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An exponential inequality for negatively associated random variables

H. Jabbari* **and M. Jabbari***

Department of Statistics, Ferdowsi University of Mashhad, Mashhad, Iran
e-mail: jabbarinh@um.ac.ir; jabbarinm@yahoo.com

H.A. Azarnoosh*

Department of Statistics, Ferdowsi University of Mashhad, Mashhad, Iran
e-mail: azarnush@math.um.ac.ir
url: http://mathstat.um.ac.ir

Abstract: We prove an exponential inequality for negatively associated and strictly stationary random variables. A condition is given for almost sure convergence and the associated rate of convergence is specified in terms of the underlying covariance function.

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1. Introduction

In nonparametric estimation, exponential inequalities of Bernstein type represent a powerful tool for proving convergence rates. The significance of exponential inequalities toward several probability and statistical applications is well known. There exist several versions available in the literature for independent sequences of variables. Sometimes, the random variables are positively associated. Ioannides and Roussas [4] proved an exponential inequality for positively associated random variables under some assumptions of uniform boundedness

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and some conditions on the covariance structure of the variables. Oliveira [9] extended these results by dropping the boundedness assumption.

One of dependent structure of random variables (r.v.'s) that has attracted the interest of probabilists and statisticians is negative association (NA). This concept is one qualitative version of negative dependence among random variables. For other versions of negative dependence, such as upper (lower) orthant dependence, reverse regular of order two in pairs, conditionally decreasing in sequence and negatively dependent in sequence, we refer to Lehmann [8], Block et al. [1], Ebrahimi and Ghosh [3], Joag-dev and Patil [6], and Karlin and Rinott [7]. Among those types of negative dependence, only the NA class enjoys the important property of being closed under formation of increasing functions of disjoint sets of random variables. As pointed out and proved by Joag-dev and Proschan [5], a number of well known multivariate distributions possess the NA property, such as (a) multinomial, (b) convolution of unlike multinomial, (c) multivariate hypergeometric, (d) Dirichlet, (e) Dirichlet compound multinomial, (f) negatively correlated normal distribution, (g) permutation distribution, (h) random sampling without replacement, and (i) joint distribution of ranks. Because of their wide applications in multivariate statistical analysis and reliability theory, the notations of negatively associated random variables have received more and more attention recently.

The article is organized as follows: in the next section, the necessary notation and terminology are introduced before the main result. In addition to the basic assumption of negative association, we require that the r.v.'s are uniformly bounded and we impose some further conditions on the covariance structure, see assumptions (A1)-(A3). The proof of the theorem rests on Lemma 3.4 which is formulated and proved in Section 3. Actually, all preliminary results needed are taken care of in the same section, as is the proof of the theorem. The optimal associated rate of convergence depends on the underlying covariance function and is explicitly calculated by way of formulas (3.34) and (3.31) for given covariance function. In Section 4, we discuss two classes of covariances falling into scope of Theorem 2.1. To avoid unnecessary repetitions, it is stated at the outset that all limits are taken as $n \to \infty$.

2. Definitions, notations and formulation of main results

At first, we introduce two definitions for ND and NA random variables.

Definition 2.1. Two random variables X and Y are negatively quadrant dependent (NQD) if for every $x, y \in R$ we have

$$\mathbf{P}(X \leq x, Y \leq y) \leq \mathbf{P}(X \leq x)\mathbf{P}(Y \leq y). \tag{2.1}$$

Definition 2.2. A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets A and B of $\{1, 2, \dots, n\}$,

$$Cov(f_1(X_i, i\epsilon A), f_2(X_j, j\epsilon B)) \leq 0. \tag{2.2}$$

whenever f_1 and f_2 are coordinatewise increasing and such that the covariance exists. An infinite family of random variables is NA if every finite subfamily is NA.

Consider a sequence of natural numbers p_n such that, for each $n \geq 1$, $1 \leq p_n < n$ and $p_n \to \infty$. Then, we divide the set $\{1, 2, \ldots, n\}$ into successive groups each containing p_n elements. Define r_n as the greatest integer less or equal to $n/2p_n$, which implies that $n/2r_n p_n \to 1$. Thus the set $\{1, 2, \ldots, n\}$ is split into $2r_n$ groups, each consisting of p_n elements; the remaining $n - 2r_n p_n < p_n$ elements constitute a set which may be empty.

For easy reference, we consider some assumptions that the main result in this paper remained valid.

Assumptions

(A1) The basic assumption is that the r.v.'s $\{X_i, i \geq 1\}$ are NA.

(A2) The r.v.'s are bounded, $|X| \leq M/2,\ i \geq 1$ (M is independent of i) and covariance invariant,

$$Cov(X_i, X_{i+k}) = Cov(X_1, X_{k+1}).\ i \geq 1,\ k \geq 1 \tag{2.3}$$

(A3) Without loss of generality, it is assumed $Cov(X_1, X_{k+1})$ is nondecreasing as $k \to \infty$.

Remark 2.1. Covariance invariant in assumption (A2) can be dropped, if we assumed

$$Cov(X_1, X_{k+1}) = inf\{Cov(X_i, X_{i+k}); i \geq 1\},\ k \geq 1.$$

Define $\bar{S}_n$ and ε_n by

$$\bar{S}_n = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mathrm{E}X_i), \quad \varepsilon_n = \left(\frac{\alpha M^2}{2}\right)^{1/2}\left(\frac{\log n}{r_n}\right)^{1/2}, \tag{2.4}$$

where M is as in assumption (A2) and α is an arbitrary constant greater than one. Then the main result obtained in following theorem.

Theorem 2.1. *Let $\bar{S}_n$ and ε_n be defined by (2.4). Then, under assumptions (A1) and (A2), and the proviso*

$$Cov(X_1, X_{k+1}) \leq \exp\left\{-\frac{4(M+1)}{3}M\left(\frac{\alpha}{2}\right)^{1/2}(r_n \log n)^{1/2}\right\}, \tag{2.5}$$

it holds

$$P(|\bar{S}_n| \geq \varepsilon_n) \leq C_0 \exp(-c r_n {\varepsilon_n}^2), \quad c = 2/9M^2 \tag{2.6}$$

for all sufficiently large n, $n \geq n_0$, where C_0 is a constant.

Furthermore, $\bar{S}_n \to 0$ a.s. at the rate $1/\varepsilon_n$. The optimal specification of r_n is given by (3.33) or (3.34) and the respective $1/\varepsilon_n$ is given by (3.31).

3. Preliminary results

Let $Y_i = X_i - \mathrm{E}X_i$, so that $|Y_i| \le M$, $i \ge 1$ are NA and $\bar{S}_n = n^{-1}\sum_{i=1}^n Y_i$. Define the r.v.'s U_i, V_i, $i = 1, \ldots, r_n$ and W_n by

$$U_i = Y_{2(i-1)p_n+1} + \cdots + Y_{(2i-1)p_n}, \quad V_i = Y_{(2i-1)p_n+1} + \cdots + Y_{2ip_n}, \tag{3.1}$$

$$W_n = Y_{2p_n r_n+1} + \cdots + Y_n, \tag{3.2}$$

where p_n and r_n are as in the previous section and

$$\bar{U}_n = \frac{1}{n}\sum_{i=1}^{r_n} U_i, \quad \bar{V}_n = \frac{1}{n}\sum_{i=1}^{r_n} V_i, \quad \bar{W}_n = \frac{W_n}{n}. \tag{3.3}$$

So that

$$\bar{S}_n = \bar{U}_n + \bar{V}_n + \bar{W}_n. \tag{3.4}$$

Lemma 3.1. *Suppose X and Y are NQD random variables with finite variance and f, g are complex valued functions on R^1 with f' and g' bounded. Then*

$$|Cov(f(X), g(Y))| \le -\|f'\|_\infty \|g'\|_\infty Cov(X, Y), \tag{3.5}$$

where $\|.\|$ denotes the sup norm on R^1; in particular, for any real s and t,

$$|\mathrm{E}(e^{isX+itY}) - \mathrm{E}(e^{isX})\mathrm{E}(e^{itY})| \le -|s||t|Cov(X, Y). \tag{3.6}$$

Proof. Define

$$\mathrm{H}(x, y) = \mathbf{P}(X \le x, Y \le y) - \mathbf{P}(X \le x)\mathbf{P}(Y \le y). \tag{3.7}$$

By Hoeffding lemma [8],

$$Cov(X, Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{H}(x, y)dxdy. \tag{3.8}$$

This equation can be easily generalized (see, for more information Newman, 1980) to yield

$$Cov(f(X), g(Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f'(x)g'(y)\mathrm{H}(x, y)dxdy.$$

Since X and Y are NQD random variables $\mathrm{H}(x, y) \le 0$, thus

$$\begin{aligned} |Cov(f(X), g(Y))| &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f'(x)|.|g'(y)||\mathrm{H}(x, y)|dxdy. \\ &\le \|f'\|_\infty.\|g'\|_\infty Cov(X, Y), \end{aligned} \tag{3.9}$$

as desired. ∎

Lemma 3.2. *Let $X_1, X_2, \dots, X_n$ be a sequence of NA random variables bounded by constant δ'. Then, for every $\lambda > 0$,*

$$|Cov(e^{\lambda \sum_{i=1}^{n-1} X_i}, e^{\lambda X_n})| \leq -\lambda^2 e^{n\lambda\delta'} \sum_{1\leq i<j\leq n} Cov(X_i, X_j). \tag{3.10}$$

Proof. By Lemma 3.1 for $\lambda > 0$, we have

$$|Cov(e^{\lambda X_1}, e^{\lambda X_2})| \leq \lambda^2 e^{2\lambda\delta'} Cov(X_1, X_2). \tag{3.11}$$

The result follows by induction and using the fact that if X, Y and Z are NA then so are X and $Y + Z$ as they are increasing functions of NA r.v.'s. □

We quote next a general lemma used to control some of the terms appearing in the course of proof.

Lemma 3.3 ([2]). *Let W be a central random variable. If there exist $a, b \in R$ such that $\mathrm{P}(a \leq W \leq b) = 1$ then, for every $\lambda > 0$*

$$|\mathrm{E}(e^{\lambda W})| \leq \exp\left(\frac{\lambda^2(b-a)^2}{8}\right). \tag{3.12}$$

Lemma 3.4. *Let $\varepsilon_n > 0$ and $\bar{U}_n$ be defined by (3.3) and suppose assumptions (A1)-(A3) hold. Then, for an appropriate constant C_0,*

$$\mathrm{P}(\bar{U}_n \geq \varepsilon_n) \leq C_0 \exp(-2r_n{\varepsilon_n}^2/M^2),$$

provided

$$-Cov(X_1, X_{k+1}) \leq \exp(-4(M+1)r_n\varepsilon_n/M^2).$$

Proof. The r.v.'s $U_1, \dots, U_{r_n}$ are NA and $|U_i| \leq p_n M$ for all i. For some $\lambda > 0$, set $h(x) = e^{\frac{\lambda}{n}x}$, $-p_n M < x < p_n M$ so that $h'(x) = \frac{\lambda}{n} e^{\frac{\lambda}{n}x}$ and for their sup-norms, it holds: $\|h\|_\infty \leq e^{\lambda p_n \frac{M}{n}}$, $\|h'\|_\infty \leq \frac{\lambda}{n} e^{\lambda p_n \frac{M}{n}}$. With this h and $A = \{1, \dots, r_n - 1\}$, $B = \{r_n\}$, so that $\sharp A + \sharp B - 2 = r_n - 2$, apply Lemma 3.2 to obtain

$$Cov(e^{\frac{\lambda}{n}\sum_{i=1}^{r_n-1} U_i}, e^{\frac{\lambda}{n}U_{r_n}}) \leq -\frac{\lambda^2}{n^2} e^{\lambda p_n r_n \frac{M}{n}} \sum_{i=1}^{r_n-1} Cov(U_i, U_{r_n}),$$

so that

$$\begin{aligned} \mathrm{E}(e^{\lambda \bar{U}_n}) &= \mathrm{E}(e^{\frac{\lambda}{n}\sum_{i=1}^{r_n-1} U_i} . e^{\frac{\lambda}{n}U_{r_n}}) \\ &\leq \mathrm{E}e^{\frac{\lambda}{n}\sum_{i=1}^{r_n-1} U_i} \mathrm{E}e^{\frac{\lambda}{n}U_{r_n}} - \frac{\lambda^2}{n^2} e^{\lambda p_n r_n \frac{M}{n}} \sum_{i=1}^{r_n-1} Cov(U_i, U_{r_n}). \end{aligned} \tag{3.13}$$

However

$$\begin{aligned}\sum_{i=1}^{r_n-1} Cov(U_i, U_{r_n}) &= \sum_{i=1}^{r_n-1} \sum_{j=2(i-1)p_n+1}^{(2i-1)p_n} \sum_{k=2(r_n-1)p_n+1}^{(2r_n-1)p_n} Cov(Y_j, Y_k) \\ &\leq \sum_{j\epsilon A} \sum_{k\epsilon B} Cov(Y_j, Y_k) \\ &= \sum_{i=1}^{r_n-1} \sum_{j\epsilon A_i} \sum_{k\epsilon B} Cov(Y_j, Y_k), \end{aligned} \tag{3.14}$$

where

$$\begin{aligned}\mathrm{A} &= \{\underbrace{1,\ldots,p_n};\underbrace{2p_n+1,\ldots,3p_n};\ldots;\underbrace{2(r_n-2)p_n+1,\ldots,(2r_n-3)p_n}\} \\ &= \{A_1; A_2; \ldots; A_{r_n-1}\}\end{aligned}$$

and $\mathrm{B} = \{2(r_n-2)p_n+1,\ldots,(2r_n-3)p_n\}$. By assumptions (A2) and (A3),

$$\begin{aligned}\sum_{j\epsilon A_1} \sum_{k\epsilon B} Cov(Y_j, Y_k) &\geq p_n Cov(Y_1, Y_{2(r_n-1)p_n+1}) + \cdots \\ &\quad + p_n Cov(Y_{p_n}, Y_{2(r_n-1)p_n+1}) \\ &\geq p_n^2 Cov(Y_{p_n}, Y_{2(r_n-1)p_n+1}).\end{aligned}$$

Similarly,

$$\sum_{j\epsilon A_2} \sum_{k\epsilon B} Cov(Y_j, Y_k) \geq p_n^2 Cov(Y_{p_n}, Y_{2(r_n-2)p_n+1})$$

and continuing like this

$$\sum_{j\epsilon A_{r_n-1}} \sum_{k\epsilon B} Cov(Y_j, Y_k) \geq p_n^2 Cov(Y_{p_n}, Y_{2p_n+1}).$$

Thus,

$$\begin{aligned}\sum_{j\epsilon A} \sum_{k\epsilon B} Cov(Y_j, Y_k) &\geq p_n^2[Cov(Y_{p_n}, Y_{2p_n+1}) + \cdots + Cov(Y_{p_n}, Y_{2(r_n-2)p_n+1}) \\ &\quad + Cov(Y_{p_n}, Y_{2(r_n-1)p_n+1})] \\ &\geq p_n^2 r_n Cov(Y_{p_n}, Y_{2p_n+1}).\end{aligned} \tag{3.15}$$

By means of (3.15) and (3.14), inequality (19) becomes

$$\mathrm{E}(e^{\lambda \bar{U}_n}) \leq \mathrm{E}e^{\frac{\lambda}{n}\sum_{i=1}^{r_n-1} U_i} \mathrm{E}e^{\frac{\lambda}{n}U_{r_n}} - \frac{\lambda^2}{n^2} e^{\lambda p_n r_n \frac{M}{n}} p_n^2 r_n Cov(Y_{p_n}, Y_{2p_n+1}). \tag{3.16}$$

By applying the inequality $1+x \leq e^x$, $x \in R$, for $x = \frac{\lambda}{n}U_{r_n}$ and $x = \frac{\lambda}{n}\sum_{i=1}^{r_n-1} U_i$ and getting expectations, we have

$$1 \leq \mathrm{E}e^{\frac{\lambda}{n}\sum_{i=1}^{r_n-1} U_i} \mathrm{E}e^{\frac{\lambda}{n}U_{r_n}}. \tag{3.17}$$

From $2p_nr_n \leq n$, we get $\frac{p_n^2 r_n}{n^2} \leq 1/4r_n$. Therefore, inequality (3.16) becomes

$$\mathrm{E}(e^{\lambda \bar{U}_n}) \leq \mathrm{E}e^{\frac{\lambda}{n}\sum_{i=1}^{r_n-1} U_i}\mathrm{E}e^{\frac{\lambda}{n}U_{r_n}}[1-\frac{\lambda^2}{4r_n}e^{\lambda M}Cov(Y_{p_n}, Y_{2p_n+1})]. \tag{3.18}$$

The inequality $xe \leq e^x$, $x \in R$ gives $\frac{\lambda^2}{4} \leq e^{-2}e^{\lambda} < e^{\lambda}$, so that

$$1-\frac{\lambda^2}{4r_n}e^{\lambda M}Cov(Y_{p_n}, Y_{2p_n+1}) \leq 1-\frac{1}{r_n}e^{\lambda(1+M)}Cov(Y_{p_n}, Y_{2p_n+1}), \tag{3.19}$$

and we wish to have $-e^{\lambda(1+M)}Cov(Y_{p_n}, Y_{2p_n+1}) \leq 1$ or

$$\lambda \leq -\frac{1}{M+1}\log(-Cov(Y_{p_n}, Y_{2p_n+1})). \tag{3.20}$$

On account of (3.19) and (3.20), inequality (3.18) yields

$$\mathrm{E}(e^{\lambda \bar{U}_n}) \leq \mathrm{E}e^{\frac{\lambda}{n}\sum_{i=1}^{r_n-1} U_i}\mathrm{E}e^{\frac{\lambda}{n}U_{r_n}}\left[1+\frac{1}{r_n}\right]. \tag{3.21}$$

Repeating the process which led to (3.21) another $r_n - 1$ times, we obtain, under condition (3.20),

$$\mathrm{E}(e^{\lambda \bar{U}_n}) \leq \left[1+\frac{1}{r_n}\right]^{r_n}\prod_{i=1}^{r_n}\mathrm{E}e^{\frac{\lambda}{n}U_i}. \tag{3.22}$$

By applying Lemma 3.3 and taking $W = U_i$, we have $|U_i| \leq p_nM$ and $b - a = 2p_nM$. Then

$$\prod_{i=1}^{r_n}\mathrm{E}e^{\frac{\lambda}{n}U_i} \leq e^{\lambda^2 M^2\frac{p_n^2 r_n}{2n^2}} \leq e^{\frac{\lambda^2 M^2}{8r_n}}. \tag{3.23}$$

Since also $(1+\frac{1}{r_n})^{r_n} \leq C_1$ for any positive constant C_1, inequality (3.22) becomes

$$\mathrm{E}e^{\lambda \bar{U}_n} \leq C_1 e^{\frac{\lambda^2 M^2}{8r_n}}. \quad \textit{subject to } (3.20) \tag{3.24}$$

Thus, for $\varepsilon_n > 0$ and under (3.20)

$$\mathrm{P}(\bar{U}_n \geq \varepsilon_n) \leq C_1\exp(-\lambda\varepsilon_n + \frac{\lambda^2 M^2}{8r_n}). \tag{3.25}$$

By minimizing the right-hand side in (3.25) with respect to λ, we have

$$\mathrm{P}(\bar{U}_n \geq \varepsilon_n) \leq C_1\exp\left(-2\frac{r_n{\varepsilon^2}_n}{M^2}\right), \tag{3.26}$$

for $\lambda_0 = 4\frac{r_n\varepsilon_n}{M^2}$ subject to (3.20).

For λ_0 as in (3.26), (3.20) is equivalent to

$$-Cov(Y_{p_n}, Y_{2p_n+1}) \leq \exp\left\{-\frac{4(M+1)}{M^2}r_n\varepsilon_n\right\}. \tag{3.27}$$

This completes the proof of the lemma. ☐

Remark 3.1. As stated in assumption (A3), the condition that the covariance function $Cov(X_1, X_{k+1})$ be nondecreasing is not, really, necessary, although it would not be easy to envision cases where it does not occur. This can be justified in the process of arriving at inequality (3.15) by way of (3.14). All one has to do is to produce some more refined bounds for the covariances, but such a result does not appear worth the effort.

For almost sure convergence purposes, we wish to have $2\frac{r_n\varepsilon_n^2}{M^2} = \log n^\alpha$ (for any arbitrary $\alpha > 1$), or equivalently,

$$\varepsilon_n = \left(\frac{\alpha M^2}{2}\right)^{1/2}\left(\frac{\log n}{r_n}\right)^{1/2}. \tag{3.28}$$

Then, λ_0 becomes

$$\lambda_0 = \left(\frac{8\alpha}{M^2}\right)^{1/2}(r_n \log n)^{1/2} \tag{3.29}$$

and condition (3.27) yields

$$-Cov(Y_{p_n}, Y_{2p_n+1}) \leq \exp\left\{-\frac{4(M+1)}{M}\left(\frac{\alpha}{2}\right)^{1/2}(r_n \log n)^{1/2}\right\}. \tag{3.30}$$

The following lemma summarizes these results.

Lemma 3.5. *Suppose assumptions (A1)-(A3) hold. With ε_n specified by (3.28), we have*

$$\mathrm{P}(\bar{U}_n \geq \varepsilon_n) \leq C_1 \exp\left(-2\frac{r_r{\varepsilon^2}_n}{M^2}\right),$$

provided $Cov(Y_{p_r}, Y_{2p_r+1})$ satisfies condition (3.30).

Remark 3.2. It is obvious that $\bar{V}_n$, as defined in (3.3), satisfies the same inequalities as $\bar{U}_n$ in Lemmas 3.4 and 3.5.

We may now dispense $\bar{W}_n$ as defined in (3.3).

Lemma 3.6. *Under assumptions (A1)-(A3) and with ε_n defined by (3.28), $\mathrm{P_r}(|\bar{W}_n| \geq \varepsilon_n) = 0$ for any large enough n.*

Proof. W_n consists of $n - 2p_n r_n$ terms and $n - 2p_n r_n < p_n$. Then, $|\bar{W}_n| < p_n\frac{M}{n} = 0$. So that $\mathrm{P}(|\bar{W}_n| \geq \varepsilon_n) \leq \mathrm{P}(M \geq \frac{n\varepsilon_n}{p_n})$. However, for any large enough n, this last expression is 0. □

Proof of Theorem 2.1. From Lemma 3.5, Remark 3.2 and Lemma 3.6, we obtain that the r.v.'s $-Y_i$, $i = 1, 2, \ldots, n$ have the same properties as the r.v.'s Y_i, $i = 1, 2, \ldots, n$. Thus, under condition (3.30) we have

$$\mathrm{P}(|\bar{U}_n| \geq \varepsilon_n) = \mathrm{P}(\bar{U}_n \geq \varepsilon_n) + \mathrm{P}(-\bar{U}_n \geq \varepsilon_n) \leq 2C_0 \exp\left(-2\frac{r_n\varepsilon_n^2}{M^2}\right),$$

and similarly for $\mathrm{P_r}(|\bar{V}_n| \geq \varepsilon_n)$. Then,

$$\begin{aligned} \mathrm{P}(|\bar{S}_n| \geq 3\varepsilon_n) &\leq \mathrm{P}(|\bar{U}_n| \geq \varepsilon_n) + \mathrm{P}(|\bar{V}_n| \geq \varepsilon_n) + \mathrm{P}(|\bar{W}_n| \geq \varepsilon_n) \\ &\leq \mathrm{P}(|\bar{U}_n| \geq \varepsilon_n) + \mathrm{P}(|\bar{V}_n| \geq \varepsilon_n) \quad (for\ n \geq n_0,\ say) \\ &\leq 4C_1 \exp\left(-2\frac{r_n\varepsilon_n^2}{M^2}\right). \end{aligned}$$

Finally, we obtain

$$\mathrm{P}(|\bar{S}_n| \geq \varepsilon_n) \leq C_0 \exp(-cr_n\varepsilon_n^2), \qquad c = 2/9M^2, \quad n \geq n_0,$$

where $C_0 = 4C_1$ provided

$$-\,Cov(Y_{p_n}, Y_{2p_n+1}) \leq \exp\left\{-\frac{4(M+1)}{3M^2}\left(\frac{\alpha}{2}\right)^{1/2}(r_n \log n)^{1/2}\right\}.$$

Then, $\bar{S}_n$ convergence to zero a.s. at the rate of $1/\varepsilon_n$.

For the value of n specified in (3.28), the rate of convergence is given by

$$\frac{1}{\varepsilon_n} = \left(\frac{2}{\alpha M^2}\right)^{1/2}\left(\frac{r_n}{\log n}\right)^{1/2}. \tag{3.31}$$

Inequality (3.30) becomes, equivalently:

$$r_n \leq \frac{1}{8\alpha}\left(\frac{M}{M+1}\right)^2 \frac{\log^2(-Cov(X_{p_n}, X_{2p_n+1}))}{\log n}. \tag{3.32}$$

Expression (3.31) shows that the maximum rate is attained for the maximum allowed value of r_n. This maximum value is obtained from (3.32) and is

$$r_n = \frac{1}{8\alpha}\left(\frac{M}{M+1}\right)^2 \frac{\log^2(-Cov(X_{p_n}, X_{2p_n+1}))}{\log n}. \tag{3.33}$$

This is so, because when r_n increases, p_n decreases, due to the fact that r_n and p_n are inverse proportional and also to the assumption that $Cov(X_1, X_{k+1})$ is an increasing function. Again, by the fact that $n/2p_nr_n \to 1$, it follows that $p_n = \frac{1}{2x_n}\frac{n}{r_n}$, some $0 < x_n \to 1$. So that (3.31) becomes

$$\begin{aligned} r_n &= \frac{1}{8\alpha}\left(\frac{M}{M+1}\right)^2 \frac{\log^2(-Cov(X_{p_n}, X_{2p_n+1}))}{\log n}, \\ p_n &= \frac{1}{2x_n}\frac{n}{r_n},\ 0 < x_n \to 1 \end{aligned} \tag{3.34}$$

and all one has to do is solve for r_n. Then, the corresponding rate of convergence is obtained from (3.31). □

Remark 3.3. From (3.31), it follows that the optimal convergence rate is obtained by taking $r_n = n$. However, such a choice is not allowed here. Consequently, the convergence rate $(n/\log n)^{1/2}$ is unattainable in the present framework.

Remark 3.4. From (3.31), by taking $r_n = \sqrt{n}$ we have the convergence rate $(\sqrt{n}/\log n)^{1/2}$.

4. Examples

In this section, two examples with common covariance functions are discussed. In each case, the optimal choice of r_n, provided by formula (3.34) is given as well as the corresponding best rate of almost sure convergence through formula (3.31).

Example 4.1. Suppose that $Cov(X_1, X_{k+1}) = \rho_0 \rho^k$, $0 < \rho < 1$, $\rho_0 < 0$. Then relation (3.34) is of the form:

$$r_n = C_{1n} \frac{1}{\log n} + C_{2n} \frac{n}{r_n \log n} + C_{3n} \frac{n^2}{r_n^2 \log n}$$

or

$$r_n{}^3 = C_{1n} \frac{r_n^2}{\log n} + C_{2n} \frac{n r_n}{\log n} + C_{3n} \frac{n^2}{\log n}, \tag{4.1}$$

and the last term on the right-hand side in (4.1) is of highest order. Therefore $r_n{}^3$ is of the order $n^2 / \log n$ and r_n is of the order $(n^2/\log n)^{1/3}$. Then, by (3.31), it turns out that $1/\varepsilon_n$ is of the order $(n/(\log n)^2)^{1/3}$.

Example 4.2. Suppose $Cov(X_1, X_{k+1}) = a_0 k^{-\lambda}$, $\lambda > 0$, $a_0 < 0$.

Then, we have

$$\begin{aligned} r_n \quad = \quad & C_{1n} \frac{1}{\log n} + C_{2n} \frac{\log r_n}{\log n} + C_{3n} \frac{(\log r_n)^2}{\log n} + C_{4n} + C_{5n} (\log r_n) \\ & + C_{6n} (\log n), \end{aligned} \tag{4.2}$$

and the last term on the right-hand side in (4.2) is of highest order. Thus, r_n is of order $\log n$ and then, $1/\varepsilon_n$ is a constant. Therefore, in this case, we do have almost sure convergence without rates.

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