The Annals of Applied Probability
2008, Vol. 18, No. 5, 1944–1969
DOI: 10.1214/07-AAP510

GEODESICS IN FIRST PASSAGE PERCOLATION

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We consider a wide class of ergodic first passage percolation processes on $\mathbb{Z}^2$ and prove that there exist at least four one-sided geodesics a.s. We also show that coexistence is possible with positive probability in a four-color Richardson's growth model. This improves earlier results of Häggström and Pemantle [*J. Appl. Probab.* **35** (1995) 683–692], Garet and Marchand [*Ann. Appl. Probab.* **15** (2005) 298–330] and Hoffman [*Ann. Appl. Probab.* **15** (2005) 739–747] who proved that first passage percolation has at least two geodesics and that coexistence is possible in a two-color Richardson's growth model.

1. Introduction.

1.1. *First passage percolation.* First passage percolation is a process introduced by Hammersley and Welsh as a time-dependent model for the passage of a fluid through a porous medium which has provided a large number of problems of probabilistic interest with excellent physical motivation [11]. Study of this model led to the development of the ergodic theory of subadditive processes by Kingman [15]. It also has links to mathematical biology through Richardson's growth model [10]. A good overview of first passage percolation is contained in [14].

Let μ be a stationary measure on $[0,\infty)^{\mathrm{Edges}(\mathbb{Z}^d)}$ and let ω be a realization of μ. For any x and y we define $\tau(x,y)$, the *passage time from x to y*, by

$$\tau(x,y) = \inf \sum \omega(v_i, v_{i+1})$$

where the sum is taken over all of the edges in the path and the inf is taken over all paths connecting x to y. The time-minimizing path from x to y is called a *geodesic*.

An infinite path $v_1, v_2, \ldots$ is called a *geodesic* if for all $0 < i < j$

$$\tau(v_i, v_j) = \sum_{k=i}^{j-1} \omega(v_k, v_{k+1}).$$

In this paper we prove that for a very general class of first passage percolation processes, there exist at least four disjoint infinite geodesics a.s.

Received April 2006; revised December 2007.
AMS 2000 subject classifications. 60K35, 82B43.
Key words and phrases. First passage percolation, Richardson's growth model.

For notational reasons it will often be convenient to think of τ as a function defined on $\mathbb{R}^2 \times \mathbb{R}^2$ by setting

$$\tau(x+u, y+v) = \tau(x, y)$$

for any $x, y \in \mathbb{Z}^2$ and any $u, v \in [-\frac{1}{2}, \frac{1}{2})^2$. For any $x \in \mathbb{R}^2$ and $S \subset R^2$ we write

$$\tau(x, S) = \inf_{y \in S} \tau(x, y).$$

The most basic result from first passage percolation is the shape theorem. Define

$$R(t) = \{v : \tau(\mathbf{0}, v) \leq t\}.$$

The shape theorem says that there is a nonempty set R such that (modulo the boundary) $\frac{R(t)}{t}$ converges to R a.s.

THEOREM 1.1 [3]. *Let μ be stationary and ergodic, where the distribution on any edge has finite $d + \varepsilon$ moment for $\varepsilon > 0$. There exists a closed set R which is nonempty, convex and symmetric about reflection through the coordinate axis such that for every $\varepsilon > 0$*

$$P\left(\exists T : (1-\varepsilon)R \subset \frac{R(t)}{t} \subset (1+\varepsilon)R \text{ for all } t > T\right) = 1.$$

This theorem is an example of a subadditive ergodic theorem. In general, little is known about the shape of R other than it is convex and symmetric. Cox and Durrett have shown that there are nontrivial product measures such that the boundary of R contains a flat piece yet it is neither a square nor a diamond [6]. However, for any nonempty, convex and symmetric set R there exists a stationary measure μ such that the shape for μ is R [9].

Another widely studied aspect of first passage percolation is geodesics. We let $G(x, y)$ be the geodesic connecting x and y. Define

$$\Gamma(x) = \bigcup_{y \in \mathbb{Z}^d} \{e \in G(x, y)\}.$$

We refer to this as the *tree of infection of x*. We define $K(\Gamma(x))$ to be the number of topological ends in $\Gamma(x)$.

Newman has conjectured that for a large class of μ, $|K(\Gamma(\mathbf{0}))| = \infty$ a.s. [16]. Häggström and Pemantle proved that if $d = 2$, μ is i.i.d. and $\omega(e)$ has exponential distribution, then with positive probability $|K(\Gamma(\mathbf{0}))| > 1$. In independent work Garet and Marchand [7] and Hoffman [12] extended this result in two directions. Their results apply to a wide class of ergodic measures μ on any $d \geq 2$.

Newman has proved that if μ is i.i.d. and R has certain properties, then $|K(\Gamma(\mathbf{0}))| = \infty$ a.s. [16]. Although these conditions are plausible there are no known measures μ with S that satisfy these conditions. In this paper we prove an analogous theorem but with a much weaker condition on R. Unfortunately even this weaker condition, that ∂R is not a polygon, has not been verified for any version of i.i.d. first passage percolation.

Now we will introduce some more notation which will let us list the conditions that we place on μ for the rest of this paper. We say that μ has *unique passage times* for all x and $y \neq z$:

$$\mathbf{P}\big(\tau(x, y) \neq \tau(x, z)\big) = 1.$$

Now we are ready to define the class of measures that we will work with. We say that μ is *good* if:

1. μ is ergodic,
2. μ has all the symmetries of $\mathbb{Z}^d$,
3. μ has unique passage times,
4. the distribution of μ on any edge has finite $2 + \varepsilon$ moment for some $\varepsilon > 0$,
5. R is bounded.

Throughout the rest of the paper we will assume that μ is good. Unfortunately there is no general necessary and sufficient condition to determine when the shape R is bounded and therefore there is no general condition for μ to be good. See [9] for examples. However, if μ is i.i.d. and the distribution on any edge is continuous with finite $2 + \varepsilon$ moment, then μ is good. See Theorem 4.3 in [8] for more information about conditions that imply μ is good in the case that μ is stationary but not i.i.d.

1.2. *Spatial growth models.* Richardson's growth model, a simple competition model between diseases, was introduced by Häggström and Pemantle [10]. The rules for this model are as follows. Each vertex $z \in \mathbb{Z}^2$ at each time $t \geq 0$ is either infected by one of k diseases ($z_t \in \{1, \dots, k\}$) or is uninfected ($z_t = 0$). Initially for each disease there is one vertex which is infected by that disease. All other vertices are initially uninfected. Once a vertex is infected by one of the diseases it stays infected by that disease for all time and is not infected by any disease. All of the diseases spread from sites they have already infected to neighboring uninfected sites at some rate.

We now explain the relationship between first passage percolation and Richardson's growth models. For any $\omega \in [0, \infty)^{\mathrm{Edges}(\mathbb{Z}^d)}$ with unique passage times and any $x_1, \dots, x_k \in \mathbb{Z}^d$, we can project ω to $\tilde{\omega}_{x_1,\dots,x_k} \in (\{0, 1, \dots, k\}^{\mathbb{Z}^d})^{[0,\infty)}$ by

$$\tilde{\omega}_{x_1,\dots,x_k}(z, t) = \begin{cases} i, & \text{if } \tau(x_i, z) \leq t \text{ and } \tau(x_i, z) < \tau(x_j, z) \text{ for all } i \neq j, \\ 0, & \text{else.} \end{cases}$$

If μ has unique passage times, then μ projects onto a measure on $(\{0, 1, \dots, k\}^{\mathbb{Z}^d})^{[0,\infty)}$. It is clear that the models start with a single vertex in states 1 through k. Vertices in states $i > 0$ remain in their states forever, while vertices in state 0 which are adjacent to a vertex in state i can switch to state i. We think of the vertices in states $i > 0$ as infected with one of k infections while the vertices in state 0 are considered uninfected.

For this model it is most common to choose μ to be i.i.d. with an exponential distribution on each edge. This makes the spatial growth process Markovian.

As each $z \in \mathbb{Z}^d$ eventually changes to some state $i > 0$ and then stays in that state for the rest of time, we can define the limiting configuration

$$\tilde{\omega}_{x_1,\dots,x_k}(z) = \lim_{t\to\infty} \tilde{\omega}_{x_1,\dots,x_k}(z,t).$$

We say that *mutual unbounded growth* or *coexistence* occurs if the limiting configuration has infinitely many z in state i for all $i \le k$. More precisely we define $\mathrm{C}(x_1,\dots,x_k)$ to be the event that

$$|\{z : \tilde{\omega}_{x_1,\dots,x_k}(z) = 1\}| = \cdots = |\{\tilde{\omega}_{x_1,\dots,x_k}(z) = k\}| = \infty.$$

We refer to this event as *coexistence* or *mutual unbounded growth*.

1.3. *Results*. Our results depend on the geometry of R. Let $\mathrm{Sides}(\mu)$ be the number of sides of ∂R if ∂R is a polygon and infinity if ∂R is not a polygon. Note that by symmetry we have $\mathrm{Sides}(\mu) \ge 4$ for any good measure μ. Let $G(x_1,\dots,x_k)$ be the event that there exist disjoint geodesics g_i starting at x_i. In this paper we prove the following theorem about general first passage percolation.

THEOREM 1.2. *Let μ be good. For any $\varepsilon > 0$ and $k \le \mathrm{Sides}(\mu)$ there exists $x_1,\dots,x_k$ such that*

$$\mathbf{P}(G(x_1,\dots,x_k)) > 1 - \varepsilon.$$

We also get two closely related theorems. Let A be a finite subset of $\mathrm{Edges}(\mathbb{Z}^2)$. Let $\{(c_a,d_a)\}_{a\in A}$ be a collection of intervals with $0 < c_a < d_a$ for all $a \in A$. Let A' be the event that $\omega(a) \in (c_a,d_a)$ for all $a \in A$. Let B' be any event such that $P(B') > 0$ and B' does not depend on $\omega|_A$ (if $\omega \in B'$ and $\omega|_{A^c} = \omega'|_{A^c}$, then $\omega' \in B'$). We say that μ has *finite energy* if

$$\mathbf{P}(A' \cap B') > 0$$

for all such events A' and B'.

THEOREM 1.3. *If μ is good and has finite energy, then for any $k \le \mathrm{Sides}(\mu)$*

$$\mathbf{P}\big(|K(\Gamma(\mathbf{0}))| \ge k\big) > 0.$$

THEOREM 1.4. *Let μ be good. For any $k \le \mathrm{Sides}(\mu)/2$*

$$\mathbf{P}\big(|K(\Gamma(\mathbf{0}))| \ge k\big) = 1.$$

Theorem 1.3 extends a theorem of Häggström and Pemantle [10]. They proved that under the same hypothesis,

$$\mathbf{P}\big(|K(\Gamma(\mathbf{0}))| > 1\big) > 0.$$

Garet and Marchand [7] and Hoffman [12] extended the results of Häggström and Pemantle to a general class of first passage percolation processes in any dimension.

As an easy consequence of Theorem 1.4 we get

COROLLARY 1.5. *There exists a good measure μ such that*

$$\mathbf{P}\big(|K(\Gamma(\mathbf{0}))| = \infty\big) = 1.$$

PROOF. This follows easily from Theorem 1.4 and [9] where it is proven that there is a good measure μ such that R is the unit disk. $\square$

Our main result on a multiple-color Richardson's growth model is that with positive probability coexistence occurs.

THEOREM 1.6. *If μ is good and $k \leq \mathrm{Sides}(\mu)$, then for any $\varepsilon > 0$ there exist $x_1, \dots, x_k$ such that*

$$\mathbf{P}(C(x_1, \dots, x_k)) > 1 - \varepsilon.$$

Häggström and Pemantle [10] proved that if μ is i.i.d. with exponential distribution, then

$$\mathbf{P}\big(C((0,0),(0,1))\big) > 0.$$

Garet and Marchand [7] and Hoffman [12] proved that in any dimension mutual unbounded growth is possible when $k = 2$. Our result extends the previous results in two ways. First it shows that coexistence is possible with four colors. It also shows that the points $x_1, \dots, x_k$ can be chosen such that the probability of coexistence approaches 1. None of the three proofs that coexistence is possible in the two-color Richardson's growth model were able to show that the probability of coexistence went to 1 as the initial sites x_1 and x_2 moved farther apart.

COROLLARY 1.7. *There exists a nontrivial i.i.d. measure μ and $x_1, \dots, x_8$ such that*

$$\mathbf{P}(C(x_1, \dots, x_8)) > 0.$$

PROOF. By [4] there exists a μ which is i.i.d. such that ∂R is neither a square nor a diamond. As R is symmetric $\mathrm{Sides}(\mu) \geq 8$. Thus the result follows from Theorem 1.6. $\square$

2. Notation. Much of the notation that we introduce is related to the shape R. For $v \in \mathbb{R}^2 \setminus (0, 0)$ let

$$T^*(v) = \frac{1}{\sup\{k : kv \in R\}}.$$

It is not hard to check that T^* is a norm on $\mathbb{R}^2$ whose unit ball is R. Thus it is equivalent with Euclidean distance. It might be helpful to note that Theorem 1.1 implies that

$$\lim \frac{1}{n}\tau(\mathbf{0}, nv) = T^*(v) \qquad \text{a.s.} \tag{1}$$

Also we have that $T^*(\alpha v) = \alpha T^*(v)$ and $T^*(v) = 1$ for all $v \in \partial R$ and all $\alpha \in \mathbb{R}$. We use $\|v\| = \sqrt{v_1^2 + v_2^2}$ to represent the length of v.

Let the set V consist of all $v \in \partial R$ such that there is a unique line L_v which is tangent to R through v. For such a v let $w(v)$ be a unit vector parallel to L_v. Let $L_{n,v}$ be the line through nv in the direction of $w(v)$. We start with two lemmas about the set V.

LEMMA 2.1. *For any $k \le \mathrm{Sides}(\mu)$ there exist points $v_1, \dots, v_k \in V$ such that the lines L_{v_i} are distinct for all i.*

PROOF. If $\mathrm{Sides}(\mu) < \infty$, then ∂R is a polygon and the lemma is obvious. For v in the first quadrant define $w^+(v)$ to be the largest angle (measured counterclockwise) between the positive x-axis and a line through v that does not intersect the interior of R. (At least one such line exists by the convexity of R.) Define $w^-(v)$ to be the smallest angle (measured counterclockwise) between the positive x-axis and a line through v that does not intersect the interior of R.

As v rotates from being parallel to the positive x-axis to being parallel to the positive y-axis both w^+ and w^- are nondecreasing. Thus they are continuous almost everywhere. It is easy to check that ∂R has a unique tangent at v if and only if $w^+(v) = w^-(v)$. As R is convex there is a unique tangent line at almost every point in ∂R and the two functions are equal for almost every v.

If ∂R is not a polygon, then $w^+(v)$ takes on infinitely many values. For every $i \in \mathbb{N}$ choose v_i such that $w^+(v_{i+1}) > w^+(v_i)$ for all i. For each i choose v_i' such that v_i' is in the arc of ∂R from v_{2i} to v_{2i+1} and $w^+(v_i') = w^-(v_i')$. This is possible because the two functions are equal almost everywhere so there exists a point of equality on every arc of positive length. Thus at each v_i' there is a unique tangent line to ∂R. For any $i > j$ we have that

$$w^+(v_i') \ge w^+(v_{2i}) > w^+(v_{2j+1}) \ge w^+(v_j')$$

and the tangent lines at v_i' and v_j' are distinct. $\square$

LEMMA 2.2. *There is a unique line tangent to ∂R at the point v if and only if*

$$\lim_{b\to 0} \frac{T^*(v+w(v)b)-1}{|b|}=0. \tag{2}$$

PROOF. Fix $v \in \partial R$. For $v' \in \partial R$ and v' not parallel to $w(v)$ we can find a and b such that $v' = av + abw(v)$. Then we have

$$T^*\big(v+bw(v)\big) = T^*\left(\frac{1}{a}v'\right) = \frac{1}{a}T^*(v') = \frac{1}{a}.$$

As v' approaches v we have $a \to 1$ and $b \to 0$.

It is easy to check that ∂R having a unique tangent line at v is equivalent to

$$\lim_{v'\to v, v'\in\partial R} \frac{\|v+bw(v)-v'\|}{\|bw(v)\|}=0,$$

as

$$\frac{\|v+bw(v)-v'\|}{\|bw(v)\|} = \frac{(1/a-1)\|v'\|}{|b|\cdot\|w(v)\|}$$

having a unique tangent line is equivalent to

$$\lim_{v'\to v, v'\in\partial R} \frac{(1/a-1)\|v'\|}{|b|\cdot\|w(v)\|}=0.$$

Since $\|v'\| \to \|v\| \neq 0$ as $v' \to v$ having a unique tangent line is equivalent to

$$\lim_{v'\to v, v'\in\partial R} \frac{1/a-1}{|b|}=0,$$

we have that $b \to 0$ is equivalent to $v' \to v$ for $v' \in \partial R$. Thus ∂R having a unique tangent at v is equivalent to

$$\lim_{v'\to v, v'\in\partial R} \frac{1/a-1}{|b|} = \lim_{b\to 0}\frac{T^*(v+bw(v))-1}{|b|}=0. \qquad \square$$

Let $S \subset \mathbb{R}^2$. We define the function

$$B_S(x,y) = \inf_{z\in S}\tau(x,z) - \inf_{z\in S}\tau(y,z).$$

LEMMA 2.3. *For any set $S \subset \mathbb{Z}^2$ and any $x, y, z \in \mathbb{Z}^2$:*

1. $B_S(x,y) \le \tau(x,y)$, *and*
2. $B_S(x,y) + B_S(y,z) = B_S(x,z)$.

PROOF. These properties follow easily from the subadditivity of τ and the definition of B_S. $\square$

These functions are useful in analyzing the growth model because of the following fact.

LEMMA 2.4. *If there exist $c > 0$, and $x_1, \dots, x_k \in V$ such that*

$$\mathbf{P}\big(B_{L_{n,v_i}}(x_j, x_i) > 0\ \forall i \neq j\big) \geq 1 - c$$

for infinitely many n, then

$$\mathbf{P}(C(x_1, \dots, x_k)) \geq 1 - c.$$

PROOF. If for a fixed i and all $j \neq i$

$$B_{L_{n,v_i}}(x_j, x_i) > 0,$$

then there exists $z \in L_{n,v_i}$ such that $\tau(z, x_i) < \tau(z, x_j)$ for all $j \neq i$. Thus there is a $z \in L_{n,v_i}$ such that

$$\tilde{\omega}_{x_1,\dots,x_k}(z) = i.$$

For a fixed i each z is in only one L_{n,v_i}, so if there exist infinitely many n such that for all i and $j \neq i$

$$B_{L_{n,v_i}}(x_j, x_i) > 0,$$

then for every i there are infinitely many z such that

$$\tilde{\omega}_{x_1,\dots,x_k}(z) = i.$$

By assumption we have that there exist infinitely many n such that

$$\mathbf{P}\big(B_{L_{n,v_i}}(x_j, x_i) > 0 \text{ for all } i \text{ and } j \neq i\big) \geq 1 - c.$$

Thus we have that with probability at least $1 - c$ there exist infinitely many n such that for all i and $j \neq i$

$$B_{L_{n,v_i}}(x_j, x_i) > 0.$$

In conjunction with the previous paragraph this proves the lemma. $\square$

LEMMA 2.5.

$$\mathbf{P}(G(x_1, \dots, x_k)) \geq \mathbf{P}(C(x_1, \dots, x_k)).$$

PROOF. For any

$$y \in \{z : \tilde{\omega}_{x_1,\dots,x_k}(z) = i\}$$

the geodesic from x_i to y lies entirely in

$$\{z : \tilde{\omega}_{x_1,\dots,x_k}(z) = i\}.$$

By compactness, if

$$|\{z : \tilde{\omega}_{x_1,\dots,x_k}(z) = i\}| = \infty,$$

then there exists an infinite geodesic g_i which is contained in the vertices

$$\{z : \tilde{\omega}_{x_1,\dots,x_k}(z) = i\}. \qquad \square$$

LEMMA 2.6. *For all $v \in \mathbb{R}^2 \setminus \mathbf{0}$ and $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon, v) > 0$ and $M_0 = M_0(\varepsilon, v)$ such that for all $M > M_0$, all events E with $\mathbf{P}(E) < \delta$ and any $r \in \mathbb{R}^2$ we have that*

$$\mathbf{E}\big(\tau(r, r + Mv)\mathbf{1}_E\big) < M\varepsilon.$$

PROOF. Consider the space $\bar{\Omega} = [0,\infty)^{\mathrm{Edges}(\mathbb{Z}^d)} \times [0,1) \times [0,1)$ with measure $\bar{\mu}$ the direct product of μ with Lebesgue measure. We write $\bar{\mathbf{P}}$ and $\mathbf{E}_{\bar{\mu}}$ for probability and expectation with respect to $\bar{\mu}$. Any vector $v \in \mathbb{R}^2 \setminus 0$ acts on $\bar{\Omega}$ in the following manner.

For any $v \in \mathbb{R}^2 \setminus \mathbf{0}$ and $(\omega, a, b) \in \bar{\Omega}$ we have

$$\bar{\sigma}_v(\omega, a, b) = (\sigma_{v'}(\omega), c, d)$$

where $v + (a,b) = v' + (c,d)$, $v' \in \mathbb{Z}^2$ and $c, d \in [0,1) \times [0,1)$. For convenience we often write for $(a,b) \in \mathbb{R}^2$

$$(\omega, a, b) = (\sigma_{v'}(\omega), c, d),$$

where $(a,b) = v' + (c,d)$, $v' \in \mathbb{Z}^2$ and $c, d \in [0,1) \times [0,1)$. For any $a, b, c, d \in \mathbb{R}$ and ω we write

$$\tau((\omega,(a,b)),(\omega,(c,d))) = \tau^{\omega}((a,b),(c,d)).$$

We also define the function

$$f(\omega, a, b) = \tau\big((\omega,(a,b)),\big(\omega,(a,b)+v\big)\big).$$

Note that f is in L^1.

For any set E with $\mathbf{P}(E) < \delta$ we write $\bar{E} = E \times [0,1) \times [0,1)$ and we have $\bar{\mathbf{P}}(E) < \delta$. For any $M \in \mathbb{R}$ choose k such that $k \le M \le k+1$:

$$\begin{aligned}
&\frac{1}{M}\mathbf{E}\big(\tau(r, r+Mv)\big)\mathbf{1}_E \\
&\qquad \le \frac{1}{k}\bigg(\mathbf{E}\sup_{a,b\in[0,1)} \tau\big((a,b), kv + (a,b)\big)\mathbf{1}_E \\
&\qquad\qquad + \mathbf{E}\sup_{M\in[k,k+1)}\big(\tau(r+kv, r+Mv)\big)\bigg) \\
&\qquad \le \frac{1}{k}\bigg(\mathbf{E}_{\bar{\mu}}\sup_{a,b\in[0,1)} \tau\big((\omega,(a,b)),\big(\omega, kv+(a,b)\big)\big)\mathbf{1}_{\bar{E}} \\
&\qquad\qquad\qquad + \mathbf{E}\sup_{M\in[k,k+1)}\big(\tau(r+kv, r+Mv)\big)\bigg) \\
&\qquad \le \frac{1}{k}\bigg(\mathbf{E}_{\bar{\mu}}\tau\big((\omega,(a,b)),\big(\omega, kv+(a,b)\big)\big)\mathbf{1}_{\bar{E}}
\end{aligned}$$

$$
\begin{aligned}
&\qquad + \mathbf{E} \sup_{a,b,c,d \in [0,1)} \tau\big((a,b),(c,d)\big) \\
&\qquad + \mathbf{E} \sup_{a,b,c,d \in [0,1)} \tau\big(kv+(a,b), kv+(c,d)\big) \\
&\qquad\qquad + \mathbf{E} \sup_{M \in [k,k+1)} \big(\tau(r+kv, r+Mv)\big)\Big) \\
&\le \frac{1}{k}\Big(\mathbf{E}_{\bar{\mu}} \tau\big((\omega,(a,b)),\big(\omega, kv+(a,b)\big)\big)\mathbf{1}_{\bar{E}} \\
&\qquad + \mathbf{E} \sup_{a,b,c,d \in [0,1)} \tau\big((a,b),(c,d)\big) + \mathbf{E} \sup_{a,b \in [0,1)} \tau\big(kv, kv+(a,b)\big) \\
&\qquad\qquad\qquad + \mathbf{E} \sup_{M \in [k,k+1)} \big(\tau(r+kv, r+Mv)\big)\Big).
\end{aligned}
$$

The second, third and fourth terms in the last inequality are bounded independent of k. Thus their contribution to the right-hand side goes to zero as k goes to infinity. Then we have that

$$
\begin{aligned}
&\frac{1}{k}\tau\big((\omega,(a,b)),\big(\omega, kv+(a,b)\big)\big) \\
&\qquad \le \frac{1}{k}\sum_{0}^{k-1} \tau\big(\omega, jv+(a,b), \omega, (j+1)v+(a,b)\big) \qquad (3)\\
&\qquad \le \frac{1}{k}\sum_{0}^{k-1} f(\bar{\sigma}_v^j(\omega,a,b)).
\end{aligned}
$$

By the ergodic theorem the sum on the right-hand side of (3) is converging to an L^1 function almost everywhere and in L^1. Thus we can choose δ such that $\bar{\mathbf{P}}(\bar{E}) < \delta$ implies

$$
\frac{1}{k}\mathbf{E}_{\bar{\mu}} \tau\big((\omega,(a,b)),\big(\omega, kv+(a,b)\big)\big)\mathbf{1}_{\bar{E}} \le \mathbf{E}_{\bar{\mu}}\left(\frac{1}{k}\sum_{0}^{k-1} f(\bar{\sigma}_v^j(\omega,a,b))\mathbf{1}_{\bar{E}}\right) < \varepsilon.
$$

This proves the lemma. $\square$

We use this lemma in two contexts.

COROLLARY 2.7. *For all $v \in \mathbb{R}^2 \setminus \mathbf{0}$, $r \in \mathbb{R}^2$, $\varepsilon > 0$ and $M \in \mathbb{R}$ let $E = E(M, v, r, \varepsilon)$ be the event that*

$$
\tau(r, r+Mv) > (1+\varepsilon/2)M.
$$

There exists $M_0 = M_0(v, r, \varepsilon)$ such that for all $M > M_0$

$$
\mathbf{E}\big(\tau(r, r+Mv)\mathbf{1}_E\big) \le M\varepsilon.
$$

PROOF. Fix v, r, m and ε. By Theorem 1.1 we have that $\mathbf{P}(E) \to 0$ as $M \to \infty$. Thus we can apply Lemma 2.6 to prove the corollary. □

COROLLARY 2.8. *For any $v \in V$, $r \in \mathbb{R}^2$, $m \in \mathbb{R}$ and $\varepsilon > 0$ there exists $M_0 = M_0(v, r, m, \varepsilon)$ such that for all $M > M_0$*

$$\mathbf{E}\big(\tau(r - mv, r + Mv)\big) \le M(1 + \varepsilon).$$

PROOF. Fix v, r, m and ε. Let $E = E(M)$ be the event that $\tau(r - mv, r + Mv) > M(1 + \varepsilon/2)$. By Theorem 1.1 we have that $\mathbf{P}(E) \to 0$ as $M \to \infty$. Thus we can apply Lemma 2.6 to prove

$$\mathbf{E}\big(\tau(r - mv, r + Mv)\mathbf{1}_E\big) \le M(\varepsilon/2).$$

By the definition of E

$$\mathbf{E}\big(\tau(r - mv, r + Mv)\mathbf{1}_{E^c}\big) \le M(1 + \varepsilon/2).$$

Putting those two together proves the corollary. □

3. Outline. We start by outlining a possible method to prove that there are infinitely many geodesics starting at the origin. Then we show the portion of this plan that we cannot prove. Finally we show how to adapt this method to get the results in this paper.

It is easy to construct geodesics beginning at $\mathbf{0}$. We can take any sequence $W_1, W_2, \ldots$ of disjoint subsets of $\mathbb{Z}^2$ and consider $G(\mathbf{0}, W_n)$, the geodesic from $\mathbf{0}$ to W_n. [The finite geodesic $G(0, W_n)$ is well defined a.s. because the measure μ is good so R is bounded. Then Theorem 1.1 implies the existence of the finite geodesic.] Using compactness it is easy to show that there exists a subsequence n_k such that $G(\mathbf{0}, W_{n_k})$ converges to an infinite geodesic.

If we take two sequences of sets W_n and W_n' we can construct a geodesic for each sequence. It is difficult to determine whether or not the two sequences produce the same or different geodesics. The tool that we use to distinguish the geodesics is Busemann functions. Every geodesic generates a Busemann function as follows.

For any $x, y \in \mathbb{Z}^2$ and infinite geodesic $G = (v_0, v_1, v_2, \ldots)$ we can define

$$\hat{B}_G^{\omega}(x, y) = \hat{B}_G(x, y) = \lim_{n\to\infty} \tau(x, v_n) - \tau(y, v_n).$$

To see the limit exists first note that

$$\begin{aligned}\hat{B}_G(x, y) &= \lim_{n\to\infty} \tau(x, v_n) - \tau(y, v_n)\\ &= \lim_{n\to\infty} \tau(x, v_n) - \tau(v_0, v_n) + \tau(v_0, v_n) - \tau(y, v_n)\\ &= \lim_{n\to\infty} \big(\tau(x, v_n) - \tau(v_0, v_n)\big) + \lim_{n\to\infty} \big(\tau(v_0, v_n) - \tau(y, v_n)\big).\end{aligned}$$

As G is a geodesic the two sequences in the right-hand side of the last line are bounded and monotonic so they converge. Thus $\hat{B}_G(x, y)$ is well defined.

Two distinct geodesics may generate the same Busemann function but distinct Busemann functions mean that there exist distinct geodesics.

To construct a geodesic we pick $(a, b) \in \mathbb{Z}^2$ and we set W_n to be

$$W_n = \{w \in \mathbb{Z}^2 : w \cdot (a, b) \geq n\}.$$

If we could show for every $z \in \mathbb{Z}^2$ that $G_n(z)$, the geodesic from z to W_n, converges, then it would be possible to show that

$$\lim_{M\to\infty} \frac{1}{M} B\big(\mathbf{0}, (bM, -aM)\big) = 0 \tag{4}$$

and

$$\lim_{M\to\infty} \frac{1}{M} B\big(\mathbf{0}, (aM, -bM)\big) = \inf_{v\cdot(a,b)=a^2+b^2} T^*(v). \tag{5}$$

Thus for any (a', b') which is not a scalar multiple of (a, b) we would be able to show that

$$\lim_{M\to\infty} \frac{1}{M} B\big(\mathbf{0}, (b'M, -a'M)\big) \neq 0. \tag{6}$$

Thus for any (a, b) and (a', b') which are not scalar multiples we get distinct geodesics. In this way it would be possible to construct an infinite sequence of distinct geodesics. We are unable to show that the geodesics $G(z, W_n)$ converge. But for some $(a, b) \in \mathbb{Z}^2$ we can establish versions of (4) and (5). These are Lemmas 4.5 and 4.4. These lemmas form the heart of our proof.

4. Proofs. Although it is convenient to write $\tau(x, y)$ for $x, y \in \mathbb{R}^2$, the distribution of $\tau(x, y)$ is equal to the distribution of $\tau(x + z, y + z)$ only if $z \in \mathbb{Z}^2$. For $z \notin \mathbb{Z}^2$ the distribution of $\tau(x, y)$ may not be equal to the distribution of $\tau(x + z, y + z)$ which will make the notation more complicated. But the distributions are close enough so that this lack of shift invariance for noninteger translations will not cause any significant problems. To deal with this lack of translation invariance we let

$$I(a, b) = I_v(a, b) = \sup_{x\in L_{a,v}} \big(\mathbf{E}(\tau(x, L_{b,v}))\big).$$

In the next three lemmas we show

$$\sup_{x\in L_{a,v}} |I(a, b) - \mathbf{E}(\tau(x, L_{b,v}))|$$

is bounded uniformly in a, b and v.

LEMMA 4.1. *For any $v \in V$ and $u \in \mathbb{R}^2 \setminus 0$ such that $u = \alpha v + \gamma w(v)$,*

$$\frac{\alpha \|v\|}{\|u\|} < \sqrt{2}.$$

PROOF. First we show that for any $v \in V$ that lies in the first octant (between the lines $y = 0$ and $y = x$ with $x > 0$), $w(v)$ then points in one of the octants between the lines $x = 0$ and $x = -y$.

Let $\tilde{v}$ be the image of v under reflection about the line $x = y$ and let v^* be the image of v under reflection about the line $x = 0$. The line from v to $\tilde{v}$ is parallel to the line $x = -y$ while the line from v to v^* is parallel to the line $x = 0$.

By the convexity of R we have that for any two points in R and any line tangent to ∂R the two points lie on the same side of the line (or in one closed halfplane). Thus $\mathbf{0}$, v^* and $\tilde{v}$ all lie on the same side of $L_{1,v}$. This implies $L_{1,v}$ does not intersect the interior of the line segment between v^* and $\tilde{v}$ and $w(v)$ points in the octants between the lines $x = 0$ and $x = -y$.

Then

$$|v \cdot w(v)| \leq \frac{\sqrt{2}}{2} \|v\| \cdot \|w(v)\|$$

and the angle between v and $w(v)$ is at least $45°$. By the symmetry of R this inequality holds for all $v \in V$. For a fixed α and v the value of γ which minimizes $\|u\|$ occurs when the points $\mathbf{0}$, αv and u form a right triangle. As the angle between v and $w(v)$ is at least $45°$ we have that

$$\frac{\alpha \|v\|}{\|u\|} < \sqrt{2}.$$ □

LEMMA 4.2. *Let $v \in V$, $x_1, x_3 \in \mathbb{R}^2$ and $n_1, n_2, n_3, n_4 \in \mathbb{R}$ with $x_1 \in L_{n_1,v}$, $x_3 \in L_{n_3,v}$, $n_1 < n_2$, $n_3 < n_4$ and*

$$(n_2 - n_1) - (n_4 - n_3) \geq 2/\|v\|.$$

Then

$$\mathbf{E}(\tau(x_1, L_{n_2,v})) \geq \mathbf{E}(\tau(x_3, L_{n_4,v})).$$

This implies that for any $m > 2/\|v\|$ and any $r \in L_{n_1,v}$,

$$\mathbf{E}\big(\tau(r - mv, L_{n_2,v})\big) \geq I(n_1, n_2). \tag{7}$$

PROOF. First we define $\tilde{x}_1$ and $\tilde{x}_3$ to be the points in $\mathbb{Z}^2$ closest to x_1 and x_3, respectively (i.e., $x_1 \in \tilde{x}_1 + [-1/2, 1/2)^2$ and $x_3 \in \tilde{x}_3 + [-1/2, 1/2)^2$).

Next define $\alpha_1, \alpha_3, \gamma_1, \gamma_3 \in \mathbb{R}$ such that

$$\tilde{x}_1 - x_1 = \alpha_1 v + \gamma_1 w(v) \quad \text{and} \quad \tilde{x}_3 - x_3 = \alpha_3 v + \gamma_3 w(v).$$

By the definition of the $\tilde{x}_i$ we have that $\|x_i - \tilde{x}_i\| \le \sqrt{2}/2$. Thus by Lemma 4.1 we have that $|\alpha_1|, |\alpha_3| \le 1/\|v\|$.

Then define $\tilde{\alpha}$ and $\tilde{\gamma}$ such that

$$\begin{aligned}\tilde{\alpha}v + \tilde{\gamma}w(v) &= \tilde{x}_3 - \tilde{x}_1\\ &= x_3 - x_1 + (\alpha_3 - \alpha_1)v + (\gamma_3 - \gamma_1)w(v)\\ &= (n_3 - n_1 + \alpha_3 - \alpha_1)v + Cw(v)\end{aligned}$$

for some $C \in \mathbb{R}$. Also

$$|\tilde{\alpha} - (n_3 - n_1)| = |\alpha_3 - \alpha_1| \le 2/\|v\|$$

or

$$(n_3 - n_1) - 2/\|v\| \le \tilde{\alpha}.$$

Then

$$L_{n_2,v} + (\tilde{x}_3 - \tilde{x}_1) = L_{n_2+\tilde{\alpha},v}.$$

Also

$$\mathbf{E}(\tau(x_1, L_{n_2,v})) = \mathbf{E}(\tau(\tilde{x}_1, L_{n_2,v})) = \mathbf{E}(\tau(\tilde{x}_3, L_{n_2+\tilde{\alpha},v})) = \mathbf{E}(\tau(x_3, L_{n_2+\tilde{\alpha},v})).$$

The first and third inequalities are due to the definition of $\tilde{x}_1$ and $\tilde{x}_3$, respectively, while the second is due to the shift invariance of the distribution under shifts in $\mathbb{Z}^2$. (The image of $L_{n_2,v}$ under translation by $\tilde{x}_3 - \tilde{x}_1$ is $L_{n_2+\tilde{\alpha},v}$.) Thus

$$\mathbf{E}(\tau(x_1, L_{n_2,v})) = \mathbf{E}(\tau(x_3, L_{n_2+\tilde{\alpha},v})) \ge \mathbf{E}(\tau(x_3, L_{n_4,v})) \tag{8}$$

if and only if $n_4 \le n_2 + \tilde{\alpha}$. As

$$\begin{aligned}2/\|v\| &\le (n_2 - n_1) - (n_4 - n_3),\\ 0 &\le (n_2 - n_4) + (n_3 - n_1) - 2/\|v\|,\\ 0 &\le (n_2 - n_4) + \tilde{\alpha},\\ n_4 &\le n_2 + \tilde{\alpha},\end{aligned}$$

thus by (8)

$$\mathbf{E}(\tau(x_1, L_{n_2,v})) \ge \mathbf{E}(\tau(x_3, L_{n_4,v})) \tag{9}$$

and the first part of the lemma is true.

For the second statement, for any $n_3 \in \mathbb{R}$ take any r and w in $L_{n_1,v}$. Apply (9) with $x_1 = w$, $x_3 = r - mv$, $n_3 = n_1 - \alpha$ and $n_2 = n_4$ to get

$$\mathbf{E}\big(\tau(r - mv, L_{n_2,v})\big) = \mathbf{E}\big(\tau(r - mv, L_{n_4,v})\big) \ge \mathbf{E}(\tau(w, L_{n_2,v})).$$

As this holds for all $w \in L_{n_1,v}$ we have

$$\mathbf{E}\big(\tau(r - mv, L_{n_2,v})\big) \ge \sup_{w \in L_{n_1,v}} \mathbf{E}(\tau(w, L_{n_2,v})) = I(n_1, n_2).$$

□

LEMMA 4.3. *There exists $\beta \in \mathbb{R}$ such that for all $v \in V$ and $n_1, n_2 \in \mathbb{R}$ with $n_2 - n_1 > 2/\|v\|$ and for all $w, y \in L_{n_1,v}$,*

$$|\mathbf{E}(\tau(w, L_{n_2,v})) - \mathbf{E}(\tau(y, L_{n_2,v}))| < \beta.$$

We also have that for any $n_1, n_2, \alpha \in \mathbb{R}$,

$$|I(n_1, n_2) - I(n_1 + \alpha, n_2 + \alpha)| < \beta. \tag{10}$$

PROOF. Pick β such that for all x, z with $\|x - z\| = 2$ we have $\mathbf{E}(\tau(x, z)) < \beta$. Define x and z by $x = y - 2v/\|v\|$ and $z = y + 2v/\|v\|$. By Lemma 4.2 we have

$$\mathbf{E}(\tau(z, L_{n_2,v})) \le \mathbf{E}(\tau(y, L_{n_2,v})), \mathbf{E}(\tau(w, L_{n_2,v})) \le \mathbf{E}(\tau(x, L_{n_2,v})).$$

Thus

$$\begin{aligned} 0 &\le |\mathbf{E}(\tau(y, L_{n_2,v})) - \mathbf{E}(\tau(w, L_{n_2,v}))| \\ &\le \mathbf{E}(\tau(x, L_{n_2,v})) - \mathbf{E}(\tau(z, L_{n_2,v})) \le \mathbf{E}(\tau(x, z)) < \beta. \end{aligned}$$

For the second part choose $z \in L_{n_1+\alpha,v}$ and $r \in L_{n_1,v}$. Also choose $m > 2/\|v\|$ such that $\mathbf{E}(\tau(r - mv, r)) < \beta$. There exist $\tilde{r} \in \mathbb{Z}^2$ and $r - mv \in \tilde{r} + [1/2, 1/2)^2$. Let $\tilde{n}_1$ be such that $\tilde{r} \in L_{\tilde{n}_1,v}$. There exist $\tilde{z} \in \mathbb{Z}^2$ and $z \in \tilde{z} + [1/2, 1/2)^2$. Let $\hat{n}_1$ be such that $\tilde{r} \in L_{\hat{n}_1,v}$. Then we have $n_2 - \tilde{n}_1 > n_2 + \alpha - \hat{n}_1$. This implies

$$\mathbf{E}\big(\tau(r - mv, L_{n_2,v})\big) = \mathbf{E}(\tau(\tilde{r}, L_{n_2,v})) > \mathbf{E}(\tau(\tilde{z}, L_{n_2+\alpha,v})) = \mathbf{E}(\tau(z, L_{n_2+\alpha,v})).$$

As this holds for all $z \in L_{n_1+\alpha,v}$ we have

$$\mathbf{E}(\tau(r, L_{n_2,v})) \ge I(n_1 + \alpha, n_2 + \alpha). \tag{11}$$

As $\mathbf{E}(\tau(r - mv, r)) < \beta$ we also have

$$I(n_1, n_2) + \beta \ge \mathbf{E}(\tau(r, L_{n_2,v})) + \beta > \mathbf{E}\big(\tau(r - mv, L_{n_2,v})\big). \tag{12}$$

Thus combining (11) and (12),

$$I(n_1, n_2) + \beta > \mathbf{E}\big(\tau(r - mv, L_{n_2,v})\big) \ge I(n_1 + \alpha, n_2 + \alpha).$$

An analogous argument gives

$$I(n_1 + \alpha, n_2 + \alpha) + \beta > I(n_1, n_2)$$

which completes the proof. □

Now we show that for a typical choice of $n, M \in \mathbb{R}$, $v \in V$ and $r \in \mathbb{R}^2$ we have that $B_{L_{n,v}}(r, r + Mv)$ is close to $\tau(r, r + Mv)$ (which is close to M because $v \in \partial R$). This (along with Lemma 4.5) is one of two key steps in showing that for distinct $v, v' \in V$ we will get distinct Busemann functions.

We define the lower density of $A \subset \mathbb{N}$ to be

$$\underline{\text{density}(A)} = \liminf_{N \to \infty} \frac{1}{N} |A \cap [1, 2, \ldots, N]|.$$

Similarly we define

$$\overline{\text{density}(A)} = \limsup_{N\to\infty} \frac{1}{N} |A \cap [1, 2, \dots, N]|.$$

We will assume the reader is familiar with all of the normal properties of density of sets, for example,

$$\underline{\text{density}(A)} + \overline{\text{density}(A^c)} = 1$$

and

$$\overline{\text{density}(A \cup B)} \le \overline{\text{density}(A)} + \overline{\text{density}(B)}.$$

We often shorten lower density to density as it will not cause confusion.

LEMMA 4.4. *For any $v \in V$, any $\varepsilon > 0$, there exists $M_0 = M_0(\varepsilon, v)$ such that for all $M > M_0$ and all $r \in \mathbb{R}^2$ the density of n such that*

$$\mathbf{P}\big(M(1-\varepsilon) < B_{L_{n,v}}(r, r+Mv) < M(1+\varepsilon)\big) > 1-\varepsilon \tag{13}$$

is at least $1-\varepsilon$.

PROOF. By Lemma 2.3 for any r, n, M and v,

$$B_{L_{n,v}}(r, r+Mv) \le \tau(r, r+Mv) \tag{14}$$

and by Theorem 1.1 for any r, v and sufficiently large M,

$$\mathbf{P}\big(\tau(r, r+Mv) < M(1+\varepsilon)\big) > 1-\varepsilon. \tag{15}$$

Thus for sufficiently large M the upper bound on $B_{L_{n,v}}(r, r+Mv)$ is satisfied for all n with probability at least $1-\varepsilon$.

Now we bound the probability that $B_{L_{n,v}}(r, r+Mv)$ is too small. Let d be such that $r \in L_{d,v}$ and let $m \in \mathbb{R}$ be such that $m\|v\| > 2$. For any sufficiently large M and any $n \ge d + M$,

$$\begin{aligned}
&I(d,n) - I(d+M,n) \\
&\quad \le \mathbf{E}\Big(\inf_{y\in L_{n,v}} \tau(r-mv, y)\Big) - \sup_{x\in L_{d+M,v}} \Big(\mathbf{E}\Big(\inf_{y\in L_{n,v}} \tau(x,y)\Big)\Big) && (16)\\
&\quad \le \mathbf{E}\Big(\inf_{y\in L_{n,v}} \tau(r-mv, y)\Big) - \mathbf{E}\Big(\inf_{y\in L_{n,v}} \tau(r+Mv, y)\Big) \\
&\quad \le \mathbf{E}\Big(\inf_{y\in L_{n,v}} \tau(r-mv, y) - \inf_{y\in L_{n,v}} \tau(r+Mv, y)\Big) \\
&\quad \le \mathbf{E}\big(B_{L_{n,v}}(r-mv, r+Mv)\big) && (17)\\
&\quad \le \mathbf{E}\big(\tau(r-mv, r+Mv)\big) && (18)\\
&\quad \le M(1+\varepsilon). && (19)
\end{aligned}$$

Equation (16) follows from (7) in Lemma 4.2 and the definition of $I(d+M,n)$, (17) follows from the definition of $B_{L_{n,v}}$, (18) follows from Lemma 2.3 and (19) follows from Corollary 2.8.

Let k be such that

$$d+kM \le n < d+(k+1)M.$$

For k and M by Theorem 1.1 and (10),

$$\begin{aligned}
(k+1)M(1-\varepsilon) &\le I(d,n)\\
&\le I(d,n)+\left(\sum_{l=1}^{k} -I(d+lM,n)+I(d+lM,n)\right)\\
&\le \left(\sum_{l=0}^{k-1} I(d+lM,n)-I\big(d+(l+1)M,n\big)\right)\\
&\quad + I(d+kM,n)\\
&\le \left(\sum_{l=0}^{k-1} I(d,n-lM)-I(d+M,n-lM)+2\beta\right)\\
&\quad + I(d,n-kM),
\end{aligned}\tag{20}$$

$$\begin{aligned}
(k+1)M(1-2\varepsilon) &\le \left(\sum_{l=0}^{k-1} I(d,n-lM)-I(d+M,n-lM)\right)\\
&\quad + I(d,n-kM)\\
&\le \left(\sum_{l=0}^{k-1} I(d,n-lM)-I(d+M,n-lM)\right)\\
&\quad + I(d,d+M).
\end{aligned}\tag{21}$$

By (19) the sum in the right-hand side of (21) is the sum of $k+1$ terms bounded above by $M(1+\varepsilon)$. Thus the number of $l<k$ such that

$$I(d,n-lM)-I(d+M,n-lM) > M\big(1-\sqrt{\varepsilon}\big)$$

is at least $k(1-4\sqrt{\varepsilon})$. The above result holds for all $\varepsilon>0$ and all $M=M(\varepsilon)$ sufficiently large. Thus we get that for any $\varepsilon>0$ and any $M\in\mathbb{N}$ sufficiently large and any $j\in[0,1,2,\dots,M-1]$ the density of n such that

$$I(d,j+Mn)-I(d+M,j+Mn) > M(1-\varepsilon)\tag{22}$$

is at least $1-\varepsilon$. Combining this result for all $j\in[0,1,2,\dots,M-1]$ we get that for any $\varepsilon>0$ and any $M\in\mathbb{N}$ sufficiently large the density of n such that

$$I(d,n)-I(d+M,n) > M(1-\varepsilon)\tag{23}$$

is at least $1-\varepsilon$.

Now we show that for any M, n, r and v such that (23) is satisfied we have that with high probability $B_{L_{n,v}}(r, r+Mv)$ is large. Let E be the event that

$$B_{L_{n,v}}(r, r+Mv) > M(1+\varepsilon/2).$$

By Lemma 2.3 and Theorem 1.1 we can make $\mathbf{P}(E)$ arbitrarily small by making M sufficiently large. Then we get

$$\begin{aligned}
&\mathbf{E}\big(B_{L_{n,v}}(r, r+Mv)\big)\\
&\quad = \mathbf{E}\Big(\inf_{z\in L_{n,v}} \tau(r,z)\Big) - \mathbf{E}\Big(\inf_{z\in L_{n,v}} \tau(r+Mv,z)\Big)\\
&\quad > I(d,n) - \beta - I(d+M,n) \qquad (24)\\
&\quad > M(1-\varepsilon) - \beta \qquad (25)\\
&\quad > M(1-2\varepsilon), \qquad (26)\\
&\mathbf{E}\big(B_{L_{n,v}}(r, r+Mv)\mathbf{1}_E\big) + \mathbf{E}\big(B_{L_{n,v}}(r, r+Mv)\mathbf{1}_{E^C}\big)\\
&\quad > M(1-2\varepsilon), \qquad (27)\\
&\mathbf{E}\big(B_{L_{n,v}}(r, r+Mv)\mathbf{1}_{E^C}\big)\\
&\quad > M(1-2\varepsilon) - \mathbf{E}\big(B_{L_{n,v}}(r, r+Mv)\mathbf{1}_E\big)\\
&\quad > M(1-2\varepsilon) - \mathbf{E}\big(\tau(r, r+Mv)\mathbf{1}_E\big) \qquad (28)\\
&\quad > M(1-3\varepsilon). \qquad (29)
\end{aligned}$$

Equation (24) follows from Lemma 4.3. Equation (25) follows from (22). Equation (26) holds for large M. Equation (28) follows from Lemma 2.3 and Theorem 1.1. Equation (29) is due to Corollary 2.7.

As the expected value of the function

$$B_{L_{n,v}}(r, r+Mv)\mathbf{1}_{E^C}$$

is close to its maximum, $M(1+\varepsilon)$, we get that with high probability the function is close to its maximum. Thus we get that for any $\varepsilon > 0$ (possibly larger than the previous ε but still arbitrarily small) and all sufficiently large M, the set of n such that

$$\mathbf{P}\big(M(1-\varepsilon) < B_{L_{n,v}}(r, r+Mv)\big) > 1-\varepsilon \qquad (30)$$

has density at least $1-\varepsilon$. Putting together (14), (15) and (30) proves the lemma. □

LEMMA 4.5. *For any $v \in V$, $\varepsilon > 0$, there exists $M_0 = M_0(\varepsilon, v)$ such that for any $M \in \mathbb{R}$ with $|M| > M_0$ and any $r \in \mathbb{R}^2$ the density of n such that*

$$\mathbf{P}\big(\big|B_{L_{n,v}}(r, r+Mw(v))\big| < \varepsilon|M|\big) > 1-\varepsilon$$

is at least $1-\varepsilon$.

PROOF. First we prove the upper bound in the case that M is positive. Fix $\varepsilon > 0$. Since R has a unique tangent line at v by (2) we can find $b > 0$ such that

$$T^*\bigl(v + bw(v)\bigr)(1 + \varepsilon b) < (1 + 2\varepsilon b). \tag{31}$$

By Lemma 4.4 for any $\varepsilon, b > 0$ and all sufficiently large M the density of n such that

$$\mathbf{P}\bigl(B_{L_{n,v}}(r - Mv, r) \geq M(1 - \varepsilon b)\bigr) > 1 - \varepsilon \tag{32}$$

is at least $1 - \varepsilon$. By Theorem 1.1 for any $\varepsilon, b > 0$ and all sufficiently large M

$$\mathbf{P}\bigl(\tau\bigl(r - Mv, r + Mbw(v)\bigr) \leq MT^*\bigl(v + bw(v)\bigr)(1 + \varepsilon b)\bigr) > 1 - \varepsilon. \tag{33}$$

Choose M large enough such that both (32) and (33) are satisfied.

Thus with probability at least $1 - 2\varepsilon$ the density of n such that the following inequalities are satisfied is at least $1 - \varepsilon$:

$$\begin{aligned}
\tau\bigl(r - Mv, r + Mbw(v)\bigr) &\geq B_{L_{n,v}}\bigl(r - Mv, \\
&\qquad\qquad r + Mbw(v)\bigr) \\
&\geq B_{L_{n,v}}(r - Mv, r) \\
&\qquad + B_{L_{n,v}}\bigl(r, r + bMw(v)\bigr), \\
\tau\bigl(r - Mv, r + Mbw(v)\bigr) - B_{L_{n,v}}(r - Mv, r) &\geq B_{L_{n,v}}\bigl(r, r + bMw(v)\bigr), \\
MT^*\bigl(v + bw(v)\bigr)(1 + \varepsilon b) - M(1 - \varepsilon b) &\geq B_{L_{n,v}}\bigl(r, r + bMw(v)\bigr), \qquad (34) \\
M(1 + 2\varepsilon b) - M(1 - \varepsilon b) &\geq B_{L_{n,v}}\bigl(r, r + bMw(v)\bigr), \qquad (35) \\
3bM\varepsilon &\geq B_{L_{n,v}}\bigl(r, r + bMw(v)\bigr).
\end{aligned}$$

The first two lines follow deterministically from Lemma 2.3. Equation (34) is true with probability at least $1 - 2\varepsilon$. This follows from (33) and (32). Equation (35) follows from (31). Thus we have that for any sufficiently large M the density of n such that

$$\mathbf{P}\bigl(B_{L_{n,v}}\bigl(r, r + bMw(v)\bigr) \leq 3bM\varepsilon\bigr) > 1 - 2\varepsilon$$

is at least $1 - \varepsilon$. By replacing $w(v)$ with $-w(v)$ and interchanging r and $r + bMw(v)$ we get that for any sufficiently large M the density of n such that

$$\mathbf{P}\bigl(B_{L_{n,v}}\bigl(r, r + bMw(v)\bigr) \geq -3bM\varepsilon\bigr) > 1 - 2\varepsilon$$

is at least $1 - \varepsilon$. The case that M is negative follows in the same manner by replacing $w(v)$ with $-w(v)$. As ε was arbitrary the lemma follows. $\square$

LEMMA 4.6. *Let $v \in V$. For all $y \in \mathbb{R}^2$ let $s = s(v, y)$ and $t = t(v, y)$ be such that*

$$v + sw(v) = y + tv. \tag{36}$$

Then for all $\varepsilon > 0$ and all sufficiently large M the density of n with

$$\mathbf{P}\big(B_{L_{n,v}}(My, Mv) > M(t-\varepsilon)\big) > 1-\varepsilon \tag{37}$$

is at least $1-\varepsilon$.

PROOF. Fix $\varepsilon > 0$. If

$$B_{L_{n,v}}\big(My, M(y+tv)\big) > M(t-\varepsilon) \tag{38}$$

and

$$B_{L_{n,v}}\big(M\big(v+sw(v)\big), Mv\big) > -\varepsilon M, \tag{39}$$

then

$$\begin{aligned} B_{L_{n,v}}(My, Mv) &= B_{L_{n,v}}\big(My, M\big(v+sw(v)\big)\big) + B_{L_{n,v}}\big(M\big(v+sw(v)\big), Mv\big) \\ &= B_{L_{n,v}}\big(My, M(y+tv)\big) + B_{L_{n,v}}\big(M\big(v+sw(v)\big), Mv\big) \\ &> M(t-\varepsilon) - \varepsilon M \\ &> M(t-2\varepsilon). \end{aligned}$$

The first line follows from Lemma 2.3, the second from (36), and the third from (38) and (39).

By Lemma 4.4 for any sufficiently large M the density of n such that (38) is satisfied with probability at least $1-\varepsilon$ is at least $1-\varepsilon$. If $s \neq 0$, then by Lemma 4.5 for any sufficiently large M the density of n such that (39) is satisfied with probability at least $1-\varepsilon$ is at least $1-\varepsilon$. If $s = 0$, then $M(v+sw(v)) = Mv$ and (39) is satisfied for all M and n. As ε is arbitrary the lemma is true. □

PROOF OF THEOREM 1.6. By the definition of $\mathrm{Sides}(\mu)$ for any $k \leq \mathrm{Sides}(\mu)$ we can find $v_1, \dots, v_k$ such that $v_i \in V$ for all i and the lines L_{v_i} are all distinct. The fact that all $v_i \in \partial R$ and that the tangent lines are distinct implies that $t(v_i, v_j) > 0$ for any $i \neq j$. By multiple applications of Lemma 4.6 there exists $c > 0$ such that for all $\varepsilon > 0$ there exists M such that the density of n with

$$\mathbf{P}\big(B_{L_{n,v_i}}(Mv_j, Mv_i) > cM \ \forall i \neq j\big) > 1-\varepsilon \tag{40}$$

is at least $1-\varepsilon$. We then choose x_i to be the point in $\mathbb{Z}^2$ nearest to Mv_i. Thus by Lemma 2.4 and (40) we have coexistence with probability at least $1-\varepsilon$. □

PROOF OF THEOREM 1.2. This follows from Lemma 2.5 and Theorem 1.6. □

For the following proofs we will use the following notation. For $(w,z) \in \mathbb{R}^2$ we use the notation $|(w,z)| = \sqrt{w^2+z^2}$ and $\mathrm{Ball}(c,r) = \{a \in \mathbb{R}^2 : |c-a| < r\}$. Let $x, v, y \in V$ have distinct tangent lines. Let $A = A(x,v,y) \subset \partial R$ be the (open) arc of ∂R from x to y that contains v. Remember that $G(\mathbf{0}, L_{n,v})$ is the unique geodesic from $\mathbf{0}$ to $L_{n,v}$. For any $x, v, y \in V$ we consider the event

$$G(\mathbf{0}, L_{n,v}) \cap \partial(MR) \subset MA. \tag{41}$$

LEMMA 4.7. *Let $x, v, y \in V$ have distinct tangent lines and let $\varepsilon > 0$. There exists $M_0 = M_0(\varepsilon, x, v, y)$ such that for any $M > M_0$ we have that the density of n such that*

$$\mathbf{P}((41) \text{ is satisfied}) > 1 - \varepsilon$$

is at least $1 - \varepsilon$.

PROOF. First we claim that for fixed M, n, ε and $z \in \partial(R) \setminus A$, if

$$\tau(0, Mz) \geq \tau(0, Mv) - \varepsilon M \tag{42}$$

and $Mz \in G(0, L_{n,v})$, then

$$\begin{aligned}
\tau(0, L_{n,v}) &= \tau(0, Mz) + \tau(Mz, L_{n,v}) \\
&= \tau(0, Mz) + \tau(Mz, L_{n,v}) - \tau(Mv, L_{n,v}) + \tau(Mv, L_{n,v}) \\
&\geq \tau(0, Mv) - \varepsilon M + B_{L_{n,v}}(Mz, Mv) + \tau(Mv, L_{n,v}) \\
&\geq \tau(0, Mv) + \tau(Mv, L_{n,v}) + B_{L_{n,v}}(Mz, Mv) - \varepsilon M \\
&\geq \tau(0, L_{n,v}) + B_{L_{n,v}}(Mz, Mv) - \varepsilon M.
\end{aligned}$$

The first line is true because $Mz \in G(0, L_{n,v})$. The third line is true because of (42) and the definition of $B_{L_{n,v}}$. The last line is true because of the subadditivity of τ. Thus

$$B_{L_{n,v}}(Mz, Mv) \leq \varepsilon M.$$

Fix $\{y_i\}_{i=1}^k$, $y_i \in \partial(R) \setminus A$ for all i, such that for every $z \in \partial(R) \setminus A$ there exists y_i with $|y_i - z| < \varepsilon/10(T^*(1,0) + T^*(0,1))$. Next we note that if

$$B_{L_{n,v}}(My_i, Mv) \geq 10\varepsilon M$$

and

$$\tau(My_i, Mz) \leq 2\varepsilon M,$$

then

$$\begin{aligned}
B_{L_{n,v}}(Mz, Mv) &= B_{L_{n,v}}(Mz, My_i) + B_{L_{n,v}}(My_i, Mv) \\
&\geq 10\varepsilon M - \tau(My_i, Mz) > 2\varepsilon M.
\end{aligned}$$

Thus to bound the probability that there exists $z \in \partial(R) \setminus A$ such that $Mz \in G(0, L_{n,v})$ we need only to bound the probabilities of:

1. $\tau(0, Mz) \geq \tau(0, Mv) - \varepsilon M$ for all $z \in \partial R$,
2. $B_{L_{n,v}}(My_i, Mv) \geq 10\varepsilon M$ for all y_i, and
3. $\tau(My_i, Mz) \leq 2\varepsilon M$ for all y_i and $Mz \in \mathrm{Ball}(My_i, \varepsilon M/(T^*(1,0)+T^*(0,1)))$.

For sufficiently large M the first and third events happen with probability $1-\varepsilon$ by Theorem 1.1. As $y_i \in \partial R \setminus A$ we can write $y = (1-t)v + sw(v)$ with $t > 0$. Thus by Lemma 4.6 we have that the density of n such that the second event happens with probability at least $1-\varepsilon$ is at least $1-\varepsilon$. □

PROOF OF THEOREM 1.4. Let $v_1, \dots, v_k \in V$ have distinct tangent lines. By Lemma 4.7 we see that for $i = 1, \dots, k/2$ there exist M and infinitely many n such that the finite geodesics $G(\mathbf{0}, L_{n,v_{2i}})$ are pairwise disjoint in $M\partial R$. They all intersect at $\mathbf{0}$ so for infinitely many n the geodesics are pairwise disjoint in the complement of MR. Thus by compactness we can take weak limits to get at least k infinite geodesics that are pairwise disjoint in the complement of MR. Thus $|K(\Gamma(0))| \geq k$. □

To prove Theorem 1.3 we consider the event

$$G(Mv, L_{v,n}) \cap MR \subset \mathrm{Ball}(Mv, \varepsilon M). \tag{43}$$

LEMMA 4.8. *Let $\varepsilon > 0$. There exists $M_0 = M_0(\varepsilon)$ such that for any $M > M_0$ we have that the density of n such that*

$$\mathbf{P}((43) \textit{ is satisfied}) > 1 - \varepsilon$$

is at least $1 - \varepsilon$.

PROOF. If there exists $z \in MR \setminus \mathrm{Ball}(Mv, \varepsilon M)$ and $z \in G(Mv, L_{nv})$, then there exists $z \in G(Mv, L_{nv})$ such that

$$z \in Z = \partial\bigl(MR \setminus \mathrm{Ball}(Mv, \varepsilon M)\bigr).$$

Choose $\{y_i\}_{i=1}^k$ such that for any $z \in Z$ there exists y_i such that

$$|z - y_i| < \varepsilon/100\bigl(T^*(1,0) + T^*(0,1)\bigr).$$

Suppose the following events happen:

1. $B_{L_{n,v}}(Mv, My_i) < \varepsilon M(T^*(1,0) + T^*(0,1))/10$ for all i,
2. $\tau(Mv, Mz) > \varepsilon M(T^*(1,0) + T^*(0,1))/3$ for all z such that $|z - v| \geq \varepsilon$, and
3. $\tau(My_i, Mz) < \varepsilon M(T^*(1,0) + T^*(0,1))/10$ for all i and z such that $|z - y_i| < \varepsilon/100(T^*(1,0) + T^*(0,1))$.

Then we claim that (43) is satisfied. To see this we note that by conditions 1 and 3

$$\begin{aligned} B_{L_{n,v}}(Mv, Mz) &= B_{L_{n,v}}(Mv, My_i) + B_{L_{n,v}}(My_i, Mz) \\ &< \varepsilon M\bigl(T^*(1,0) + T^*(0,1)\bigr)/10 + \tau(My_i, Mz) \\ &< \varepsilon M\bigl(T^*(1,0) + T^*(0,1)\bigr)/5 \end{aligned} \tag{44}$$

for all $z \in Z$. If $Mz \in G(Mv, L_{n,v})$, then

$$B_{L_{n,v}}(Mv, Mz) = \tau(Mv, Mz).$$

Thus by condition 2 if $z \in Z$ and $Mz \in G(Mv, L_{n,v})$, then

$$B_{L_{n,v}}(Mv, Mz) = \tau(Mv, Mz) > \varepsilon M\bigl(T^*(1,0) + T^*(0,1)\bigr)/3$$

which contradicts (44) and establishes the claim.

Thus to prove the lemma we need to show that the density of n such that the probability of all of the events in conditions 1, 2 and 3 occurring is greater than $1-\varepsilon$. By the argument in Lemma 4.6 for sufficiently large M with probability at least $1-\varepsilon/3$ the density of n such that condition 1 occurs is at least $1-\varepsilon$. By Theorem 1.1 the probabilities of the last two events can be made greater than $1-\varepsilon/3$. $\square$

For the final proof we will be dealing with multiple realizations of first passage percolation. To deal with this we will use the notation $\tau^{\omega}(x, y)$, $B^{\omega}_S(x, y)$ and $G^{\omega}(x, y)$ to represent the quantities $\tau(x, y)$, $B_S(x, y)$ and $G(x, y)$ in ω.

PROOF OF THEOREM 1.3. Given k, by Theorem 1.2 we can choose M and $x_1, \dots, x_k \in \partial MR$ such that with positive probability there exist disjoint geodesics G_i starting at each x_i. By Lemma 4.6 we have that there exists a measurable choice of geodesics G_i and vertices x_i such that for any $i \neq j$

$$\hat{B}_{G_i}(x_j, x_i) > 100. \tag{45}$$

There exist finite paths $\tilde{G}_i \subset MR$ and an event E of positive probability that satisfy the following condition. For each $\omega \in E$ and i, the paths G_i and $\tilde{G}_i$ agree in MR. Let $y_i \in \mathbb{Z}^2$ be the first vertex in G_i after G_i exits MR for the last time. We can find $a_i > 0$ and restrict to a smaller event of positive probability where

$$a_i < \tau(x_i, y_i) < a_i + 1. \tag{46}$$

We can pick some large K and further restrict our event as follows. Let $z, z' \in \mathbb{Z}^2 \setminus MR$ be such that there exist $x, x' \in \mathbb{Z}^2 \cap MR$ and $|z - z'| = |x - x'| = 1$. We require that for any such z and z' that there exists a path from z to z' that lies entirely outside of MR and has passage time less than or equal to K. For K large enough the resulting event $\hat{E}$ will have positive probability. We now create a new event E' by taking any $\omega \in \hat{E}$ and altering the passage times in MR. We will do this in a way such that E' has positive probability and the inequality $|K(\Gamma(\mathbf{0}))| \geq k$ is satisfied for all $\omega \in E'$.

First we choose paths $\hat{G}_i \subset MR$ that connect $\mathbf{0}$ to y_i such that $\hat{G}_i \cap \tilde{G}_j = \varnothing$ for all $i \neq j$. This is possible by Lemma 4.8. A configuration $\omega' \in E'$ if:

1. There exists an $\omega \in \hat{E}$ such that $\omega(e) = \omega'(e)$ for all edges e with both endpoints in $(MR)^C$ and

$$a_i < \tau^{\omega'}(\mathbf{0}, y_i) < a_i + 2. \tag{47}$$

2. For every $z \in \mathbb{Z}^2 \setminus MR$ there exists i such that

$$G(\mathbf{0}, z)|_{MR} = \hat{G}_i.$$

3. For every $z \in \mathbb{Z}^2 \setminus MR$ and every i

$$G(y_i, z) \subset (MR)^C.$$

Note that the first and last conditions imply that for every $z \in \mathbb{Z}^2 \setminus MR$ and every i

$$\tau^{\omega'}(y_i, z) \geq \tau^{\omega}(y_i, z). \tag{48}$$

Also note that the first condition implies if $v \in G_i \setminus MR$, then

$$\tau^{\omega'}(y_i, v) = \tau^{\omega}(y_i, v). \tag{49}$$

Fix $\omega' \in E'$ and $v \in G_i \setminus MR$. We claim that $G^{\omega'}(\mathbf{0}, v) = \hat{G}^i \cup G^{\omega}(y_i, v)$. By condition 2 we know that $G^{\omega'}(\mathbf{0}, v)$ must pass through some y_l. Then we calculate

$$\begin{aligned}
\tau^{\omega'}(\mathbf{0}, y_i) + \tau^{\omega'}(y_i, v) &< a_i + 2 + \tau^{\omega}(y_i, v) \\
&< \tau^{\omega}(x_i, y_i) + 2 + \tau^{\omega}(y_i, v) \\
&< \tau^{\omega}(x_i, v) + 2.
\end{aligned} \tag{50}$$

The first inequality follows from (47) and (49). The second inequality follows from (46). The third is true because x_i, y_i and v are all on G_i. Next we calculate

$$\begin{aligned}
\tau^{\omega'}(\mathbf{0}, y_j) + \tau^{\omega'}(y_j, v) &> a_j + \tau^{\omega}(y_j, v) \\
&> \tau^{\omega}(x_j, y_j) - 1 + \tau^{\omega}(y_j, v) \\
&> \tau^{\omega}(x_j, v) - 1 \\
&> \tau^{\omega}(x_i, v) - 1 + \hat{B}_{G_i}(x_j, x_i) \\
&> \tau^{\omega}(x_i, v) + 99.
\end{aligned} \tag{51}$$

The first inequality follows from (47) and (48). The second inequality follows from (46), the third from the subadditivity of τ, the fourth from the definition of $\hat{B}_{G_i}$ and the final from (45).

Combining (50) and (51) we get that for every $v \in G_i$ the geodesic $G^{\omega'}(\mathbf{0}, v)$ passes through y_i. As this holds true for every i we have that

$$\mathbf{P}\big(|K(\Gamma(\mathbf{0}))| \geq k\big) \geq \mathbf{P}(E').$$

The conditions on E' can be satisfied by picking the passage times through the edges in $\bigcup \hat{G}_i$ to be in some appropriate interval to satisfy (47) and by choosing the edges not in $\bigcup \hat{G}_i$ to have passage times larger than $10(K + 2 + \max a_i)$. As μ has finite energy we see that

$$\mathbf{P}\big(|K(\Gamma(\mathbf{0}))| \geq k\big) \geq \mathbf{P}(E') > 0. \qquad \square$$

Acknowledgment. I would like to thank Itai Benjamini for introducing me to this problem and Oded Schramm for noticing a error in an earlier version of this paper.

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