta}_{-t})$  at that time. By Proposition 10 the densities of  $\eta_0 \Delta \eta'_0$  and  $\tilde{\eta}_0 \Delta \eta'_0$  are both smaller than  $2c(t)$  for all  $t$ . Hence the density of  $\tilde{\eta}_0 \Delta \eta_0$  is bounded by  $4c(t)$  for all  $t$ . Since  $c(t)$  converges to 0, this implies that  $\eta_0 = \tilde{\eta}_0$  and by the same argument  $\eta_s = \tilde{\eta}_s$  for all  $s \in \mathbb{R}$ .  $\square$

**Proof of Proposition 10.** We use an argument of Ekhaus and Gray [7] as developed in Section 2 of [19]. Let  $\xi_t^- = \eta_t \cap \tilde{\eta}_t$ ,  $\xi_t^+ = \tilde{\eta}_t \setminus \eta_t$  and  $\xi_t^- = \eta_t \setminus \tilde{\eta}_t$ . As in [19] we call them *yellow*, *blue* and *red* particles, respectively. For each  $t$  the process  $(\xi_t^-, \xi_t^+, \xi_t^-)$  is ergodic and  $\xi_t^+$  has the same density as  $\xi_t^-$ . We want to show that the density of  $\xi_t^+$  goes to zero as  $t \rightarrow \infty$ . Label the particles of  $\eta_t$  and  $\tilde{\eta}_t$  as follows: Call  $\eta_t(i)$  the position of the  $i$ th particle of  $\eta$ . Initially  $\eta_0(i) < \eta_0(i + 1)$  for all  $i$ ; the same for  $\tilde{\eta}$ . The labels evolve in time depending on the color. At the time of an  $\omega$  Poisson point at  $x$ , proceed as follows:

- (1) If the closest left  $\eta$  and  $\tilde{\eta}$  particles are in  $\xi^-$ , then both of them jump to  $x$  carrying their labels.
- (2) If the closest left  $\eta$  particle is blue localized in  $x_1 < x$  and the closest  $\tilde{\eta}$  particle is red localized in  $y_1 < x_1$ , then the  $\eta$  particle at  $x_1$  and the  $\tilde{\eta}$  particle at  $y_1$  jump to  $x$  carrying their labels and change their color to yellow. If furthermore there are blue particles in  $y_1 < x_k < \dots < x_2 < x_1$ , the blue particle in  $x_i$  jumps to  $x_{i+1}$ ,  $i = 2, \dots, k - 1$  carrying the label and the color.
- (2') The same as (2) by interchanging  $\eta$  with  $\tilde{\eta}$  and blue with red.
- (3) If the closest  $\eta$  particle is blue at position  $x_1 < x$  and the closest  $\tilde{\eta}$  particle is yellow at position  $y_1 < x_1$ , then the yellow particles in  $y_1$  jump to  $x$  carrying the label and keeping the color. If furthermore there are blue particles in  $y_1 < x_k, \dots, x_2 < x_1$ , then the  $\eta$  blue particle at  $x_i$  jumps to  $x_{i+1}$ ,  $i = 2, \dots, k - 1$  carrying the label and the color.
- (3') The same as (3) by interchanging  $\eta$  with  $\tilde{\eta}$  and blue with red.

In other words, when a labeled particle becomes yellow it keeps the color forever. Yellow particles behave as first-class particles while blue and red particles do as second-class particles. When blue and red particles coalesce they change the color to yellow; this happens in cases (2) and (2'). Since there is no creation of new  $\xi^\pm$  particles, the density must be non-increasing. If a labeled  $\eta$  or  $\tilde{\eta}$  particle overpasses another particle, then it becomes yellow. As a consequence blue and red particles can be overpassed but cannot overpass other particles.

Call *everblue* those particles in the initial configuration  $\eta$  that will be blue at all  $t \geq 0$  and *everred* those particles in  $\tilde{\eta}$  that will be red at all times. The configuration of everblue particles has a translation invariant distribution but not necessarily ergodic. Let  $\gamma_t^+ \subset \eta$  be the set of  $\eta$  particles that at time  $t$  will be blue. The law of  $\gamma_t^+$  is ergodic with a (deterministic) density  $c(t)$ , the density of  $\xi_t^+$ , which decreases to a value  $c$  as  $t \rightarrow \infty$ . Since the configuration of everblue particles is the intersection in  $t$  of  $\gamma_t^+$ , it has density  $c$ . For the same reason the configuration of everred  $\tilde{\eta}$  particles has also ergodic law with density  $c$ .

**Lemma 11.** *Assume the conditions of Proposition 10. If the density of everblue particles is strictly positive, then there exists an  $M < \infty$ , an  $n < \infty$  and a  $\delta > 0$  such that for each  $t$ , the density of red particles  $W$  at time  $t$  satisfying*

- (i) *there exist blue particles at time  $t$  in  $(W, W + M]$ ,*
- (ii) *there are at most  $n$   $\eta_t$  particles in  $(W, W + M]$ ,*
- (iii) *there are no  $\eta_t$  particles in  $(W + M, W + M + \delta]$ ,*

*is at least  $1/(4M)$ .*

**Proof.** The existence of an  $M$  such that the the red particles  $W$  at time  $t$  satisfying (i) have at least density  $1/M$  is proven in Lemma 3.1 of [19] for another process. This is the hard part of the Ekhaus–Gray argument. The proof applies here because it only uses the fact that red and blue particles cannot overpass, so everred and everblue particles maintain the order. Since  $\eta_t$  is a Poisson process, the density  $a(k)$  of  $\eta_t$  particles  $U$  having at least  $k$   $\eta_t$  particles in the interval  $(U, U + M)$  decreases exponentially with  $k$ . For  $n$  sufficiently large so that  $a(n) < 1/(2M)$  the density of red particles at time  $t$  satisfying (i) and (ii) for this  $n$  is at least  $1/(2M)$ . Finally, as  $\eta_t$  is a Poisson process, the density

$b(\delta)$  of  $\eta_t$  particles  $U$  such that there is at least one  $\eta$  particle in the interval  $(U + M, U + M + \delta]$  is bounded above by  $\delta\rho$  (the mean number). Take  $\delta$  sufficiently small such that  $b(\delta) < 1/(4M)$ . Then the density of red particles  $W$  at time  $t$  satisfying (i) and (ii) and such that there is another  $\eta$  particle in  $(W + M, W + M + \delta]$  is smaller than  $1/(4M)$  and the red particles  $W$  at time  $t$   $\xi_t^-$  satisfying (i–iii) have at least density  $1/(4M)$ .  $\square$

**Lemma 12.** *Under the conditions of Lemma 11 there exists a positive  $\varepsilon'$  such that for all  $t$  the density of  $\xi_t^-$  minus the density of  $\xi_{t+1}^-$  is at least  $\varepsilon'$ .*

**Proof.** Take integer times  $t$  and for each  $\eta_t$  particle  $U$  consider the following event in the space–time Poisson process  $\omega$ :

- (iv) there are no  $\omega$  points in  $(U, U + M] \times [t, t + 1/2]$ .
- (v)  $\omega$  has exactly  $n + 1$  points  $(x_1, t_1), \dots, (x_{n+1}, t_{n+1})$  in  $(U + M, U + M + \delta] \times [t, t + 1]$  and they are increasing in the time coordinate and decreasing in the space coordinate.

This event has a positive probability  $\varepsilon''$  and it is independent of the past up to  $t$ . This implies that the density of red particles at time  $t$  satisfying conditions (i)–(v) has some positive density at least  $\varepsilon' = \varepsilon''/8M$ .

The red particles at time  $t$  satisfying (i)–(v) will collide with a blue particle between  $t$  and  $t + 1$ . Hence the density of  $\xi_{t+1}^- - \xi_t^-$  is not smaller than  $\varepsilon'$ .  $\square$

To conclude the proof of Proposition 10 we argue by contradiction. If we assume that the density of  $\xi_t^+$  decreases to a non-negative constant  $c$ , then by Lemma 12 this same density decreases by a fixed amount  $\varepsilon' > 0$  at each unit of time.  $\square$

### 5. Regeneration properties of the multiclass invariant measure

Assume that there is a second-class particle at the origin. This corresponds to an unused service in the queue, which in turn implies there are no customers in the queue at time 0. Since the queue is Markovian, the future depends only on the number of customers at time 0. The attempted departure to the right of the origin is a Poisson process with rate  $\rho$ . We conclude that conditioned on having a second-class particle at the origin, the first plus second-class particles to the right of the origin form a Poisson process of rate  $\rho$ . By reversing time, the arrivals in the reverse process also form a Poisson process of rate  $\lambda$ . The arrivals in the reverse queue are just the effective departures in the forward queue; that is, the first-class particles. Hence, conditioned on having a second-class particle at the origin, the first-class particles to the left of the origin form a Poisson process of rate  $\lambda$  and independent of the right process.

For  $n \geq 3$  classes there are not regeneration events but as in the discrete case, there are *regeneration strings*. Fix a vector of classes  $(c_0, \dots, c_\ell)$  such that  $c_0 = n$ ,  $c_\ell = 2$  and for each class  $m \in \{3, \dots, n - 1\}$  there exists a position  $j_m < \ell$  with  $c_{j_m} = m$  and with  $c_j \leq m$  for all  $j \in \{j_m + 1, \dots, \ell\}$ . For instance  $c = (4, 1, 2, 3, 1, 2)$  qualifies for  $\ell = 5$  and  $j_3 = 3$ . We call  $c$  a *regeneration string*. Let  $\mathcal{X}^n(c)$  be the set of configurations  $\xi$  in  $\mathcal{X}^n$  such that there are positions  $0 = x_0 < \dots < x_\ell$  such that  $x_i \in \xi^{c_i}$  and  $\xi^k \cap (x_i, x_{i+1}) = \emptyset$  for  $k = 1, \dots, n$ . In other words, for  $\xi \in \mathcal{X}^n(c)$  there is a  $\xi^n$  particle at the origin and looking at the classes of the first  $\ell$  particles to the right of the origin, their classes follow the vector  $c$ . A translation invariant measure  $\mu$  on  $\mathcal{X}^n$  with densities  $\rho^1, \dots, \rho^n$  conditioned on  $\mathcal{X}^n(c)$  is the Palm measure  $\hat{\mu}$  further conditioned on  $\mathcal{X}^n(c)$ ; here the Palm measure is the measure conditioned on having a  $\xi^n$  particle at the origin defined by  $\hat{\mu}f = (1/|I|\rho^n) \int d\mu(\xi) \sum_{r \in \xi^n \cap I} f(\theta_r \xi)$  for any measurable bounded set  $I \subset \mathbb{R}$ . As in the discrete case one can prove the following proposition.

**Proposition 13.** *Let  $c$  be a regeneration string of length  $\ell$  and  $x_0 < \dots < x_\ell$ . Then, under the multiclass invariant measure  $M\nu$  conditioned on  $\mathcal{X}^n(c)$  the string  $\xi \cap (-\infty, x_0)$  is independent of  $\xi \cap (x_\ell, \infty)$ .*

The proof is analogous to the two-classes case. One has to verify that after a regeneration string the queue is empty, so whatever will happen in the future (of  $x_\ell$ ) depends on the Poisson processes of the future, so it is independent of the past of  $x_\ell$ . In particular the superposition of the classes to the right of the regeneration string  $\xi^1 \cup \dots \cup \xi^n \cup (x_\ell, \infty)$  under the conditioned measure of the proposition is a Poisson process of rate  $\rho^n$ . See more details in [11].



## 6. Shocks in HAD

Liggett [16] introduced the coupled process to prove ergodic properties of the exclusion process; he called “discrepancies” what we call second-class particles. The interest in the study of invariant measures for the coupled process was renewed when its association to shock measures become clear. A shock related to a process  $\eta_t$  is a (possibly random) position depending on  $t$  with the property that *uniformly in time* the asymptotic densities to the right and left of the shock are different. Wick [23] showed the existence of that position for the totally asymmetric zero range process with densities 0 and  $\lambda$  to the left and right of the origin, respectively. In fact this is equivalent to the position of the leftmost particle in the TASEP with densities 0 and  $\lambda$  to the left and right of the origin, respectively. Then [8] and [10] (see also [18]) used the existence of an invariant measure for  $\xi_t$  for  $n = 2$  to show the existence of a shock in the TASEP. In a similar way Garcia [14] proved the existence of a shock measure for the HAD process.

We show now a way to construct a shock measure for the HAD with asymptotic densities  $\lambda > \rho$  to the left and right of the origin, respectively. Start with the invariant measure  $\mu$  of Theorem 1 for the coupled process with two marginals,  $\nu^\lambda$  marginal for  $\xi^1$  and  $\nu^\rho$  marginal for  $\xi^1 \cup \xi^2$ . Let  $\hat{\mu}$  be the measure  $\mu$  conditioned to have a second-class particle at the origin, and  $\xi$  be a two-class configuration chosen according to  $\hat{\mu}$ . From  $\xi$  construct a configuration  $\eta$  by superposing the negative particles of  $\xi^1$  and all the particles of  $\xi^2$  except the one at the origin:

$$\eta = [\xi^1 \cap (-\infty, 0)] \cup \xi^2 \setminus \{0\}. \quad (30)$$

Call  $S$  the map that transforms  $\xi$  in  $\eta$  and  $\mu_{\lambda,\rho}$  the law of  $\eta$  so constructed. For the coupled process with initial configurations  $\eta$  and  $\eta' := \eta \cup \{0\}$ , there is only a discrepancy for all times. This discrepancy behaves like a second-class particle. Call  $X_t$  its position. The process  $(\eta_t, X_t)$  is Markovian but the marginal process  $X_t$  is not. Let  $\eta'_t$  be the process defined as the translation by  $X_t$  of  $\eta_t$ . Then  $\mu'$  is invariant for  $\eta'_t$ . The proof is based on the fact that  $S$  commutes with the dynamics of  $\xi_t$  in the following sense:

$$S\Phi^2(t, \omega, \xi) = \Phi'(t, \omega, S\xi), \quad (31)$$

where  $\Phi^2$  is the operator that transforms a time  $t$ , points  $\omega$  and initial configuration  $\xi$  in the configuration of the two-class process at time  $t$  as seen from the second-class particle  $\tau_{X_t}\xi_t$ . Here  $\tau_x$  is the translation operator defined by  $\tau_x A = \{y - x, y \in A\}$ . The map  $\Phi'$  takes a time  $t$ , points  $\omega$  and an initial (one-class) configuration  $\eta$  into the process as seen from a second-class particle  $\tau_{X_t}\eta_t$ . This works in the same way as for the TASEP [8,10] and for the HAD [14], so we omit the proof.

## Acknowledgements

Thanks to Eric Cator, Sheldon Goldstein and Herbert Spohn.

This paper is partially supported by FAPESP, CNPq, PRONEX. PAF thanks hospitality and support from IHES and Laboratoire de Probabilités of Université de Paris 7. JBM thanks IME–USP and the FAPESP. This paper was partially supported by the USP–COFECUB agreement and the Brazil–France agreement.

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