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Changing the branching mechanism of a continuous state branching process using immigration

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Abstract. We consider an initial population whose size evolves according to a continuous state branching process. Then we add to this process an immigration (with the same branching mechanism as the initial population), in such a way that the immigration rate is proportional to the whole population size. We prove this continuous state branching process with immigration proportional to its own size is itself a continuous state branching process. By considering the immigration as the apparition of a new type, this construction is a natural way to model neutral mutation. It also provides in some sense a dual construction of the particular pruning at nodes of continuous state branching process introduced by the authors in a previous paper. For a critical or sub-critical quadratic branching mechanism, it is possible to explicitly compute some quantities of interest. For example, we compute the Laplace transform of the size of the initial population conditionally on the non-extinction of the whole population with immigration. We also derive the probability of simultaneous extinction of the initial population and the whole population with immigration.

Résumé. Nous considérons une population initiale dont la taille évolue selon un processus de branchement continu. Nous ajoutons ensuite à ce processus une population migrante (qui évolue selon le même mécanisme de branchement que la population initiale), avec un taux d'immigration propotionnel à la taille de la population totale. Nous montrons que ce processus de branchement continu avec immgration proportionnelle à sa taille est encore un processus de branchement continu. En voyant cette immigration comme l'apparition d'un nouveau type, cette construction est un moyen naturel de modéliser des mutations, neutres vis à vis de l'évolution. Elle peut être également vue comme la construction duale de l'élagage aux noeuds de l'arbre généalogique associé à la population totale, introduit par les auteurs dans un article précédent. Lorsque le mécanisme de branchement est quadratique et critique ou sous-critique, il est possible de calculer explicitement certaines quantités intéressantes. Par example, nous calculons la transformée de Laplace de la taille de la population initiale conditionnellement à la non-extinction de la population totale. Nous en déduisons également la probabilité d'extinction simultanée de la population initiale et de la population totale.

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1. Introduction

We consider an initial Eve-population of type 0 whose size evolves as a continuous state branching process (CB), $Y^0 = (Y^0_t, t \geq 0)$, with branching mechanism ψ^0 defined by

$$\psi^0(\lambda) = \alpha^0\lambda + \beta\lambda^2 + \int_{(0,\infty)} \pi(\mathrm{d}\ell)\big[\mathrm{e}^{-\lambda\ell} - 1 + \lambda\ell\mathbf{1}_{\{\ell\leq 1\}}\big], \tag{1}$$

where $\alpha^0 \in \mathbb{R}$, $\beta \geq 0$ and π is a Radon measure on $(0, \infty)$ such that $\int_{(0,\infty)}(1 \wedge \ell^2)\pi(\mathrm{d}\ell) < \infty$. See [9] for a definition of CB as limit of Galton–Watson processes. We assume that this population undergoes some irreversible mutations with constant rate, giving birth either to one individual of type 1 (with rate $\bar{\alpha}$), or to infinitely many offspring of type 1 (with rate and mutant offspring size described by a measure ν). This second population of type 1 evolves according to the same branching mechanism as the Eve-population (i.e. the mutations are neutral). The population of type 1 undergoes also some mutations and gives birth to a population of type 2 with the same rules, and so on.

If we lose track of the genealogy, the new population of type 1 can be seen as an immigration process with rate proportional to the size of the Eve-population, the population of type 2 is an immigration process with rate proportional to the size of the population of type 1, and so on. We are interested in the law of the total population size $X = (X_t, t \geq 0)$, which is a CB with immigration (CBI) rate proportional to its own size. If the mutations are neutral, we expect X to be a CB. This is indeed the case: if ψ^0 is the branching mechanism of the Eve-population and

$$\phi(\lambda) = \bar{\alpha}\lambda + \int_{(0,+\infty)} \nu(\mathrm{d}x)\left(1 - \mathrm{e}^{-\lambda x}\right)$$

is the immigration mechanism, then the total population size is a CB with branching mechanism $\psi = \psi^0 - \phi$, see Theorem 3.3.

Another approach is to associate with critical or sub-critical CBs a genealogical structure, i.e. an infinite continuous random tree (CRT), see [5] or [10]. In that context, each individual of the CB can be followed during its lifetime and mutations can be added as marks on its lineage. Pruning the CRT associated with the total population (of branching mechanism $\psi = \psi^0 - \phi$) at these marks allow to recover the Eve-population from the total population. This construction has been used in [2] and the construction given here via immigration proportional to the size of the population can be seen as the dual of the pruning construction of [2], see Section 4 and more precisely Corollary 4.2. However [2] considers only the case where the branching mechanism of X is given by a shift of the branching mechanism of the Eve-population. We shall give in a forthcoming paper [3] a more general pruning procedure which will correspond to the general proportional immigration presented here.

Natural questions then arise from a population genetics point of view, where only the whole population X_t is observed at time t. In order to compute some quantities related to the Eve-population, given the total population, we compute the joint law of the Eve-population and the whole population at a given time: (Y_t^0, X_t). For the quadratic critical or sub-critical branching mechanism, we provide the explicit Laplace transform of the joint distribution of (Y_t^0, X_t). In particular, we compute $\mathbb{P}(Y_t^0 = 0 | X_t > 0)$, the probability for the Eve-type to have disappeared at time t, conditionally on the survival of the total population at time t, see Remark 5.3. We also compute the Laplace transform of Y_t^0 conditionally on the population to never be extinct, see Proposition 5.6. In Lemma 5.5, we compute the probability of simultaneous extinction of the Eve-population and the whole population, in other words, the probability for the last individual alive to have undergone no mutation. The techniques used here did not lead us to an explicit formula but for the quadratic branching mechanism. For the general critical or sub-critical case, we use, in [1], a Williams decomposition of the genealogical tree to give a very simple formula for the probability of simultaneous extinction of the Eve-population and the whole population.

In the particular case of CB with quadratic branching mechanism ($\psi(u) = \beta u^2$, $\beta > 0$), similar results are given in [15] (using genealogical structure for CB) and in [16] (using a decomposition of Bessel bridges from [12]). In the critical ($\psi'(0^+) = 0$) or sub-critical ($\psi'(0^+) > 0$) case one could have used the genealogical process associated to CB introduced by [10] and to CBI developed by [8] to prove the present result. This presentation would have been more natural in view of the pruning method used in [2]. Our choice not to rely on this presentation was motivated by the possibility to consider super-critical cases ($\psi'(0^+) < 0$).

The paper is organized as follows: In Section 2, we recall some well-known facts on CB and CBI. In Section 3, we build a CBI X whose branching mechanism is ψ^0 and immigration rate at time t proportional to X_t and prove that this process is again a CB. We give in Section 4 some links with the pruning at nodes of CB introduced in [2]. Eventually, we compute the joint law of the Eve-population and the whole population in Section 5, as well as some related quantities.

2. CB and CB with immigration

The results from this section can be found in [7] (see also [11] for a survey on CB and CBI, and the references therein). Let ψ be a branching mechanism of a CB: for $\lambda \geq 0$,

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} \pi(\mathrm{d}\ell)\big[\mathrm{e}^{-\lambda\ell} - 1 + \lambda\ell \mathbf{1}_{\{\ell\leq 1\}}\big], \tag{2}$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$ and π is a Radon measure on $(0,\infty)$ such that $\int_{(0,\infty)}(1\wedge\ell^2)\pi(\mathrm{d}\ell) < \infty$. Notice ψ is smooth on $(0,\infty)$ and convex. We have $\psi'(0^+) \in [-\infty, +\infty)$, and $\psi'(0^+) = -\infty$ if and only if $\int_{(1,\infty)} \ell\pi(\mathrm{d}\ell) = \infty$. In order to consider only conservative CB, we shall also assume that for all $\varepsilon > 0$

$$\int_0^\varepsilon \frac{1}{|\psi(u)|}\,\mathrm{d}u = \infty. \tag{3}$$

Notice that $\psi'(0^+) > -\infty$ implies (3).

2.1. *CB*

Let $\mathbb{P}_x$ be the law of a CB $Z = (Z_t, t \geq 0)$ started at $x \geq 0$ and with branching mechanism ψ. The process Z is a Feller process and thus càdlàg. Thanks to (3), the process is conservative, that is a.s. for all $t \geq 0$, $Z_t < +\infty$. For every $\lambda > 0$, for every $t \geq 0$, we have

$$\mathbb{E}_x\big[\mathrm{e}^{-\lambda Z_t}\big] = \mathrm{e}^{-xu(t,\lambda)}, \tag{4}$$

where the function u is the unique non-negative solution of

$$u(t,\lambda) + \int_0^t \psi\big(u(s,\lambda)\big)\,\mathrm{d}s = \lambda, \quad \lambda \geq 0, t \geq 0. \tag{5}$$

This equation is equivalent to

$$\int_{u(t,\lambda)}^{\lambda} \frac{\mathrm{d}r}{\psi(r)} = t, \quad \lambda \geq 0, t \geq 0. \tag{6}$$

The process Z is infinitely divisible. Let Q be its canonical measure. The σ-finite measure Q is defined on the set of càdlàg functions. Intuitively, it gives the "distribution" of the size process for a population generated by an infinitesimal individual. In particular, by the Lévy–Itô decomposition (see for instance [14]), Z is distributed under $\mathbb{P}_x$ as $\sum_{i\in I} Z^i$, where $\sum_{i\in I}\delta_{Z^i}$ is a Poisson point measure with intensity $x\,Q(\mathrm{d}Z)$. Thus, for any non-negative measurable function F defined on the set of càdlàg functions, we have the following exponential formula for Poisson point measure

$$\mathbb{E}_x\big[\mathrm{e}^{-\sum_{i\in I} F(Z^i)}\big] = \exp\big(-x\,Q\big[1-\mathrm{e}^{-F(Z)}\big]\big).$$

The CB is called critical (resp. super-critical, resp. sub-critical) if $\psi'(0^+) = 0$ (resp. $\psi'(0^+) < 0$, resp. $\psi'(0^+) > 0$).

We shall need inhomogeneous notation. For $t < 0$, we set $Z_t = 0$. Let $\mathbb{P}_{x,t}$ denote the law of $(Z_{s-t}, s \in \mathbb{R})$ under $\mathbb{P}_x$, and let Q_t be the distribution of $(Z_{s-t}, s \in \mathbb{R})$ under Q.

For μ a positive measure on $\mathbb{R}$, we set $H^\mu = \sup\{r \in \mathbb{R};\, \mu([r,\infty)) > 0\}$ the maximal element of its support.

Proposition 2.1. *Let μ be a finite positive measure on $\mathbb{R}$ with support bounded from above (i.e. H^μ is finite). Then we have for all $s \in \mathbb{R}$, $x \geq 0$,*

$$\mathbb{E}_x\big[\mathrm{e}^{-\int Z_{r-s}\mu(\mathrm{d}r)}\big] = \mathrm{e}^{-xw(s)}, \tag{7}$$

where the function w is a measurable locally bounded non-negative solution of the equation

$$w(s) + \int_s^\infty \psi\big(w(r)\big)\,\mathrm{d}r = \int_{[s,\infty)} \mu(\mathrm{d}r), \quad s \le H^\mu \quad \textit{and} \quad w(s) = 0, \quad s > H^\mu. \tag{8}$$

If $\psi'(0^+) > -\infty$ or if $\mu(\{H^\mu\}) > 0$, then (8) has a unique measurable locally bounded non-negative solution.

This result is well known for the critical and sub-critical branching mechanism (see e.g. [6]). As we did not find a reference for the super-critical branching mechanism, we give a short proof of this proposition.

Proof of Proposition 2.1. Let $n \ge 1$. We set $Z_t^{(n),s} = Z_{(i+1)/2^n - s}$ for $t \in [i/2^n, (i+1)/2^n)$. Using that Z is càdlàg, we get a.s. $\lim_{n\to\infty} Z_t^{(n),s} = Z_{t-s}$ for all $t, s \in \mathbb{R}$. Since the process Z is finite, we get by the dominated convergence theorem a.s. for all $s \in \mathbb{R}$

$$\int_{[-s,H^\mu]} Z_{r-s}\mu(\mathrm{d}r) = \lim_{n\to\infty} \int_{[-s,H^\mu]} Z_r^{(n),s}\mu(\mathrm{d}r).$$

Using the Markov property of Z, we get that

$$\mathbb{E}_x\big[\mathrm{e}^{-\int Z_r^{(n),s}\mu(\mathrm{d}r)}\big] = \mathrm{e}^{-x w^{(n)}(s)},$$

where $w^{(n)}$ is the unique non-negative solution of

$$w^{(n)}(s) + \int_s^{([H^\mu 2^n]+1)/2^n} \psi\big(w^{(n)}(r)\big)\,\mathrm{d}r = \int_{[k/2^n,\infty)} \mu(\mathrm{d}r),$$

with k s.t. $k/2^n < s \le (k+1)/2^n$.

Let $T > H^\mu + 1$. Notice that for all $s \in [-T, T]$, we have $\int Z_r^{(n),s}\mu(\mathrm{d}r) \le \sup\{Z_t, t \in [0, 2T]\}\mu([-T, H^\mu]) < \infty$ a.s. Let C be defined by $\mathrm{e}^{-C} = \mathbb{E}_x[\mathrm{e}^{-\sup\{Z_t, t\in[0,2T]\}\mu([-T,H^\mu])}]$. Notice $C < \infty$. This implies that for all $n \ge 1$, $s \in [-T, T]$,

$$0 \le w^{(n)}(s) \le C < \infty.$$

By the dominated convergence theorem, $w^{(n)}(s)$ converges to $w(s) = -\log(\mathbb{E}_1[\mathrm{e}^{-\int Z_{r-s}\mu(\mathrm{d}r)}])$, which lies in $[0, C]$, for all $s \in [-T, T]$. By the dominated convergence theorem, we deduce that w solves (8). Since T is arbitrary, the proposition is proved but for the uniqueness of solutions of (8).

If $\psi'(0^+) > -\infty$, then ψ is locally Lipschitz. This implies there exists a unique locally bounded non-negative solution of (8).

If $\psi'(0^+) = -\infty$, and $\mu(\{H^\mu\}) > 0$, we get that $\int Z_{r-s}\mu(\mathrm{d}r) \ge a Z_{H^\mu - s}$, where $a = \mu(\{H^\mu\}) > 0$. This implies that $w(s) \ge u(H^\mu - s, a) > 0$ for $s \in \mathbb{R}$. The function $u(\cdot, a)$ is strictly positive on $\mathbb{R}_+$ because of condition (3) and Eq. (6). Since ψ is locally Lipschitz on $(0, \infty)$, we deduce there exists a unique locally bounded non-negative solution of (8). □

2.2. *CBI*

Let $x > 0$, $\bar\alpha \ge 0$, ν be a Radon measure on $(0, \infty)$ such that $\int_{(0,\infty)} (1 \wedge x)\nu(\mathrm{d}x) < \infty$. Let $\mathcal{B}_+$ denote the set of non-negative measurable functions defined on $\mathbb{R}$. Let $h \in \mathcal{B}_+$ be locally bounded. We consider the following independent processes.

- $\sum_{i\in I} \delta_{t^i, x^i, Z^i}$, a Poisson point measure with intensity $h(t)\mathbf{1}_{\{t\ge 0\}}\,\mathrm{d}t\,\nu(\mathrm{d}x)\mathbb{P}_{x,t}(\mathrm{d}Z)$.
- $\tilde Z$, distributed according to $\mathbb{P}_x$.
- $\sum_{j\in J} \delta_{t^j, \hat Z^j}$, a Poisson point measure with intensity $\bar\alpha h(t)\mathbf{1}_{\{t\ge 0\}}\,\mathrm{d}t\,Q_t(\mathrm{d}Z)$.

For $t \in \mathbb{R}$, let $Y_t = \tilde{Z}_t + \sum_{i\in I} Z_t^i + \sum_{j\in J} \hat{Z}_t^j \in [0,\infty]$. We say $Y = (Y_t, t \geq 0)$ is a continuous state branching process with immigration (CBI) started at x, whose branching mechanism is ψ and immigration is characterized with (h, ϕ) where the immigration mechanism, ϕ, is defined by

$$\phi(\lambda) = \bar{\alpha}\lambda + \int_{(0,\infty)} \nu(\mathrm{d}x)\big(1 - \mathrm{e}^{-\lambda x}\big), \quad \lambda \geq 0, \tag{9}$$

where $\bar{\alpha} \geq 0$, and ν is a Radon measure on $(0,\infty)$ such that $\int_{(0,\infty)} (1 \wedge x)\nu(\mathrm{d}x) < \infty$.

One gets Y is a conservative Hunt process when h is constant, see [7]. Notice that Y is a non-homogeneous Markov processes. We also have $Y_0 = x$ and $Y_t = 0$ for $t < 0$.

Using the Poisson point measure property, one can construct on the same probability space two CBI, Y^1 and Y^2, with the same branching process ψ, the same starting point and immigration characterized by (h^1, ϕ) and (h^2, ϕ) such that $Y_t^1 \leq Y_t^2$ for all $t \leq T$ as soon as $h^1(t) \leq h^2(t)$ for all $t \leq T$. We can apply this with $h^1 = h$ and $h^2(t) = \sup\{h(s); s \in [0, T]\}$ for $t \in \mathbb{R}$ and some $T > 0$, and use that Y^2 is conservative (see [7]) to get that Y^1 has a locally bounded version over $[0, T]$. Since T is arbitrary, this implies that any CBI has a locally bounded version. We shall work with this version.

The following lemma is a direct consequence of the exponential formula for Poisson point measures (see e.g. [13], Chapter XII).

Lemma 2.2. *Let μ be a finite positive measure on $\mathbb{R}$ with support bounded from above (i.e. H^μ is finite). We have for $s \in \mathbb{R}$:*

$$\mathbb{E}\big[\mathrm{e}^{-\int Y_{r-s}\mu(\mathrm{d}r)}\big] = \mathrm{e}^{-xw(s) - \int_0^\infty h(t)\phi(w(s+t))\,\mathrm{d}t}, \tag{10}$$

where the function w is defined by (7).

3. State dependent immigration

3.1. *Induction formula*

Let $(x_k, k \in \mathbb{N})$ be a sequence of non-negative real numbers. Let Y^0 be a CB with branching mechanism ψ^0, defined by (1), starting at x_0. We shall assume that Y^0 is conservative, that is condition (3) holds for ψ^0. We construct by induction Y^n, $n \geq 1$, as the CBI started at x_n, with branching mechanism ψ^0 and immigration characterized by (Y^{n-1}, ϕ), with ϕ given by (9).

Lemma 3.1. *Let $(\mu_k, k \in \mathbb{N})$ be a family of finite measures on $\mathbb{R}$ with support bounded from above. We have for all $n \in \mathbb{N}$, $s \in \mathbb{R}$,*

$$\mathbb{E}\big[\mathrm{e}^{-\sum_{k=0}^n \int Y_{r-s}^k \mu_k(\mathrm{d}r)}\big] = \mathrm{e}^{-\sum_{k=0}^n x_{n-k} w_k^{(n)}(s)},$$

where $w_0^{(n)}$ is defined by (7) with μ replaced by μ_n, and for $k \geq 1$, $w_k^{(n)}$ is defined by (7) with μ replaced by $\mu_{n-k}(\mathrm{d}r) + \phi(w_{k-1}(r))\,\mathrm{d}r$. In particular, w_k is a locally bounded non-negative solution of the equation

$$w(s) + \int_s^\infty \psi^0\big(w(r)\big)\,\mathrm{d}r = \int_{[s,\infty)} \mu_{n-k}(\mathrm{d}r) + \int_s^\infty \phi\big(w_{k-1}^{(n)}(r)\big)\,\mathrm{d}r, \quad s \in \mathbb{R}. \tag{11}$$

(*Notice $w_k(s) = 0$ for $s > \max\{H^{\mu_{k'}}, k' \in \{0, \ldots, k\}\}$.*)

Proof. This is a consequence of the computation of $\mathbb{E}[\mathrm{e}^{-\sum_{k=0}^n \int Y_{r-s}^k \mu_k(\mathrm{d}r)} | Y^0, \ldots, Y^{n-1}]$, using Proposition 2.1. This also implies that (11) holds. Then, by induction, one deduces from (11) that w_k is locally bounded. □

3.2. *Convergence of the total mass process*

We consider the sequence $(Y^n, n \geq 0)$ defined in the previous section with $x_0 = x \geq 0$ and $x_n = 0$ for $n \geq 1$. We set $X_t^n = \sum_{k=0}^n Y_t^k$ for $t \in \mathbb{R}$. Let X_t be the increasing limit of X_t^n as $n \to +\infty$, for all $t \in \mathbb{R}$. We have $X_t \in [0, +\infty]$. We call $X = (X_t, t \in \mathbb{R})$ a CBI with branching mechanism ψ^0 and immigration process (X, ϕ). We set $\psi = \psi^0 - \phi$.

Remark 3.2. *For $\lambda \geq 0$, we have*

$$\psi^0(\lambda) - \phi(\lambda) = \left(\alpha^0 - \bar{\alpha} - \int_{(0,1]} \ell \nu(\mathrm{d}\ell)\right)\lambda + \beta\lambda^2 + \int_{(0,\infty)} \big(\pi(\mathrm{d}\ell) + \nu(\mathrm{d}\ell)\big)\big[\mathrm{e}^{-\lambda\ell} - 1 + \lambda\ell \mathbf{1}_{\{\ell \leq 1\}}\big].$$

This gives that $\psi = \psi^0 - \phi$ is a branching mechanism.

The process Y^0 describes the size process of the Eve-population, Y^1 the size process of the population of mutants born from the Eve-population Y^0, Y^2 the size process of the population of mutants born from mutant population Y^1, and so on. The size process of the total population is given by $X = \sum_{k \geq 0} Y^k$. In a neutral mutation case, it is natural to assume that all the processes Y^k have the same branching mechanism. Since we assume $x_k = 0$ for all $k \geq 1$, this means only the Eve-population is present at time 0.

Theorem 3.3. *We assume that ψ is conservative, i.e. satisfies (3). The process X, which is a CBI with branching mechanism ψ^0 and immigration process (X, ϕ), is a CB with branching mechanism $\psi = \psi^0 - \phi$.*

Remark 3.4. *As a consequence of Theorem 3.3, X is a Markov process. Notice that the process $(Y^0, \ldots, Y^n)$ is also Markov but not $(X_t^n, t \geq 0)$ for $n \geq 1$.*

Proof of Theorem 3.3. Let μ be a finite measure on $\mathbb{R}$ with support bounded from above (i.e. $H^\mu < \infty$). We shall assume that $\mu(\{H^\mu\}) = a > 0$.

We keep the notations of Lemma 3.1, with $\mu_k = \mu$. In particular we see from (11) that $w_k^{(n)}$ does not depend on n. We shall denote it by w_k. By monotone convergence, we have

$$\mathbb{E}\big[\mathrm{e}^{-\int X_{r-s}\mu(\mathrm{d}r)}\big] = \lim_{n\to\infty} \mathbb{E}\big[\mathrm{e}^{-\sum_{k=0}^n \int Y_{r-s}^k \mu(\mathrm{d}r)}\big] = \lim_{n\to\infty} \mathrm{e}^{-x w_n(s)},$$

where the limits are non-increasing. This implies that $(w_n, n \geq 0)$ increases to a non-negative function w_∞. By monotone convergence theorem (for $\int_s^{H^\mu} \psi^0(w(r)) \mathbf{1}_{\{w_n(r) > 0\}}\, \mathrm{d}r$ and the integral with ϕ) and dominated convergence theorem (for $\int_s^{H^\mu} \psi^0(w(r)) \mathbf{1}_{\{w_n(r) \leq 0\}}\, \mathrm{d}r$), we deduce from (11), that w_∞ solves $w(s) = 0$ for $s > H^\mu$ and

$$w(s) + \int_s^{H^\mu} \psi^0\big(w(r)\big)\, \mathrm{d}r = \int_{[s,\infty)} \mu(\mathrm{d}r) + \int_s^{H^\mu} \phi\big(w(r)\big)\, \mathrm{d}r, \quad s \leq H^\mu. \tag{12}$$

Notice that $w_\infty(s) \in [0, \infty]$ and the two sides of the previous equality may be infinite.

Thanks to Proposition 2.1, and since $\psi^0 - \phi$ is a branching mechanism (see Remark 3.2), there exists a unique locally bounded non-negative solution of (12), which we shall call $\bar{w}$. Therefore to prove that $w_\infty = \bar{w}$, it is enough to check that w_∞ is locally bounded. This will be the case if we check that $w_\infty \leq \bar{w}$. In particular, we get $w_\infty = \bar{w}$, if we can prove that $w_n \leq \bar{w}$ for all $n \in \mathbb{N}$. We shall prove this by induction.

We consider the measure $\mu^0(\mathrm{d}r) = \mu(\mathrm{d}r) + \phi(\bar{w}(r)) \mathbf{1}_{\{r \leq H^\mu\}}\, \mathrm{d}r$. Notice $H^{\mu_0} = H^\mu$ and $\mu^0(\{H^{\mu^0}\}) = \mu(\{H^\mu\}) = a > 0$. We define $\bar{w}_0$ by

$$\mathrm{e}^{-x\bar{w}_0(s)} = \mathbb{E}\big[\mathrm{e}^{-\int Y_{r-s}^0 \mu^0(\mathrm{d}r)}\big].$$

The function $\bar{w}_0$ is a locally bounded non-negative function which solves

$$w(s) + \int_s^{H^\mu} \psi^0\big(w(r)\big)\, \mathrm{d}r = \int_{[s,\infty)} \mu(\mathrm{d}r) + \int_s^{H^\mu} \phi\big(\bar{w}(r)\big)\, \mathrm{d}r, \quad s \leq H^\mu.$$

Thanks to Proposition 2.1, $\bar{w}_0$ is unique. Since $\bar{w}$ solves the same equation, we deduce that $\bar{w}_0 = \bar{w}$. We also have

$$\mathrm{e}^{-x w_0(s)} = \mathbb{E}\big[\mathrm{e}^{-\int Y^0_{r-s}\mu(\mathrm{d}r)}\big] \geq \mathbb{E}\big[\mathrm{e}^{-\int Y^0_{r-s}\mu^0(\mathrm{d}r)}\big].$$

This implies that $w_0 \leq \bar{w}_0 = \bar{w}$.

Assume we proved that $w_{n-1} \leq \bar{w}$ for some $n \geq 1$. Then we can consider the measure $\mu^n(\mathrm{d}r) = \mu(\mathrm{d}r) + [\phi(\bar{w}(r)) - \phi(w_{n-1}(r))]\mathbf{1}_{\{r \leq H^\mu\}}\,\mathrm{d}r$. Notice $H^{\mu^n} = H^\mu$ and $\mu^n(\{H^{\mu^n}\}) = a > 0$. Recall $x = x_0 \geq 0$ and $x_k = 0$ for $k \geq 1$. We define $\bar{w}_n$ by

$$\mathrm{e}^{-x\bar{w}_n(s)} = \mathbb{E}\big[\mathrm{e}^{-\int Y^n_{r-s}\mu^n(\mathrm{d}r)}\big].$$

The function $\bar{w}_n$ is a locally bounded non-negative function which solves for $s \leq H^\mu$

$$\begin{aligned} w(s) + \int_s^{H^\mu} \psi^0\big(w(r)\big)\,\mathrm{d}r &= \int_{[s,\infty)} \mu^n(\mathrm{d}r) + \int_s^{H^\mu} \phi\big(w_{n-1}(r)\big)\,\mathrm{d}r \\ &= \int_{[s,\infty)} \mu(\mathrm{d}r) + \int_s^{H^\mu} \phi\big(\bar{w}(r)\big)\,\mathrm{d}r. \end{aligned}$$

Thanks to Proposition 2.1, $\bar{w}_n$ is unique. Since $\bar{w}$ solves the same equation, we deduce that $\bar{w}_n = \bar{w}$. We also have

$$\mathrm{e}^{-x w_n(s)} = \mathbb{E}\big[\mathrm{e}^{-\int Y^n_{r-s}\mu(\mathrm{d}r)}\big] \geq \mathbb{E}\big[\mathrm{e}^{-\int Y^n_{r-s}\mu^n(\mathrm{d}r)}\big].$$

This implies that $w_n \leq \bar{w}$. Therefore, this holds for all $n \geq 0$, which according to our previous remark entails that $w_\infty = \bar{w}$.

By taking $\mu(\mathrm{d}r) = \sum_{k=1}^K \lambda_k \delta_{t_k}(\mathrm{d}r)$ for $K \in \mathbb{N}^*$, $\lambda_1, \ldots, \lambda_K \in [0,\infty)$ and $0 \leq t_1 \leq \cdots \leq t_K$, we deduce that X has the same finite marginal distribution as a CB with branching mechanism $\psi^0 - \phi$. Hence X is a CB with branching mechanism $\psi^0 - \phi$. □

4. The dual to the pruning at node

For $\theta \in \mathbb{R}$, we consider the group of operators $(T_\theta, \theta \in \mathbb{R})$ on the set of real measurable functions defined by

$$T_\theta(f)(\cdot) = f(\theta + \cdot) - f(\theta).$$

Let ψ^0 be given by (1) with Lévy measure π. Using the previous section, for $\theta > 0$, we can give a probabilistic interpretation to $T_{-\theta}(\psi^0)$ as a branching mechanism of a CBI with proportional immigration. Let $\theta_0 = \sup\{\theta \geq 0;\ \int_{(1,\infty)} \mathrm{e}^{\theta\ell}\pi(\mathrm{d}\ell) < \infty\}$. Notice that $\theta_0 = 0$ if ${\psi^0}'(0^+) = -\infty$, as ${\psi^0}'(0^+) = -\infty$ is equivalent to $\int_{(1,\infty)} \ell\pi(\mathrm{d}\ell) = +\infty$. We assume $\theta_0 > 0$ and we set $\Theta = (0,\theta_0]$ if $\int_{(1,\infty)} \mathrm{e}^{\theta_0\ell}\pi(\mathrm{d}\ell) < \infty$ and $\Theta = (0,\theta_0)$ otherwise. Let $\theta \in \Theta$. We define

$$\phi_\theta(\lambda) = 2\beta\theta\lambda + \int_{(0,\infty)} \big(\mathrm{e}^{\theta x} - 1\big)\big(1 - \mathrm{e}^{-\lambda x}\big)\pi(\mathrm{d}x).$$

It is straightforward to check that $T_{-\theta}(\psi^0) = \psi^0 - \phi_\theta$ and that ϕ_θ is an immigration mechanism. Notice that for $\theta < \theta_0$, we have $T_{-\theta}(\psi^0)'(0^+) > -\infty$, that is $T_{-\theta}(\psi^0)$ is a conservative branching mechanism.

The next corollary is a direct consequence of the previous section.

Corollary 4.1. *Let $\theta \in \Theta$. If $\theta = \theta_0$ assume furthermore that $T_{-\theta_0}(\psi^0)$ is conservative. A CBI process X with branching mechanism ψ and immigration (X, ϕ_θ) is a CB with branching mechanism $T_{-\theta}(\psi)$.*

On the other end, for $\theta > 0$, $T_\theta(\psi)$ can be seen as the branching mechanism of a pruned CB. The following informal presentation relies on the pruning procedure developed in [2]. Let us consider a CRT associated with a critical or sub-critical branching mechanism ψ with no Brownian part, which we shall write in the following form:

$$\psi(\lambda) = \alpha_1\lambda + \int_{(0,+\infty)} \big(\mathrm{e}^{-\lambda r} - 1 - \lambda r\big)\pi(\mathrm{d}r),$$

with $\int_{[1,\infty)} r\pi(\mathrm{d}r) < \infty$, $\alpha_1 = \alpha + \int_{[1,\infty)} r\pi(\mathrm{d}r) \geq 0$ and $\int_{(0,1)} r\pi(\mathrm{d}r) = +\infty$. In that case, the CB process with branching mechanism ψ has no diffusion part ($\beta = 0$) and increases only by positive jumps.

Let us recall that a CRT can be coded by the so-called height process $H = (H_t, t \geq 0)$, see [5]. A CRT is a random rooted real tree, i.e. a compact metric space (τ, d) such that there is only one continuous path from a point to another, with a distinguished vertex called the root. To each $t \geq 0$ is associated an individual in the tree, and H_t is the distance of this individual to the root. Intuitively, for the individual $t \geq 0$, $H_t \geq 0$ represents its generation. Conversely, it is possible to reconstruct the CRT given the height process H. The individual t is called an ancestor of s if $H_t = \min\{H_u, u \in [s \wedge t, s \vee t]\}$, and we shall write $s \succcurlyeq t$. This describes a genealogical structure that can be coded by a real tree.

Informally, for t fixed, the "size" of the population at generation $a \geq 0$ of all individuals $r \leq t$ is given by the local time of H at level a up to time t, L_t^a say. For the CRT associated with the branching mechanism ψ, the process $L = (L^a_{T_x}, a \geq 0)$, where $T_x = \inf\{t \geq 0, L_t^0 \geq x\}$ is a CB with branching mechanism ψ. The height process codes for the genealogy of the CB process L.

An individual t is called a node of the CRT if the height process corresponding to its descendants, $(H_s - H_t, s \succcurlyeq t)$ has a positive local time at level 0, say Δ_t. (If $t \leq T_x$, then Δ_t corresponds to a jump of the CB process L at level H_t; reciprocally to a jump Δ of the CB process L at level a there corresponds an individual $t \leq T_x$ such that $H_t = a$ and $\Delta_t = \Delta$.) Intuitively Δ_t corresponds to the "size" of the offspring population of individual t. Let $\theta > 0$ be fixed. A node t of size Δ_t is marked with probability $1 - \mathrm{e}^{-\theta\Delta_t}$, independently of the other nodes. To prune the CRT, we just remove all individuals who have a marked ancestor. The height process of the pruned CRT is then given by $H^\theta = (H_{C_t}, t \geq 0)$, where C is the inverse of the Lebesgue measure of the set of individuals whose ancestors have no mark:

$$C_t = \inf\left\{r_0 \geq 0; \int_0^{r_0} \mathbf{1}_{\{\forall s, r \succcurlyeq s, s \text{ is not marked}\}}\, \mathrm{d}r \geq t\right\}.$$

Theorem 1.5 in [2] shows that this pruned CRT is itself a CRT associated with the branching mechanism $T_\theta(\psi)$.

By looking at the local time of the pruned process, we get a nice construction of a CB process of branching mechanism $T_\theta(\psi)$, which we shall call a pruned CB with intensity $\theta > 0$, from a CB process of branching mechanism ψ. Notice this construction was done under the assumption that $\beta = 0$ (see also [4] when $\beta > 0$ and $\pi = 0$). The general pruning procedure in the general case $\beta > 0$ and $\pi \neq 0$ will be presented in a forthcoming paper [3].

In a certain sense the immigration is the dual to the pruning at node: to build a CB process of branching mechanism ψ from a CB process of branching mechanism $T_\theta(\psi)$, with $\theta > 0$, one has to add an immigration at time t which rate is proportional to the size of the population at time t and immigration mechanism $\tilde{\phi}_\theta$ defined by:

$$\tilde{\phi}_\theta(\lambda) = T_\theta(\psi)(\lambda) - \psi(\lambda) = 2\beta\theta\lambda + \int_{(0,\infty)} \left(1 - \mathrm{e}^{-\theta x}\right)\left(1 - \mathrm{e}^{-\lambda x}\right)\pi(\mathrm{d}x) \quad \text{for } \lambda \geq 0.$$

In other words, we get the following result, whose first part comes from Theorem 1.5 in [2]. As in [2], we assume only for the next corollary that $\beta = 0$ and $\int_{(0,1)} \ell\pi(\mathrm{d}\ell) = +\infty$.

Corollary 4.2. *Let X be a critical or sub-critical CB process with branching mechanism ψ. Let $X^{(\theta)}$ be the pruned CB of X with intensity $\theta > 0$: $X^{(\theta)}$ is a CB process with branching mechanism $T_\theta(\psi)$. The CBI process, $\tilde{X}$, with branching mechanism $T_\theta(\psi)$ and immigration $(\tilde{X}, \tilde{\phi}_\theta)$ is distributed as X.*

5. Application: Law of the initial process

We consider a population whose size evolves as $X = (X_t, t \geq 0)$, a CB with branching mechanism ψ given by (2). We assume ψ satisfies the hypothesis of Section 2. This population undergoes some irreversible mutations with constant rate. Each mutation produces a new type of individuals. In the critical or sub-critical quadratic case ($\pi = 0$), the genealogy of the associated CB corresponds to the limit of the genealogy of a Wright–Fisher model, up to some time change due to non-constant population size.

We assume the population at time 0 has the same original Eve-type. We are interested in the law of $Y^0 = (Y_t^0, t \geq 0)$, the "size" of the sub-population with the original type knowing the size of the whole population. In particular, we

shall compute $\mathbb{P}(Y_t^0 = 0 | X_t > 0)$, the probability for the Eve-type to have disappeared, conditionally on the survival of the total population at time t.

We shall assume Y^0 is a CB with branching mechanism ψ^0 and X is the CBI with immigration (X, ϕ), with $\phi = \psi^0 - \psi$, considered in Section 3.2. Thus, we model the mutations by an immigration process with rate proportional to the size of the population.

The joint law of (X_t, Y_t^0) can be easily characterized by the following lemma.

Lemma 5.1. *Let $t \geq 0$, $\lambda_1, \lambda_2 \in \mathbb{R}_+$. We assume $X_0 = Y_0^0 = x \geq 0$. We have*

$$\mathbb{E}\big[\mathrm{e}^{-\lambda_1 X_t - \lambda_2 Y_t^0}\big] = \mathrm{e}^{-x w(0)},$$

where (w, w^*) is the unique measurable non-negative solution on $(-\infty, t]$ of

$$w(s) + \int_s^t \psi^0\big(w(r)\big)\,\mathrm{d}r = \lambda_1 + \lambda_2 + \int_s^t \phi\big(w^*(r)\big)\,\mathrm{d}r,$$
$$w^*(s) + \int_s^t \psi\big(w^*(r)\big)\,\mathrm{d}r = \lambda_1.$$

Proof. Recall the notation of Section 3.2. In particular $x_0 = x$ and $x_n = 0$ for all $n \geq 1$. Let us apply Lemma 3.1 with

$$\mu_0(\mathrm{d}r) = (\lambda_1 + \lambda_2)\delta_t(\mathrm{d}r),$$
$$\mu_k(\mathrm{d}r) = \lambda_1 \delta_t(\mathrm{d}r) \quad \text{for } k \geq 1.$$

We get

$$\mathbb{E}\big[\mathrm{e}^{-(\lambda_1 X_t^n + \lambda_2 Y_t^0)}\big] = \mathrm{e}^{-x w_n^{(n)}(0)},$$

where for $s \leq t$,

$$w_0^{(n)}(s) + \int_s^t \psi^0\big(w_0^{(n)}(r)\big)\,\mathrm{d}r = \lambda_1,$$
$$w_k^{(n)}(s) + \int_s^t \psi^0\big(w_k^{(n)}(r)\big)\,\mathrm{d}r = \lambda_1 + \int_s^t \phi\big(w_{k-1}^{(n)}(r)\big)\,\mathrm{d}r \quad \text{for } 1 \leq k \leq n-1,$$
$$w_n^{(n)}(s) + \int_s^t \psi^0\big(w_n^{(n)}(r)\big)\,\mathrm{d}r = \lambda_1 + \lambda_2 + \int_s^t \phi\big(w_{n-1}^{(n)}(r)\big)\,\mathrm{d}r.$$

We let n go to infinity and use similar arguments as in the proof of Theorem 3.3 to get the result. □

Some more explicit computations can be made in the case of quadratic branching mechanism (see also [16] when $\alpha = 0$). Let $\alpha \geq 0$ and $\theta > 0$ and set

$$\psi(u) = \alpha u + u^2, \qquad \psi^0(u) = T_\theta(\psi)(u) = (\alpha + 2\theta)u + u^2.$$

The CB which models the total population is critical ($\alpha = 0$) or sub-critical ($\alpha > 0$). The immigration mechanism is $\phi(u) = \psi^0(u) - \psi(u) = 2\theta u$.

We set $b = (\alpha + 2\theta)$ and for $t \geq 0$,

$$h(t) = \begin{cases} 1 + \lambda_1 \frac{1 - \mathrm{e}^{-\alpha t}}{\alpha} & \text{if } \alpha > 0, \\ 1 + \lambda_1 t & \text{if } \alpha = 0. \end{cases} \tag{13}$$

Proposition 5.2. *Let $t \geq 0$, $\lambda_1, \lambda_2 \in \mathbb{R}_+$. We have*

$$\mathbb{E}\big[\mathrm{e}^{-\lambda_1 X_t - \lambda_2 Y_t^0}\big] = \mathrm{e}^{-x v_0(t)},$$

where

$$v_0(t) = \mathrm{e}^{-bt} h(t)^{-2} \left(\frac{1}{\lambda_2} + \int_0^t \mathrm{e}^{-br} h(r)^{-2}\, \mathrm{d}r \right)^{-1} + \lambda_1 \mathrm{e}^{-\alpha t} h(t)^{-1}.$$

Proof. By the previous lemma, we have

$$\mathbb{E}\big[\mathrm{e}^{-\lambda_1 X_t - \lambda_2 Y_t^0}\big] = \mathrm{e}^{-x w(0)}, \tag{14}$$

where for $s \leq t$,

$$w(s) + \int_s^t w(r)\big(w(r) + b\big)\, \mathrm{d}r = \lambda_1 + \lambda_2 + 2\theta \int_s^t w^*(r)\, \mathrm{d}r,$$

$$w^*(s) + \int_s^t w^*(r)\big(w^*(r) + \alpha\big)\, \mathrm{d}r = \lambda_1. \tag{15}$$

The last equation is equivalent to

$$\big(w^*\big)' - w^*\big(w^* + \alpha\big) = 0 \quad \text{on } (-\infty, t],\ w^*(t) = \lambda_1. \tag{16}$$

The function $z^* := \frac{1}{w^*}$ is thus the unique solution of

$$\big(z^*\big)' + \alpha z^* + 1 = 0 \quad \text{on } (-\infty, t],\ z^*(t) = \frac{1}{\lambda_1}.$$

If $\alpha > 0$, this leads to

$$z^*(s) = \frac{1}{\alpha}\big(\mathrm{e}^{\alpha(t-s)} - 1\big) + \frac{1}{\lambda_1}\mathrm{e}^{\alpha(t-s)}.$$

If $\alpha = 0$, we have $z^*(s) = t - s + \frac{1}{\lambda_1}$. We get

$$w^*(s) = h'(t-s)h(t-s)^{-1} = \lambda_1 \mathrm{e}^{-\alpha(t-s)} h(t-s)^{-1}. \tag{17}$$

Equation (15) is equivalent to

$$w' - w(w + b) = -2\theta w^* \quad \text{on } (-\infty, t],\ w(t) = \lambda_1 + \lambda_2.$$

Set $y = w - w^*$ and use the differential equation (16), to get that y solves

$$y' - y^2 - y\big(2w^* + b\big) = 0 \quad \text{on } (-\infty, t],\ y(t) = \lambda_2.$$

Then the function $z := 1/y$ is the unique solution of

$$z' + \big(2w^* + b\big)z + 1 = 0 \quad \text{on } (-\infty, t],\ z(t) = \frac{1}{\lambda_2}.$$

One solution of the homogeneous differential equation $z_0' = -(2w^* + b)z_0$ is $z_0(s) = \mathrm{e}^{b(t-s)} h(t-s)^2$. Looking for solutions of the form $z(s) = C(s)z_0(s)$ gives

$$z(s) = z_0(s)\left(\frac{1}{\lambda_2} + \int_s^t z_0(u)^{-1}\, \mathrm{d}u \right).$$

We conclude using (14) and $w = w^* + z^{-1}$. □

Remark 5.3. *We can compute the conditional probability of the non-extinction of the Eve-population*: $\mathbb{P}(Y_t^0 > 0 | X_t > 0)$. *However, this computation can be done without the joint law of* (X_t, Y_t^0) *as*

$$\mathbb{P}\big(Y_t^0 > 0 | X_t > 0\big) = \frac{\mathbb{P}(Y_t^0 > 0, X_t > 0)}{\mathbb{P}(X_t > 0)} = \frac{\mathbb{P}(Y_t^0 > 0)}{\mathbb{P}(X_t > 0)} = \frac{1 - \mathbb{P}(Y_t^0 = 0)}{1 - \mathbb{P}(X_t = 0)},$$

with $\mathbb{P}(X_t = 0) = \lim_{\lambda_1 \to \infty} \mathbb{E}[\mathrm{e}^{-\lambda_1 X_t}] = \mathrm{e}^{-x g(\alpha,t)^{-1}}$ *and* $\mathbb{P}(Y_t^0 = 0) = \lim_{\lambda_2 \to \infty} \mathbb{E}[\mathrm{e}^{-\lambda_2 Y_t^0}] = \mathrm{e}^{-x g(b,t)^{-1}}$, *where*

$$g(a,t) = \begin{cases} \frac{\mathrm{e}^{at}-1}{a} & \text{if } a > 0, \\ t & \text{if } a = 0. \end{cases} \tag{18}$$

The same kind of computation allows also to compute the joint law at different times.

Proposition 5.4. *Let* $0 \le u < t$, $\lambda_1, \lambda_2 \in \mathbb{R}_+$. *We have*

$$\mathbb{E}\big[\mathrm{e}^{-\lambda_1 X_t - \lambda_2 Y_u^0}\big] = \mathrm{e}^{-x v_1(u,t)},$$

where

$$v_1(u,t) = \mathrm{e}^{-bt} h(t)^{-2} \left(\frac{\mathrm{e}^{-b(t-u)} h(t-u)^{-2}}{\lambda_2} + \int_{t-u}^{t} \mathrm{e}^{-br} h(r)^{-2} \,\mathrm{d}r \right)^{-1} + \lambda_1 \mathrm{e}^{-\alpha t} h(t)^{-1}.$$

Proof. Recall the notation of Section 3.2. In particular $x_0 = x$ and $x_n = 0$ for all $n \ge 1$. Let us apply Lemma 3.1 with

$$\mu_0 = \lambda_1 \delta_t + \lambda_2 \delta_u,$$
$$\mu_k = \lambda_1 \delta_t \quad \text{for } k \ge 1.$$

Let n go to infinity as in the proof of Lemma 5.1 to get that

$$\mathbb{E}\big[\mathrm{e}^{-\lambda_1 X_t - \lambda_2 Y_u^0}\big] = \mathrm{e}^{-x w(0)}, \tag{19}$$

where (w, w^*) is the unique non-negative solution on $(-\infty, t]$ of

$$\begin{aligned} w(s) + \int_s^t \psi^0\big(w(r)\big) \,\mathrm{d}r &= \lambda_1 + \lambda_2 \mathbf{1}_{\{s \le u\}} + \int_s^t \phi\big(w^*(r)\big) \,\mathrm{d}r, \\ w^*(s) + \int_s^t \psi\big(w^*(r)\big) \,\mathrm{d}r &= \lambda_1. \end{aligned} \tag{20}$$

Notice w^* is still given by (17). For $s > u$, we have $w(s) = w^*(s)$ and, for $s \le u$, Eq. (20) is equivalent to

$$w' - w(w+b) = -2\theta w^* \quad \text{on } (-\infty, u],\ w(t) = w^*(u) + \lambda_2.$$

From the proof of Proposition 5.2, we get

$$\frac{1}{w(s) - w^*(s)} = \mathrm{e}^{b(t-s)} h(t-s)^2 \left(\frac{\mathrm{e}^{-b(t-u)} h(t-u)^{-2}}{\lambda_2} + \int_s^u \mathrm{e}^{-b(t-r)} h(t-r)^{-2} \,\mathrm{d}r \right).$$

We conclude using (19). □

At this stage, we can give the joint distribution of the extinction time of X, $\tau_X = \inf\{t > 0; X_t = 0\}$, and of Y^0, $\tau_{Y^0} = \inf\{t > 0; Y_t^0 = 0\}$. For $u \le t$, we have $\mathbb{P}(\tau_X \le t, \tau_{Y^0} \le u) = \lim_{\lambda_1 \to \infty, \lambda_2 \to \infty} \exp -x v_1(u,t)$ that is

$$\mathbb{P}(\tau_X \le t, \tau_{Y^0} \le u) = \exp -x \left(\mathrm{e}^{-bt + 2\alpha t} \left(\int_{t-u}^{t} \mathrm{e}^{-br + 2\alpha r} g(\alpha, t)^2 g(\alpha, r)^{-2} \,\mathrm{d}r \right)^{-1} + g(\alpha, t)^{-1} \right).$$

We can compute the probability of simultaneous extinction of the Eve-population and the whole population, see also Proposition 5 in [16], where $\alpha = 0$. In [1], using different techniques we derive this formula for general critical or sub-critical branching mechanisms.

Lemma 5.5. *We have* $\mathbb{P}(\tau_{Y^0} = \tau_X | \tau_X = t) = \mathrm{e}^{-2\theta t}$.

Proof. We have $\mathbb{P}(\tau_{Y^0} = \tau_X | \tau_X = t) = 1 - \frac{\lim_{u \uparrow t} \partial_t \mathbb{P}(\tau_{Y^0} \le u, \tau_X \le t)}{\partial_t \mathbb{P}(\tau_X \le t)} = \mathrm{e}^{-2\theta t}$. □

We can deduce from the latter proposition the law of Y_u^0 conditionally on the non-extinction of the whole population. We set

$$A(b,u) = \frac{1}{\lambda_2}\mathrm{e}^{bu} + g(b,u).$$

Proposition 5.6. *Let* $u \ge 0$, $\lambda_2 \in \mathbb{R}_+$. *We have*

$$\lim_{t \to +\infty} \mathbb{E}\big[\mathrm{e}^{-\lambda_2 Y_u^0} | X_t > 0\big] = \mathrm{e}^{-x A(b,u)^{-1}} \big(1 - A(b,u)^{-2} G(\alpha, u)\big),$$

where

$$G(a,u) = \frac{2}{\lambda_2}\mathrm{e}^{bu} g(a,u) + \begin{cases} 2\frac{g(b+a,u)-g(b,u)}{a} & \text{if } a > 0, \\ 2\partial_1 g(b,u) & \text{if } a = 0. \end{cases}$$

Proof. We have

$$\mathbb{E}\big[\mathrm{e}^{-\lambda_2 Y_u^0} | X_t > 0\big] = \frac{\mathbb{E}[\mathrm{e}^{-\lambda_2 Y_u^0}] - \mathbb{E}[\mathrm{e}^{-\lambda_2 Y_u^0} \mathbf{1}_{\{X_t = 0\}}]}{\mathbb{P}(X_t > 0)}.$$

Using Proposition 5.4

$$\mathbb{E}\big[\mathrm{e}^{-\lambda_2 Y_u^0} \mathbf{1}_{\{X_t = 0\}}\big] = \lim_{\lambda_1 \to +\infty} \mathbb{E}\big[\mathrm{e}^{-\lambda_2 Y_u^0 - \lambda_1 X_t}\big] = \mathrm{e}^{-x \bar{v}_1(u,t)},$$

with $\bar{v}_1(u,t) = \lim_{\lambda_1 \to +\infty} v_1(u,t)$.

Definition (18) implies

$$\bar{v}_1(u,t) = \left(\frac{\mathrm{e}^{bu} g(\alpha,t)^2}{\lambda_2 g(\alpha, t-u)^2} + \mathrm{e}^{bt} \int_{t-u}^{t} \mathrm{e}^{-br} \frac{g(\alpha,t)^2}{g(\alpha,r)^2} \,\mathrm{d}r \right)^{-1} + \mathrm{e}^{-\alpha t} g(\alpha,t)^{-1}.$$

Performing an asymptotic expansion of $\bar{v}_1$ as t goes to ∞ leads to the result. □

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