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A lattice gas model for the incompressible Navier–Stokes equation

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Abstract. We recover the Navier–Stokes equation as the incompressible limit of a stochastic lattice gas in which particles are allowed to jump over a mesoscopic scale. The result holds in any dimension assuming the existence of a smooth solution of the Navier–Stokes equation in a fixed time interval. The proof does not use nongradient methods or the multi-scale analysis due to the long range jumps.

Résumé. Nous retrouvons l'équation de Navier–Stokes comme limite incompressible d'un gas sur réseau où les particules peuvent sauter sur des distances mésoscopiques. Le résultat est valable en toute dimension supposant l'existence d'une solution lisse de l'équation de Navier–Stokes en un intervale de temps donné. La démonstration ne dépend pas des méthodes non-gradients ou l'analyse multi-échelle grâce aux sauts de longue portée.

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1. Introduction

A major open problem in nonequilibrium statistical mechanics is the derivation of the hydrodynamical equations from microscopic Hamiltonian dynamics. The main difficulty in this project lies in the poor knowledge of the ergodic properties of such systems. To overcome this obstacle, deterministic Hamiltonian dynamics have been successfully replaced by interacting particle systems (cf. [3] and references therein).

Following this approach, in the sequel of the development of the nongradient method by Quastel [5] and Varadhan [6], Esposito, Marra and Yau [1,2] derived the incompressible Navier–Stokes equation for stochastic lattice gases in dimension $d \geq 3$.

The main step of their proof relies on a sharp estimate of the spectral gap of the jump part of the generator of the process and on the characterization of the germs of the exact and closed forms in a Hilbert space of local functions. The characterization of the closed forms as the sum of exact forms and currents allows, through a multi-scale analysis, the decomposition of the current as a sum of a gradient part and a local function in the range of the generator.

In this article, we consider a stochastic lattice gas with long range jumps. The dynamics are built in a way that the density and the momentum are the only conserved quantities. Choosing appropriately the size and the rates of the jumps, we are able to show that a small perturbation of a constant density and momentum profile evolves in a diffusive time scale as the solution of the incompressible Navier–Stokes equation.

In contrast with [1,2], the mesoscopic range of the jumps permits to consider perturbations around the constant profile of order N^{-b}, for b small, where N is a scaling parameter proportional to the inverse of the distance between particles. This choice has two important consequences. On the one hand, in order to close the equation, one does not need to replace currents by averages of conserved quantities over macroscopic boxes, but only over mesoscopic cubes, whose size depends on the parameter b. In particular, there is no need to recur to the multi-scale analysis or to the closed and exact forms, simplifying considerably the proof. On the other hand, choosing b small enough ($b < 1/2$), one can avoid in dimensions 1 and 2 the Gaussian fluctuations around the hydrodynamic limit and prove a law of large numbers for the conserved quantities in this regime. We are thus able to derive the incompressible Navier–Stokes equation even in low dimension, where the usual approach is intrinsically impossible since it involves scales in which fluctuations appear.

The main drawback of the approach presented is that it requires a bound on the spectral gap of the full dynamics restricted to finite cubes. The bound needs only to be a polynomial in the volume of the cube, but the generator includes the collision part. This problem, already mentioned in [2], is rather difficult in general. We prove such a bound in Section 6 for a specific choice of velocities.

The model can be informally described as follows. Let $\mathcal{V}$ be a finite set of velocities in $\mathbb{R}^d$, invariant under reflections and exchange of coordinates. For each v in $\mathcal{V}$, consider a long-range asymmetric exclusion process on $\mathbb{Z}^d$ whose mean drift is $vN^{-(1-b)}$. Superposed to these dynamics, there is a collision process which exchange velocities of particles in the same site in a way that momentum is conserved.

Under diffusive time scaling, assuming local equilibrium, it is not difficult to show that the evolution of the conserved quantities is described by the parabolic equations

$$\begin{cases} \partial_t \rho + N^b \sum_{v\in\mathcal{V}} v \cdot \nabla F_0(\rho, \mathbf{p}) = \Delta\rho, \\ \partial_t p_j + N^b \sum_{v\in\mathcal{V}} v_j v \cdot \nabla F_j(\rho, \mathbf{p}) = \Delta p_j, \end{cases}$$

where ρ stands for the density and $\mathbf{p} = (p_1, \ldots, p_d)$ for the momentum. $F_0, \ldots, F_d$ are thermodynamical quantities determined by the ergodic properties of the dynamics.

Consider an initial profile given by $(\rho, \mathbf{p}) = (\alpha, \boldsymbol{\beta}) + N^{-b}(\varphi_0, \boldsymbol{\varphi})$, where $(\alpha, \boldsymbol{\beta})$ are appropriate constants. Expanding the solution of the previous equations around $(\alpha, \boldsymbol{\beta})$ and assuming that the first component φ_0 does not depend on space, we obtain that the momentum should evolve according to the incompressible Navier–Stokes equation

$$\begin{cases} \operatorname{div} \boldsymbol{\varphi} = 0, \\ \partial_t \varphi_\ell = A_0 \partial_\ell \varphi_\ell^2 + A_1 \varphi \cdot \nabla\varphi_\ell + A_2 \partial_\ell |\varphi|^2 + \Delta\varphi_\ell, \end{cases}$$

for $1 \le \ell \le d$, where A_0, A_1, A_2 are model-dependent constants. This is the content of the main theorem of the article. We prove that under an appropriate time scale, the normalized empirical measures associated to the momentum converge to the solution of the above incompressible Navier–Stokes equation.

The proof relies on the relative entropy method introduced by Yau [7]. We show that the entropy of the state of the process with respect to a slowly varying parameter Gibbs state is small in a finite time interval provided the solution of the incompressible Navier–Stokes equation is smooth in this interval.

To obtain such a bound on the entropy, we compute its time derivative which can be expressed in terms of currents. A one block estimate, which requires a polynomial bound on the spectral gap of the generator of the process, permits to express the currents in terms of the empirical density and momenta. The linear part of the functions of the density and momenta cancel; while the second order terms can be estimated by the entropy. We obtain in this way a Gronwall inequality for the relative entropy, which in turn, gives the required bound.

The article is organized as follows. In Section 2, we establish the notation and state the main results of the article. In Sections 3 and 4, we examine the incompressible limit of an asymmetric long range exclusion process. We state in this simpler context some ergodic theorems needed in the proof of the incompressible limit of the stochastic lattice gas. In Section 5, we prove the main result of the article, while in Section 6, we prove a spectral gap, polynomial in the volume, for the generator of a stochastic lattice gas restricted to a finite cube and in Section 7, we state an equivalence of ensembles for the canonical measures of lattice gas models.

2. Notation and results

Denote by $\mathbb{T}_N^d = \{0, \dots, N-1\}^d$ the d-dimensional torus with N^d points and let $\mathcal{V} \subset \mathbb{R}^d$ be a finite set of velocities $v = (v_1, \dots, v_d)$. Assume that $\mathcal{V}$ is invariant under reflexions and permutations of the coordinates:

$$(v_1, \dots, v_{i-1}, -v_i, v_{i+1}, \dots, v_d) \quad \text{and} \quad (v_{\sigma(1)}, \dots, v_{\sigma(d)})$$

belong to $\mathcal{V}$ for all $1 \le i \le d$ and all permutations σ of $\{1, \dots, d\}$ provided $(v_1, \dots, v_d)$ belongs to $\mathcal{V}$.

On each site of the discrete d-dimensional torus $\mathbb{T}_N^d$ at most one particle for each velocity is allowed. A configuration is denoted by $\eta = \{\eta_x,\ x \in \mathbb{T}_N^d\}$ where $\eta_x = \{\eta(x, v),\ v \in \mathcal{V}\}$ and $\eta(x, v) \in \{0, 1\},\ x \in \mathbb{T}_N^d,\ v \in \mathcal{V}$, is the number of particles with velocity v at x. The set of particle configurations is $X_N = (\{0, 1\}^{\mathcal{V}})^{\mathbb{T}_N^d}$.

The dynamics consist of two parts: long range asymmetric random walks with exclusion among particles of the same velocity and binary collisions between particles of different velocities. The first part of the dynamics corresponds to the evolution of a mesoscopic asymmetric simple exclusion process. The jump law and the waiting times are chosen so that the rate of jumping from site x to site $x + z$ for a particle with velocity v is $p_N(z, v)$, where

$$p_N(z, v) = \frac{A_M}{M^{d+2}} \left\{ 2 + \frac{1}{N^a} q_M(z, v) \right\} \mathbf{1}\{z \in \Lambda_M\}.$$

In this formula, $a > 0$ is a fixed parameter, M is a function of N to be chosen later, Λ_M is the cube $\{-M, \dots, M\}^d$, A_M is given by

$$\frac{A_M}{M^{d+2}} \sum_{z \in \Lambda_M} z_i z_j = \delta_{i,j} \tag{2.1}$$

for $1 \le i, j \le d$, and $q_M(z, v)$ is any bounded nonnegative rate such that

$$\frac{A_M}{M^{d+1}} \sum_{z \in \Lambda_M} q_M(z, v) z_i = v_i$$

for $1 \le i \le d$ and $M \ge 1$. A possible choice is $q_M(z, v) = M^{-1}(z \cdot v)$, where $u \cdot v$ stands for the inner product in $\mathbb{R}^d$. Note that particles with velocity v have mean displacement $M^{-1} N^{-a} v$.

The generator $\mathcal{L}_N^{ex}$ of the random walk part of the dynamics acts on local functions f of the configuration space X_N as

$$\left(\mathcal{L}_N^{ex} f\right)(\eta) = \sum_{v \in \mathcal{V}} \sum_{\substack{x \in \mathbb{T}_N^d \\ z \in \Lambda_M}} \eta(x, v)\left[1 - \eta(x + z, v)\right] p_N(z, v) \left[f\left(\eta^{x, x+z, v}\right) - f(\eta)\right],$$

where

$$\eta^{x,y,v}(z, w) = \begin{cases} \eta(y, v) & \text{if } w = v \text{ and } z = x, \\ \eta(x, v) & \text{if } w = v \text{ and } z = y, \\ \eta(z, w) & \text{otherwise.} \end{cases}$$

The collision part of the dynamics is described as follows. Denote by $\mathcal{Q}$ the set of all collisions which preserve momentum:

$$\mathcal{Q} = \left\{\left(v, w, v', w'\right) \in \mathcal{V}^4 :\ v + w = v' + w'\right\}.$$

Particles of velocities v and w at the same site collide at rate one and produce two particles of velocities v' and w' at that site. The generator $\mathcal{L}_N^c$ is, therefore,

$$\mathcal{L}_N^c f(\eta) = \sum_{y \in \mathbb{T}_N^d} \sum_{q \in \mathcal{Q}} p(y, q, \eta)\left[f\left(\eta^{y,q}\right) - f(\eta)\right],$$

where the rate $p(y,q,\eta)$, $q=(v,w,v',w')$, is given by

$$p(y,q,\eta)=\eta(y,v)\eta(y,w)\big[1-\eta\big(y,v'\big)\big]\big[1-\eta\big(y,w'\big)\big]$$

and where the configuration $\eta^{y,q}$, $q=(v_0,v_1,v_2,v_3)$, after the collision is defined as

$$\eta^{y,q}(z,u)=\begin{cases}\eta(y,v_{j+2}) & \text{if } z=y \text{ and } u=v_j \quad \text{for some } 0\le j\le 3,\\ \eta(z,u) & \text{otherwise,}\end{cases}$$

where the index of v_{j+2} should be understood modulo 4.

The generator $\mathcal{L}_N$ of the stochastic lattice gas we examine in this article is the superposition of the exclusion dynamics with the collisions just introduced:

$$\mathcal{L}_N=N^2\big\{\mathcal{L}_N^{ex}+\mathcal{L}_N^{c}\big\}.$$

Note that time has been *sped up* diffusively. Let $\{\eta(t)\colon t\ge 0\}$ be the Markov process with generator $\mathcal{L}_N$ and denote by $\{S_t^N\colon t\ge 0\}$ the semi-group associated to $\mathcal{L}_N$.

For a probability measure μ on X_N, denote by $\mathbb{P}_\mu$ the measure on the path space $D(\mathbb{R}_+,X_N)$ induced by $\{\eta(t)\colon t\ge 0\}$ and the initial measure μ. Expectation with respect to $\mathbb{P}_\mu$ is denoted by $\mathbb{E}_\mu$.

2.1. *The invariant states*

For each configuration $\xi\in\{0,1\}^{\mathcal{V}}$, denote by $I_0(\xi)$ the mass of ξ and by $I_k(\xi)$, $k=1,\dots,d$, the momentum of ξ:

$$I_0(\xi)=\sum_{v\in\mathcal{V}}\xi(v),\qquad I_k(\xi)=\sum_{v\in\mathcal{V}}v_k\xi(v).$$

Set $\mathbf{I}(\xi):=(I_0(\xi),\dots,I_d(\xi))$. Assume that the set of velocities $\mathcal{V}$ is chosen in such a way that the unique quantities conserved by the dynamics $\mathcal{L}_N$ are mass and momentum: $\sum_{x\in\mathbb{T}_N^d}\mathbf{I}(\eta_x)$.

Two examples of sets of velocities with this property were proposed by Esposito, Marra and Yau [2]. In Model I, $\mathcal{V}=\{\pm e_1,\dots,\pm e_d\}$, where $\{e_j,\ j=1,\dots,d\}$ stands for the canonical basis of $\mathbb{R}^d$. In Model II, $d=3$, w is a root of w^4-6w^2-1 and $\mathcal{V}$ contains $(1,1,w)$, all reflections of this vector and all permutations of the coordinates, performing a total of 24 vectors since $w\ne\pm 1$.

For each chemical potential $\boldsymbol{\lambda}=(\lambda_0,\lambda_1,\dots,\lambda_d)$ in $\mathbb{R}^{d+1}$, denote by $m_{\boldsymbol{\lambda}}$ the measure on $\{0,1\}^{\mathcal{V}}$ given by

$$m_{\boldsymbol{\lambda}}(\xi)=\frac{1}{Z(\boldsymbol{\lambda})}\exp\big\{\boldsymbol{\lambda}\cdot\mathbf{I}(\xi)\big\},$$

where $Z(\boldsymbol{\lambda})$ is a normalizing constant. Notice that $m_{\boldsymbol{\lambda}}$ is a product measure on $\{0,1\}^{\mathcal{V}}$, i.e., that the variables $\{\xi(v)\colon v\in\mathcal{V}\}$ are independent under $m_{\boldsymbol{\lambda}}$.

Denote by $\mu_{\boldsymbol{\lambda}}^N$ the product measure on $(\{0,1\}^{\mathcal{V}})^{\mathbb{T}_N^d}$ with marginals given by

$$\mu_{\boldsymbol{\lambda}}^N\big\{\eta\colon\ \eta(x,\cdot)=\xi\big\}=m_{\boldsymbol{\lambda}}(\xi)$$

for each ξ in $\{0,1\}^{\mathcal{V}}$ and x in $\mathbb{T}_N^d$. Notice that $\{\eta(x,v)\colon x\in\mathbb{T}_N^d, v\in\mathcal{V}\}$ are independent variables under $\mu_{\boldsymbol{\lambda}}^N$.

For each $\boldsymbol{\lambda}$ in $\mathbb{R}^{d+1}$ a simple computation shows that $\mu_{\boldsymbol{\lambda}}^N$ is an invariant state for the Markov process with generator $\mathcal{L}_N$, that the generator $\mathcal{L}_N^c$ is symmetric with respect to $\mu_{\boldsymbol{\lambda}}^N$ and that $\mathcal{L}_N^{ex}$ has an adjoint $\mathcal{L}_N^{ex,*}$ in which $p_N(z,v)$ is replaced by $p_N^*(z,v)=p_N(-z,v)$. In particular, if we denote by $\mathcal{L}_N^{ex,s}$, $\mathcal{L}_N^{ex,a}$ the symmetric and the antisymmetric part of $\mathcal{L}_N^{ex}$, we have that

$$\big(\mathcal{L}_N^{ex,s}f\big)(\eta)=\frac{2A_M}{M^{d+2}}\sum_{v\in\mathcal{V}}\sum_{\substack{x\in\mathbb{T}_N^d\\ z\in\Lambda_M}}(T_{x,x+z,v}f)(\eta),$$

$$\left(\mathcal{L}_N^{ex,a} f\right)(\eta) = \frac{A_M}{M^{d+2} N^a} \sum_{v \in \mathcal{V}} \sum_{\substack{x \in \mathbb{T}_N^d \\ z \in \Lambda_M}} q_M(z,v) (T_{x,x+z,v} f)(\eta),$$

where

$$(T_{x,x+z,v} f)(\eta) = \eta(x,v)\big[1 - \eta(x+z,v)\big]\big[f\big(\eta^{x,x+z,v}\big) - f(\eta)\big].$$

The expectation under the invariant state $\mu_{\boldsymbol{\lambda}}^N$ of the mass and momentum are given by

$$\rho(\boldsymbol{\lambda}) := E_{m_{\boldsymbol{\lambda}}}\big[I_0(\xi)\big] = \sum_{v \in \mathcal{V}} \theta_v(\boldsymbol{\lambda}),$$

$$p_k(\boldsymbol{\lambda}) := E_{m_{\boldsymbol{\lambda}}}\big[I_k(\xi)\big] = \sum_{v \in \mathcal{V}} v_k \theta_v(\boldsymbol{\lambda}).$$

In this formula, $\theta_v(\boldsymbol{\lambda})$ denotes the expected value of the density of particles with velocity v under $m_{\boldsymbol{\lambda}}$:

$$\theta_v(\boldsymbol{\lambda}) := E_{m_{\boldsymbol{\lambda}}}\big[\xi(v)\big] = \frac{\exp\{\lambda_0 + \sum_{k=1}^d \lambda_k v_k\}}{1 + \exp\{\lambda_0 + \sum_{k=1}^d \lambda_k v_k\}}. \tag{2.2}$$

Denote by $(\rho, \mathbf{p})(\boldsymbol{\lambda}) := (\rho(\boldsymbol{\lambda}), p_1(\boldsymbol{\lambda}), \dots, p_d(\boldsymbol{\lambda}))$ the map which associates the chemical potential to the vector of density and momentum. Note that $(\rho, \mathbf{p})$ is the gradient of the strictly convex function $\log Z(\boldsymbol{\lambda})$. In particular, $(\rho, \mathbf{p})$ is one to one. In fact, it is possible to prove that $(\rho, \mathbf{p})$ is a diffeomorphism onto $\mathfrak{A} \subset \mathbb{R}^{d+1}$, the interior of the convex envelope of $\{\mathbf{I}(\xi), \xi \in \{0,1\}^{\mathcal{V}}\}$. Denote by $\boldsymbol{\Lambda} = (\Lambda_0, \dots, \Lambda_d) : \mathfrak{A} \to \mathbb{R}^{d+1}$ the inverse of $(\rho, \mathbf{p})$. This correspondence permits to parameterize the invariant states by the density and the momentum: for each $(\rho, \mathbf{p})$ in $\mathfrak{A}$, we have a product measure $\nu_{\rho,\mathbf{p}}^N = \mu_{\boldsymbol{\Lambda}(\rho,\mathbf{p})}^N$ on $(\{0,1\}^{\mathcal{V}})^{\mathbb{T}_N^d}$.

2.2. *Spectral gap*

For $L \geq 1$ and a configuration η, let $\mathbf{I}^L(x) = (I_0^L(x), \dots, I_d^L(x))$ be the average of the conserved quantities in a cube of length L centered at x:

$$\mathbf{I}^L(x) := \mathbf{I}^L(x,\eta) = \frac{1}{|\Lambda_L|} \sum_{z \in x + \Lambda_L} \mathbf{I}(\eta_z). \tag{2.3}$$

Let $\mathfrak{V}_L$ be the set of all possible values of $\mathbf{I}^L(0)$ when η runs over $(\{0,1\}^{\mathcal{V}})^{\Lambda_L}$. Obviously, $\mathfrak{V}_L$ is a finite subset of the convex envelope of $\{\mathbf{I}(\xi) \colon \xi \in \{0,1\}^{\mathcal{V}}\}$. The set of configurations $(\{0,1\}^{\mathcal{V}})^{\Lambda_L}$ splits in invariant subsets: For each $\mathbf{i}$ in $\mathfrak{V}_L$, let

$$\mathcal{H}_L(\mathbf{i}) := \big\{\eta \in \big(\{0,1\}^{\mathcal{V}}\big)^{\Lambda_L} \colon \mathbf{I}^L(0) = \mathbf{i}\big\}.$$

For each $\mathbf{i}$ in $\mathfrak{V}_L$, define the canonical measure $\nu_{\Lambda_L,\mathbf{i}}$ as the uniform probability measure on $\mathcal{H}_L(\mathbf{i})$.

Denote by $\mathcal{L}_{\Lambda_M}$ the generator $\mathcal{L}_N$ restricted to the cube Λ_M without acceleration. More precisely, on the state space $(\{0,1\}^{\mathcal{V}})^{\Lambda_M}$ consider the generator $\mathcal{L}_{\Lambda_M} = \mathcal{L}_{\Lambda_M}^{ex} + \mathcal{L}_{\Lambda_M}^{c}$, which acts on local functions $f : (\{0,1\}^{\mathcal{V}})^{\Lambda_M} \mapsto \mathbb{R}$ as

$$\left(\mathcal{L}_{\Lambda_M}^{ex} f\right)(\eta) = \sum_{v \in \mathcal{V}} \sum_{\substack{x,y \in \Lambda_M \\ |x-y| < M}} \eta(x,v)\big[1 - \eta(y,v)\big] p_N(y-x,v)\big[f\big(\eta^{x,y,v}\big) - f(\eta)\big],$$

$$\left(\mathcal{L}_{\Lambda_M}^{c} f\right)(\eta) = \sum_{y \in \Lambda_M} \sum_{q \in \mathcal{Q}} p(y,q,\eta)\big[f\big(\eta^{y,q}\big) - f(\eta)\big].$$

Since the only conserved quantities are the total mass and momentum, the process restricted to each component $\mathcal{H}_M(\mathbf{i})$ is ergodic. It has therefore a finite spectral gap: For each $\mathbf{i}$ in $\mathfrak{V}_M$, there exists a finite constant $C(M,\mathbf{i})$ such that

$$\langle f; f\rangle_{\nu_{\Lambda_M,\mathbf{i}}} \le C(M,\mathbf{i})\bigl\langle f, (-\mathcal{L}_{\Lambda_M} f)\bigr\rangle_{\nu_{\Lambda_M,\mathbf{i}}}$$

for all functions f in $L^2(\nu_{\Lambda_M,\mathbf{i}})$. Here and below $\langle f; f\rangle_\nu$ stands for the variance of f with respect to a measure ν and $\langle\cdot,\cdot\rangle_\nu$ for the scalar product in $L^2(\nu)$.

We shall assume that the inverse of the spectral gap increases polynomially in the length of the cube: There exists $C_0 > 0$ and $\kappa > 0$ such that

$$\max_{\mathbf{i}\in\mathfrak{V}_M} C(M,\mathbf{i}) \le C_0 M^\kappa. \tag{2.4}$$

We prove this hypothesis in Section 6 for Model I.

2.3. *Incompressible limit*

For $k = 0,\dots,d$, denote by $\pi^{k,N}$ the empirical measure associated to the kth conserved quantity:

$$\pi^{k,N} = N^{-d}\sum_{x\in\mathbb{T}_N^d} I_k(\eta_x)\delta_{x/N},$$

where δ_u stands for the Dirac measure concentrated on u.

Denote by $\langle\pi^{k,N}, H\rangle$ the integral of a test function H with respect to an empirical measure $\pi^{k,N}$. To compute $\mathcal{L}_N\langle\pi^{k,N}, H\rangle$, note that $\mathcal{L}_N^c I_k(\eta_x)$ vanishes for $k = 0,\dots,d$ because the collision operators preserve local mass and momentum. In particular, $\mathcal{L}_N\langle\pi^{k,N}, H\rangle = N^2\mathcal{L}_N^{ex}\langle\pi^{k,N}, H\rangle$. To compute $\mathcal{L}_N^{ex}\langle\pi^{k,N}, H\rangle$, consider separately the symmetric and the antisymmetric part of $\mathcal{L}_N^{ex}$. After two summations by parts and a Taylor expansion, we obtain that

$$N^2\mathcal{L}_N^{ex,s}\bigl\langle\pi^{0,N}, H\bigr\rangle = \bigl\langle\pi^{0,N}, \Delta H\bigr\rangle + \mathrm{O}\biggl(\frac{M}{N}\biggr),$$

$$N^2\mathcal{L}_N^{ex,a}\bigl\langle\pi^{0,N}, H\bigr\rangle = \frac{N^{1-a}}{M}\frac{1}{N^d}\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}(\partial_{u_j}H)\biggl(\frac{x}{N}\biggr)\tau_x W_j^M + \mathrm{O}\bigl(N^{-a}\bigr),$$

for every smooth function H. In this formula, Δ stands for the Laplacian. τ_x stands for the translation by x on the state space X_N so that $(\tau_x\eta)(y,v) = \eta(x+y,v)$ for all x, y in $\mathbb{Z}^d$, v in $\mathcal{V}$, and W_j^M, $j = 1,\dots,d$, is the current given by

$$W_j^M = \frac{A_M}{M^{d+1}}\sum_{v\in\mathcal{V}}\sum_{z\in\Lambda_M} q_M(z,v)z_j\eta(0,v)\bigl\{1-\eta(z,v)\bigr\}. \tag{2.5}$$

In the same way, for $1\le k\le d$, a long but simple computation shows that

$$N^2\mathcal{L}_N^{ex,s}\bigl\langle\pi^{k,N}, H\bigr\rangle = \bigl\langle\pi^{k,N}, \Delta H\bigr\rangle + \mathrm{O}\biggl(\frac{M}{N}\biggr),$$

$$N^2\mathcal{L}_N^{ex,a}\bigl\langle\pi^{k,N}, H\bigr\rangle = \frac{N^{1-a}}{M}\frac{1}{N^d}\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}(\partial_{u_j}H)\biggl(\frac{x}{N}\biggr)\tau_x W_{k,j}^M + \mathrm{O}\bigl(N^{-a}\bigr),$$

where $W_{k,j}^M$ is the current defined by

$$W_{k,j}^M = \frac{A_M}{M^{d+1}}\sum_{v\in\mathcal{V}} v_k\sum_{z\in\Lambda_M} q_M(z,v)z_j\eta(0,v)\bigl\{1-\eta(z,v)\bigr\}. \tag{2.6}$$

The explicit formulas for $\mathcal{L}_N\langle\pi^{k,N},H\rangle$ permit to predict the hydrodynamic behavior of the system under diffusive scaling assuming local equilibrium. By (2.1), the expectation of the currents W_j^M, $W_{k,j}^M$ under the invariant state $\mu_{\boldsymbol{\lambda}}^N$ are given by

$$E_{\mu_{\boldsymbol{\lambda}}^N}\big[W_j^M\big]=\sum_{v\in\mathcal{V}}\chi\big(\theta_v(\boldsymbol{\lambda})\big)v_j,\qquad E_{\mu_{\boldsymbol{\lambda}}^N}\big[W_{k,j}^M\big]=\sum_{v\in\mathcal{V}}\chi\big(\theta_v(\boldsymbol{\lambda})\big)v_jv_k.$$

In this formula and below, $\chi(a)=a(1-a)$. In view of the previous computation, if the conservation of local equilibrium holds, the limiting equation in the diffusive regime is expected to be

$$\begin{cases}\partial_t\rho+\frac{N^{1-a}}{M}\sum_{v\in\mathcal{V}}v\cdot\nabla\chi\big(\theta_v\big(\Lambda(\rho,\mathbf{p})\big)\big)=\Delta\rho,\\ \partial_tp_j+\frac{N^{1-a}}{M}\sum_{v\in\mathcal{V}}v_jv\cdot\nabla\chi\big(\theta_v\big(\Lambda(\rho,\mathbf{p})\big)\big)=\Delta p_j,\end{cases}\tag{2.7}$$

where ∇F stands for the gradient of F.

We turn now to the incompressible limit. Note that $\theta_v(\mathbf{0})=1/2$ for all v and that

$$\rho(\mathbf{0})=\frac{|\mathcal{V}|}{2}=:a_0,\qquad p_k(\mathbf{0})=\frac{1}{2}\sum_{v\in\mathcal{V}}v_k=0,$$

where the last identity follows from the symmetry assumptions made on $\mathcal{V}$. Therefore, $\boldsymbol{\Lambda}(a_0,\mathbf{0})=\mathbf{0}$ and by Taylor expansion,

$$\chi\big(\theta_v\big(\Lambda(a_0+\epsilon\varphi_0,\epsilon\boldsymbol{\varphi})\big)\big)=\chi\bigg(\frac{1}{2}\bigg)-\epsilon^2\Bigg\{\frac{1}{4}\sum_{\ell=0}^d\partial_\ell\Lambda_0(a_0,0)\varphi_\ell+\frac{1}{4}\sum_{k=1}^d\sum_{\ell=0}^d v_k\partial_\ell\Lambda_k(a_0,0)\varphi_\ell\Bigg\}^2+\mathrm{O}\big(\epsilon^3\big)$$

because $\chi'(1/2)=0$, $\partial_0\theta_v(\mathbf{0})=(1/4)$, $\partial_k\theta_v(\mathbf{0})=(1/4)v_k$, $1\le k\le d$. Here ∂_ℓ stands for the partial derivative with respect to the ℓth coordinate. It follows from the previous explicit formulas for $\partial_k\theta_v(\mathbf{0})$ that

$$\partial_\ell\Lambda_0(a_0,\mathbf{0})=4\delta_{0,\ell}|\mathcal{V}|^{-1},\qquad \partial_\ell\Lambda_k(a_0,\mathbf{0})=4\delta_{k,\ell}\bigg\{\sum_{v\in\mathcal{V}}v_\ell^2\bigg\}^{-1}$$

for $1\le k\le d$, $0\le\ell\le d$. In particular, $\chi(\theta_v(\Lambda(a_0+\epsilon\varphi_0,\epsilon\boldsymbol{\varphi})))$ is equal to

$$\chi\bigg(\frac{1}{2}\bigg)-\epsilon^2\Bigg\{\frac{\varphi_0}{|\mathcal{V}|}+\sum_{k=1}^d\frac{v_k\varphi_k}{\sum_{v\in\mathcal{V}}v_k^2}\Bigg\}^2+\mathrm{O}\big(\epsilon^3\big)$$

Due to the symmetry properties of $\mathcal{V}$,

$$\sum_{v\in\mathcal{V}}v_kv_j=B\delta_{k,j},\tag{2.8}$$

where $B=\sum_{v\in\mathcal{V}}v_1^2$. The denominator in the expression inside braces is thus equal to B.

To investigate the incompressible limit around $(a_0,\mathbf{0})$, fix $b>0$ and assume that a solution of (2.7) has the form $\rho(t,u)=a_0+N^{-b}\varphi_0(t,u)$, $p_k(t,u)=N^{-b}\varphi_k(t,u)$. Then to obtain a nontrivial limit, we need to set $M=N^{1-a-b}$ to obtain that $(\varphi_0,\boldsymbol{\varphi})(t,u)$ is the solution of

$$\begin{cases}\partial_t\varphi_0=\sum_{v\in\mathcal{V}}v\cdot\nabla\big\{\frac{\varphi_0}{|\mathcal{V}|}+\frac{1}{B}\sum_{k=1}^dv_k\varphi_k\big\}^2+\Delta\varphi_0,\\ \partial_t\varphi_\ell=\sum_{v\in\mathcal{V}}v_\ell v\cdot\nabla\big\{\frac{\varphi_0}{|\mathcal{V}|}+\frac{1}{B}\sum_{k=1}^dv_k\varphi_k\big\}^2+\Delta\varphi_\ell,\end{cases}\tag{2.9}$$

$1\le\ell\le d$.

To recover the Navier–Stokes equation, we need to introduce some notation related to the velocity space $\mathcal{V}$. Let $R_{k,\ell,m,n}(v) = v_k v_\ell v_m v_n$. By the symmetry properties of $\mathcal{V}$, if $k \neq \ell$, we have that

$$\sum_{v\in\mathcal{V}} R_{k,\ell,m,n}(v) = \{\delta_{m,k}\delta_{n,\ell} + \delta_{m,\ell}\delta_{n,k}\}C, \tag{2.10}$$

where $C = \sum_{v\in\mathcal{V}} v_k^2 v_\ell^2 = \sum_{v\in\mathcal{V}} v_1^2 v_2^2$. On the other hand,

$$\sum_{v\in\mathcal{V}} R_{k,k,m,n}(v) = \delta_{m,k}\delta_{n,k} D + \delta_{m,n}\{1-\delta_{m,k}\}C, \tag{2.11}$$

where $D = \sum_{v\in\mathcal{V}} v_k^4 = \sum_{v\in\mathcal{V}} v_1^4$.

Assume that $\varphi_0(0,u)$ is constant and that $\operatorname{div}\boldsymbol{\varphi}(0,\cdot) = 0$. Since $\partial_t\varphi_0$ and $\Delta\varphi_0$ vanish and since $\mathcal{V}$ is invariant by reflexion around the origin, replacing v by $-v$, the first equation in (2.9) can be rewritten as

$$\sum_{k=1}^{d}\sum_{v\in\mathcal{V}} v_k v\cdot\nabla\varphi_k = 0.$$

By (2.8), this equation becomes

$$\operatorname{div}\boldsymbol{\varphi} = 0.$$

The same argument permits to rewrite the second equations in (2.9) as

$$\partial_t\varphi_\ell = B^{-2}\sum_{v\in\mathcal{V}} v_\ell v\cdot\nabla\left\{\sum_{k=1}^{d} v_k\varphi_k\right\}^2 + \Delta\varphi_\ell.$$

The first term on the right-hand side of this expression is equal to

$$2B^{-2}\sum_{k,m,n=1}^{d}\varphi_k\partial_m\varphi_n\sum_{v\in\mathcal{V}} R_{k,\ell,m,n}(v).$$

It follows from (2.10), (2.11) and elementary algebra that this expression is equal to B^{-2} times

$$(D-3C)\partial_\ell\varphi_\ell^2 + 2C\varphi\cdot\nabla\varphi_\ell + C\partial_\ell|\varphi_\ell|^2$$

because $\operatorname{div}\boldsymbol{\varphi} = 0$. We recover in this way the Navier–Stokes equation

$$\begin{cases}\operatorname{div}\boldsymbol{\varphi} = 0,\\ \partial_t\varphi_\ell = A_0\partial_\ell\varphi_\ell^2 + A_1\varphi\cdot\nabla\varphi_\ell + A_2\partial_\ell|\varphi|^2 + \Delta\varphi_\ell,\end{cases} \tag{2.12}$$

where $A_0 = (D-3C)/B^2$, $A_1 = 2C/B^{-2}$ and $A_2 = C/B^2$. For Model I, we get $A_0 = 1$, $A_1 = A_2 = 0$, while for Model II, $B = 16+8w^2$, $C = 8+16w^2$, $D = 16+8w^4$ and A_0 vanishes because w is chosen as a root of w^4-6w^2-1.

2.4. *Statement of the result*

Recall that κ stands for the polynomial growth rate of the spectral gap. Assume that $b < a$,

$$a+b > 1-\frac{2}{d+\kappa}, \qquad a+\left(\frac{\kappa-2}{\kappa}\right)b > 1-\frac{2}{\kappa}, \qquad a+\left(1+\frac{2}{d}\right)b < 1. \tag{2.13}$$

The first two displayed conditions are needed in the proof of the one-block estimate, where the size of the cube cannot be too large. The last condition appears in the replacement of expectations with respect to canonical measures by expectations with respect to grand canonical measures, where the volume $|\Lambda_M|$ has to be large. It is easy to produce constants $a, b > 0$ meeting the above requirements. It is enough to choose first $0 < a < 1$, close enough to 1, and then to find b small enough.

Let $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_d) : \mathbb{T}^d \to \mathbb{R}^d$ be a smooth divergence free vector field. Denote by $\boldsymbol{\varphi}(t)$ the solution of (2.12) with initial condition $\boldsymbol{\varphi}$, assumed to be smooth in a time interval $[0, T]$. Denote by ν_t^N the product measure on X_N with chemical potential chosen so that

$$E_{\nu_t^N}\big[I_0(\eta_x)\big] = a_0 + \frac{\varphi_0}{N^b}, \qquad E_{\nu_t^N}\big[I_k(\eta_x)\big] = \frac{\varphi_k(t, x/N)}{N^b}$$

for $1 \le k \le d$ with φ_0 being a constant. This is possible for N large enough since $\boldsymbol{\varphi}$ is bounded and $\boldsymbol{\Lambda}(a_0, \mathbf{0}) = \mathbf{0}$.

For two probability measures μ, ν on X_N, denote by $H_N(\mu|\nu)$ the entropy of μ with respect to ν:

$$H_N(\mu|\nu) = \sup_f \left\{ \int f \, \mathrm{d}\mu - \log \int e^f \, \mathrm{d}\nu \right\},$$

where the supremum is carried over all bounded continuous functions on X_N. We are now in a position to state the main theorem of this article.

Theorem 2.1. *Assume conditions (2.4) and (2.13). Let $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_d) : \mathbb{T}^d \to \mathbb{R}^d$ be a smooth divergence free vector field. Denote by $\boldsymbol{\varphi}(t)$ the solution of (2.12) with initial condition $\boldsymbol{\varphi}$ and assume $\boldsymbol{\varphi}(t, u)$ to be smooth in $[0, T] \times \mathbb{T}^d$ for some $T > 0$. Let $\{\mu^N : N \ge 1\}$ be a sequence of measures on X_N such that $H_N(\mu^N | \nu_0^N) = \mathrm{o}(N^{d-2b})$. Then, $H_N(\mu^N S_t^N | \nu_t^N) = \mathrm{o}(N^{d-2b})$ for $0 \le t \le T$.*

Corollary 2.2. *Under the assumptions of Theorem 2.1, for every $0 \le t \le T$ and every continuous function $F : \mathbb{T}^d \to \mathbb{R}$,*

$$\lim_{N\to\infty} \frac{N^b}{N^d} \sum_{x \in \mathbb{T}_N^d} F\left(\frac{x}{N}\right) \big\{ I_0\big(\eta_x(t)\big) - a_0 \big\} = \varphi_0 \int_{\mathbb{T}^d} F(u) \, \mathrm{d}u,$$

$$\lim_{N\to\infty} \frac{N^b}{N^d} \sum_{x \in \mathbb{T}_N^d} F\left(\frac{x}{N}\right) I_k\big(\eta_x(t)\big) = \int_{\mathbb{T}^d} F(u) \varphi_k(t, u) \, \mathrm{d}u$$

in $L^1(\mathbb{P}_{\mu^N})$.

The corollary is an elementary consequence of the theorem and of the entropy inequality.

3. Mesoscopic asymmetric exclusion processes

We start with a model with no velocities. The proof is simpler in this context and the results stated will needed for the stochastic lattice gas. Denote by η the configurations of the state space $\mathcal{X}_N = \{0, 1\}^{\mathbb{T}_N^d}$ so that $\eta(x)$ is either 0 or 1 if site x is vacant or not. We consider a mesoscopic asymmetric exclusion process on $\mathcal{X}_N$. This is the Markov process whose generator is given by

$$(L_N f)(\eta) = \sum_{\substack{x \in \mathbb{T}_N^d \\ z \in \mathbb{Z}^d}} \eta(x) \big[1 - \eta(x+z)\big] p_N(z) \big[f\big(\sigma^{x,x+z}\eta\big) - f(\eta) \big],$$

where,

$$p_N(z) = \frac{1}{M^{d+2}} \left\{ 2A_M + \frac{1}{N^a} q(z) \right\} \mathbf{1}\{z \in \Lambda_M\}.$$

In this formula $a > 0$, M, A_M are chosen as in the previous section and $q(y) = \mathrm{sign}(y \cdot v)$ for a fixed vector $v \in \mathbb{R}^d$. On the other hand, $\sigma^{x,y}\eta$ is the configuration obtained from η by interchanging the occupation variables $\eta(x)$, $\eta(y)$:

$$\big(\sigma^{x,y}\eta\big)(z) = \begin{cases} \eta(z) & \text{if } z \neq x, y, \\ \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y. \end{cases}$$

For a probability measure μ on $\mathcal{X}_N$, $\mathbb{P}_\mu$ stands for the measure on the path space $D(\mathbb{R}_+, \mathcal{X}_N)$ induced by the Markov process with generator L_N *sped up* by N^2 and the initial measure μ. Expectation with respect to $\mathbb{P}_\mu$ is denoted by $\mathbb{E}_\mu$. Denote by $\{S_t^N\colon t \geq 0\}$ the semi-group associated to the generator $N^2 L_N$.

For $0 \leq \alpha \leq 1$, denote by μ_α^N the Bernoulli product measure on $\mathcal{X}_N$ with density α. An elementary computation shows that μ_α^N is an invariant state for the Markov process with generator L_N. Moreover, the symmetric and the antisymmetric part of the generator L_N, respectively, denoted by L_N^s, L_N^a, are given by:

$$\big(L_N^s f\big)(\eta) = \frac{2A_M}{M^{d+2}} \sum_{\substack{x \in \mathbb{T}_N^d \\ z \in \mathbb{Z}^d}} \eta(x)\big[1 - \eta(x+z)\big]\big[f\big(\sigma^{x,x+z}\eta\big) - f(\eta)\big],$$

$$\big(L_N^a f\big)(\eta) = \frac{1}{M^{d+2} N^a} \sum_{\substack{x \in \mathbb{T}_N^d \\ z \in \mathbb{Z}^d}} q(z)\eta(x)\big[1 - \eta(x+z)\big]\big[f\big(\sigma^{x,x+z}\eta\big) - f(\eta)\big].$$

We investigate in this and in the next section the incompressible limit of this model. Consider first the hydrodynamic behavior of the process under diffusive scaling. Denote by π^N the empirical measure associated to a configuration:

$$\pi^N = \pi^N(\eta) = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta(x)\delta_{x/N}.$$

Denote by $\langle \pi^N, H\rangle$ the integral of a test function H with respect to an empirical measure π^N. To compute $N^2 L_N \langle \pi^N, H\rangle$, we consider separately the symmetric and the antisymmetric part of the generator. After two summations by parts and a Taylor expansion, we obtain that

$$N^2 L_N^s \langle \pi^N, H\rangle = \langle \pi^N, \Delta H\rangle + \mathrm{O}\left(\frac{M}{N}\right).$$

On the other hand, after a summation by parts, $N^2 L_N^a \langle \pi^N, H\rangle$ becomes

$$\frac{N^{1-a}}{M} \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \sum_{j=1}^d (\partial_{u_j} H)\left(\frac{x}{N}\right) \tau_x W_j^M + \mathrm{O}\big(N^{-a}\big),$$

where τ_x stands for the translation by x on the state space $\mathcal{X}_N$ so that $(\tau_x \eta)(y) = \eta(x+y)$ for all x, y in $\mathbb{T}_N^d$, and $\mathbf{W}^M = (W_1^M, \ldots, W_d^M)$ is the current given by

$$W_j^M = \frac{1}{M^{d+1}} \sum_{z \in \Lambda_M} q(z) z_j \eta(0)\big\{1 - \eta(z)\big\}. \tag{3.1}$$

The expectation of the current under the invariant state μ_α^N is

$$\alpha(1-\alpha)\frac{1}{M^{d+1}} \sum_{z \in \Lambda_M} q(z) z_j. \tag{3.2}$$

Since $M = N^{1-a-b}$, the limiting equation in the diffusive regime is, therefore, expected to be

$$\partial_t \rho + N^b \gamma \cdot \nabla \rho(1-\rho) = \Delta\rho,$$

where $\gamma_j = \int_{[-1,1]^d} u_j q(u)\,\mathrm{d}u$.

To investigate the incompressible limit around density $1/2$, suppose that a solution of the previous equation has the form $\rho(t,u) = (1/2) + N^{-b}\varphi(t,u)$. An elementary computation shows that

$$\partial_t \varphi = \gamma \cdot \nabla\varphi^2 + \Delta\varphi.$$

Assume the following conditions on a and b, which could certainly be relaxed:

$$\begin{aligned} &\frac{d}{d+2} < a+b, \qquad a + \max\left\{2, 1+\frac{2}{d}\right\} b < 1, \\ &2\left(1+\frac{2}{d}\right) b < 1. \end{aligned} \tag{3.3}$$

The first assumption, which forbids a large mesoscopic range M, is used in the proof of the one-block estimate. The second and third assumptions, which require a not too small range M, are used throughout the proof to discard error terms.

By the same reasons of the previous section, there exist positive constants a, b satisfying these assumptions.

Fix a continuous function $\varphi_0 : \mathbb{T}^d \to \mathbb{R}$. Denote by $\varphi = \varphi(t,u)$ the solution of the nonlinear parabolic equation

$$\begin{cases} \partial_t \varphi = \gamma \cdot \nabla\varphi^2 + \Delta\varphi, \\ \varphi(0,\cdot) = \varphi_0(\cdot). \end{cases} \tag{3.4}$$

For $t \geq 0$, let ν_t^N be the product measure on $\mathcal{X}_N$ with marginals given by

$$E_{\nu_t^N}\big[\eta(x)\big] = \frac{1}{2} + \frac{1}{N^b}\varphi\left(t, \frac{x}{N}\right).$$

This is possible for N large enough because φ is bounded. Recall that we denote by $H_N(\nu|\mu)$ the relative entropy of a probability measure ν with respect to μ.

Theorem 3.1. *Assume conditions* (3.3). *Fix a smooth function $\varphi_0 : \mathbb{T}^d \to \mathbb{R}$ and denote by $\varphi = \varphi(t,u)$ the solution of* (3.4) *with initial condition φ_0. Assume φ to be smooth in the layer $[0,T] \times \mathbb{T}^d$. Let $\{\mu^N : N \geq 1\}$ be a sequence of measures on $\mathcal{X}_N$, such that $H_N(\mu^N|\nu_0^N) = \mathrm{o}(N^{d-2b})$. Then $H_N(\mu^N S_t^N|\nu_t^N) = \mathrm{o}(N^{d-2b})$ for all $0 \leq t \leq T$.*

Fix two bounded functions $\varphi_i : \mathbb{T}^d \to \mathbb{R}$, $i = 1, 2$, and denote by $\nu^{N,i}$ the product measures associated to the density profile $(1/2) + N^{-b}\varphi_i$. A second order Taylor expansion shows that

$$H_N\big(\nu^{N,2}\big|\nu^{N,1}\big) = \mathrm{O}\big(N^{d-2b}\big).$$

The assumption on the entropy formulated in the theorem permits, therefore, to distinguish between N^{-b}-perturbations of a constant density profile.

A law of large numbers for the corrected empirical measure follows from this result. For a configuration η, denote by $\Pi^N(\eta)$ the corrected empirical measure defined by

$$\Pi^N = \Pi^N(\eta) = \frac{N^b}{N^d} \sum_{x \in \mathbb{T}_N^d} \left\{\eta(x) - \frac{1}{2}\right\} \delta_{x/N}$$

considered as an element of $\mathcal{M}(\mathbb{T}^d)$, the space of Radon measures on $\mathbb{T}^d$ endowed with the weak topology. For $t \geq 0$, let $\Pi_t^N = \Pi^N(\eta_t)$.

Corollary 3.2. *Under the assumptions of Theorem 3.1, for every $0 \le t \le T$ and every continuous function $F : \mathbb{T}^d \to \mathbb{R}$,*

$$\lim_{N\to\infty} \langle \Pi_t^N, F\rangle = \int_{\mathbb{T}^d} \varphi(t,u) F(u)\, \mathrm{d}u$$

in $L^1(\mathbb{P}_{\mu^N})$.

The corollary is an elementary consequence of Theorem 3.1 and the entropy inequality.

4. Incompressible limit of mesoscopic exclusion processes

We prove in this section Theorem 3.1. Fix a smooth function $\varphi_0 : \mathbb{T}^d \to \mathbb{R}$ and denote by $\varphi = \varphi(t,u)$ the solution of (3.4) with initial condition φ_0, supposed to be smooth in the time interval $[0,T]$. Let $\{\mu^N :\ N \ge 1\}$ be a sequence of measures on $\mathcal{X}_N$ satisfying the assumptions of Theorem 3.1.

4.1. *Entropy, Dirichlet form and Ergodic constants*

An elementary computation shows that the entropy $H_N(\mu^N|\mu^N_{1/2})$ of μ^N with respect to $\mu^N_{1/2}$ is of order N^{d-2b}. Indeed, by the explicit formula for the entropy and by the entropy inequality,

$$H_N\big(\mu^N|\mu^N_{1/2}\big) \le \left(1+\frac{1}{A}\right) H_N\big(\mu^N|\nu^N_0\big) + \frac{1}{A}\log\int \left(\frac{\mathrm{d}\nu^N_0}{\mathrm{d}\mu^N_{1/2}}\right)^{1+A} \mathrm{d}\mu^N_{1/2}$$

for all $A > 0$. A Taylor expansion shows that the second term on the right-hand side is of order N^{d-2b}. In particular,

$$N^{2b-d} H_N\big(\mu^N|\mu^N_{1/2}\big) \le C_0 \tag{4.1}$$

for some finite constant C_0 depending only on φ_0.

Let f^N_t be the Radon–Nikodym derivative $\mathrm{d}\mu^N S^N_t/\mathrm{d}\mu^N_{1/2}$ so that

$$\partial_t f^N_t = N^2 L^*_N f^N_t,$$

where L^*_N stands for the adjoint of L_N in $L^2(\mu^N_{1/2})$. It follows from (4.1) and a well-known estimate on the entropy production (cf. [3], Section V.2) that

$$\frac{N^{2b}}{N^d} H_N\big(\mu^N S^N_t|\mu^N_{1/2}\big) + \frac{N^{2b}}{N^d}\int_0^t D_N\big(\mu^N_{1/2}, f^N_s\big)\,\mathrm{d}s \le C_0 \tag{4.2}$$

for all $N \ge 0$ and $t \ge 0$. In this formula, D_N stands for the Dirichlet form defined as

$$D_N\big(\mu^N_{1/2}, f\big) = N^2\big\langle -L_N\sqrt{f}, \sqrt{f}\big\rangle_{\mu^N_{1/2}},$$

where $\langle\cdot,\cdot\rangle_{\mu^N_{1/2}}$ is the scalar product in $L^2(\mu^N_{1/2})$. An elementary computation shows that

$$D_N\big(\mu^N_{1/2}, f\big) = \frac{A_M N^2}{M^{d+2}} \sum_{|x-y|\le M} \big\langle \big\{\nabla^{x,y}\sqrt{f}\big\}^2\big\rangle_{\mu^N_{1/2}},$$

where

$$\big(\nabla^{x,y} g\big)(\eta) = g\big(\sigma^{x,y}\eta\big) - g(\eta),$$

and that D_N is a convex, lower semicontinuous functional.

Let L_{Λ_M} be the symmetric part of the generator L_N restricted to the cube Λ_M:

$$(L_{\Lambda_M} f)(\eta) = \frac{2A_M}{M^{d+2}} \sum_{\substack{x,y\in\Lambda_M\\|x-y|\le M}} \eta(x)\big[1-\eta(y)\big]\big[f\big(\sigma^{x,y}\eta\big) - f(\eta)\big], \tag{4.3}$$

and denote by $\mu_{\Lambda_M,K}$, $0 \le K \le |\Lambda_M|$, the canonical measure on $\{0,1\}^{\Lambda_M}$ concentrated on the hyperplane with K particles. In the case of the exclusion process, $\mu_{\Lambda_M,K}$ is just the uniform measure over all configurations of $\{0,1\}^{\Lambda_M}$ with K particles. Denote by D_{Λ_M} the Dirichlet form associated to L_{Λ_M}:

$$D_{\Lambda_M}(\mu, f) = \frac{A_M}{M^{d+2}} \sum_{\substack{x,y\in\Lambda_M\\|x-y|\le M}} \big\langle \big\{\nabla^{x,y}\sqrt{f}\big\}^2\big\rangle_\mu,$$

where μ stands either for the marginal on Λ_M of the grand canonical measure $\mu^N_{1/2}$ or for a canonical measure $\mu_{\Lambda_M,K}$.

By comparing the Dirichlet form D_{Λ_M} with the Bernoulli–Laplace Dirichlet form, in which all jumps are allowed with rate $|\Lambda_M|^{-1}$ and which is known to have a spectral gap of order 1 (cf. [5]), we can prove that the spectral gap of D_{Λ_M} is of order M^{-2}.

4.2. *The relative entropy method*

The proof of Theorem 3.1 is based on the relative entropy method introduced by Yau [7]. Let $\psi_t^N = \mathrm{d}\nu_t^N/\mathrm{d}\mu^N_{1/2}$. It follows from the explicit formulas for the product measure ν_t^N that

$$\log\psi_t^N = \sum_{x\in\mathbb{T}_N^d} \log\frac{[(1/2)+N^{-b}\varphi(t,x/N)]}{[(1/2)-N^{-b}\varphi(t,x/N)]}\eta(x) + \sum_{x\in\mathbb{T}_N^d} \log\left\{1-2N^{-b}\varphi\left(t,\frac{x}{N}\right)\right\}.$$

Let $H_N(t) = N^{2b-d} H_N(\mu^N S_t^N|\nu_t^N)$ and recall that we denote the Radon–Nikodym derivative $\mathrm{d}\mu^N S_t^N/\mathrm{d}\mu^N_{1/2}$ by f_t^N. With the notation just introduced, we have that

$$H_N(t) = N^{2b-d}\int f_t^N \log\frac{f_t^N}{\psi_t^N}\,\mathrm{d}\mu^N_{1/2}.$$

Theorem 3.1 follows from Gronwall lemma and the following estimate.

Proposition 4.1. *Fix a sequence of measures $\{\mu^N\colon N\ge 1\}$ satisfying the assumptions of Theorem 3.1. There exists $\gamma>0$, such that*

$$H_N(t) \le \gamma\int_0^t H_N(s)\,\mathrm{d}s + o_N(1)$$

for all $t\le T$.

The proof of Proposition 4.1 is divided in several steps. We begin with a well known upper bound for the entropy production (see e.g. [3], Lemma 6.1.4).

$$\frac{\mathrm{d}}{\mathrm{d}t} H_N(t) \le N^{2b-d}\int f_t^N \frac{(N^2 L_N^* - \partial_t)\psi_t^N}{\psi_t^N}\,\mathrm{d}\mu^N_{1/2}. \tag{4.4}$$

A long and tedious computation gives that $(\psi_t^N)^{-1}(N^2 L_N^* - \partial_t)\psi_t^N$ is equal to

$$
\begin{aligned}
&\frac{4}{N^b}\sum_{x\in\mathbb{T}_N^d}(\Delta\varphi)\left(t,\frac{x}{N}\right)\left\{\eta(x)-\frac{1}{2}\right\}+\frac{16}{N^{2b}}\sum_{i,j=1}^{d}\sum_{x\in\mathbb{T}_N^d}(\partial_{u_i}\varphi)\left(t,\frac{x}{N}\right)(\partial_{u_j}\varphi)\left(t,\frac{x}{N}\right)\tau_x V_{i,j}^M(\eta)\\
&\quad+4(1+\varepsilon_N)\sum_{j=1}^{d}\sum_{x\in\mathbb{T}_N^d}(\partial_{u_j}\varphi)\left(t,\frac{x}{N}\right)\tau_x W_j^{*,M}(\eta)\\
&\quad-\frac{4}{N^b}\sum_{x\in\mathbb{T}_N^d}(\partial_t\varphi)\left(t,\frac{x}{N}\right)\left\{\eta(x)-\left(\frac{1}{2}\right)-N^{-b}\varphi\left(t,\frac{x}{N}\right)\right\}+\mathrm{o}\big(N^{d-2b}\big).
\end{aligned}
\tag{4.5}
$$

In this formula, $W_j^{*,M}$ and $V_{i,j}^M$ stand for

$$
\begin{aligned}
W_j^{*,M}(\eta)&=\frac{1}{M^{d+1}}\sum_{z\in\Lambda_M}q(-z)z_j\eta(0)\big\{1-\eta(z)\big\},\\
V_{i,j}^M(\eta)&=\frac{A_M}{M^{d+2}}\sum_{z\in\Lambda_M}z_i z_j\eta(0)\big\{1-\eta(z)\big\}.
\end{aligned}
$$

We used the inequalities $a>b$, $a+2b<1$, which follow from assumptions (3.3), to estimate several terms in the above computation by $\mathrm{o}(N^{d-2b})$. The expression $(1+\varepsilon_N)(\partial_{u_j}\varphi)$ in the third line stands for $(\partial_{u_j}\varphi)\{1-4N^{-2b}\varphi^2\}^{-1}$. Keep in mind that ε_N is of order N^{-2b}.

If we replace $\eta(x)-1/2$ in the first term of (4.5) by $\eta(x)-(1/2)-N^{-b}\varphi(t,\frac{x}{N})$ and $V_{i,j}^M$ by $V_{i,j}^M-(1/4)\delta_{i,j}$, as $N\uparrow\infty$, the expressions added multiplied by $(1/4)N^{2b-d}$ converge to

$$
\int_{\mathbb{T}^d}(\Delta\varphi)\varphi+\sum_{j=1}^{d}\int_{\mathbb{T}^d}(\partial_{u_j}\varphi)^2=0.
$$

Therefore, in view of (4.4) and (4.5), the time derivative of the renormalized entropy $H_N(t)$ is bounded above by

$$
\begin{aligned}
&\mathbb{E}_{\mu^N}\left[\frac{4(1+\varepsilon_N)N^{2b}}{N^d}\sum_{j=1}^{d}\sum_{x\in\mathbb{T}_N^d}(\partial_{u_j}\varphi)\left(t,\frac{x}{N}\right)\tau_x W_j^{*,M}(\eta_t)\right]\\
&\quad+\mathbb{E}_{\mu^N}\left[\frac{16}{N^d}\sum_{i,j=1}^{d}\sum_{x\in\mathbb{T}_N^d}(\partial_{u_i}\varphi)\left(t,\frac{x}{N}\right)(\partial_{u_j}\varphi)\left(t,\frac{x}{N}\right)\tau_x\hat{V}_{i,j}^M(\eta_t)\right]\\
&\quad+\mathbb{E}_{\mu^N}\left[\frac{4N^b}{N^d}\sum_{x\in\mathbb{T}^d}(\Delta\varphi-\partial_t\varphi)\left(t,\frac{x}{N}\right)\left\{\eta_t(x)-\left(\frac{1}{2}\right)-N^{-b}\varphi\left(t,\frac{x}{N}\right)\right\}\right]+\mathrm{o}_N(1),
\end{aligned}
\tag{4.6}
$$

where $\hat{V}_{i,j}^M(\eta)=V_{i,j}^M(\eta)-(1/4)\delta_{i,j}$.

We now use the ergodicity to replace the functions $W_j^{*,M}$ and $\hat{V}_{i,j}^M$ by their projections on the conserved quantity over mesoscopic cubes.

4.3. *One block estimate*

Recall from the end of Section 4.1 that $\mu_{\Lambda_M,K}$ stands for the uniform measure over all configurations of $\{0,1\}^{\Lambda_M}$ with K particles. For $1\le j\le d$, denote by $F_j(K/|\Lambda_M|)$ the expected value of the current $W_j^{*,M}$ with respect to $\mu_{\Lambda_M,K}$. An elementary computation shows that

$$F_j(\beta)=E_{\mu_{\Lambda_M,K}}\big[W_j^{*,M}\big]=-\gamma_j^M\left\{1+\frac{1}{|\Lambda_M|-1}\right\}\beta(1-\beta),$$

provided $\beta=K/|\Lambda_M|$ and $\gamma_j^M=M^{-(d+1)}\sum_{z\in\Lambda_M}z_jq(z)=\gamma_j+\mathrm{O}(M^{-1})$.

For a positive integer $\ell\ge 1$, let $\eta^\ell(x)$ be the average number of particles in a cube of size ℓ around x:

$$\eta^\ell(x)=\frac{1}{|\Lambda_\ell|}\sum_{y\in x+\Lambda_\ell}\eta(y).$$

For $M\ge 1$, $1\le j\le d$, let

$$V_{j,M}=W_j^{*,M}-F_j\big(\eta^M(0)\big).$$

Lemma 4.2. *For every $t>0$, $1\le j\le d$ and continuous function $G:\mathbb{T}^d\to\mathbb{R}$,*

$$\lim_{N\to\infty}\mathbb{E}_{\mu^N}\left[\left|\int_0^t \mathrm{d}s\,\frac{N^{2b}}{N^d}\sum_{x\in\mathbb{T}_N^d}G\left(\frac{x}{N}\right)\tau_xV_{j,M}(\eta_s)\right|\right]=0.$$

Proof. By the entropy inequality and Jensen inequality, the expectation appearing in the statement of the lemma is bounded above by

$$\frac{N^{2b}}{AN^d}H_N\big(\mu^N\big|\mu_{1/2}^N\big)+\frac{N^{2b}}{AN^d}\log\mathbb{E}_{\mu_{1/2}^N}\left[\exp A\left|\int_0^t\mathrm{d}s\sum_{x\in\mathbb{T}_N^d}G\left(\frac{x}{N}\right)\tau_xV_{j,M}(\eta_s)\right|\right]$$

for every $A>0$. In view of (4.1), to prove the lemma, it is enough to show that the second term vanishes, as $N\uparrow\infty$, for any $A>0$. Since $\mathrm{e}^{|x|}\le \mathrm{e}^x+\mathrm{e}^{-x}$, it is enough to estimate the previous expectation without the absolute value.

By Feynman–Kac formula and by the variational formula for the largest eigenvalue of an operator, the second term without the absolute value is bounded above by

$$\frac{tN^{2b}}{AN^d}\sup_f\left\{\sum_{x\in\mathbb{T}_N^d}AG\left(\frac{x}{N}\right)\int\tau_xV_{j,M}f\,\mathrm{d}\mu_{1/2}^N-D_N\big(\mu_{1/2}^N,f\big)\right\},\tag{4.7}$$

where the supremum is carried over all density functions f with respect to $\mu_{1/2}^N$.

Since the measure $\mu_{1/2}^N$ is translation invariant and since $V_{j,M}$ depends on the configuration only through $\{\eta(z):\ z\in\Lambda_M\}$,

$$\int(\tau_xV_{j,M})f\,\mathrm{d}\mu_{1/2}^N=\int V_{j,M}(\tau_{-x}f)\,\mathrm{d}\mu_{1/2}^N=\int V_{j,M}f_{x,M}\,\mathrm{d}\mu_{1/2}^N,$$

where $f_{x,M}=E_{\mu_{1/2}^N}[\tau_{-x}f|\eta(z),z\in\Lambda_M]$.

On the other hand, by convexity of the Dirichlet form and by translation invariance of $\mu_{1/2}^N$, for any x, y in Λ_M such that $|x-y|\le M$,

$$\big\langle\big\{\nabla^{x,y}\sqrt{f_{z,M}}\big\}^2\big\rangle_{\mu_{1/2}^N}\le\big\langle\big\{\nabla^{x,y}\sqrt{\tau_{-z}f}\big\}^2\big\rangle_{\mu_{1/2}^N}=\big\langle\big\{\nabla^{x+z,y+z}\sqrt{f}\big\}^2\big\rangle_{\mu_{1/2}^N}.$$

Therefore, summing over x, y in Λ_M, $|x-y| \le M$, and in z in $\mathbb{T}_N^d$, we obtain that

$$\sum_{z\in\mathbb{T}_N^d} \sum_{\substack{x,y\in\Lambda_M\\|x-y|\le M}} \big\langle \big\{\nabla^{x,y}\sqrt{f_{z,M}}\big\}^2\big\rangle_{\mu_{1/2}^N} \le C_0 M^d \sum_{|x-y|\le M} \big\langle \big\{\nabla^{x,y}\sqrt{f}\big\}^2\big\rangle_{\mu_{1/2}^N}$$

for some universal constant C_0.

Recall the definition of the Dirichlet forms D_N and D_{M,Λ_M} introduced above. It follows from the previous estimates that the expression (4.7) is bounded above by

$$\frac{tN^{2b}}{AN^d} \sum_{x\in\mathbb{T}_N^d} \sup_f \left\{ AG\left(\frac{x}{N}\right) \int V_{j,M} f \, \mathrm{d}\mu_{1/2}^N - \frac{C_0 N^2}{M^d} D_{M,\Lambda_M}\big(\mu_{1/2}^N, f\big) \right\}, \tag{4.8}$$

where the supremum is carried over all densities f with respect to the marginal of $\mu_{1/2}^N$ on the cube Λ_M.

In particular, by projecting the density over each hyperplane with a fixed total number of particles and recalling the perturbation theorem on the largest eigenvalue of a symmetric operator (Theorem 1.1 of Appendix 3 in [3]), in view of (4.8), we obtain that (4.7) is less than or equal to

$$C(\gamma) A t \|G\|_\infty^2 \frac{N^{2b} M^d}{N^2} \big\langle (-L_{M,\Lambda_M})^{-1} V_{j,M}, V_{j,M} \big\rangle_{\mu_{1/2}^N}$$

for some finite constant $C(\gamma)$ depending only on γ. Here we need the assumption that $M^{d+2} \ll N^2$ to be allowed to apply the Rayleigh expansion.

Since the generator L_{M,Λ_M} has a spectral gap of order M^{-2}, $\langle (-L_{M,\Lambda_M})^{-1} V_{j,M}, V_{j,M}\rangle_{\mu_{1/2}^N}$ is bounded by $C_0 M^2 \langle V_{j,M}; V_{j,M}\rangle_{\mu_{1/2}^N}$, which is less than or equal to $C_0\ M^{2-d}$. Thus, (4.7) is bounded by $C_0 A t \|G\|_\infty^2 N^{-2a}$ because $M = N^{1-a-b}$. This concludes the proof of the lemma. □

For $1 \le i, j \le d$, let

$$F_{i,j}(\beta) = E_{\mu_{\Lambda_M,K}}\big[\hat{V}_{i,j}^M\big] = \beta(1-\beta)\left\{1 + \frac{1}{|\Lambda_M| - 1}\right\}\delta_{i,j} - \left(\frac{1}{4}\right)\delta_{i,j},$$

with the same convention that $\beta = K/|\Lambda_M|$. Let $w_{i,j}^M(\eta) = \hat{V}_{i,j}^M - F_{i,j}(\eta^M(0))$. The arguments of the proof of Lemma 4.2 shows that for every $t > 0$, $1 \le i, j \le d$ and continuous function $G : \mathbb{T}^d \to \mathbb{R}$,

$$\lim_{N\to\infty} \mathbb{E}_{\mu^N}\left[\left|\int_0^t \mathrm{d}s \, \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} G\left(\frac{x}{N}\right) \tau_x w_{i,j}^M(\eta_s)\right|\right] = 0. \tag{4.9}$$

The arguments are even simpler due to the absence of the factor N^{2b} multiplying the sum.

By Lemma 4.2 and (4.9), integrating in time (4.6), we obtain that the entropy $H_N(t)$ is less than or equal to

$$\begin{aligned}
&\frac{4(1+\varepsilon_N)N^{2b}}{N^d} \int_0^t \mathrm{d}s \, \mathbb{E}_{\mu^N}\left[\sum_{j=1}^d \sum_{x\in\mathbb{T}_N^d} (\partial_{u_j}\varphi)\left(s, \frac{x}{N}\right) F_j\big(\eta_s^M(x)\big)\right] \\
&\quad + \frac{16}{N^d} \int_0^t \mathrm{d}s \, \mathbb{E}_{\mu^N}\left[\sum_{i,j=1}^d \sum_{x\in\mathbb{T}_N^d} (\partial_{u_i}\varphi)\left(s, \frac{x}{N}\right) (\partial_{u_j}\varphi)\left(s, \frac{x}{N}\right) F_{i,j}\big(\eta_s^M(x)\big)\right] \\
&\quad + \frac{4N^b}{N^d} \int_0^t \mathrm{d}s \, \mathbb{E}_{\mu^N}\left[\sum_{x\in\mathbb{T}^d} (\Delta\varphi - \partial_s\varphi)\left(s, \frac{x}{N}\right) \left\{\eta_s(x) - \left(\frac{1}{2}\right) - N^{-b}\varphi\left(s, \frac{x}{N}\right)\right\}\right]
\end{aligned} \tag{4.10}$$

plus an error term of order $o_N(1)$ for every $t \le T$.

Recall that $\chi(a) = a(1-a)$. Since $N^{2b} \ll M^d$, we may replace in the previous formula $F_j(\eta_s^M(x))$ by $-\gamma_j^M \chi(\eta_s^M(x))$ and $F_{i,j}(\eta_s^M(x))$ by $\{\chi(\eta_s^M(x)) - \chi(1/2)\}\delta_{i,j}$. Moreover, since $N^{-d}\sum_{x\in\mathbb{T}_N^d}(\partial_{u_j}\varphi)(t,x/N)$ is of order N^{-1} and since $b<1/2$, we may further replace $\chi(\eta_t^M(x))$ by $\chi(\eta_t^M(x)) - \chi(1/2)$ in the first term. Finally, since for a smooth function G, $M^{-d}\sum_{y:\,|y-x|\le M}[G(y/N) - G(x/N)]$ is of order $(M/N)^2$ and since M^2N^{b-2} vanishes as $N\uparrow\infty$, we may replace $\eta_s(x)$ by $\eta_s^M(x)$ in the third term. After all these replacements and since $\chi(b) - \chi(1/2) = -[b-(1/2)]^2$, (4.10) becomes

$$
\begin{aligned}
&\frac{4(1+\varepsilon_N)N^{2b}}{N^d}\int_0^t \mathrm{d}s\,\mathbb{E}_{\mu^N}\Bigg[\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}\gamma_j^M(\partial_{u_j}\varphi)\Big(s,\frac{x}{N}\Big)\Big\{\eta_s^M(x)-\frac12\Big\}^2\Bigg]\\
&\quad-\frac{16}{N^d}\int_0^t \mathrm{d}s\,\mathbb{E}_{\mu^N}\Bigg[\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}(\partial_{u_i}\varphi)^2\Big(s,\frac{x}{N}\Big)\Big\{\eta_s^M(x)-\frac12\Big\}^2\Bigg]\\
&\quad+\frac{4N^b}{N^d}\int_0^t \mathrm{d}s\,\mathbb{E}_{\mu^N}\Bigg[\sum_{x\in\mathbb{T}^d}(\Delta\varphi-\partial_s\varphi)\Big(s,\frac{x}{N}\Big)\Big\{\eta_s^M(x)-\Big(\frac12\Big)-N^{-b}\varphi\Big(s,\frac{x}{N}\Big)\Big\}\Bigg].
\end{aligned}\tag{4.11}
$$

The second line of the previous formula is easy to estimate. One can argue that it is negative or one can add $N^{-b}\varphi(s,x/N)$ inside the braces and apply Lemma 4.3 below. The first term in (4.11) without the factor $(1+\varepsilon_N)$ can be written as

$$
\begin{aligned}
&\frac{4N^{2b}}{N^d}\int_0^t \mathrm{d}s\,\mathbb{E}_{\mu^N}\Bigg[\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}\gamma_j^M(\partial_{u_j}\varphi)\Big(s,\frac{x}{N}\Big)\Big\{\eta_s^M(x)-\frac12-N^{-b}\varphi\Big(s,\frac{x}{N}\Big)\Big\}^2\Bigg]\\
&\quad+\frac{4N^b}{N^d}\int_0^t \mathrm{d}s\,\mathbb{E}_{\mu^N}\Bigg[\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}\gamma_j^M(\partial_{u_j}\varphi^2)\Big(s,\frac{x}{N}\Big)\Big\{\eta_s^M(x)-\frac12-N^{-b}\varphi\Big(s,\frac{x}{N}\Big)\Big\}\Bigg]\\
&\quad+\frac{4}{N^d}\int_0^t \mathrm{d}s\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}\gamma_j^M(\partial_{u_j}\varphi)\Big(s,\frac{x}{N}\Big)\varphi^2\Big(s,\frac{x}{N}\Big).
\end{aligned}
$$

As $N\uparrow\infty$, for each fixed j, s, the last term of this expression converges to $4\int_{\mathbb{T}^d}\mathrm{d}u\,\gamma_j(\partial_{u_j}\varphi)(s,u)\varphi^2(s,u)=0$. By Lemma 4.3 below, the first term is bounded by $\gamma_0\int_0^t \mathrm{d}s\,H_N(s)+o_N(1)$ for some finite constant γ_0. In the second term, since $\varepsilon_N N^b$ vanishes as $N\uparrow\infty$ and since, by (3.3), $N^b\ll M$, we may replace $(1+\varepsilon_N)\gamma_j^M$ by γ_j. The resulting expression cancels with the third term of (4.11) because φ is the solution of (3.4). This proves Proposition 4.1, and, therefore, Theorem 3.1.

We conclude this section with an estimate on the variance of the density in terms of the relative entropy.

Lemma 4.3. *There exists $\gamma_0>0$ such that*

$$
\mathbb{E}_{\mu^N}\Bigg[\frac{N^{2b}}{N^d}\sum_{x\in\mathbb{T}_N^d}\Big\{\eta_t^M(x)-\Big(\frac12\Big)-N^{-b}\varphi\Big(t,\frac{x}{N}\Big)\Big\}^2\Bigg]\le\gamma_0 H_N(t)+o_N(1)
$$

for $0\le t\le T$.

Proof. By the entropy inequality, the expectation appearing in the statement of the lemma is bounded above by

$$
\frac{1}{\gamma}H_N(t)+\frac{N^{2b}}{\gamma N^d}\log\mathbb{E}_{\nu_t^N}\Bigg[\exp\Bigg\{\gamma\sum_{x\in\mathbb{T}_N^d}\Big(\eta_t^M(x)-\Big(\frac12\Big)-N^{-b}\varphi\Big(t,\frac{x}{N}\Big)\Big)^2\Bigg\}\Bigg]
$$

for every $\gamma > 0$. By Hölder inequality, the second term is less than or equal to

$$\frac{N^{2b}}{\gamma N^d M^d} \sum_{x \in \mathbb{T}_N^d} \log \mathbb{E}_{\nu_t^N} \left[\exp\left\{ \gamma |\Lambda_M| \left(\eta_t^M(x) - \left(\frac{1}{2}\right) - N^{-b} \varphi\left(t, \frac{x}{N}\right)\right)^2 \right\}\right].$$

The above expectation is bounded uniformly in N provided γ is small enough. The expression is thus bounded by $\gamma^{-1} N^{2b} M^{-d}$, which concludes the proof of the lemma. □

5. Proof of the Incompressible limit

Fix the reference measure $\nu_*^N = \nu_{(a_0,\mathbf{0})}^N$. Consider a sequence of probability measures $\{\mu^N\colon N \ge 1\}$ satisfying the assumptions of Theorem 2.1. A straightforward argument, similar to the one which led to (4.1), shows that

$$N^{2b-d} H_N\big(\mu^N \big| \nu_*^N\big) \le C_0$$

for some finite constant depending only on $a_0, \varphi_0, \boldsymbol{\varphi}$.

Denote by f_t^N the Radon–Nikodym derivative $\mathrm{d}\mu^N S_t^N / \mathrm{d}\nu_*^N$ and recall that f_t^N solves the equation

$$\partial_t f_t^N = \mathcal{L}_N^* f_t^N,$$

where $\mathcal{L}_N^*$ stands for the adjoint of $\mathcal{L}_N$ in $L^2(\nu_*^N)$. By the previous estimate on the relative entropy of μ^N with respect to ν_*^N, we get that

$$H_N\big(\mu^N S_t^N \big| \nu_*^N\big) + \int_0^t \mathrm{d}s\, D_N\big(f_s^N\big) \le C_0 N^{d-2b}, \tag{5.1}$$

where D_N stands for the Dirichlet form: $D_N(f) = \langle f^{1/2}, (-\mathcal{L}_N) f^{1/2}\rangle_{\nu_*^N}$.

Let $\psi_t^N = \mathrm{d}\nu_t^N / \mathrm{d}\nu_*^N$. It follows from the explicit formulas for the product measures ν_t^N that

$$\log \psi_t^N = \sum_{x \in \mathbb{T}_N^d} \boldsymbol{\lambda}(t,x) \cdot \mathbf{I}(\eta_x) - \sum_{x \in \mathbb{T}_N^d} \log \frac{Z(\boldsymbol{\lambda}(t,x))}{Z(\mathbf{0}))},$$

where $\boldsymbol{\lambda}(t,x) := \boldsymbol{\Lambda}(a_0 + N^{-b}\varphi_0, N^{-b}\boldsymbol{\varphi}(t, x/N))$ and

$$Z(\boldsymbol{\lambda}) = \sum_{\xi \in \{0,1\}^{\mathcal{V}}} \exp\big\{\boldsymbol{\lambda} \cdot \mathbf{I}(\xi)\big\}.$$

Let $H_N(t) = N^{2b-d} H_N(\mu^N S_t^N | \nu_t^N)$. With the notation just introduced, we have that

$$H_N(t) = N^{2b-d} \int f_t^N \log \frac{f_t^N}{\psi_t^N} \, \mathrm{d}\nu_*^N.$$

Theorem 2.1 follows from Gronwall lemma and the following estimate.

Proposition 5.1. *Fix a sequence of measures $\{\mu^N\colon N \ge 1\}$ satisfying the assumptions of Theorem 2.1. There exists $\gamma > 0$ such that*

$$H_N(t) \le \gamma \int_0^t H_N(s)\, \mathrm{d}s + o_N(1)$$

for all $0 \le t \le T$.

The proof of Proposition 5.1 is divided in several steps. We begin with a well-known upper bound for the entropy production.

$$\frac{\mathrm{d}}{\mathrm{d}t} H_N(t) \le N^{2b-d} \int f_t^N \frac{(\mathcal{L}_N^* - \partial_t)\psi_t^N}{\psi_t^N} \, \mathrm{d}\nu_*^N. \tag{5.2}$$

Next result is needed in to discard irrelevant terms on the right-hand side of the previous expression.

Lemma 5.2. *Let $G : \mathbb{T}^d \to \mathbb{R}$ a continuous function and $\{\mu^N : N \ge 1\}$ a sequence of measures satisfying the assumptions of Theorem 2.1. Then*

$$\mathbb{E}_{\mu^N}\left[N^{-d} \sum_{x \in \mathbb{T}_N^d} G\left(\frac{x}{N}\right) I_k\big(\eta_x(t)\big)\right] \le H_N(t) + \mathrm{O}\big(N^{-b}\big)$$

for $0 \le t \le T$, $1 \le k \le d$. The lemma remains in force for $k = 0$, if we replace $I_k(\eta_x(t))$ by $I_0(\eta_x(t)) - a_0$.

Proof. Fix $1 \le k \le d$. We may replace $I_k(\eta_x(t))$ by $I_k(\eta_x(t)) - N^{-b}\varphi_k(t, x/N)$ paying a price of order N^{-b}. It remains to apply the entropy inequality with respect to measure ν_t, which is product, and perform a second order Taylor expansion. $\square$

A long and tedious computation gives that $(\psi_t^N)^{-1}(\mathcal{L}_N^* - \partial_t)\psi_t^N$ is equal to

$$\begin{aligned}
&\frac{4}{BN^b} \sum_{k=1}^d \sum_{x \in \mathbb{T}_N^d} (\Delta\varphi_k)\left(t, \frac{x}{N}\right) I_k(\eta_x) + \frac{4}{B} \sum_{j,k=1}^d \sum_{x \in \mathbb{T}_N^d} (\partial_{u_j}\varphi_k)\left(t, \frac{x}{N}\right) \tau_x W_{k,j}^{*,M} \\
&\quad + \frac{16}{B^2 N^{2b}} \sum_{i,j=1}^d \sum_{k,\ell=1}^d \sum_{x \in \mathbb{T}_N^d} (\partial_{u_i}\varphi_k)\left(t, \frac{x}{N}\right) (\partial_{u_j}\varphi_\ell)\left(t, \frac{x}{N}\right) \tau_x V_{i,j}^{k,\ell,M} \\
&\quad - \frac{4}{BN^b} \sum_{k=1}^d (\partial_t\varphi_k)\left(t, \frac{x}{N}\right) \left\{ I_k(\eta_x) - \frac{\varphi_k(t, x/N)}{N^b} \right\} + R_N(t) + \mathrm{o}\big(N^{d-2b}\big).
\end{aligned} \tag{5.3}$$

In this formula, $W_{k,j}^{*,M}$ and $V_{i,j}^{k,\ell,M}$ stand for

$$\begin{aligned}
W_{k,j}^{*,M} &= \frac{A_M}{M^{d+1}} \sum_{v \in \mathcal{V}} v_k \sum_{z \in \Lambda_M} q_M(-z, v) z_j \eta(0, v)\big\{1 - \eta(z, v)\big\}, \\
V_{i,j}^{k,\ell,M} &= \frac{A_M}{M^{d+2}} \sum_{v \in \mathcal{V}} v_k v_\ell \sum_{z \in \Lambda_M} z_i z_j \eta(0, v)\big\{1 - \eta(z, v)\big\}.
\end{aligned}$$

Since the density ψ_t^N is a function of the conserved quantities $\mathbf{I}$, the collision part of the generator is irrelevant in the previous computation. We used repeatedly Lemma 5.2 and the fact that $b < a$, which follows from (2.13), to discard superfluous terms. The remainder $\mathrm{o}(N^{d-2b})$ should be understood as an expression whose expectation with respect to $\mu^N S_t^N$ integrated in time is of order $\mathrm{o}(N^{d-2b})$, while $R_N(t)$ is an expression which multiplied by N^{2b-d} is bounded by $H_N(t) + \mathrm{O}(N^{-b})$ in virtue of Lemma 5.2.

If we replace $I_k(\eta_x)$ in the first term of (5.3) by $I_k(\eta_x) - \varphi_k(t, x/N)N^{-b}$ and $V_{i,j}^{k,\ell,M}$ by $V_{i,j}^{k,\ell,M} - (B/4)\delta_{i,j}\delta_{k,\ell}$, as $N \uparrow \infty$, the expressions added when multiplied by $(B/4)N^{2b-d}$ converge to

$$\sum_{k=1}^d \int_{\mathbb{T}^d} (\Delta\varphi_k)\varphi_k + \sum_{j,k=1}^d \int_{\mathbb{T}^d} (\partial_{u_j}\varphi_k)^2 = 0.$$

Therefore, in view of (5.2), (5.3) and Lemma 5.2, the time derivative of the renormalized entropy $H_N(t)$ is bounded above by

$$
\begin{aligned}
& H_N(t) + \mathbb{E}_{\mu^N}\left[\frac{4N^{2b}}{BN^d}\sum_{j,k=1}^{d}\sum_{x\in\mathbb{T}_N^d}(\partial_{u_j}\varphi_k)\left(t,\frac{x}{N}\right)\tau_x W_{k,j}^{*,M}(t)\right] \\
& \quad + \mathbb{E}_{\mu^N}\left[\frac{16}{B^2N^d}\sum_{i,j=1}^{d}\sum_{k,\ell=1}^{d}\sum_{x\in\mathbb{T}_N^d}(\partial_{u_i}\varphi_k)\left(t,\frac{x}{N}\right)(\partial_{u_j}\varphi_\ell)\left(t,\frac{x}{N}\right)\tau_x \hat{V}_{i,j}^{k,\ell,M}(t)\right] \\
& \quad + \mathbb{E}_{\mu^N}\left[\frac{4N^b}{BN^d}\sum_{k=1}^{d}\sum_{x\in\mathbb{T}^d}(\Delta\varphi_k - \partial_t\varphi_k)\left(t,\frac{x}{N}\right)\left\{I_k(\eta_x(t)) - \frac{\varphi_k(t,x/N)}{N^b}\right\}\right] + o_N(1),
\end{aligned}
\tag{5.4}
$$

where $\hat{V}_{i,j}^{k,\ell,M} = V_{i,j}^{k,\ell,M} - (B/4)\delta_{i,j}\delta_{k,\ell}$.

We now use the ergodicity to replace the functions $W_{k,j}^{*,M}$ and $\hat{V}_{i,j}^{k,\ell,M}$ by their projections on the conserved quantities. For $s \ge 0$ and x in $\mathbb{Z}^d$, denote by $\mathbf{I}^M(s,x)$ the average at time s of the conserved quantities over a cube Λ_M centered at x:

$$
\mathbf{I}^M(s,x) = \frac{1}{|\Lambda_M|}\sum_{y\in x+\Lambda_M}\mathbf{I}\big(\eta_y(s)\big).
$$

To keep notation simple, let $\mathbf{I}_s^M := \mathbf{I}^M(s,0)$.

Recall the definition of the canonical measures $\nu_{\Lambda_M,\mathbf{i}}$ presented in Section 2.2. Since we assumed in (2.4) the global dynamics restricted to a cube of length M to have a spectral gap of order M^κ and since $a+b > 1-[2/(d+\kappa)]$, $a+(\kappa-2/\kappa)b > 1-(2/\kappa)$, repeating the arguments presented in the proof of Lemma 4.2 and taking advantage of the estimate (5.1) we derive the so-called one block estimate. In this lemma, the collision part of the dynamics, also sped up by N^2, plays an important role.

Lemma 5.3. *For every $t \ge 0$, every $1 \le j,k \le d$ and every continuous function $G:[0,T]\times\mathbb{T}^d \to \mathbb{R}$,*

$$
\limsup_{N\to\infty}\mathbb{E}_{\mu^N}\left[\left|\int_0^t \mathrm{d}s\,\frac{N^{2b}}{N^d}\sum_{x\in\mathbb{T}_N^d}G\left(s,\frac{x}{N}\right)\tau_x\big\{W_{k,j}^{*,M}(s) - E_{\nu_{\Lambda_M,\mathbf{I}_s^M}}\big[W_{k,j}^{*,M}\big]\big\}\right|\right] = 0.
$$

Since $\nu_{\Lambda_M,\mathbf{i}}$ is the counting measure,

$$
E_{\nu_{\Lambda_M,\mathbf{i}}}\big[W_{k,j}^{*,M}\big] = -\sum_{v\in\mathcal{V}}v_k v_j E_{\nu_{\Lambda_M,\mathbf{i}}}\big[\eta(0,v)\big[1-\eta(e_1,v)\big]\big]
$$

because $A_M M^{-(d+1)}\sum_{z\in\Lambda_M}q_M(z,v)z_j = v_j$. In the previous formula, site e_1 can be replaced by any site of Λ_M different from the origin. Since $N^{2b} \ll M^d$, by the equivalence of ensembles, stated in Proposition 7.1 below, we can replace the expectation with respect to the canonical measure by the expectation with respect to the grand canonical measure paying a price of order $o_N(1)$.

For $1 \le j,k \le d$, let

$$
R_{j,k}(\rho,\mathbf{p}) := E_{\mu^N_{\Lambda(\rho,\mathbf{p})}}\big[W_{k,j}^{*,M}\big] = -\sum_{v\in\mathcal{V}}v_k v_j \chi\big(\theta_v\big(\boldsymbol{\Lambda}(\rho,\mathbf{p})\big)\big),
$$

where $\theta_v(\cdot)$ is defined in (2.2). Up to this point, we replaced the first expectation in (5.4) by

$$
\mathbb{E}_{\mu^N}\left[\frac{4N^{2b}}{BN^d}\sum_{j,k=1}^{d}\sum_{x\in\mathbb{T}_N^d}(\partial_{u_j}\varphi_k)\left(t,\frac{x}{N}\right)R_{j,k}\big(\mathbf{I}^M(t,x)\big)\right] + o_N(1).
$$

Since $\boldsymbol{\Lambda}(a_0, \mathbf{0}) = \mathbf{0}$ and $\theta_v(\mathbf{0}) = 1/2$, $R_{j,k}(a_0, \mathbf{0}) = -(B/4)\delta_{j,k}$. On the other hand, since φ is divergence free,

$$\frac{4N^{2b}}{BN^d} \sum_{j,k=1}^{d} \sum_{x \in \mathbb{T}_N^d} (\partial_{u_j} \varphi_k)\left(t, \frac{x}{N}\right) R_{j,k}(a_0, \mathbf{0}) = \frac{-N^{2b}}{N^d} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_N^d} (\partial_{u_j} \varphi_j)\left(t, \frac{x}{N}\right)$$

vanishes for each fixed N. We may, therefore, add this expression to the previous expectation to obtain that the first term in (5.4) is equal to

$$\mathbb{E}_{\mu^N}\left[\frac{4N^{2b}}{BN^d} \sum_{j,k=1}^{d} \sum_{x \in \mathbb{T}_N^d} (\partial_{u_j} \varphi_k)\left(t, \frac{x}{N}\right) \left\{R_{j,k}\left(\mathbf{I}^M(t,x)\right) - R_{j,k}(a_0, \mathbf{0})\right\}\right] + o_N(1). \tag{5.5}$$

The same arguments show that we can replace $\hat{V}_{i,j}^{k,\ell,M}$ in the second term of (5.4) by its expectation with respect to the grand canonical measure. The proof is even simpler due to the absence of the factor N^{2b} in front of the sum. Since $R_{j,k}(a_0, \mathbf{0}) = -(1/4)\delta_{j,k} B$,

$$E_{\mu^N_{\boldsymbol{\Lambda}(\rho,\mathbf{p})}}\left[\hat{V}_{i,j}^{k,\ell,M}\right] = -\delta_{i,j}\left[R_{k,\ell}(\rho, \mathbf{p}) - R_{k,\ell}(a_0, \mathbf{0})\right].$$

The one-block estimate permits therefore to replace the second expectation in (5.4) by

$$-\mathbb{E}_{\mu^N}\left[\frac{16}{B^2 N^d} \sum_{i,k,\ell=1}^{d} \sum_{x \in \mathbb{T}_N^d} (\partial_{u_i} \varphi_{k,\ell})\left(t, \frac{x}{N}\right) \left\{R_{k,\ell}\left(\mathbf{I}^M(t,x)\right) - R_{k,\ell}(a_0, \mathbf{0})\right\}\right], \tag{5.6}$$

where $(\partial_{u_i} \varphi_{k,\ell})(t, x/N) = (\partial_{u_i} \varphi_k)(t, x/N)(\partial_{u_i} \varphi_\ell)(t, x/N)$.

It is now clear that (5.6) is a term of lower order than (5.5). We, therefore, only need to estimate the latter. Fix an arbitrary $\epsilon > 0$. Since $R_{j,k}$ is a bounded function, the integral in (5.5) when restricted to $|\mathbf{I}^M(t,x) - (a_0, \mathbf{0})| > \epsilon$ is bounded above by

$$\mathbb{E}_{\mu^N}\left[\frac{C_0 N^{2b}}{\epsilon^3 N^d} \sum_{x \in \mathbb{T}_N^d} \left|\mathbf{I}^M(t,x) - (a_0, \mathbf{0})\right|^3\right], \tag{5.7}$$

where C_0 is a constant depending on $\mathcal{V}$ and $\boldsymbol{\varphi}$. In the expression above, we may replace $(a_0, \mathbf{0})$ by $(a_0 + N^{-b}\varphi_0, N^{-b}\boldsymbol{\varphi}(t, x/N))$ paying a price of order N^{-b}. Since $I^M(t,x)$ belongs to a compact set the expression obtained after replacing is bounded above by

$$\mathbb{E}_{\mu^N}\left[\frac{C(\mathcal{V}) C_0 N^{2b}}{\epsilon^3 N^d} \sum_{x \in \mathbb{T}_N^d} \left|\mathbf{I}^M(t,x) - \left(a_0 + \frac{\varphi_0}{N^b}, \frac{\boldsymbol{\varphi}(t, x/N)}{N^b}\right)\right|^2\right].$$

By Lemma 5.4 below, this expression is bounded by $\gamma_0 H_N(t) + o_N(1)$ for some $\gamma_0 > 0$.

In order to deal with the integral (5.5) on $|\mathbf{I}^M(t,x) - (a_0, \mathbf{0})| \le \epsilon$, we perform a Taylor expansion of $R_{j,k}$. The first term in the expansion vanishes because the gradient of $R_{j,k}$ vanishes at $(a_0, \mathbf{0})$. The contribution of the second order terms is

$$\frac{4N^{2b}}{BN^d} \sum_{j,k=1}^{d} \sum_{x \in \mathbb{T}_N^d} (\partial_{u_j} \varphi_k)\left(t, \frac{x}{N}\right) \sum_{v \in \mathcal{V}} v_k v_j \left\{\frac{I_0^L - a_0}{|\mathcal{V}|} + \frac{1}{B} \sum_{\ell=1}^{d} v_\ell I_\ell^L\right\}^2.$$

Expanding the square, the term in I_0^L vanishes because $\boldsymbol{\varphi}$ is divergence free and the cross product vanishes because $\mathcal{V}$ is symmetric. This sum is, therefore, equal to

$$\frac{4N^{2b}}{B^3N^d}\sum_{j,k,\ell,m=1}^{d}\sum_{x\in\mathbb{T}_N^d}(\partial_{u_j}\varphi_k)\left(t,\frac{x}{N}\right)\sum_{v\in\mathcal{V}}v_kv_jv_\ell v_m I_\ell^L I_m^L.$$

Replacing I_ℓ^L by $I_\ell^L-\varphi_\ell(t,x/N)N^{-b}$, we may rewrite the previous expression as the sum of three kind of terms. The first one, the 0 order term in I, consists simply in replacing I_ℓ^L by $\varphi_\ell N^{-b}$. As N tends to infinity, this term converges to

$$\frac{4(D-3C)}{B^3}\sum_{j=1}^{d}\int_{\mathbb{T}^d}(\partial_{u_j}\varphi_j)\varphi_j^2+\frac{4C}{B^3}\sum_{j,k=1}^{d}\int_{\mathbb{T}^d}\left(\partial_{u_j}\varphi_k^2\right)\varphi_j.$$

An integration by parts shows that this expression vanishes because $\boldsymbol{\varphi}$ is divergence free. The linear term in I cancels with the last term of (5.4) because $\boldsymbol{\varphi}$ is the solution of the Navier–Stokes equation (2.12). Remains the quadratic term in I, equal to

$$\frac{4(D-3C)N^{2b}}{B^3N^d}\sum_{j=1}^{d}\sum_{x\in\mathbb{T}_N^d}(\partial_{u_j}\varphi_j)\left(t,\frac{x}{N}\right)\left\{I_j^L(t,x)-\frac{\varphi_j(t,x/N)}{N^b}\right\}^2$$
$$+\frac{8CN^{2b}}{B^3N^d}\sum_{j,k=1}^{d}\sum_{x\in\mathbb{T}_N^d}(\partial_{u_j}\varphi_k)\left(t,\frac{x}{N}\right)\left\{I_j^L(t,x)-\frac{\varphi_j(t,x/N)}{N^b}\right\}\left\{I_k^L(t,x)-\frac{\varphi_k(t,x/N)}{N^b}\right\}$$

because $\boldsymbol{\varphi}$ is divergence free. By Lemma 5.4 below, this expression is bounded by $\gamma_0H_N(t)+\mathrm{o}_N(1)$ for some $\gamma_0>0$.

Finally, we consider the remainder in the Taylor expansion. Since $R_{j,k}$ is smooth, we can choose ϵ small enough for the third derivative of $R_{j,k}$ to be bounded in an ϵ-neighborhood of $(a_0,\mathbf{0})$ by a finite constant C_0 depending on $\mathcal{V}$ and $\boldsymbol{\varphi}$. In particular, the remainder is bounded above by

$$\frac{C_0N^{2b}}{N^d}\sum_{x\in\mathbb{T}_N^d}\left|\mathbf{I}^M(t,x)-(a_0,\mathbf{0})\right|^3.$$

The same arguments used to estimate (5.7) prove that this expression is bounded by $\gamma_0H_N(t)+\mathrm{o}_N(1)$ for some $\gamma_0>0$. This concludes the proof of Proposition 5.1.

We conclude the section with an estimate repeatedly used in the proof of Proposition 5.1. We need here again the assumption that $N^{2b}\ll M^d$.

Lemma 5.4. *There exists* $\gamma_0>0$, *such that*

$$\mathbb{E}_{\mu^N}\left[\frac{N^{2b}}{N^d}\sum_{x\in\mathbb{T}_N^d}\left\{I_j^M(t,x)-\frac{\varphi_j(t,x/N)}{N^b}\right\}^2\right]\le\gamma_0H_N(t)+\mathrm{o}_N(1)$$

for $1\le j\le d$, $0\le t\le T$. *The statement remains in force for* $j=0$ *if* $\varphi_j(t,x/N)N^{-b}$ *is replaced by* $a_0+\varphi_0N^{-b}$.

Proof. By the entropy inequality the expectation appearing in the statement of the lemma is bounded above by

$$\frac{1}{\gamma}H_N(t)+\frac{N^{2b}}{\gamma N^d}\log\mathbb{E}_{\nu_t^N}\left[\exp\left\{\gamma\sum_{x\in\mathbb{T}_N^d}\left(I_j^M(t,x)-\frac{\varphi_j(t,x/N)}{N^b}\right)^2\right\}\right]$$

for every $\gamma > 0$. By Hölder inequality, the second term is less than or equal to

$$\frac{N^{2b}}{\gamma N^d M^d} \sum_{x \in \mathbb{T}_N^d} \log \mathbb{E}_{\nu_t^N} \left[\exp \left\{ \gamma |\Lambda_M| \left\{ I_j^M(t,x) - \frac{\varphi_j(t,x/N)}{N^b} \right\}^2 \right\} \right].$$

The above expectation is bounded uniformly in N provided γ is small enough. The expression is thus bounded by $\gamma^{-1} N^{2b} M^{-d}$, which concludes the proof of the lemma. □

6. Spectral gap for stochastic lattice gases

We prove in this section a spectral gap of polynomial order for the generator of the stochastic lattice gas. We consider a slightly different process, in which the exclusion dynamics allows particles to jump to any site of Λ_M at rate $M^{-(d+2)}$. We do not require, therefore, the jump to be of size smaller than M. Of course, the Dirichlet forms of both dynamics are equivalent and the result stated in Proposition 6.1 extends to the original dynamics.

Fix $M \ge 1$ and consider the process restricted to the cube Λ_M without the factor N^2. The generator of the process, denoted by $\mathcal{L}_M$, can be written as $\mathcal{L}_M^{ex} + \mathcal{L}_M^c$, where

$$\left(\mathcal{L}_M^{ex} f\right)(\eta) = \frac{1}{M^{d+2}} \sum_{v \in \mathcal{V}} \sum_{x,z \in \Lambda_M} \eta(x,v)\left[1 - \eta(z,v)\right]\left[f\left(\eta^{x,z,v}\right) - f(\eta)\right],$$

$$\mathcal{L}_M^c f(\eta) = \sum_{y \in \Lambda_M} \sum_{q \in \mathcal{Q}} p(y,q,\eta)\left[f\left(\eta^{y,q}\right) - f(\eta)\right]$$

and $p(y,q,\eta)$ is defined at the beginning of Section 2.

For each fixed $\mathbf{i}$ in $\mathfrak{V}_M$, recall that we denote by $\nu_{\Lambda_M,\mathbf{i}}$ the invariant measure concentrated on configurations η of $(\{0,1\}^{\mathcal{V}})^{\Lambda_M}$ such that $\mathbf{I}^M(\eta) = \mathbf{i}$. An elementary computation shows that

$$\begin{aligned} \left\langle f, -\mathcal{L}_M^{ex} f\right\rangle_{\nu_{\Lambda_M,\mathbf{i}}} &= \frac{1}{4M^{d+2}} \sum_{v \in \mathcal{V}} \sum_{x,y \in \Lambda_M} E_{\nu_{\Lambda_M,\mathbf{i}}}\left[\left\{f\left(\xi^{x,y,v}\right) - f(\xi)\right\}^2\right], \\ \left\langle f, -\mathcal{L}_M^{c} f\right\rangle_{\nu_{\Lambda_M,\mathbf{i}}} &= \frac{1}{2} \sum_{q \in \mathcal{Q}} \sum_{x \in \Lambda_M} E_{\nu_{\Lambda_M,\mathbf{i}}}\left[p(x,q,\xi)\left\{f\left(\xi^{x,q}\right) - f(\xi)\right\}^2\right]. \end{aligned} \tag{6.1}$$

Denote by $E_\nu[f;f]$ the variance of f with respect to a measure ν and by $\langle\cdot,\cdot\rangle_\nu$ the inner product in $L^2(\nu)$.

Proposition 6.1. *There exists a finite constant C_2, depending only on $\mathcal{V}$, $\mathcal{Q}$, such that*

$$E_{\nu_{\Lambda_M,\mathbf{i}}}[f;f] \le C_2 M^{2+3d+2d^2} \langle f, -\mathcal{L}_M f\rangle_{\nu_{\Lambda_M,\mathbf{i}}}$$

for all f in $L^2(\nu_{\Lambda_M,\mathbf{i}})$, all $\mathbf{i}$ in $\mathfrak{V}_M$ and all $M \ge 1$.

The proof of this proposition relies on estimates on the Dirichlet forms associated to $\mathcal{L}_M^{ex}$ and $\mathcal{L}_M^c$. Denote by $\tilde{\mathcal{L}}_M^c$ the generator of a dynamics in which collisions between particles at different sites are allowed:

$$\left(\tilde{\mathcal{L}}_M^c f\right)(\eta) = \frac{1}{|\Lambda_M|^3} \sum_{x_1,\dots,x_4 \in \Lambda_M} \sum_{q \in \mathcal{Q}} p(\mathbf{x},q,\eta)\left\{f\left(\eta^{\mathbf{x},q}\right) - f(\eta)\right\},$$

where, for $q = (u,v,u',v')$, $\mathbf{x} = (x_1,\dots,x_4)$,

$$p(\mathbf{x},q,\eta) = \eta_{x_1}(u)\eta_{x_2}(v)\left[1 - \eta_{x_3}(u')\right]\left[1 - \eta_{x_4}(v')\right]$$

and where $\eta^{\mathbf{x},q}$ is the configuration η in which the occupation variables $\eta_{x_1}(u)$, $\eta_{x_2}(v)$, $\eta_{x_3}(u')$, $\eta_{x_4}(v')$ are flipped.

The first lemma of this section states that the Dirichlet forms associated to $\mathcal{L}^c_M$ and to $\tilde{\mathcal{L}}^c_M$ are comparable and that the Dirichlet form of the conditional expectation of a function with respect to the total number of particles with fixed velocity can be estimated by the Dirichlet form of the original function. For each v in $\mathcal{V}$, let K_v be the total number of particles with velocity v in Λ_M:

$$K_v = K_v(\eta) = \sum_{x\in\Lambda_M} \eta(x, v).$$

Lemma 6.2. *There exists a finite constant C_2, depending only on $\mathcal{V}$, such that*

$$\langle -\tilde{\mathcal{L}}^c_M f, f\rangle_{\nu_{\Lambda_M,\mathbf{i}}} \le C_2 M^2 \langle -\mathcal{L}^{ex}_M f, f\rangle_{\nu_{\Lambda_M,\mathbf{i}}} + C_2\langle -\mathcal{L}^c_M f, f\rangle_{\nu_{\Lambda_M,\mathbf{i}}}$$

for every f in $L^2(\nu_{\Lambda_M,\mathbf{i}})$. Moreover, let

$$F = E_{\nu_{\Lambda_M,\mathbf{i}}}\big[f \,|\, \{K_v, v\in\mathcal{V}\}\big].$$

Then,

$$\langle -\mathcal{L}^c_M F, F\rangle_{\nu_{\Lambda_M,\mathbf{i}}} \le \langle -\tilde{\mathcal{L}}^c_M f, f\rangle_{\nu_{\Lambda_M,\mathbf{i}}}$$

for every f in $L^2(\nu_{\Lambda_M,\mathbf{i}})$.

Proof. An elementary computation shows that

$$\langle -\tilde{\mathcal{L}}^c_M f, f\rangle_{\nu_{\Lambda_M,\mathbf{i}}} = \frac{1}{2|\Lambda_M|^3}\sum_{q\in\mathcal{Q}}\sum_{x_1,\dots,x_4\in\Lambda_M} E_{\nu_{\Lambda_M,\mathbf{i}}}\big[p(\mathbf{x},q,\xi)\big\{f\big(\xi^{\mathbf{x},q}\big)-f(\xi)\big\}^2\big]$$

for f in $L^2(\nu_{\Lambda_M,\mathbf{i}})$. Fix q and $\mathbf{x}$. We construct a path from ξ to $\xi^{\mathbf{x},q}$ with jumps and collisions of particles in the same site in the following way. Assume that the set $\mathcal{V}$ has been ordered: $\mathcal{V} = \{v_1,\dots,v_n\}$ and, without loss of generality that $q = (v_1,\dots,v_4)$. We first exchange the occupation variable $\xi_{x_2}(v_2)$, $\xi_{x_1}(v_2)$; then $\xi_{x_3}(v_3)$, $\xi_{x_1}(v_3)$ and finally $\xi_{x_4}(v_4)$, $\xi_{x_1}(v_4)$. At this point, we may perform the collision at site x_1 and move back the particles and holes to their final positions in the reversed order.

The total length of the path is at most 7. Denote by $\zeta_0 = \xi,\dots,\zeta_\ell = \xi^{\mathbf{x},q}$ the successive configurations. Writing $\{f(\xi^{\mathbf{x},q}) - f(\xi)\}$ as $\sum_j\{f(\zeta_{j+1}) - f(\zeta_j)\}$, applying Schwarz inequality, reversing the order of the summations and estimating the total number of configurations whose path jumps from ζ_j to ζ_{j+1}, we obtain that for each $q = (v_1,\dots,v_4)$,

$$\begin{aligned}
&\sum_{x_1,\dots,x_4\in\Lambda_M} E_{\nu_{\Lambda_M,\mathbf{i}}}\big[p(\mathbf{x},q,\xi)\big\{f\big(\xi^{\mathbf{x},q}\big)-f(\xi)\big\}^2\big]\\
&\quad\le C_0|\Lambda_M|^2\sum_{i=1}^4\sum_{x,y\in\Lambda_M} E_{\nu_{\Lambda_M,\mathbf{i}}}\big[\big\{f\big(\xi^{x,y,v_i}\big)-f(\xi)\big\}^2\big]\\
&\qquad + C_0|\Lambda_M|^3\sum_{x\in\Lambda_M} E_{\nu_{\Lambda_M,\mathbf{i}}}\big[p(x,q,\xi)\big\{f\big(\xi^{x,q}\big)-f(\xi)\big\}^2\big]
\end{aligned}$$

for some finite constant C_0. In particular, summing over q in $\mathcal{Q}$ and dividing by $2|\Lambda_M|^3$, we obtain that

$$\begin{aligned}
\langle -\tilde{\mathcal{L}}^c_M f, f\rangle_{\nu_{\Lambda_M,\mathbf{i}}} &\le \frac{C_2}{|\Lambda_M|}\sum_{v\in\mathcal{V}}\sum_{x,y\in\Lambda_M} E_{\nu_{\Lambda_M,\mathbf{i}}}\big[\big\{f\big(\xi^{x,y,v}\big)-f(\xi)\big\}^2\big]\\
&\quad + C_2\sum_{q\in\mathcal{Q}}\sum_{x\in\Lambda_M} E_{\nu_{\Lambda_M,\mathbf{i}}}\big[p(x,q,\xi)\big\{f\big(\xi^{x,q}\big)-f(\xi)\big\}^2\big]
\end{aligned}$$

for some finite constant C_2 depending on $\mathcal{V}$. This expression is bounded by

$$C_2\big\{\big\langle -\mathcal{L}^c_M f, f\big\rangle_{\nu_{\Lambda_M,\mathbf{i}}} + M^2\big\langle -\mathcal{L}^{ex}_M f, f\big\rangle_{\nu_{\Lambda_M,\mathbf{i}}}\big\}$$

in view of (6.1). This concludes the proof of the first statement of the lemma. We turn now to the second.

Fix x in Λ_M and $q=(u,v,u',v')$ in $\mathcal{Q}$. An elementary computation shows that

$$F\big(\eta^{x,q}\big)-F(\eta)=\frac{1}{Z_q(\eta)}\sum_{x_1,\dots,x_4\in\Lambda_M} E\big[p(\mathbf{x},q,\xi)\big\{f\big(\xi^{\mathbf{x},q}\big)-f(\xi)\big\}\big|\big\{K_v(\eta)\colon\ v\in\mathcal{V}\big\}\big],$$

where

$$Z_q(\eta)=K_u(\eta)K_v(\eta)\big(|\Lambda_M|-K_{u'}\big)\big(|\Lambda_M|-K_{v'}\big).$$

In particular, by Schwarz inequality,

$$\begin{aligned}&E_{\nu_{\Lambda_M,\mathbf{i}}}\big[p(x,q,\eta)\big\{F\big(\eta^{x,q}\big)-F(\eta)\big\}^2\big]\\ &\quad\le\sum_{x_1,\dots,x_4\in\Lambda_M}E_{\nu_{\Lambda_M,\mathbf{i}}}\bigg[\frac{p(x,q,\eta)}{Z_q(\eta)}E\big[p(\mathbf{x},q,\xi)\big\{f\big(\xi^{\mathbf{x},q}\big)-f(\xi)\big\}^2\big|\big\{K_v(\eta)\colon\ v\in\mathcal{V}\big\}\big]\bigg].\end{aligned}$$

Taking conditional expectation with respect to $\{K_v(\eta)\colon\ v\in\mathcal{V}\}$, summing over x q and dividing by 2, the previous expression becomes

$$\frac{1}{2|\Lambda_M|^3}\sum_{q\in\mathcal{Q}}\sum_{x_1,\dots,x_4\in\Lambda_M}E_{\nu_{\Lambda_M,\mathbf{i}}}\big[p(\mathbf{x},q,\xi)\big\{f\big(\xi^{\mathbf{x},q}\big)-f(\xi)\big\}^2\big]=\big\langle-\tilde{\mathcal{L}}^c_M f,f\big\rangle_{\nu_{\Lambda_M,\mathbf{i}}}.$$

This concludes the proof of the lemma. □

Proof of Proposition 6.1. Fix $M\ge 1$, $\mathbf{i}$ in $\mathfrak{V}_M$ and a function f in $L^2(\nu_{\Lambda_M,\mathbf{i}})$. Denote the conditional expectation of f with respect to $\mathbf{K}=\{K_v,v\in\mathcal{V}\}$ by $F(\mathbf{K})$:

$$F(\mathbf{K})=E_{\nu_{\Lambda_M,\mathbf{i}}}\big[f|\{K_v,v\in\mathcal{V}\}\big].$$

By orthogonality,

$$E_{\nu_{\Lambda_M,\mathbf{i}}}[f;f]=E_{\nu_{\Lambda_M,\mathbf{i}}}\big[\big\{f-F(\mathbf{K})\big\}^2\big]+E_{\nu_{\Lambda_M,\mathbf{i}}}[F;F].\tag{6.2}$$

Using only the exclusion part of the dynamics, since particles jump uniformly over the cube Λ_M with rate $M^{-(d+2)}$, by [5],

$$E_{\nu_{\Lambda_M,\mathbf{i}}}\big[\big\{f-F(\mathbf{K})\big\}^2\big]\le C_0M^2\big\langle f,-\mathcal{L}^{ex}_M f\big\rangle_{\nu_{\Lambda_M,\mathbf{i}}}$$

for some finite universal constant C_0.

To estimate the second piece on the right-hand side of (6.2), note that the exclusion part is irrelevant, while the Dirichlet form associated to the collision part can be written as

$$\big\langle F,-\mathcal{L}^c_M F\big\rangle_{\nu_{\Lambda_M,\mathbf{i}}}=|\Lambda_M|^{-3}\sum_{q\in\mathcal{Q}}\big\langle K_vK_w\big(|\Lambda_M|-K_{v'}\big)\big(|\Lambda_M|-K_{v'}\big)\big\{F\big(\mathbf{K}^q\big)-F(\mathbf{K})\big\}^2\big\rangle_{\nu_{\Lambda_M,\mathbf{i}}}.$$

Denote by $\Omega_{M,\mathbf{i}}$ the state space of velocities on Λ_M,

$$\Omega_{M,\mathbf{i}}=\bigg\{\mathbf{K}=(K_1,\dots,K_v)\colon\ \sum_v\mathbf{K}=|\Lambda_M|\mathbf{i}\bigg\},$$

and by $\bar{\nu}_{\Lambda_M,\mathbf{i}}$ the invariant state $\nu_{\Lambda_M,\mathbf{i}}$ projected on $\Omega_{M,\mathbf{i}}$. An elementary computation shows that

$$\bar{\nu}_{\Lambda_M,\mathbf{i}}(\mathbf{K}) = \frac{1}{Z_{M,\mathbf{i}}} \prod_{v\in\mathcal{V}} \binom{|\Lambda_M|}{K_v}$$

for some renormalizing constant $Z_{M,\mathbf{i}}$.

Consider from now on Model I. Set $|\Lambda_M|\mathbf{i} := (I_0, \dots, I_d)$. Suppose without loss of generality that $K_{-e_j} \le K_{e_j}$ for $1 \le j \le d$ and let $K_j = K_{-e_j}$. Since $I_j = K_{e_j} - K_{-e_j}$, $\mathbf{K}$ can be recovered from $\mathbf{i}$, K_j. We may, therefore, ignore $(K_{e_1}, \dots, K_{e_d})$ and assume that $(K_1, \dots, K_d)$ is evolving on the hyperplane

$$\mathcal{H} = \mathcal{H}_{M,\mathbf{i}} = \left\{(K_1, \dots, K_d):\ K_j \ge 0, 2\sum_j K_j = I_0 - \sum_j I_j\right\}.$$

On the set $\mathcal{H}$ the measure $\bar{\nu}_{\Lambda_M,\mathbf{i}}$ becomes

$$\bar{\nu}_{\Lambda_M,\mathbf{i}}(K_1, \dots, K_d) = \frac{1}{Z_{M,\mathbf{i}}} \prod_{j=1}^{d} \binom{|\Lambda_M|}{K_j} \binom{|\Lambda_M|}{K_j + I_j}.$$

For $0 \le n \le |\Lambda_M|$ let

$$h_n(a) = \left(\frac{|\Lambda_M| - a}{1 + a}\right)\left(\frac{|\Lambda_M| - (a+n)}{1 + (a+n)}\right). \tag{6.3}$$

h_n is a strictly convex, strictly decreasing function in the interval $[0, |\Lambda_M| - n]$. Moreover, for $n < m$, $h_n(a) < h_m(a)$ on the interval $[0, |\Lambda_M| - m]$.

Denote by $\mathfrak{d}_j$ the configuration of $\mathbb{N}^d$ with a unique particle at coordinate j. An elementary computation shows that

$$\frac{\bar{\nu}(\mathbf{K} + \mathfrak{d}_j)}{\bar{\nu}(\mathbf{K} + \mathfrak{d}_k)} \le 1 \quad \text{if and only if } h_{I_j}(K_j) \le h_{I_k}(K_k), \tag{6.4}$$

where summation is understood componentwise.

Denote by $\tilde{\mathbf{K}}$ an ordered solution of (6.7) below and fix a function F in $L^2(\bar{\nu}_{\Lambda_M,\mathbf{i}})$. We have that

$$E_{\bar{\nu}_{\Lambda_M,\mathbf{i}}}[F; F] \le E_{\bar{\nu}_{\Lambda_M,\mathbf{i}}}\left[\left\{F(\mathbf{K}) - F(\tilde{\mathbf{K}})\right\}^2\right]. \tag{6.5}$$

For each $\mathbf{K}$ in the hyperplane $\mathcal{H}$, consider the following infinite path. Let $\mathbf{K}^0 = \mathbf{K}$ and assume that $\mathbf{K}^0, \dots, \mathbf{K}^\ell$ have been defined. Let j_0, k_0 such that

$$h_{I_{k_0}}\left(K^\ell_{k_0} - 1\right) = \min_k h_{I_k}\left(K^\ell_k - 1\right), \qquad h_{I_{j_0}}\left(K^\ell_{j_0}\right) = \max_j h_{I_j}\left(K^\ell_j\right). \tag{6.6}$$

If $\mathbf{K}^\ell$ is a solution of (6.7) ($h_{I_{k_0}}(K^\ell_{k_0} - 1) \ge h_{I_{j_0}}(K^\ell_{j_0})$), let $\mathbf{K}^{\ell+1} = \mathbf{K}^\ell$; otherwise, let $\mathbf{K}^{\ell+1} = \mathbf{K}^\ell - \mathfrak{d}_{k_0} + \mathfrak{d}_{j_0}$. In this latter case, by (6.8), $\bar{\nu}_{\Lambda_M,\mathbf{i}}(\mathbf{K}^\ell) < \bar{\nu}_{\Lambda_M,\mathbf{i}}(\mathbf{K}^{\ell+1})$. Since $\mathcal{H}$ is finite and since $\bar{\nu}_{\Lambda_M,\mathbf{i}}(\mathbf{K}^\ell)$ decreases whenever $\mathbf{K}^\ell$ is not a solution of (6.7), the path reaches eventually a solution. The path can, therefore, be written as $(\mathbf{K}^0, \dots, \mathbf{K}^{\ell_0}, \mathbf{K}^{\ell_0}, \dots)$, where $\mathbf{K}^{\ell_0}$ solves (6.7) and $\bar{\nu}_{\Lambda_M,\mathbf{i}}(\mathbf{K}^m) < \bar{\nu}_{\Lambda_M,\mathbf{i}}(\mathbf{K}^n)$ for $0 \le m < n \le \ell_0$.

By the end of the proof of Lemma 6.3 below, there is a path from $\mathbf{K}^{\ell_0}$ to $\tilde{\mathbf{K}}$ of length less than or equal to $d/2$, passing only by solutions of (6.7), and such that all configurations visited have the same probability. Juxtaposing the two previous paths, we obtain the path $\Gamma(\mathbf{K}, \tilde{\mathbf{K}}) = (\mathbf{K} = \mathbf{K}^0, \dots, \mathbf{K}^{\ell_0}, \dots, \mathbf{K}^{\ell_\mathbf{K}} = \tilde{\mathbf{K}})$, where $\ell = \ell_\mathbf{K}$ stands for the total length of the path. By construction, the probability of the configurations visited is nondecreasing.

We are now ready to estimate the right-hand side of (6.5). By Schwarz inequality,

$$E_{\bar{\nu}_{\Lambda_M,\mathbf{i}}}\left[\left\{F(\mathbf{K}) - F(\tilde{\mathbf{K}})\right\}^2\right] \le \sum_{\mathbf{K}} \bar{\nu}_{\Lambda_M,\mathbf{i}}(\mathbf{K}) \ell_K \sum_{j=0}^{\ell_K - 1} \left\{F(\mathbf{K}_{j+1}) - F(\mathbf{K}_j)\right\}^2.$$

Since we just need a polynomial bound on the spectral gap and since this method can not provide a sharp estimate, we bound the length of a path ℓ_K by the total number of configurations $|\mathcal{H}| \le |\Lambda_M|^d$. On the other hand, since $\bar{\nu}_{\Lambda_M,\mathbf{i}}(\mathbf{K}) \le \bar{\nu}_{\Lambda_M,\mathbf{i}}(\mathbf{K}_j)$, we may replace the former by the latter. Finally, inverting the order of summations and estimating the total number of configurations which contains in its path to $\tilde{\mathbf{K}}$ a fixed couple $\mathbf{K}_j$, $\mathbf{K}_{j+1}$ by the total number of configurations, we get that the previous expression is less than or equal to

$$C_0 M^{2d^2} \sum_{\mathbf{K}} \sum_{\mathbf{L}\sim K} \bar{\nu}_{\Lambda_M,\mathbf{i}}(\mathbf{K}) \big\{F(\mathbf{L}) - F(\mathbf{K})\big\}^2$$

for some universal constant C_0. In this formula, the second sum is carried over all configurations $\mathbf{L}$ which can be obtained from $\mathbf{K}$ by letting a particle jump from a site to another: $\mathbf{L} = \mathbf{K} - \mathfrak{d}_j + \mathfrak{d}_k$ for some $j \neq k$. By the explicit formula for the Dirichlet form of F derived above, this expression is less than or equal to

$$C_0 M^{3d+2d^2} \langle F, -\mathcal{L}_M^c F\rangle_{\bar{\nu}_{\Lambda_M,\mathbf{i}}}.$$

It remains to apply Lemma 6.2 to conclude the proof of the spectral gap. □

We conclude this section with a result used in the proof of Proposition 6.1.

Lemma 6.3. *Fix* $\mathbf{i}$ *in* $\mathfrak{V}_M$ *such that* $I_1 \le \cdots \le I_d$, $|\Lambda_M|\mathbf{i} := (I_0, \ldots, I_d)$. *The system of equations*

$$\begin{cases} h_{I_k}(K_k - 1) \ge h_{I_j}(K_j) & \text{for all } 1 \le k, j \le d, \\ 2\sum_j K_j = I_0 - \sum_j I_j, & \\ 0 \le K_j \le |\Lambda_M|, & 1 \le j \le d, \end{cases} \tag{6.7}$$

has a solution such that $K_d \le \cdots \le K_1$. *Moreover, if* $\mathbf{K}$, $\mathbf{L}$ *are two solutions of* (6.7), *then* $|K_j - L_j| \le 1$ *for all* $1 \le j \le d$ *and* $\bar{\nu}_{\Lambda_M,\mathbf{i}}(\mathbf{K}) = \bar{\nu}_{\Lambda_M,\mathbf{i}}(\mathbf{L})$.

Proof. To prove the existence of a solution, recall from (6.4) that

$$\frac{\bar{\nu}(\mathbf{K} + \mathfrak{d}_j - \mathfrak{d}_k)}{\bar{\nu}(\mathbf{K})} \le 1 \quad \text{if and only if } h_{I_j}(K_j) \le h_{I_k}(K_k - 1). \tag{6.8}$$

Consider a configuration $\mathbf{K}^*$ which maximizes the probability $\bar{\nu}_{\Lambda_M,\mathbf{i}}$. The inequality on the left-hand side of the previous displayed formula is satisfied for all j, k. In particular, $\mathbf{K}^*$ solves (6.7).

Fix a solution of (6.7). We claim that $K_j \le K_i$ if $I_i < I_j$. Assume by contradiction that $K_i < K_j$. In this case

$$h_{I_i}(K_i) \le h_{I_j}(K_j - 1) < h_{I_i}(K_j - 1) \le h_{I_i}(K_i), \tag{6.9}$$

which is a contradiction. Here, the first inequality follows from the first property in (6.7) of $\mathbf{K}$, the second from the fact that $h_{I_j} < h_{I_i}$ and the last from the relation $K_i \le K_j - 1$.

Suppose that $I_i = I_j$ for some $i < j$ and that $K_i < K_j$ for a solution $\mathbf{K}$ of (6.7). Let $\tilde{\mathbf{K}}$ be such that $\tilde{\mathbf{K}}_k = \mathbf{K}_k$ for $k \neq i, j$; $\tilde{\mathbf{K}}_i = \mathbf{K}_j$, $\tilde{\mathbf{K}}_j = \mathbf{K}_i$. It is easy to check that $\tilde{\mathbf{K}}$ is also a solution of (6.7). This observation together with the estimate derived in the previous paragraph show that there exists a solution $\mathbf{K}$ of (6.7) with $K_d \le \cdots \le K_1$.

Finally, let $\mathbf{K}$, $\mathbf{L}$ be two solutions of (6.7). Suppose by contradiction that $L_j \le K_j - 2$ for some j. Since $\sum_j K_j = \sum_j L_j$, there exists i such that $K_i < L_i$. In particular,

$$h_{I_i}(L_i - 1) \le h_{I_i}(K_i) \le h_{I_j}(K_j - 1) < h_{I_j}(L_j) \le h_{I_i}(L_i - 1). \tag{6.10}$$

The first and third inequalities follow from the fact that h_{I_i}, h_{I_j} are strictly decreasing functions and the relations $K_i < L_i$, $L_j < K_j - 1$; while the second and fourth inequalities follow from the property of $\mathbf{K}$, $\mathbf{L}$. This proves the first property of $\mathbf{K}$, $\mathbf{L}$.

To prove the second property of $\mathbf{K}$, $\mathbf{L}$, consider a path from $\mathbf{K}$ to $\mathbf{L}$: $\mathbf{K} = \boldsymbol{M}^0, \dots, \boldsymbol{M}^\ell = \mathbf{L}$, for each $0 \le i < \ell$, $\boldsymbol{M}^{i+1} = \boldsymbol{M}^i + \mathfrak{d}_j - \mathfrak{d}_k$ for some $j = j(i)$, $k = k(i)$. It is not difficult to show that there exists such a path with $\ell \le d/2$, $\boldsymbol{M}^i$ solving (6.7) for all i.

Fix i and let $\boldsymbol{M}^i = \boldsymbol{M}$, $\boldsymbol{M}^{i+1} = \boldsymbol{M} + \mathfrak{d}_j - \mathfrak{d}_k$, $\boldsymbol{M}^i$, $\boldsymbol{M}^{i+1}$ solving (6.7). Since $\boldsymbol{M}$ solves (6.7), $h_{I_j}(M_j) \le h_{I_k}(M_k - 1)$. Using now that $\boldsymbol{M} + \mathfrak{d}_j - \mathfrak{d}_k$ solves (6.7), we obtain the reverse inequality so that $h_{I_j}(M_j) = h_{I_k}(M_k - 1)$. In particular, in view of (6.8), $\bar{\nu}_{\Lambda_M,\mathbf{i}}(\boldsymbol{M}^i) = \bar{\nu}_{\Lambda_M,\mathbf{i}}(\boldsymbol{M}^{i+1})$. This concludes the proof of the lemma. □

7. Equivalence of ensembles

We prove in this section the equivalence of ensembles for the stochastic lattice gas introduced in Section 2. Recall the definition of the set $\mathfrak{V}_L$ and of the canonical measures $\nu_{\Lambda_L,\mathbf{i}}$. Notice that for every $\boldsymbol{\lambda}$ in $\mathbb{R}^{d+1}$

$$\nu_{\Lambda_L,\mathbf{i}}(\cdot) = \mu_{\boldsymbol{\lambda}}^{\Lambda_L}\big(\cdot|\mathbf{I}^L = \mathbf{i}\big).$$

For $(\rho, \mathbf{p})$ in $\mathfrak{A}$ the expectation of the one site random variable $\mathbf{I}(\eta_x)$ under the product measure $\mu^{\Lambda_L}_{\Lambda(\rho,\mathbf{p})}$ is equal to $(\rho, \mathbf{p})$. It defines a map from $\mathfrak{A}$ to the set of probability measures on $(\{0,1\}^{\mathcal{V}})^{\Lambda_L}$. Since this map is uniformly continuous, it may be extended continuously to the closure of $\mathfrak{A}$. For each $\mathbf{i} \in \mathfrak{V}_L$ denote by $\mu^L_{\Lambda(\mathbf{i})}$ the corresponding product measure by this map. Hence, we have a one-to-one correspondence between the canonical measures $\{\nu_{\Lambda_L,\mathbf{i}}\colon \mathbf{i} \in \mathfrak{V}_L\}$ and the so-called grand canonical measures $\{\mu^N_{\boldsymbol{\Lambda}(\mathbf{i})}\colon \mathbf{i} \in \mathfrak{V}_L\}$.

Let $\langle g; f\rangle_\mu$ stand for the covariance of g, f with respect to μ: $\langle g; f\rangle_\mu = E_\mu[fg] - E_\mu[f]E_\mu[g]$ and $\langle f, g\rangle_\mu$ for the inner product in $L^2(\mu)$.

Proposition 7.1. *Fix a cube $\Lambda_\ell \subset \Lambda_L$. For each $\mathbf{i} \in \mathfrak{V}_L$ denote by ν^ℓ the projection of the canonical measure $\nu_{\Lambda_L,\mathbf{i}}$ on Λ_ℓ and by μ^ℓ the projection of the grand canonical measure $\mu^L_{\boldsymbol{\Lambda}(\mathbf{i})}$ on Λ_ℓ. Then there exists a finite constant $C(\ell, \mathcal{V})$, depending only on ℓ and $\mathcal{V}$, such that*

$$\big|E_{\mu^\ell}[f] - E_{\nu^\ell}[f]\big| \le \frac{C(\ell,\mathcal{V})}{|\Lambda_L|}\langle f; f\rangle^{1/2}_{\mu^\ell}$$

for every $f : (\{0,1\}^{\mathcal{V}})^{\Lambda_\ell} \mapsto \mathbb{R}$.

Proof. Since ν^ℓ is absolutely continuous with respect to μ^ℓ, by Schwarz inequality,

$$\begin{aligned}\big|E_{\nu^\ell}[f] - E_{\mu^\ell}[f]\big| &= \left|E_{\mu^\ell}\left[\left(\frac{\mathrm{d}\nu^\ell}{\mathrm{d}\mu^\ell}\right)\big(f - E_{\mu^\ell}[f]\big)\right]\right| \\ &\le \left\langle \frac{\mathrm{d}\nu^\ell}{\mathrm{d}\mu^\ell}; \frac{\mathrm{d}\nu^\ell}{\mathrm{d}\mu^\ell}\right\rangle^{1/2}_{\mu^\ell}\langle f; f\rangle^{1/2}_{\mu^\ell}.\end{aligned}$$

Since $\mu^L_{\boldsymbol{\Lambda}(\mathbf{i})}$ is a product measure, for any ξ in $(\{0,1\}^{\mathcal{V}})^{\Lambda_\ell}$,

$$\frac{\mathrm{d}\nu^\ell}{\mathrm{d}\mu^\ell}(\xi) = \frac{\mu^L_{\boldsymbol{\Lambda}(\mathbf{i})}[\sum_{x\in\Lambda_L\setminus\Lambda_\ell}\mathbf{I}(\eta_x) = |\Lambda_L|\mathbf{i} - |\Lambda_\ell|\mathbf{I}^\ell(\xi)]}{\mu^L_{\boldsymbol{\Lambda}(\mathbf{i})}[\sum_{x\in\Lambda_L}\mathbf{I}(\eta_x) = |\Lambda_L|\mathbf{i}]},$$

where $\mathbf{I}^\ell(\xi) = |\Lambda_\ell|^{-1}\sum_{x\in\Lambda_\ell}\mathbf{I}(\xi_x)$. Under $\mu^L_{\boldsymbol{\Lambda}(\mathbf{i})}$, $\mathbf{I}(\eta_x)$ are i.i.d. random variables taking a finite number of values. By Theorem VII.12 in [4], there exists a finite constant $C_0(\ell, \mathcal{V})$, depending only on ℓ and $\mathcal{V}$, such that,

$$\left|\frac{\mathrm{d}\nu^\ell}{\mathrm{d}\mu^\ell}(\xi) - 1\right| \le C_0(\ell,\mathcal{V})\frac{1}{|\Lambda_L|}$$

uniformly in ξ. This concludes the proof of the lemma. □

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