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DECAY OF CORRELATIONS IN NEAREST-NEIGHBOR SELF-AVOIDING WALK, PERCOLATION, LATTICE TREES AND ANIMALS

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We consider nearest-neighbor self-avoiding walk, bond percolation, lattice trees, and bond lattice animals on $\mathbb{Z}^d$. The two-point functions of these models are respectively the generating function for self-avoiding walks from the origin to $x \in \mathbb{Z}^d$, the probability of a connection from the origin to x, and the generating functions for lattice trees or lattice animals containing the origin and x. Using the lace expansion, we prove that the two-point function at the critical point is asymptotic to $const.|x|^{2-d}$ as $|x| \to \infty$, for $d \geq 5$ for self-avoiding walk, for $d \geq 19$ for percolation, and for sufficiently large d for lattice trees and animals. These results are complementary to those of [*Ann. Probab.* **31** (2003) 349–408], where spread-out models were considered. In the course of the proof, we also provide a sufficient (and rather sharp if $d > 4$) condition under which the two-point function of a random walk on $\mathbb{Z}^d$ is asymptotic to $const.|x|^{2-d}$ as $|x| \to \infty$.

1. Introduction.

1.1. *The models and results.* In this paper, we consider nearest-neighbor self-avoiding walk, bond percolation, lattice trees, and bond lattice animals on d-dimensional hypercubic lattice $\mathbb{Z}^d$, and prove that their critical two-point functions exhibit the Gaussian behavior, that is,

$$G_{p_c}(x) \sim \frac{const.}{|x|^{d-2}} \qquad \text{as } |x| \to \infty, \tag{1.1}$$

when d is large.

We first define the models we consider. A *bond* is a pair of sites $\{x, y\} \subset \mathbb{Z}^d$ with $|y-x| = 1$. For $n \geq 0$, an n-step *walk* from x to y is a mapping $\omega : \{0, 1, \ldots, n\} \to \mathbb{Z}^d$ such that $|\omega(i+1) - \omega(i)| = 1$ for $i = 0, \ldots, n-1$, with $\omega(0) = x$ and $\omega(n) = y$. Let $\mathcal{W}(x, y)$ denote the set of walks from x to y, taking any number of steps. An n-step *self-avoiding walk* (SAW) is an n-step walk ω such that $\omega(i) \neq \omega(j)$ for each pair $i \neq j$. Let $\mathcal{S}(x, y)$ denote the set of self-avoiding walks from x to y,

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taking any number of steps. A *lattice tree* (LT) is a finite connected set of bonds which has no cycles. A *lattice animal* (LA) is a finite connected set of bonds which may contain cycles. Although a tree T is defined as a set of bonds, we write $x \in T$ if x is an endpoint of some bond of T, and similarly for lattice animals. Let $\mathcal{T}(x, y)$ denote the set of lattice trees containing x and y, and let $\mathcal{A}(x, y)$ denote the set of lattice animals containing x and y. We often abbreviate lattice trees and animals as LTLA.

The random walk and self-avoiding walk *two-point functions* are defined respectively by

$$S_p(x) = \sum_{\omega \in \mathcal{W}(0,x)} p^{|\omega|}, \qquad G_p(x) = \sum_{\omega \in \mathcal{S}(0,x)} p^{|\omega|}, \tag{1.2}$$

where $|\omega|$ denotes the number of steps of the walk ω. For any $d > 0$, $\sum_x S_p(x)$ converges for $p < (2d)^{-1}$ and diverges for $p > (2d)^{-1}$, and $p = (2d)^{-1}$ plays the role of a critical point. It is well known [28] that, for $d > 2$,

$$S_{1/2d}(x) \sim const. \frac{1}{|x|^{d-2}} \qquad \text{as } |x| \to \infty. \tag{1.3}$$

A standard subadditivity argument [5, 13, 19] implies that $\sum_x G_p(x)$ converges for $p < p_c$ and diverges for $p > p_c$, for some finite positive critical value p_c.

The lattice tree and lattice animal two-point functions are defined by

$$\begin{aligned} G_p(x) &= \sum_{T \in \mathcal{T}(0,x)} p^{|T|} \qquad \text{(lattice trees)}, \\ G_p(x) &= \sum_{A \in \mathcal{A}(0,x)} p^{|A|} \qquad \text{(lattice animals)}, \end{aligned} \tag{1.4}$$

where $|T|$ and $|A|$ denote the number of bonds in T and A, respectively. A standard subadditivity argument implies that there are positive finite p_c (depending on the model) such that $\sum_x G_p(x)$ converges for $p < p_c$ and diverges for $p > p_c$ [14, 15].

Turning now to bond percolation, we associate independent Bernoulli random variables $n_{\{x,y\}}$ to each bond $\{x, y\}$ (here $|x - y| = 1$), with

$$\mathbb{P}\big(n_{\{x,y\}} = 1\big) = p, \qquad \mathbb{P}\big(n_{\{x,y\}} = 0\big) = 1 - p, \tag{1.5}$$

where $p \in [0, 1]$. A configuration is a realization of the bond variables. Given a configuration, a bond $\{x, y\}$ is called *occupied* if $n_{\{x,y\}} = 1$ and otherwise is called *vacant*. The percolation *two-point function* is defined by

$$G_p(x) = \mathbb{P}_p(0 \text{ and } x \text{ are connected by occupied bonds}), \tag{1.6}$$

where $\mathbb{P}_p$ is the probability measure on configurations induced by the bond variables. There is a critical value $p_c \in (0, 1)$ such that $\sum_x G_p(x) < \infty$ for $p \in [0, p_c)$ and $\sum_x G_p(x) = \infty$ for $p \geq p_c$ [4]. This critical point can also be characterized by the fact that the probability of existence of an infinite cluster of occupied bonds is 1 for $p > p_c$ and 0 for $p < p_c$ [2, 20].

We use G_p and p_c to denote the two-point function and the critical point of these models, although they are, of course, model-dependent. In what follows, it will be clear from the context which model is intended.

Our main result is the following theorem.

THEOREM 1.1. *For nearest-neighbor self-avoiding walk in $d \geq 5$ and for percolation and lattice trees and animals in sufficiently high dimensions, their critical two-point function $G_{p_c}(x)$ satisfies, as $|x| \to \infty$,*

$$G_{p_c}(x) = \frac{a_d A}{|x|^{d-2}} + O\left(\frac{1}{|x|^{d-2+2/d}}\right) \qquad \text{with } a_d = \frac{d\Gamma(d/2-1)}{2\pi^{d/2}}. \tag{1.7}$$

Here A is a model-dependent constant whose explicit form is given in (1.44) below, in terms of quantities appearing in the lace expansion.

REMARK 1.2. (i) The error term of (1.7) is not optimal; the error bound in the Gaussian lemma (Theorem 1.4) is responsible for the current estimate. However, the author has recently succeeded in improving the error bound of Theorem 1.4. As a result, (1.7) has now been improved to

$$G_{p_c}(x) = \frac{a_d A}{|x|^{d-2}} + O\left(\frac{1}{|x|^{d}}\right). \tag{1.8}$$

The proof of this improvement will be presented elsewhere [6].

(ii) For percolation, $d \geq 19$ is sufficient for the above theorem to hold. The restriction $d \geq 19$ comes from the fact that convergence of the lace expansion has been proved only in these dimensions. This is far from the expected limit of d ($d > 6$ should be sufficient). See, for example, [7] for the role played by the critical dimension.

(iii) The method of the present paper can also be applied to spread-out models, and reproduces the asymptotic form proved in [7], for self-avoiding walk in $d \geq 5$, for percolation in $d \geq 11$, and for lattice trees/animals in $d \geq 27$. See the explanations around (1.62) and (1.63) about how these restrictions on the dimension arises.

(iv) The method of the present paper can be applied to other models, as long as we have a suitable lace expansion. An important example is the Ising model in sufficiently high dimensions [23].

(v) The theorem provides a necessary input for a result of Aizenman [1], who proved, under certain assumptions on the decay of critical two-point function, that the largest percolation cluster present in a box of side length N are of size approximately N^4 and are approximately N^{d-6} in number. Our theorem does prove the assumptions of Aizenman, and thus establishes his result mentioned above for the nearest-neighbor percolation in $d \geq 19$. (Similar input for spread-out models has been provided by [7].)

Results similar to the above have been proven in [7], where *spread-out* models of self-avoiding walk, bond percolation, and lattice trees/animals were treated in a unified manner. (Spread-out models are defined by considering all the pairs $\{x, y\}$ with $0 < |x - y| \le L$ as bonds, for some large L. This L represents the range of the interaction, not the system size.) However, the method of [7] is not directly applicable to nearest-neighbor models, for the following reason. Critical two-point functions of spread-out models obey as $|x| \to \infty$ [7], Theorem 1.2:

$$G_{p_c}(x) \sim \frac{a_d A}{\sigma^2 |x|^{d-2}}, \tag{1.9}$$

where A is a model-dependent constant close to 1, and σ^2 is a constant which is of the order of L^2. For the spread-out model, by taking L sufficiently large (for fixed d), we can always make the coefficient $a_d A/\sigma^2$ as small as we want. Therefore, the lace expansion diagrams converge if we assume $G(x)$ is bounded by, say, twice of the right hand of (1.9). This makes it possible to prove convergence of the lace expansion in a self-consistent way based on the asymptotic form; the result of [7] was in fact proved in this manner.

In contrast, for the nearest-neighbor model, there is no σ^2 to cancel a_d, which is quite large for large d [$a_d \approx (d/2)!$]. This means the asymptotic form of (1.7) is much bigger than the true behavior of $G(x)$ for small x, and it would be difficult to prove the convergence of the lace expansion using this asymptotic form. In this paper, we bypass this difficulty by borrowing convergence results from previous works, and take a different approach from that of [7].

NOTATION. For $a, b \in \mathbb{R}$, we write $a \vee b = \max\{a, b\}$, and $a \wedge b = \min\{a, b\}$. The greatest integer n which satisfies $n \le x$ is denoted by $\lfloor x \rfloor$. The smallest integer n which satisfies $n \ge x$ is denoted by $\lceil x \rceil$.

The Euclidean norm of $x \in \mathbb{R}^d$ is denoted by $|x|$, and we write $|||x||| := |x| \vee 1$.

The indicator of an event A is denoted by $I[A]$.

A convolution on $\mathbb{Z}^d$ is denoted by $*: (f * g)(x) := \sum_{y \in \mathbb{Z}^d} f(x - y) g(y)$.

Given a function $f(x)$ on $\mathbb{Z}^d$, we define its Fourier transform as

$$\hat{f}(k) := \sum_{x \in \mathbb{Z}^d} f(x) e^{-ik\cdot x} \qquad \text{so that}$$
$$f(x) = \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} \hat{f}(k), \tag{1.10}$$

when both equations make sense. When the sum defining $\hat{f}(k)$ is not well defined [i.e., when $f(x) = G_{p_c}(x)$], we interpret $\hat{f}(k)$ by the second identity above more details are given in Appendix A.

A function $f(x)$ on $\mathbb{Z}^d$ is called $\mathbb{Z}^d$-*symmetric*, if it is invariant under the $\mathbb{Z}^d$-symmetries of reflection in coordinate hyperplanes and rotation by 90°.

We denote a positive constant by c. On each appearance c may change its value, even in a single equation. We write $f(x) \approx g(x)$ when there are finite and positive constants c_1, c_2 such that $c_1 g(x) \le f(x) \le c_2 g(x)$ for all x. We also use large-O and small-o notation: $f(x) = O(g(x))$ means $f(x)/g(x)$ remains bounded, while $f(x) = o(g(x))$ means $f(x)/g(x) \to 0$, as $x \to \infty$ (or $x \to 0$, depending on the context). Constants c and large-O/small-o's could depend on other parameters. We explain these dependencies on each occurrence if necessary.

We make use of the following quantities ($a, b \in \mathbb{Z}^d$ and $\alpha, \beta, \gamma \ge 0$, and the summations run over $\mathbb{Z}^d$):

$$G^{(\alpha)}(a) := |a|^\alpha G(a), \tag{1.11}$$

$$B(a) := \sum_{y \neq 0} G(y)G(a-y), \tag{1.12}$$

$$W^{(\beta,\gamma)}(a) := \sum_{y} G^{(\beta)}(y)G^{(\gamma)}(a-y) = \bigl(G^{(\beta)} * G^{(\gamma)}\bigr)(a), \tag{1.13}$$

$$\begin{aligned} T^{(\beta,\gamma)}(a) &:= \sum_{x,y} G^{(\beta)}(x)G^{(\gamma)}(y-x)G(a-y)\{1 - I[x=y=a=0]\} \\ &= \bigl(G^{(\beta)} * G^{(\gamma)} * G\bigr)(a) - I[a=0 \text{ and } \beta=\gamma=0]G(0)^3, \end{aligned} \tag{1.14}$$

$$S^{(\gamma)}(a) := \bigl(G^{(\gamma)} * G * G * G\bigr)(a) - I[a=0 \text{ and } \gamma=0]G(0)^4, \tag{1.15}$$

$$P(a) := (G * G * G * G * G)(a), \tag{1.16}$$

$$\begin{aligned} H^{(\beta)}(a,b) := \sum_{x,y,z,u,v} & G(z)G(u)G(x-u)G^{(\beta)}(x)G(y-x)G(v-u) \\ & \times G(z+a-v)G(y+b-v). \end{aligned} \tag{1.17}$$

Diagrammatic representations for these quantities are given in Figure 1(a) of Section 3.1. We denote suprema (over $a, b \in \mathbb{Z}^d$) of these quantities by bars, that is, $\bar{G}^{(\alpha)} := \sup_a |a|^\alpha G(a)$, $\bar{B} := \sup_a B(a)$, $\bar{W}^{(\beta,\gamma)} := \sup_a W^{(\beta,\gamma)}(a)$, and so on. These of course depend on p, but we usually omit the subscript p, because we almost always consider these quantities at criticality, $p = p_c$.

1.2. *Framework of the proof.* In this section, we explain the framework of the proof of our main result, Theorem 1.1, and reduce its proof to several propositions. We give a complete proof of Theorem 1.1 for self-avoiding walk in $d \ge 5$, but only give a proof for large d (say $d \ge 30$) for percolation. Results for percolation in $d \ge 19$ can be obtained by more detailed diagrammatic estimates which slightly improve conditions in Lemmas 1.7 and 1.8. The extra work required for percolation near $d = 19$ is essentially the same as the analysis used to prove the convergence of the lace expansion in $d = 19$ (announced in [12]), and is not reproduced here.

1.2.1. *The lace expansion.* For self-avoiding walk in $d \geq 5$, for percolation in $d \geq 19$, and for lattice trees/animals in $d \gg 1$, we have a convergent expansion, called the *lace expansion*, which provides a useful expression for two-point functions. The literature on lace expansion has increased rapidly, and we here list only a few which will be directly relevant for the present paper [3, 8–11, 25]. Good reviews will be found in [12, 19, 26].

PROPOSITION 1.3. *For self-avoiding walk in $d \geq 5$, for percolation in $d \geq 19$, and for lattice trees/animals in sufficiently high dimensions, the two-point function for $p \leq p_c$ is represented as*

$$G_p(x) = \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ikx} \hat{G}_p(k), \qquad \hat{G}_p(k) := \frac{\hat{g}_p(k)}{1 - \hat{J}_p(k)}. \tag{1.18}$$

Here

$$D(x) := \frac{1}{2d}\delta_{|x|,1}, \qquad \hat{D}(k) = \sum_x e^{-ik\cdot x} D(x) = \frac{1}{d}\sum_{j=1}^{d} \cos k_j, \tag{1.19}$$

$$\hat{J}_p(k) := \begin{cases} 2dp\hat{D}(k) + \hat{\Pi}_p(k), & (\textit{SAW}), \\ 2dp\hat{D}(k)\{1 + \hat{\Pi}_p(k)\}, & (\textit{percolation}), \\ 2dp\hat{D}(k)\{G_p(0) + \hat{\Pi}_p(k)\}, & (\textit{LTLA}), \end{cases} \tag{1.20}$$

$$\hat{g}_p(k) := \begin{cases} 1, & (\textit{SAW}), \\ 1 + \hat{\Pi}_p(k), & (\textit{percolation}), \\ G_p(0) + \hat{\Pi}_p(k), & (\textit{LTLA}), \end{cases} \tag{1.21}$$

$$\hat{\Pi}_p(k) := \sum_x \Pi_p(x) e^{-ik\cdot x}, \qquad \Pi_p(x) = \sum_{n=0}^{\infty} (-1)^n \Pi_p^{(n)}(x), \tag{1.22}$$

and $\Pi_p^{(n)}(x)$ is a nonnegative function of x. Moreover, there are positive constants c, c_1 through c_4 which are independent of p and d, a constant $\lambda \in (0, 1)$ which is independent of p, and a positive function $h^{(n)}(x)$, such that for $p \leq p_c$,

$$0 \leq \Pi_p^{(n)}(x) \leq h^{(n)}(x), \qquad \sum_x \sum_{n=0}^{\infty} h^{(n)}(x) \leq \frac{c}{d}, \tag{1.23}$$

$$\sum_x |x|^2 |\Pi_p(x)| \leq \frac{c}{d}, \qquad c_1 \frac{|k|^2}{d} \leq \hat{J}_p(0) - \hat{J}_p(k) \leq c_2 \frac{|k|^2}{d}, \tag{1.24}$$

$$0 \leq \hat{G}_p(k) \leq \frac{cd}{|k|^2} \qquad (\textit{infrared bound}) \tag{1.25}$$

and

$$\bar{G}_p^{(2)} < \lambda, \ \bar{B}_p < \lambda, \qquad (\textit{SAW in } d \ge 5), \tag{1.26}$$

$$\bar{W}_p^{(2,0)} < \lambda, \qquad \bar{T}_p^{(0,0)} < \lambda, \qquad \bar{H}_p^{(2)} < c \qquad (\textit{percolation in } d \ge 19), \tag{1.27}$$

$$\bar{T}_p^{(2,0)} < \lambda, \ \bar{S}_p^{(0)} < \lambda, \ 1 \le G_p(0) \le 4 \qquad (\textit{LTLA in } d \gg 1). \tag{1.28}$$

λ *satisfies*

$$\lambda \le \begin{cases} 0.493, & (\textit{SAW in } d \ge 5), \\ \dfrac{c_3}{d}, & (\textit{percolation in } d \ge 19 \textit{ and LTLA in } d \gg 1). \end{cases} \tag{1.29}$$

The critical point $p = p_c$ *is characterized by*

$$\hat{J}_{p_c}(0) = 1, \tag{1.30}$$

and satisfies

$$\begin{aligned} &1 \le 2dp_c \le 1 + c_4\lambda && (\textit{SAW/percolation}), \\ &1 \le 2dp_c G_{p_c}(0) \le 1 + c_4\lambda && (\textit{LTLA}). \end{aligned} \tag{1.31}$$

For self-avoiding walk, $\Pi_p^{(0)}(x) \equiv 0$ for all x, and the sum over n in (1.22) starts from $n = 1$. For percolation, our $\hat{\Pi}_p(k)$ is the same as that of [7], but differs from that of [8] by the factor $2dp\hat{D}(k)$ and is equal to $\hat{g}_p(k)$ of that paper.

The above proposition is a slightly improved version of the results obtained previously. We briefly explain how to prove the above proposition in Appendix A.

In the following, we concentrate on quantities at $p = p_c$ (*except stated otherwise*), *and omit the subscript* p *or* p_c *altogether.*

1.2.2. *Gaussian lemma.* Our main results are proved by making use of the following theorem and its corollary, which give sufficient conditions for the Gaussian behavior, $G(x) \sim \textit{const}.|x|^{2-d}$, for two-point functions of random walks and related models. For $\mathbb{Z}^d$-symmetric (not necessarily positive) functions $J(x)$ and $g(x)$, we define

$$\begin{aligned} C(x) &:= \int_{[-\pi,\pi]^d} \frac{d^dk}{(2\pi)^d} \frac{e^{ik\cdot x}}{1 - \hat{J}(k)}, \\ H(x) &:= \int_{[-\pi,\pi]^d} \frac{d^dk}{(2\pi)^d} e^{ik\cdot x} \frac{\hat{g}(k)}{1 - \hat{J}(k)}. \end{aligned} \tag{1.32}$$

THEOREM 1.4. *Let* $d \ge 3$. *Suppose* $\mathbb{Z}^d$*-symmetric* $J(x)$ *satisfies with finite* *positive K_0 through K_3*

$$\hat{J}(0) := \sum_x J(x) = 1, \qquad \hat{J}(0) - \hat{J}(k) \geq K_0 \frac{|k|^2}{2d} \qquad (k \in [-\pi, \pi]^d), \tag{1.33}$$

$$\sum_x |x|^2 J(x) := K_1, \qquad \sum_x |x|^2 |J(x)| \leq K_2, \tag{1.34}$$

$$|J(x)| \leq \frac{K_3}{|||x|||^{d+2}}. \tag{1.35}$$

Then $C(x)$ of (1.32) is well defined and satisfies as $|x| \to \infty$:

$$C(x) \sim \frac{a_d}{K_1} \frac{1}{|x|^{d-2}}. \tag{1.36}$$

Suppose further that $J(x)$ satisfies

$$\sum_x |x|^{2+\rho} |J(x)| < K_2', \qquad |J(x)| \leq \frac{K_3'}{|||x|||^{d+2+\rho}} \tag{1.37}$$

with finite positive ρ, K_2', K_3'. Then $C(x)$ satisfies

$$C(x) = \frac{a_d}{K_1} \frac{1}{|||x|||^{d-2}} + O\left(\frac{1}{|||x|||^{d-2+(\rho\wedge 2)/d}}\right). \tag{1.38}$$

Section 2 gives a complete proof of the theorem.

REMARK 1.5. (i) The above $C(x)$ is the two-point function of the Gaussian spin system whose spins at x and y interact with $J(x-y)$. When $J(x) \geq 0$, $C(x)$ can also be interpreted as the Green's function of the random walk whose transition probability from x to y is given by $J(x-y)$. We are allowing $J(x) < 0$, because $\Pi(x)$ is not necessarily positive in our lace expansion (1.18).

(ii) The pointwise bound (1.35) is sharp in $d > 4$, in the sense that there are models which mildly violate this condition and which do not exhibit the Gaussian behavior of (1.36). Details will be given in Section 2.5. For $d = 4$ and 5, the fact that (1.35) is sufficient for nonnegative J's has been pointed out by Uchiyama [28]. The author has recently learned that Lawler [18] has also shown that (1.35) is sufficient for $d > 4$ for nonnegative J.

(iii) For $d \leq 4$, the uniform bound (1.35) will not be sharp. Sharp conditions when $J \geq 0$ are $\sum_x |x|^2 J(x) < \infty$ for $d < 4$ [28], and $\sum_{x:|x|\geq r} J(x) = o(\frac{1}{r^2 \log r})$ for $d = 4$ [17].

(iv) The error bound in (1.38) is not optimal. However, the author has recently succeeded in proving a better (and hopefully optimal) error bound, according to which (1.38) is improved to

$$C(x) = \frac{a_d}{K_1} \frac{1}{|||x|||^{d-2}} + O\left(\frac{1}{|||x|||^{d-2+(\rho\wedge 2)}}\right). \tag{1.39}$$

The proof of this improvement is somewhat lengthy, and will be presented elsewhere [6]. For nonnegative J's, the error bound like (1.39) has been obtained by Lawler [18].

COROLLARY 1.6. *Let $d \ge 3$. Let $J(x)$ and $g(x)$ be $\mathbb{Z}^d$-symmetric functions. Suppose $J(x)$ satisfies (1.33)–(1.35), and $g(x)$ satisfies*

$$\sum_x |g(x)| < \infty, \qquad |g(x)| \le \frac{K_4}{|||x|||^d} \tag{1.40}$$

with finite positive K_4. Then, $H(x)$ of (1.32) is well defined and satisfies as $|x| \to \infty$

$$H(x) \sim \frac{\sum_y g(y)}{\sum_y |y|^2 J(y)} \frac{a_d}{|x|^{d-2}}. \tag{1.41}$$

Suppose further $J(x)$ satisfies (1.37) and $g(x)$ satisfies

$$|g(x)| \le \frac{K_4'}{|||x|||^{d+\rho}} \tag{1.42}$$

with finite positive ρ, K_4'. Then $H(x)$ satisfies as $|x| \to \infty$

$$H(x) = \frac{\sum_y g(y)}{\sum_y |y|^2 J(y)} \frac{a_d}{|x|^{d-2}} + O\left(\frac{1}{|x|^{d-2+(\rho\wedge 2)/d}}\right). \tag{1.43}$$

Corollary 1.6 follows immediately from Theorem 1.4 and a basic property of convolutions, Lemma B.1(iv). This is because $H(x) = (C * g)(x)$, where $*$ denotes convolution.

We intend to apply the above corollary to the representation of two-point functions by the lace expansion, (1.18). If the corollary can in fact be applied (with $\rho = 2$), then it proves Theorem 1.1 with

$$A := \frac{\sum_y g(y)}{\sum_y |y|^2 J(y)}, \tag{1.44}$$

with J and g given by (1.20). The question is whether we can really apply the proposition. For this, note that (1.33) and (1.34) follow directly from Proposition 1.3 at $p = p_c$. Therefore, it suffices to prove pointwise x-space bound (1.35) and (1.37). [Because $J(x)$ and $g(x)$ are essentially the same, (1.40) and (1.42) for $g(x)$ are automatically satisfied if $J(x)$ satisfies (1.35) and (1.37).]

1.2.3. *Reduction of the proof to an estimate on the two-point function.* The condition (1.35) is about the decay of $J(x)$, but its sufficient condition can be given in terms of $G(x)$ with the help of the following lemma, which turns an x-space bound on $G(x)$ into that on $\Pi(x)$.

LEMMA 1.7. *Consider SAW, percolation, or LTLA for which Proposition 1.3 holds. Suppose we have a bound*

$$G(x) \le \frac{\beta}{|||x|||^{\alpha}} \tag{1.45}$$

with $\beta > 0$ *and* $0 < \alpha < d$. *Then for* $x \neq 0$,

$$|\Pi(x)| \le \begin{cases} c\beta^3 |||x|||^{-3\alpha}, \\ \qquad (\textit{SAW with } \lambda < 1), \\ c\beta^2 |||x|||^{-2\alpha}, \\ \qquad (\textit{percolation in } d > 8 \textit{ with } \lambda \ll 1), \\ c(\beta^2 \vee \beta^4) |||x|||^{-(4\alpha-2d)}, \\ \qquad (\textit{LTLA in } d > 10 \textit{ with } \lambda \ll 1, \alpha > d/2) \end{cases} \tag{1.46}$$

with a λ*-dependent constant* c.

This lemma is proved in Sections 3.3, 3.5 and 3.7. The restriction $d > 8$ (for percolation) and $d > 10$ (for LTLA) is unnatural, but is present for technical reasons. Also the exponent $4\alpha - 2d$ for LTLA will not be optimal; the optimal result would give $|\Pi(x)| = O(|||x|||^{3\alpha-d})$, as proved for spread-out models in Proposition 1.8 of [7]. These facts will reflect some limitations of our current method, but the lemma still suffices for our purpose.

Employing Lemma 1.7, one can immediately conclude that a sufficient condition for (1.35) is

$$G(x) \le \frac{c}{|||x|||^{\alpha}} \qquad \text{with } \alpha = \begin{cases} \dfrac{d+2}{3}, & \text{(SAW)}, \\ \dfrac{d+2}{2}, & \text{(percolation)}, \\ \dfrac{3d+2}{4}, & \text{(LTLA)} \end{cases} \tag{1.47}$$

with some constant c, together with $d > 4$ (SAW), $d > 8$ (percolation), and $d > 10$ (LTLA) [and $\lambda < 1$ for SAW, $\lambda \ll 1$ for percolation/LTLA]. We now show that this is sufficient for (1.37) as well. Once we have (1.35), Theorem 1.4 establishes (1.36). We can then use (1.36) as an input to Lemma 1.7, and get

$$|\Pi(x)| \le \begin{cases} c|||x|||^{-3(d-2)}, & \text{(SAW)}, \\ c|||x|||^{-2(d-2)}, & \text{(percolation)}, \\ c|||x|||^{-(2d-8)}, & \text{(LTLA)}. \end{cases} \tag{1.48}$$

This in turn implies $|J(x)|, |g(x)| \le c|||x|||^{-(d+2+\rho)}$, with

$$\rho = \begin{cases} 2(d-4), & \text{(SAW)}, \\ d-6, & \text{(percolation)}, \\ d-10, & \text{(LTLA)}. \end{cases} \tag{1.49}$$

This establishes (1.37) with $\rho = 2$ (for $d \ge 5$ for SAW and for sufficiently high d for percolation/LTLA).

Our task has thus been reduced to proving (1.47).

1.2.4. *Proving the estimate* (1.47) *on two-point functions from two lemmas.* To prove (1.47), we use two lemmas. The first one is our second diagrammatic lemma, which turns bounds on weighted quantities of (1.11)–(1.17) into those on $\sum_x |x|^\alpha |\Pi(x)|$ with some power α.

LEMMA 1.8. *Consider SAW, percolation or LTLA for which Proposition* 1.3 *holds*:

(i) *For SAW with* $\lambda < 1$, *suppose* $\bar{G}^{(\alpha)}$ *and* $\bar{W}^{(\beta,\gamma)}$ *are finite for some* $\alpha, \beta, \gamma \geq 0$. *Then*,

$$\sum_x |x|^{\alpha+\beta+\gamma} |\Pi(x)| < \infty. \tag{1.50}$$

(ii) *For percolation with* λ *sufficiently small*, *suppose* $\bar{W}^{(\beta,\gamma)}$, $\bar{T}^{(0,\gamma)}$ *and* $\bar{H}^{(\beta)}$ *are finite for some* $\beta, \gamma \geq 0$. *Then*,

$$\sum_x |x|^{\beta+\gamma} |\Pi(x)| < \infty. \tag{1.51}$$

(iii) *For LTLA with* λ *sufficiently small*, *suppose* $\bar{T}^{(\beta,\gamma)}$, $\bar{S}^{(\gamma)}$ *are finite for some* $\beta, \gamma \geq 0$. *Then*,

$$\sum_x |x|^{\beta+\gamma} |\Pi(x)| < \infty. \tag{1.52}$$

This lemma is proved in Sections 3.2, 3.4 and 3.6.

Our second lemma is complementary to Lemma 1.8, and turns a bound on $\sum_x |x|^* |\Pi(x)|$ into those on $\bar{G}^{(\alpha)}$, $\bar{W}^{(\beta,\gamma)}$, $\bar{T}^{(\beta,\gamma)}$ and $\bar{S}^{(\gamma)}$.

LEMMA 1.9. *Suppose we have the expression* (1.18)–(1.22) *of* G *in terms of the lace expansion. Suppose further*

$$\sum_x |x|^{\phi} |\Pi(x)| < \infty \tag{1.53}$$

for some $\phi > 1$. *Then*, *we have for nonnegative* α, β, γ *which are not odd integers*:

$$\bar{G}^{(\alpha)} < \infty \qquad \text{if } \alpha \leq \phi \text{ and } \alpha < d-2, \tag{1.54}$$

$$\bar{W}^{(\beta,\gamma)} < \infty \qquad \text{if } \beta, \gamma \leq \lfloor \phi \rfloor, \beta+\gamma < d-4 \text{ and } \beta+\gamma-(\lfloor\beta\rfloor+\lfloor\gamma\rfloor) < 1, \tag{1.55}$$

$$\bar{T}^{(\beta,\gamma)} < \infty \qquad \text{if } \beta, \gamma \leq \lfloor \phi \rfloor, \beta+\gamma < d-6 \text{ and } \beta+\gamma-(\lfloor\beta\rfloor+\lfloor\gamma\rfloor) < 1, \tag{1.56}$$

$$\bar{S}^{(\gamma)} < \infty \qquad \text{if } \gamma \leq \lfloor \phi \rfloor \text{ and } \gamma < d-8, \tag{1.57}$$

$$\bar{H}^{(\beta)} < \infty \qquad \text{if } \beta \leq \lfloor \phi \rfloor, \beta < d-4, \text{ and } d > 6. \tag{1.58}$$

Odd integers are excluded to make the proof simpler. This restriction could be removed with some extra work, but the lemma is sufficient for our purpose in its current form.

We now explain how to prove (1.47) based on these lemmas. The basic idea is to use these lemmas repeatedly, and prove $\bar{G}^{(\alpha)}$ is finite for α required in (1.47). Consider SAW in $d \gg 1$. From (1.24) of Proposition 1.3, $\sum_x |x|^2|\Pi(x)|$ is finite. We start from this and use Lemmas 1.9 and 1.8 repeatedly, and see the quantities in the following sequence are all finite (we choose $\beta = 0$):

$$\sum_x |x|^2|\Pi(x)| \xrightarrow{\text{Lem } 1.9} \bar{G}^{(2)},\ \bar{W}^{(0,2)} \xrightarrow{\text{Lem } 1.8} \sum_x |x|^4|\Pi(x)| \xrightarrow{\text{Lem } 1.9} \bar{G}^{(4)},\ \bar{W}^{(0,4)} \xrightarrow{\text{Lem } 1.8} \cdots . \tag{1.59}$$

The exponents ϕ, α, γ are doubled in each iteration, and we can continue as far as the exponents satisfy the conditions of Lemma 1.9, that is, $\alpha < d-2$ and $\gamma < d-4$. For large d, α eventually exceeds $\frac{d+2}{3}$ required in (1.47), and we are done. For small d, it may not be so clear that α can exceed $\frac{d+2}{3}$, still satisfying $\gamma < d-4$. In the following we give a rigorous proof, focusing on this point.

PROOF OF (1.47), ASSUMING LEMMAS 1.8 AND 1.9. We begin with SAW in $d > 4$. Suppose $\sum_x |x|^{\phi_i}|\Pi(x)|$ is finite for some $\phi_i \ge 2$, and define

$$\begin{aligned} &\alpha_{i+1} = 2, \qquad \gamma_{i+1} = \{(d-4) \wedge \lfloor \phi_i \rfloor\} - \varepsilon, \\ &\phi_{i+1} = \alpha_{i+1} + \gamma_{i+1} = \{(d-2) \wedge (\lfloor \phi_i \rfloor + 2)\} - \varepsilon, \end{aligned} \tag{1.60}$$

with $0 < \varepsilon \ll 1$. Then Lemma 1.9 shows that $\bar{G}^{(\alpha_{i+1})}$ and $W^{(0,\gamma_{i+1})}$ are finite. Using this as an input to Lemma 1.8, we see that $\sum_x |x|^{\phi_{i+1}}|\Pi(x)|$ is finite as well, as long as ϕ_{i+1} is given by (1.60).

We start from $\phi_0 = 2$, and repeat the above procedure. First three iterations for ϕ_i read:

$$\begin{aligned} &\phi_0 = 2, && \phi_1 = \{(d-2) \wedge 4\} - \varepsilon, \\ &\phi_2 = \{(d-2) \wedge 5\} - \varepsilon, && \phi_3 = \{(d-2) \wedge 6\} - \varepsilon. \end{aligned} \tag{1.61}$$

As the above shows, ϕ_i is increased by one in each iteration, until it finally reaches $d-2-\varepsilon$. This in particular means $\sum_x |x|^{\phi}|\Pi(x)|$ is finite with $\phi = d-2-\varepsilon$.

Using Lemma 1.9 with $\phi = d-2-\varepsilon$ then implies that $\bar{G}^{(\alpha)}$ is finite with $\alpha = d-2-\varepsilon$, or $G(x) = O(|x|^{-(d-2-\varepsilon)})$. This is sufficient for (1.47), as long as $\frac{d+2}{3} < d-2-\varepsilon$, or $d > 4 + 3\varepsilon/2$. Because $\varepsilon > 0$ is arbitrary, this proves (1.47) for SAW in $d > 4$.

The proof proceeds in a similar fashion for percolation, using $\bar{W}^{(\beta,\gamma)}$, $\bar{T}^{(0,\gamma)}$, and $\bar{H}^{(\beta)}$. We start from $\phi_0 = 2$ and choose, instead of (1.60),

$$\begin{aligned} &\beta_{i+1} = 2, \qquad \gamma_{i+1} = \{(d-6) \wedge \lfloor \phi_i \rfloor\} - \varepsilon, \\ &\phi_{i+1} = \beta_{i+1} + \gamma_{i+1} = \{(d-4) \wedge (\lfloor \phi_i \rfloor + 2)\} - \varepsilon. \end{aligned} \tag{1.62}$$

For $d > 6$, repeating this recursion increases ϕ_i until it reaches $d - 4 - \varepsilon$. Using Lemma 1.9 with $\phi = d - 4 - \varepsilon$ implies $\bar{G}^{(\alpha)}$ is finite with $\alpha = d - 4 - \varepsilon$. This is sufficient for (1.47), as long as $\frac{d+2}{2} < d - 4 - \varepsilon$, or $d > 10$.

Finally we deal with LTLA, this time using $\bar{T}^{(\beta,\gamma)}$ and $\bar{S}^{(\gamma)}$. We start from $\phi_0 = 2$ and choose

$$\begin{aligned} &\beta_{i+1} = 2, \qquad \gamma_{i+1} = \{(d-8) \wedge \lfloor \phi_i \rfloor\} - \varepsilon, \\ &\phi_{i+1} = \beta_{i+1} + \gamma_{i+1} = \{(d-6) \wedge (\lfloor \phi_i \rfloor + 2)\} - \varepsilon. \end{aligned} \tag{1.63}$$

For $d > 8$, repeating this recursion increases ϕ_i until it reaches $d - 6 - \varepsilon$. Lemma 1.9 now implies $\bar{G}^{(\alpha)}$ is finite with $\alpha = d - 6 - \varepsilon$. This is sufficient for (1.47), as long as $\frac{3d+2}{4} < d - 6 - \varepsilon$, or $d > 26$. □

REMARK 1.10. The condition (1.47) follows immediately (for SAW in $d > 4$, for percolation in $d > 6$, and for LTLA in $d > 10$), *if* we can prove the x-space infrared bound, $G(x) \le c|||x|||^{2-d}$. Although there are models (e.g., nearest-neighbor Ising model) for which the k-space infrared bound (1.25) does imply its x-space counterpart ([27], Appendix A), it is not clear whether the same is true for models considered in this paper. The argument in this subsection has been employed to circumvent this difficulty.

2. Proof of a Gaussian lemma, Theorem 1.4. In this section, we prove Theorem 1.4. The proof is rather long, so we first present in Section 2.1 the framework of the proof, in particular that of (1.36), assuming some lemmas which are proven later in Section 2.2 through Section 2.4. In Section 2.5, we give an example which shows that the pointwise bound (1.35) is sharp in $d > 4$. Finally in Section 2.6, we comment on how to prove (1.38).

2.1. *Overview of the Proof of Theorem* 1.4, (1.36). Here we explain the framework of the proof of Theorem 1.4, in particular (1.36). The proof of (1.38) is similar, and is briefly explained in Section 2.6.

We first introduce an integral representation for $C(x)$, which was also used in [7]. The integrability of $\{1 - \hat{J}(k)\}^{-1}$ by (1.33), and a trivial identity $\frac{1}{A} = \int_0^\infty dt e^{-tA}$ $(A > 0)$ immediately imply

$$\begin{aligned} C(x) &= \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \frac{e^{ik\cdot x}}{1 - \hat{J}(k)} = \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} \int_0^\infty dt e^{-t\{1-\hat{J}(k)\}} \\ &= \int_0^\infty dt I_t(x), \end{aligned} \tag{2.1}$$

with

$$I_t(x) := \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} e^{-t\{1-\hat{J}(k)\}}. \tag{2.2}$$

Our task is to estimate this integral in detail.

We divide (2.1) into two parts. We define, depending on x,

$$T := \varepsilon |x|^2 \tag{2.3}$$

and

$$C_<(x) := \int_0^T dt\, I_t(x), \qquad C_>(x) := \int_T^\infty dt\, I_t(x) \tag{2.4}$$

so that

$$C(x) = C_<(x) + C_>(x). \tag{2.5}$$

In the above, ε is a small positive number, and will be sent to zero at the last step. The choice of T is suggested by the fact that the variable t roughly corresponds to the number of steps of random walks; compare with the method of Lawler [16], Chapter 1.

Now, for our choice of $T = \varepsilon|x|^2$, and for x satisfying $|x| \geq 1/\varepsilon$, we have the following estimates, which are proven in Sections 2.3 and 2.4, respectively:

$$\begin{aligned} C_>(x) &= \frac{a_d}{K_1} \frac{1}{|x|^{d-2}} + R_1(x) \\ &\text{with } |R_1(x)| \leq o\left(\frac{1}{|x|^{d-2}}\right) + \frac{c_1 \varepsilon^{-d/2+1} e^{-c_2/\varepsilon}}{|x|^{d-2}} + \frac{c_3 \varepsilon}{|x|^{d-2}}, \end{aligned} \tag{2.6}$$

$$|C_<(x)| \leq \frac{c_4 \varepsilon}{|x|^{d-2}}. \tag{2.7}$$

Here c_1 through c_4 (given explicitly in the proof) are finite positive constants which can be expressed in terms of d and K_i but are independent of ε and x. The error term $o(|x|^{2-d})$ *does* depend on ε.

These two estimates, together with (2.5), immediately prove (1.36). That is, for fixed $\varepsilon > 0$,

$$\begin{aligned} \limsup_{|x|\to\infty} |x|^{d-2} C(x) &\leq \frac{a_d}{K_1} + [c_1 \varepsilon^{-d/2+1} e^{-c_2/\varepsilon} + c_3\varepsilon + c_4\varepsilon], \\ \liminf_{|x|\to\infty} |x|^{d-2} C(x) &\geq \frac{a_d}{K_1} - [c_1 \varepsilon^{-d/2+1} e^{-c_2/\varepsilon} + c_3\varepsilon + c_4\varepsilon]. \end{aligned} \tag{2.8}$$

Now letting $\varepsilon \downarrow 0$ establishes

$$\limsup_{|x|\to\infty} |x|^{d-2} C(x) = \liminf_{|x|\to\infty} |x|^{d-2} C(x) = \lim_{|x|\to\infty} |x|^{d-2} C(x) = \frac{a_d}{K_1}. \tag{2.9}$$

We in the following prove (2.6) and (2.7) step by step.

2.2. *Estimates on* $\hat{J}(k)$. We start from some estimates on $1 - \hat{J}(k)$.

LEMMA 2.1. *Assume* (1.33)–(1.34) *of Theorem* 1.4 *are satisfied. Then* $\hat{J}(k)$ *satisfies for* $k \in [-\pi, \pi]^d$

$$0 \le 1 - \hat{J}(k) \le K_2 \frac{|k|^2}{2d} \tag{2.10}$$

and

$$1 - \hat{J}(k) = K_1 \frac{|k|^2}{2d} + \hat{R}_2(k) \qquad \text{with } |\hat{R}_2(k)| = o(|k|^2). \tag{2.11}$$

In the above, $o(|k|^2)$ *depends only on* d *and* K_i'*s*. *Also*, *we have*

$$\left| \frac{\partial}{\partial k_1} \hat{J}(k) \right| \le \frac{K_2}{d} |k_1|, \qquad \left| \frac{\partial^2}{\partial k_1^2} \hat{J}(k) \right| \le \frac{K_2}{d}. \tag{2.12}$$

PROOF. We first note by (1.33)

$$\begin{aligned} 1 - \hat{J}(k) &= \hat{J}(0) - \hat{J}(k) = \sum_x \{1 - \cos(k \cdot x)\} J(x) \\ &= \sum_x \frac{(k \cdot x)^2}{2} J(x) + \sum_x \left\{ 1 - \cos(k \cdot x) - \frac{(k \cdot x)^2}{2} \right\} J(x). \end{aligned} \tag{2.13}$$

Using $0 \le 1 - \cos t \le t^2/2$ and $\mathbb{Z}^d$-symmetry, we have from the first line of (2.13)

$$0 \le 1 - \hat{J}(k) \le \sum_x \frac{(k \cdot x)^2}{2} |J(x)| = \sum_x \frac{|k|^2}{2d} |x|^2 |J(x)| \le \frac{|k|^2}{2d} K_2. \tag{2.14}$$

Also, by $\mathbb{Z}^d$-symmetry of $J(x)$, the first term on the second line of (2.13) is equal to

$$\frac{|k|^2}{2d} \sum_x |x|^2 J(x) = K_1 \frac{|k|^2}{2d}. \tag{2.15}$$

Next we proceed to deal with the second term of (2.13), that is, $\hat{R}_2(k)$. We want to show that it is of smaller order than $|k|^2$, so we consider $|k|^{-2} \hat{R}_2(k)$:

$$\frac{\hat{R}_2(k)}{|k|^2} = \sum_x \frac{1 - \cos(k \cdot x) - (k \cdot x)^2/2}{|k|^2} J(x). \tag{2.16}$$

Now we note for all $t \in \mathbb{R}$

$$\left| 1 - \cos t - \frac{t^2}{2} \right| \le \frac{t^2}{2} \wedge \frac{t^4}{24}. \tag{2.17}$$

The first bound of (2.17) can be used to show that the sum in (2.16) is uniformly bounded in k:

$$\begin{aligned}\frac{|\hat{R}_2(k)|}{|k|^2} &\le \sum_x \left|\frac{1-\cos(k\cdot x)-(k\cdot x)^2/2}{|k|^2}J(x)\right| \le \sum_x \frac{(k\cdot x)^2}{2|k|^2}|J(x)| \\ &= \frac{1}{2d}\sum_x |x|^2|J(x)| \le \frac{K_2}{2d},\end{aligned} \tag{2.18}$$

where on the second line we used $\mathbb{Z}^d$-symmetry as we did in (2.14). The second bound of (2.17) shows that the summand of (2.16) goes to zero (as $|k|\to 0$) for each fixed $x\in\mathbb{Z}^d$. Therefore, by dominated convergence, the sum of (2.16) goes to zero as $|k|\to 0$, that is, we have the bound of (2.11).

To prove (2.12), we observe by $\mathbb{Z}^d$-symmetry

$$\begin{aligned}\left|\frac{\partial}{\partial k_1}\hat{J}(k)\right| &= \left|\sum_x x_1\sin(k_1x_1)\cos(k_2x_2)\cdots\cos(k_dx_d)J(x)\right| \\ &\le \sum_x |k_1||x_1|^2|J(x)| \le \frac{K_2}{d}|k_1|.\end{aligned} \tag{2.19}$$

Similarly, we note

$$\left|\frac{\partial^2}{\partial k_1^2}\hat{J}(k)\right| = \left|\sum_x \cos(k\cdot x)|x_1|^2 J(x)\right| \le \sum_x |x_1|^2|J(x)| \le \frac{K_2}{d}. \tag{2.20}$$

□

2.3. *Contribution from $t\ge T$: Proof of* (2.6). In this section, we prove (2.6), which gives an estimate on $C_>(x)$. The estimate itself is an immediate consequence of the following lemma. Note that no pointwise bound (1.35) is needed for this lemma.

LEMMA 2.2. *Fix $\varepsilon>0$ and assume* (1.33)–(1.34) *of Theorem* 1.4. *Then we have for $t\ge 1/\varepsilon$*

$$\begin{aligned} I_t(x) &= \left(\frac{d}{2\pi K_1 t}\right)^{d/2}\exp\left(-\frac{d|x|^2}{2tK_1}\right) + R_3(t) \\ &\qquad \text{with } |R_3(t)| \le o(t^{-d/2}) + c_5 e^{-c_2/\varepsilon}t^{-d/2}.\end{aligned} \tag{2.21}$$

In the above, $c_5 := 2(d/\pi K_0)^{d/2}$ and $c_2 := K_0/(4d)$. The term $o(t^{-d/2})$ may depend on ε.

PROOF OF (2.6), ASSUMING LEMMA 2.2. We just integrate (2.21) from $t = T := \varepsilon|x|^2$ to $t=\infty$. (We can apply Lemma 2.2 because of our choice of $|x|$

and T.) The integral of $R_3(t)$ is bounded as

$$\left|\int_T^\infty dt\, R_3(t)\right| \le \int_T^\infty dt[o(t^{-d/2}) + c_5 e^{-c_2/\varepsilon} t^{-d/2}]$$

$$(2.22)\qquad = o(T^{-d/2+1}) + \frac{2c_5 e^{-c_2/\varepsilon}}{d-2} T^{-d/2+1}$$

$$= \varepsilon^{-d/2+1} o\big(|x|^{-(d-2)}\big) + c_1 \varepsilon^{-d/2+1} e^{-c_2/\varepsilon} |x|^{-(d-2)},$$

$$c_1 := \frac{2c_5}{d-2},$$

where on the second line, we used our choice of T, (2.3). On the other hand, the first term of (2.21) gives

$$\int_T^\infty dt \left(\frac{d}{2\pi K_1 t}\right)^{d/2} \exp\left(-\frac{d|x|^2}{2tK_1}\right)$$

$$= \int_0^\infty dt(\cdots) - \int_0^T dt(\cdots)$$

$$(2.23)\qquad = \frac{\Gamma(d/2-1)d}{2\pi^{d/2}K_1}|x|^{2-d} - \int_0^T dt \left(\frac{1}{2\pi K_1 t}\right)^{d/2} \exp\left(-\frac{d|x|^2}{2tK_1}\right)$$

$$=: \frac{\Gamma(d/2-1)d}{2\pi^{d/2}K_1}|x|^{2-d} - R_4(x).$$

For the integrand of $R_4(x)$, we use an inequality

$$(2.24)\qquad y^{-\beta} e^{-\alpha/y} \le \left(\frac{\beta}{\alpha e}\right)^\beta \qquad \text{(valid for } \alpha, \beta, y > 0)$$

with $\alpha = d|x|^2/(2K_1)$, $\beta = d/2$ and $y = t$. The result is

$$(2.25)\qquad [\text{integrand of } R_4(x)] \le \left(\frac{1}{2\pi K_1}\right)^{d/2} \times \left(\frac{K_1}{e|x|^2}\right)^{d/2} = \left(\frac{1}{2\pi e|x|^2}\right)^{d/2},$$

and thus, $R_4(x)$ is bounded as

$$(2.26)\qquad R_4(x) \le T \times \left(\frac{1}{2\pi e|x|^2}\right)^{d/2} = \left(\frac{1}{2\pi e}\right)^{d/2} \frac{\varepsilon}{|x|^{d-2}} := \frac{c_3\varepsilon}{|x|^{d-2}}.$$

Combining (2.22)–(2.26), we get (2.6). □

PROOF OF LEMMA 2.2. We introduce $k_t > 0$ by

$$(2.27)\qquad k_t := (\varepsilon t)^{-1/2}, \qquad (\le 1)$$

and divide $I_t(x)$ into four parts:

$$(2.28)\qquad I_t(x) = I_{t,1}(x) + I_{t,2}(x) + I_{t,3}(x) + I_{t,4}(x),$$

with

$$I_{t,1}(x) := \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} e^{ik\cdot x - tK_1|k|^2/(2d)}, \tag{2.29a}$$

$$I_{t,2}(x) := -\int_{|k|>k_t} \frac{d^d k}{(2\pi)^d} e^{ik\cdot x - tK_1|k|^2/(2d)}, \tag{2.29b}$$

$$I_{t,3}(x) := \int_{|k|\le k_t} \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} \{e^{-t\{1-\hat{J}(k)\}} - e^{-tK_1|k|^2/(2d)}\}, \tag{2.29c}$$

$$I_{t,4}(x) := \int_{|k|>k_t, k\in[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} e^{-t\{1-\hat{J}(k)\}}. \tag{2.29d}$$

Integrals $I_{t,1}(x)$ through $I_{t,3}(x)$ sum up to contributions to $I_t(x)$ from $|k| \le k_t$, and $I_{t,4}(x)$ represents the contribution from $|k| > k_t$, $k \in [-\pi, \pi]^d$. The choice of k_t is motivated so that we can use (1.33) for $I_{t,3}(x)$ (because of our choice $t \ge 1/\varepsilon$, we have $|k| \le k_t \le 1$). We estimate the above integrals one by one.

The first integral $I_{t,1}(x)$ gives the main contribution, and is calculated exactly by completing the square:

$$I_{t,1}(x) = \left(\frac{d}{2\pi K_1 t}\right)^{d/2} \exp\left(-\frac{d|x|^2}{2tK_1}\right). \tag{2.30}$$

Second, for $I_{t,3}(x)$, we first change the integration variable from k to $l := \sqrt{t}k$ to obtain

$$\begin{aligned} I_{t,3}(x) = t^{-d/2} \int_{|l|^2\le 1/\varepsilon} \frac{d^d l}{(2\pi)^d} \exp\left(\frac{il\cdot x}{\sqrt{t}}\right) \\ \times \left\{\exp\left(-t\left\{1 - \hat{J}\left(\frac{l}{\sqrt{t}}\right)\right\}\right)\right. \\ \left. - \exp\left(-t\frac{K_1|l|^2}{2dt}\right)\right\} =: t^{-d/2}\tilde{I}_{t,3}(x). \end{aligned} \tag{2.31}$$

Now the integral $\tilde{I}_{t,3}(x)$ is seen to be $o(1)$ as $t \to \infty$, as follows. (i) We can get a uniform bound as

$$|\tilde{I}_{t,3}(x)| \le \int_{|l|^2\le 1/\varepsilon} \frac{d^d l}{(2\pi)^d} \left[\exp\left(-\frac{K_0|l|^2}{2d}\right) + \exp\left(-\frac{K_1|l|^2}{2d}\right)\right] < \infty, \tag{2.32}$$

where we used the lower bound (1.33) for the first term. (ii) For each fixed $l \in \mathbb{R}^d$, the integrand of $\tilde{I}_{t,3}(x)$ goes to zero (as $t \to \infty$). This is because we can write [recalling the definition (2.11) of $\hat{R}_2$]

$$\begin{aligned} \tilde{I}_{t,3}(x) = \int_{|l|^2\le 1/\varepsilon} \frac{d^d l}{(2\pi)^d} \exp\left(\frac{il\cdot x}{\sqrt{t}} - \frac{K_1|l|^2}{2d}\right) \\ \times \left\{\exp\left(-t\hat{R}_2\left(\frac{l}{\sqrt{t}}\right)\right) - 1\right\}. \end{aligned} \tag{2.33}$$

Then, in view of our bound of (2.11), the integrand goes to zero as $t \to \infty$ for fixed l. By (i) and (ii) above, we can use the dominated convergence theorem to conclude that the integral $\tilde{I}_{t,3}(x)$ of (2.31) is $o(1)$ as $t \to \infty$ [this $o(1)$ can depend on ε], and therefore

$$I_{t,3}(x) = o(t^{-d/2}) \qquad [o(t^{-d/2}) \text{ can depend on } \varepsilon]. \tag{2.34}$$

Finally we estimate $I_{t,2}(x)$ and $I_{t,4}(x)$. By definition,

$$|I_{t,2}(x)| \le \int_{|k|>k_t} \frac{d^d k}{(2\pi)^d} e^{-tK_1|k|^2/(2d)}. \tag{2.35}$$

For $I_{t,4}(x)$, we use (1.33) to bound $\hat{J}(0) - \hat{J}(k)$ for $k \in [-\pi,\pi]^d$ as

$$\begin{aligned} |I_{t,4}(x)| &:= \left| \int_{\substack{|k|>k_t \\ k\in[-\pi,\pi]^d}} \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} e^{-t\{\hat{J}(0)-\hat{J}(k)\}} \right| \\ &\le \int_{\substack{|k|>k_t \\ k\in\mathbb{R}^d}} \frac{d^d k}{(2\pi)^d} e^{-tK_0|k|^2/(2d)}. \end{aligned} \tag{2.36}$$

Therefore, using $K_0 \le K_1$ and

$$\int_{|k|\ge b} \frac{d^d k}{(2\pi)^d} e^{-a|k|^2} \le \left(\frac{1}{2\pi a}\right)^{d/2} e^{-ab^2/2} \qquad (\text{valid for } a, b > 0), \tag{2.37}$$

we get

$$\begin{aligned} |I_{t,2}(x) + I_{t,4}(x)| &\le 2\left(\frac{d}{\pi K_0 t}\right)^{d/2} e^{-tK_0 k_t^2/(4d)} \\ &= 2\left(\frac{d}{\pi K_0}\right)^{d/2} t^{-d/2} e^{-K_0/(4d\varepsilon)} =: c_5 t^{-d/2} e^{-c_2/\varepsilon}. \end{aligned} \tag{2.38}$$

The above (2.30), (2.34) and (2.38) establish (2.21) and prove the lemma. □

2.4. *Contribution from $t < T$: Proof of* (2.7). In this section, we prove (2.7), which gives an estimate on $C_<(x)$. This is the place where we have to make use of our assumption on *pointwise* x-space bound on $J(x)$, (1.35). We do need something like this, to exclude pathological examples which violate (2.7) for infinitely many x's (see, e.g., page 32 of [19] and Section 2.5 of the present paper).

The estimate (2.7) itself is an immediate consequence of the following lemma. To state the lemma, we introduce some notation. For a function $f(x)$ on $\mathbb{Z}^d$, we write $f^{(*n)}$ for the n-fold convolution of f and use $\prod_{\ell=1}^{n}{}^{(*)} f_\ell$ to denote the convolution of functions f_ℓ, $\ell = 1, 2, \dots, n$:

$$f^{(*n)}(x) := (\underbrace{f * f * \cdots * f}_{n})(x),$$
$$\prod_{\ell=1}^{n}{}^{(*)} f_\ell := (f_1 * f_2 * f_3 * \cdots * f_n)(x). \tag{2.39}$$

Also, in this subsection and for the function $J(x)$ only, we define for $j, \ell = 1, 2, \dots, d$,

$$J_j(x) := x_j J(x), \qquad J_{j,\ell}(x) := x_j x_\ell J(x). \tag{2.40}$$

$J_j^{(*n)}$ denotes the n-fold convolution of J_j, *not* x_j times $J^{(*n)}$.

LEMMA 2.3. *Under the assumption of Theorem 1.4, we have for integers $m \in [0, d]$ and $n_1, n_2, \dots, n_d \geq 0$*

$$\left|\left(I_t * \prod_{j=1}^{d}{}^{(*)} J_j^{(*n_j)}\right)(x)\right| \leq c_6(m, \vec{n}) \frac{t^{-(d+n-m)/2}}{|||x|||^m} \qquad \text{with } n := \sum_{j=1}^{d} n_j, \tag{2.41}$$

where $c_6(m, \vec{n})$ is a calculable constant depending on K_i, d, m and $\vec{n} := (n_1, n_2, \dots, n_d)$.

PROOF OF (2.7), GIVEN LEMMA 2.3. This is easy. By (2.41) with $\vec{n} = \vec{0}$ and $m = d$, we have $|I_t(x)| \leq c_6(d, \vec{0}) |||x|||^{-d}$. Integrating this from $t = 0$ to $t = T$ gives

$$|C_<(x)| \leq T \times \frac{c_6(d, \vec{0})}{|||x|||^d} = \frac{c_6(d, \vec{0})\varepsilon}{|||x|||^{d-2}}, \tag{2.42}$$

where the last equality follows from our choice of T, (2.3). This proves (2.7), with $c_4 = c_6(d, \vec{0})$. $\square$

PROOF OF LEMMA 2.3. We first prove the lemma for $m = 0$ by estimating Fourier integrals directly. We then proceed to prove the lemma for $m \geq 1$ by induction in m. To simplify notation, we abbreviate the left-hand side of (2.41) (without the absolute value) as $F_{\vec{n}}(x; t)$.

The case $m = 0$. In terms of Fourier transform, we have

$$F_{\vec{n}}(x; t) = \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ikx} e^{-t\{1-\hat{J}(k)\}} \prod_{j=1}^{d} \{i\partial_j \hat{J}(k)\}^{n_j}, \tag{2.43}$$

where (and in the following) ∂_j denotes $\partial/\partial k_j$. Using bounds (1.33) and (2.12), we can bound (2.43) as

$$
|F_{\vec{n}}(x;t)| \le \left(\frac{K_2}{d}\right)^n \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} e^{-tK_0|k|^2/(2d)} |k|^n
= ct^{-(d+n)/2}, \qquad n := \sum_{j=1}^d n_j \tag{2.44}
$$

with some constant c. This proves (2.41) for $m = 0$, if we take $c_6(0, \vec{n}) \ge c$.

The case $x = 0$. The above (2.44) also proves the lemma for $x = 0$, $t \ge 1$ and for all $m \ge 0$, because in this case the right-hand side of (2.41) increases as m increases and thus the bound for $m = 0$ takes care of those for $m > 0$ as well.

For $x = 0$ and $t \le 1$, we first note a trivial bound

$$
|F_{\vec{n}}(0)| \le \left(\frac{K_2}{d}\right)^n \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{-tK_0|k|^2/(2d)} |k|^n \le \left(\frac{\pi K_2}{\sqrt{d}}\right)^n, \tag{2.45}
$$

where we just bounded the exponential by 1 and $|k|^n$ by $(\sqrt{d}\pi)^n$. Multiplying the right-hand side by $t^{-(d+n-m)/2} \ge 1$ (valid for $0 \le m \le d+n$) proves (2.41) for $x = 0, t < 1$ and $0 \le m \le d+n$.

We have thus proved (2.41) for $x = 0$ and $n \ge 0$, $m \in (0, d+n]$. Having treated $x = 0$, we in the following focus on $x \ne 0$.

The case $m \ge 1$. Suppose we have proved (2.41) for $m-1$; we now prove it for m by induction. With the help of Fourier transform, we see for $l = 1, 2, \ldots, d$:

$$
\begin{aligned}
x_l F_{\vec{n}}(x) &= i^{n+1} \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ikx} \partial_l \left[e^{-t\{1-\hat{J}(k)\}} \prod_{j=1}^d \{\partial_j \hat{J}(k)\}^{n_j} \right] \\
&= i^{n+1} \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ikx} e^{-t\{1-\hat{J}(k)\}} \\
&\qquad \times \left[t\{\partial_l \hat{J}(k)\}^{n_l+1} \prod_{j \ne l} \{\partial_j \hat{J}(k)\}^{n_j} \right. \\
&\qquad \left. + \sum_{p=1}^d n_p \{\partial_p \hat{J}(k)\}^{n_p-1} \{\partial_l \partial_p \hat{J}(k)\} \prod_{j \ne p} \{\partial_j \hat{J}(k)\}^{n_j} \right] \\
&= tF_{\vec{n}'}(x) + \sum_{p=1}^d n_p (F_{\vec{n}''} * J_{p,l})(x),
\end{aligned} \tag{2.46}
$$

where $\vec{n}' = (n_1, n_2, \ldots, n_{l-1}, n_l + 1, n_{l+1}, \ldots)$ and $\vec{n}'' = (n_1, n_2, \ldots, n_{p-1}, n_p - 1, n_{p+1}, \ldots)$. The first term on the right-hand side is simply bounded by our inductive assumption as (note: now $\sum_j n'_j = n+1$)

$$
\begin{aligned}
|tF_{\vec{n}'}(x)| &\le t \times c_6(m-1,\vec{n}')\frac{t^{-(d+n+1-m+1)/2}}{|||x|||^{m-1}} \\
&= c_6(m-1,\vec{n}')\frac{t^{-(d+n-m)/2}}{|||x|||^{m-1}}.
\end{aligned}
\tag{2.47}
$$

For the second term, we again use our inductive assumption on $F_{\vec{n}''}$ (now $\sum_j n''_j = n-1$):

$$
\begin{aligned}
|F_{\vec{n}''}(x)| &\le c_6(m-1,\vec{n}'')\frac{t^{-(d+n-1-m+1)/2}}{|||x|||^{m-1}} \\
&= c_6(m-1,\vec{n}'')\frac{t^{-(d+n-m)/2}}{|||x|||^{m-1}}.
\end{aligned}
\tag{2.48}
$$

We have to take the convolution with $J_{p,l}$, and we argue separately for $m=1$ and $m>1$. For $m=1$, we estimate as

$$
\begin{aligned}
|(F_{\vec{n}''} * J_{p,l})(x)| &\le c_6(0,\vec{n}'')t^{-(d+n-1)/2}\sum_y |J_{p,l}(y)| \\
&\le c_6(0,\vec{n}'')t^{-(d+n-1)/2}\sum_y |y|^2|J(y)| \\
&\le c_6(0,\vec{n}'')t^{-(d+n-1)/2} \times K_2,
\end{aligned}
\tag{2.49}
$$

where we used our assumption (1.34) in the last step. For $m>1$, we take the convolution of (2.48) with

$$
|J_{p,l}(x)| \le \frac{K_3|x_p x_l|}{|||x|||^{d+2}} \le \frac{K_3}{|||x|||^{d}},
\tag{2.50}
$$

which satisfies $\sum_x |J_{p,l}(x)| \le \sum_x |x|^2|J(x)| < \infty$. The power $(m-1)$ of (2.48) is not changed by the convolution as long as $0 < m-1 < d$ [see Lemma B.1(iii)], and we get

$$
|(F_{\vec{n}''} * J_{p,l})(x)| \le cc_6(m-1,\vec{n}'')\frac{t^{-(d+n-m)/2}}{|||x|||^{m-1}}
\tag{2.51}
$$

with some constant c arising from convolution. Thus for both $m=1$ and $1<m\le d$, we get a bound of the form of (2.51) for the second term of (2.46).

Combining (2.47) and (2.51), we get

$$
\begin{aligned}
|x_l F_{\vec{n}}(x)| &\le c_6(m,\vec{n})'\frac{t^{-(d+n-m)/2}}{|||x|||^{m-1}} \quad \text{or} \\
|F_{\vec{n}}(x)| &\le c_6(m,\vec{n})'\frac{t^{-(d+n-m)/2}}{|x_l|\,|||x|||^{m-1}}
\end{aligned}
\tag{2.52}
$$

with $c_6(m, \vec{n})' = c_6(m-1, \vec{n}') + cc_6(m-1, \vec{n}'')$. Because the above holds for all $l = 1, 2, \ldots, d$, we can replace $|x_l|$ by $\|x\|_\infty$ in the above, and we get (2.41) for m [increase $c_6(m, \vec{n})'$ appropriately in order to turn $\|x\|_\infty$ into $|||x|||$]. The proof is complete. □

2.5. *We cannot do better than* $|J(x)| \le c|x|^{-(d+2)}$: *An example.* We here present a "counterexample," which mildly violates the pointwise bound $|J(x)| \le c|x|^{-(d+2)}$ and which does not exhibit the Gaussian asymptotic form of (1.36). The pointwise bound is not a necessary condition, but the following example shows that it is rather sharp for $d > 4$.

The author is grateful to Kôhei Uchiyama concerning the proof of Proposition 2.4.

PROPOSITION 2.4. *Fix $d > 4$ and $0 < \varepsilon < (d-4)/4$, and let $g(x)$ be a slowly varying, nonnegative, $\mathbb{Z}^d$-symmetric function which diverges as $|x| \to \infty$. Define*

$$h(x) = g(x)^{-(1+\varepsilon)/d}, \tag{2.53}$$

and subsets of $\mathbb{Z}^d$ as

$$\begin{aligned} \mathcal{E} &:= \{\pm l_n \boldsymbol{e}_j \,|\, 1 \le j \le d, n \ge 1\}, \\ \tilde{\mathcal{E}} &:= \{y \in \mathbb{Z}^d \,|\, \exists x \in \mathcal{E}, |y - x| \le h(x)|x|\}, \end{aligned} \tag{2.54}$$

where $\boldsymbol{e}_j$ is the unit vector in the jth coordinate axis. Finally define

$$J(x) := \frac{1-\delta}{2d} I[|x| = 1] + \frac{g(x)}{|x|^{d+2}} I[x \in \tilde{\mathcal{E}}], \tag{2.55}$$

where δ is determined so that $\sum_x J(x) = 1$. Then by choosing a sequence l_n which diverges to infinity sufficiently rapidly (*depending on g*), *we can achieve*

$$\limsup_{|x| \to \infty} |x|^{d-2} C(x) = \infty. \tag{2.56}$$

That is, *the model does not exhibit the Gaussian asymptotic form of Theorem* 1.4.

PROOF. We choose l_n which diverges sufficiently rapidly as $n \to \infty$, so that (1) $J(x) \ge 0$ for all $x \in \mathbb{Z}^d$, and (2) $\sum_x |x|^2 J(x) < \infty$. We prove, for $x \in \mathcal{E}$ with sufficiently large $|x|$,

$$C(x) \ge \frac{c}{|x|^{d-2}} g(x) h(x)^4 \exp\{-c' g(x) h(x)^d\} \tag{2.57}$$

with finite positive constants c, c' which are independent of x. This immediately implies

$$\lim_{\substack{|x| \to \infty \\ x \in \mathcal{E}}} C(x)|x|^{d-2} \ge \lim_{\substack{|x| \to \infty \\ x \in \mathcal{E}}} c g(x)^{(d-4-4\varepsilon)/d} \exp\{-c' g(x)^{-\varepsilon}\} = \infty, \tag{2.58}$$

because of our choice of g and h. In the following, we explain how to get (2.57).

First choose arbitrary but large n and define $a = l_n$. We prove (2.57) for $x = a\boldsymbol{e}_1$, which is sufficient. Define

$$\begin{aligned} q^a(y) &:= J(y) \sum_{j=1}^{d} \{I[|y + a\boldsymbol{e}_j| \le ah(a)] + I[|y - a\boldsymbol{e}_j| \le ah(a)]\}, \\ p^a(y) &:= J(y) - q^a(y). \end{aligned} \tag{2.59}$$

[With an abuse of notation, we write $g(a)$ and $h(a)$ for $g(a\boldsymbol{e}_j)$ and $h(a\boldsymbol{e}_j)$.] Because both p^a and q^a are nonnegative, we get a lower bound on $C(x)$ by discarding some terms as

$$\begin{aligned} C(x) &= \sum_{n=0}^{\infty} (p^a + q^a)^{(*n)}(x) \\ &\ge \sum_{n=0}^{\infty} n\big((p^a)^{(*(n-1))} * q^a\big)(x) = (q^a * C^a * C^a)(x) \end{aligned} \tag{2.60}$$

where we introduced locally

$$C^a(y) := \sum_{n=0}^{\infty} (p^a)^{(*n)}(y). \tag{2.61}$$

We further get a lower bound of (2.60) by restricting the sum arising from the convolution:

$$\begin{aligned} C(x) &\ge \sum_{y : |y - a\boldsymbol{e}_1| \le ah(a)} q^a(y)(C^a * C^a)(x - y) \\ &\ge \frac{g(a)}{2a^{d+2}} \sum_{|z| \le ah(a)} (C^a * C^a)(z), \end{aligned} \tag{2.62}$$

where in the last step we used $g(y) \ge g(a)/2$, because $g(x)$ is slowly varying.

To get a nice lower bound on $(C^a * C^a)(z)$, we use the following integral representation

$$\begin{aligned} (C^a * C^a)(z) &= \int_0^{\infty} dt\, t \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ikz} e^{-t(1-\hat{p}^a(k))} \\ &\ge \int_{T_1}^{T_2} dt\, t \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ikz} e^{-t(1-\hat{p}^a(k))}, \end{aligned} \tag{2.63}$$

where $T_1 := |z|^2$, $T_2 := 2|z|^2$. In the last step we used the fact that the integrand [inverse Fourier transform of $e^{-t(1-\hat{p}^a(k))}$] is nonnegative. This fact can be seen by writing it as

$$\begin{aligned}\int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ikz} e^{-t(1-\hat{p}^a(k))} &= e^{-t} \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ikz} \sum_{n=0}^{\infty} \frac{t^n}{n!} (\hat{p}^a(k))^n \\ &= e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} (p^a)^{(*n)}(z),\end{aligned} \tag{2.64}$$

and use the fact that $p^a(z)$ is nonnegative by its definition, (2.59). Because we have $\hat{p}^a(0) < 1$ now, we bound the right-hand side of (2.63) as

$$(C^a * C^a)(z) \geq \int_{T_1}^{T_2} dt\, t e^{-t(1-\hat{p}^a(0))} I_t^a(z) \geq e^{-T_2(1-\hat{p}^a(0))} \int_{T_1}^{T_2} dt\, t I_t^a(z) \tag{2.65}$$

with

$$I_t^a(z) := \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ikz} e^{-t(\hat{p}^a(0)-\hat{p}^a(k))}. \tag{2.66}$$

The first exponent of (2.65) can be bounded as

$$\begin{aligned} T_2\bigl(1-\hat{p}^a(0)\bigr) = T_2 \hat{q}^a(0) &= T_2 \sum_{y\,:\,|y \pm a\boldsymbol{e}_j| \leq ah(a)} \frac{g(y)}{|y|^{d+2}} \\ &\leq c T_2 g(a) h(a)^d a^{-2} \leq c g(a) h(a)^d,\end{aligned} \tag{2.67}$$

with some constant c, where in the last step we used $|z| \leq ah(a) \leq a$.

The remaining integral of $I_t^a(z)$ can be estimated as we did in Section 2.3. By Lemma 2.2, we have

$$I_t^a(z) = \left(\frac{d}{2\pi K_1' t}\right)^{d/2} \exp\left(-\frac{d|z|^2}{2t K_1'}\right) + o(t^{-d/2}) + c_5 e^{-c_2/\varepsilon} t^{-d/2} \tag{2.68}$$

with $K_1' \approx K_1$ for $\varepsilon > 0$ and $t > 1/\varepsilon$. For ε sufficiently small and for $t \geq |z|^2$ sufficiently large depending on ε, the first term of (2.68) dominates the rest. So we have

$$\int_{T_1}^{T_2} dt\, t I_t^a(z) \geq \int_{T_1}^{T_2} dt\, t \frac{1}{2} \left(\frac{d}{2\pi K_1' t}\right)^{d/2} \exp\left(-\frac{d|z|^2}{2t K_1'}\right) \geq \frac{c''}{|z|^{d-4}} \tag{2.69}$$

for sufficiently large z.

Combining (2.65), (2.67) and (2.69), we have for sufficiently large $|z|$

$$(C^a * C^a)(z) \geq c|z|^{4-d} \exp\{-c' g(a) h(a)^d\} \tag{2.70}$$

with positive constants c, c'. Going back to (2.62) yields, for sufficiently large $ah(a)$,

$$\begin{aligned} C(x) &\geq c \frac{g(a)}{a^{d+2}} \times \sum_{L < |z| \leq ah(a)} c|z|^{4-d} \exp\{-c' g(a) h(a)^d\} \\ &= c \exp\{-c' g(a) h(a)^d\} \frac{g(a) h(a)^4}{a^{d-2}}.\end{aligned} \tag{2.71}$$

This proves (2.57). □

2.6. *Proof of* (1.38). We here explain briefly how to prove (1.38). The framework of the proof is the same as that of (1.36), except that we choose different T and that we use explicit error bounds instead of Riemann–Lebesgue lemma. Concretely, we proceed as follows.

First, instead of (2.3), we now choose

$$T := |x|^{2-(\rho\wedge 2)/d}, \tag{2.72}$$

and use the decomposition (2.4) and (2.5).

Improved bound (1.37) on $J(x)$ improves several estimates concerning contributions from $t > T$. First, the error term $\hat{R}_2(k)$ of Lemma 2.1 now obeys $|\hat{R}_2(k)| \le \frac{K_2'}{2}|k|^{2+(\rho\wedge 2)}$. Taking $k_t = t^{-1/(2+\rho\wedge 2)}$ and using this new bound on $\hat{R}_2(k)$ improves Lemma 2.2's error bound as $|R_3(t)| \le ct^{-(d+\rho\wedge 2)/2}$. This leads, with the new choice of T, (2.72), to

$$C_>(x) = \frac{a_d}{K_1}|x|^{2-d} + O\big(|x|^{-(d-2+(\rho\wedge 2)/d)}\big). \tag{2.73}$$

Not much is improved for $t < T$, and we use Lemma 2.3 in its current form, that is, $I_t(x) = O(|x|^{-d})$. Because of the new choice of T, (2.72), this leads to slightly improved

$$C_<(x) = O\big(|x|^{-(d-2+(\rho\wedge 2)/d)}\big). \tag{2.74}$$

Combining (2.73) and (2.74) yields (1.38), and completes the proof.

3. Diagrammatic estimates. Here we prove several diagrammatic estimates, Lemmas 1.8 and 1.7 for Π of the lace expansion. These estimates are model dependent, and have to be proved individually for each model.

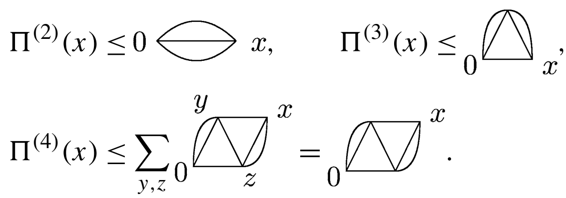
3.1. *Brief notes on diagrammatic estimates.* We first introduce some graphical notation and briefly explain basic techniques of diagrammatic estimates. These methods have been extensively used in previous works. Consult [12, 19, 26] for reviews on the lace expansion and diagrammatic estimates involved.

For self-avoiding walk, $\Pi^{(0)}(x)$ is identically zero and $\Pi^{(1)}(x)$ is nonzero only at $x = 0$. Next few terms of $\Pi^{(n)}(x)$ are bounded as follows:

$$\begin{aligned} &\Pi^{(2)}(x) \le G(x)^3, \qquad \Pi^{(3)}(x) \le \sum_{y:\, y\ne 0,x} G(y)^2 G(x-y)^2 G(x), \\ &\Pi^{(4)}(x) \le \sum_{\substack{y:\, y\ne 0\\ z:\, z\ne x}} G(y)^2 G(x-y) G(z) G(x-z)^2 G(y-z). \end{aligned} \tag{3.1}$$

We introduce diagrammatic expression to represent quantities on the right-hand side. In the diagram, a line connecting x and y represents $G(x-y)$, and unlabeled vertices with degree ≥ 2 are summed over. Bounds on $\Pi^{(n)}(x)$ ($n = 2, 3, 4$) are thus represented as

(3.2)



Diagrammatic representation for quantities defined in (1.11)–(1.17) are shown in Figure 1(a), using the above convention.

Special care is required for vertices of degree one. Vertices of degree one are not usually summed over, unless they appear in a pair—we sometimes sum over two vertices x and $x + a$, while keeping a fixed. Two examples appear in the diagrammatic representation for $H^{(\beta)}(a, b)$ of Figure 1(a), where the constant vector a and b are represented by dashed arrows.

We next turn to our basic techniques in diagrammatic estimates, which are used to estimate sums like $\sum_x \Pi^{(n)}(x)$ and $\sum_x |x|^2 \Pi^{(n)}(x)$. We perform this task by breaking the sum into products of basic units, using a simple inequality

$$\sum_x f(x)g(x) \leq \Big[\sup_x f(x)\Big]\Big[\sum_x g(x)\Big], \tag{3.3}$$

which is valid for any nonnegative functions f, g. Here x could be a group of variables.

How to use this inequality in decomposing a diagram into two small components, and finally into a product of (open) bubbles, $B(a)$, is illustrated in Figure 1(c). [In these diagrams, all horizontal lines could be of length zero; other (slant)

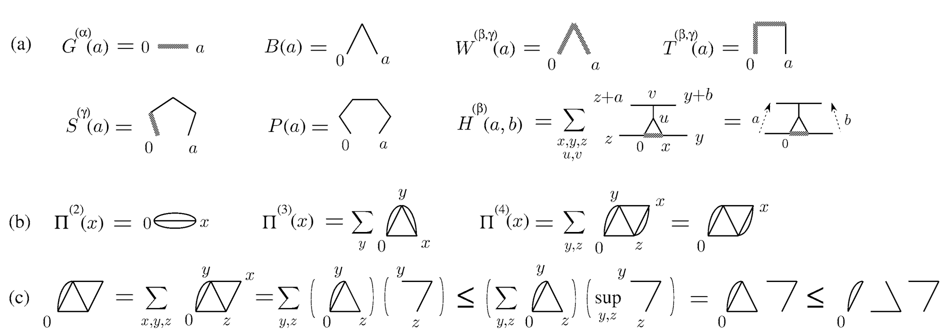


FIG. 1. (a) *Diagrammatic representation of quantities defined in* (1.12)–(1.17) *for* $a \neq 0$. *Lines weighted with* $|x|^\beta$ *and* $|x|^\gamma$ *are represented by thick shadowed lines. Two dashed arrows in* $H^{(\beta)}(a, b)$ *mean that we sum over these vertices, keeping displacement vectors* a, b *fixed.* (b) *Diagrams for* $\Pi^{(n)}(x)$ ($n = 2, 3, 4$) *for self-avoiding walk.* (c) *Using our basic inequality,* (3.3). *All unlabeled vertices with degree* ≥ 2 *are summed over.*

lines' lengths are greater than zero. Therefore, open bubbles are nothing but $B(a)$ with some a.] Graphically, we can just "peel off" open bubbles from right or left.

Arguing this way, we can bound $\sum_x \Pi^{(n)}(x)$ by a product of open bubbles, as

$$\sum_x \Pi^{(n)}(x) \le \Big(\sup_{x\neq 0} G(x)\Big)\Big(\sup_a B(a)\Big)^{n-1} \le \Big(\sup_{x\neq 0} G(x)\Big)\bar{B}^{n-1}. \tag{3.4}$$

Estimates like these will be extensively used in what follows.

3.2. *Proof of Lemma 1.8 for self-avoiding walk.* We start from the proof of Lemma 1.8 for self-avoiding walk, which is the simplest of our diagrammatic estimates. We will prove for $N \ge 3$

$$\sum_x |x|^{\alpha+\beta+\gamma}\,\Pi^{(N)} \le c N^{\alpha+\beta+\gamma+2}\lambda^{N-3} \tag{3.5}$$

with a finite constant c which is independent of N. Summing this over $N \ge 3$ (the sum converges as long as $\lambda < 1$) and noting that lowest order ($N = 2$) is bounded by $\sum_x |x|^{\alpha+\beta+\gamma} G(x)^3 = \sum_x G^{(\alpha)}(x) \times G^{(\beta)}(x) \times G^{(\gamma)}(x) \le \bar{G}^{(\alpha)}\bar{W}^{(\beta,\gamma)} = O(1)$ proves the lemma. In the following, we explain how to prove (3.5).

Step 1. *Distributing the weight* $|x|^{\alpha+\beta+\gamma}$. A typical lace expansion diagram for self-avoiding walk is shown in Figure 2(a). We want to multiply it with $|x|^{\alpha+\beta+\gamma}$ and sum over all the vertices (except 0). For this purpose, we first distribute the weight $|x|^{\alpha+\beta+\gamma}$ over suitable line segments of the diagram. Because there are three distinct lines connecting 0 and x (the uppermost line, the lowermost line, and the zigzag line), we pick a long segment out of each line.

Concretely, we proceed as follows.

- First pick the longest segment from the lowermost line connecting 0 and x. To be concrete, suppose this is ab in Figure 2(a). Because the number of segments of the lowermost line is $\lfloor N/2\rfloor$, this longest segment ab is at least as long as $|x|/\lfloor N/2\rfloor \ge 2|x|/N$.

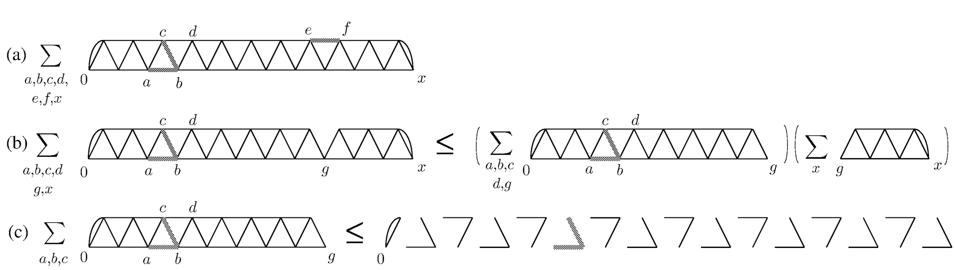


FIG. 2. (a) *A typical lace diagram for self-avoiding walk.* "*Long*" *segments are indicated by thick shadowed lines*. (b) *After extracting* $\bar{G}$ *from the diagram* (a), *decompose at* g. (c) *How to decompose the first factor of* (b) *into little bubbles and* $W^{(\beta,\gamma)}$. *Here all the unlabeled vertices with degree* ≥ 2 *are summed over.*

- Next consider the triangle which contains this longest segment. In Figure 2(a), this is triangle abc. Because the edge ab is longer than $2|x|/N$, at least one of ac or bc must be longer than $|x|/N$ (by the triangle inequality). Choose the longer one of ac and bc as our second "long" segment. (To be concrete, suppose this is ac.)
- Finally, choose the longest segment in the uppermost line connecting 0 and x. This is our third "long segment." Because the number of segments of the uppermost line is $\lceil N/2 \rceil$, the longest segment is at least as long as $|x|/\lceil N/2 \rceil \geq |x|/N$. For concreteness, suppose this is ef in Figure 2(a).

By the above choice, all three long segments are at least as long as $|x|/N$. We use this relation to bound the factor $|x|^{\alpha+\beta+\gamma} = |x|^{\alpha} \cdot |x|^{\beta} \cdot |x|^{\gamma}$. In our example, we have

$$|x|^{\alpha+\beta+\gamma} \leq N^{\alpha+\beta+\gamma} \times |e-f|^{\alpha} |a-b|^{\beta} |a-c|^{\gamma}. \tag{3.6}$$

Step 2. *Decomposition of the diagram.* Now we control the sum over all vertices of the diagram. In this example of Figure 2(a), we first peel off $\bar{G}^{(\alpha)}$ from the edge ef. This just leaves the diagram with this edge removed (and the summation over vertices are the same as before); the result is the diagram on the left-hand side of Figure 2(b). This is further bounded as in Figure 2(b), by decomposing it at vertex g. Here, the right factor looks like the one of Figure 1(c) (with more loops), and is bounded by a product of open bubbles as explained in (3.4). (For the right factor, we fix g and sum over x.)

What remains is to bound the left factor, which is decomposed as shown in Figure 2(c). As shown, this is bounded by a product of open bubbles $B(a)$, together with $W^{(\beta,\gamma)}(a)$. There are $(N-2)$ open bubbles (each of which is bounded by λ), so the example is bounded by

$$\lambda^{-(N-2)} \bar{G}^{(\alpha)} \Big(\sup_a W^{(\beta,\gamma)}(a) \Big) = \lambda^{-(N-2)} \bar{G}^{(\alpha)} \bar{W}^{(\beta,\gamma)} \leq c\lambda^{-(N-2)}. \tag{3.7}$$

Other diagrams occur, depending on which line segment is the longest—even for the diagram in Figure 2(a), we encounter $\bar{W}^{(\beta,0)} \bar{W}^{(0,\gamma)}$ instead of $\bar{W}^{(\beta,\gamma)}$ alone, if we pick bc instead of ac. These can be bounded in the same way, and all possible cases are bounded by $c\lambda^{-(N-3)}$. This is because each bubble is bounded by λ, and there are at least $(N-3)$ of them. (The diagram consists of N-loops, and at most three of them are used as $\bar{G}^{(\alpha)}$ and $\bar{W}$'s.)

Step 3. *Summary of the above.* Each of the weighted N-loop Π diagrams is bounded from above by

$$N^{\alpha+\beta+\gamma} \times \big[\bar{G}^{(\alpha)} \bar{W}^{(\beta,\gamma)} \text{ or } \bar{G}^{(\alpha)} \bar{W}^{(\beta,0)} \bar{W}^{(0,\gamma)} \big] \times c\lambda^{-(N-3)}. \tag{3.8}$$

The number of choices of long segments is bounded by $\lfloor N/2 \rfloor \times 2 \times \lceil N/2 \rceil \leq N^2$. Thus, the N-loop contribution is bounded by $cN^2 \times N^{\alpha+\beta+\gamma} \times \lambda^{(N-3)}$. This proves (3.5), and proves the lemma.

3.3. *Proof of Lemma 1.7 for self-avoiding walk.* The proof proceeds in the same spirit as that of Lemma 1.8. The diagrams look the same, but different methods are required because we are now *fixing* x.

Step 1. *Picking and extracting "long" segments.* We illustrate by a typical diagram of Figure 3(a). We first pick and extract three "long" segments exactly as we did in the proof of Lemma 1.8. The result is that we get *three* factors of

$$G_{x,N} := \sup_{y:|y|\geq|x|/N} G(y) \leq \beta\left(\frac{N}{|x|}\right)^{\alpha}, \tag{3.9}$$

and a remaining diagram, which is shown on the left of Figure 3(b). Our remaining task is to bound this diagram by $O(\lambda^{N-3})$.

Step 2. *Decomposition of the diagram.* We decompose the resulting diagram as shown on the right of Figure 3(b). The left and right factors are further decomposed into open bubbles easily (recall that we are now fixing 0 and x), and are bounded by suitable powers of λ. Our remaining task is to bound the middle factor.

Step 3. *Bounding the middle factor.* Consider the middle factor on the right of Figure 3(b) as a summation over $y, z \in \mathbb{Z}^d$ of the product of two factors, and use the Schwarz inequality as in Figure 3(c). The second factor on the right of (c) is just the bubble squared—to be more precise, one of them has nonzero lines and is bounded by λ, another is $O(1)$.

The first factor on the right of (c) is more complicated. But here we *fix only one* vertex of this diagram and sum over all others. Using translation invariance, we can move the fixed vertex from c to 0 as shown on the left of Figure 3(d). Having moved c, we can now decompose this into open bubbles as shown. (Here we are using our convention that no vertices of degree one are summed over.)

Step 4. *Summary.* We have seen that extracting three "long" lines yields $(G_{x,N})^3$, while the remaining diagram is bounded by $O(\lambda^{N-3})$. We have to sum

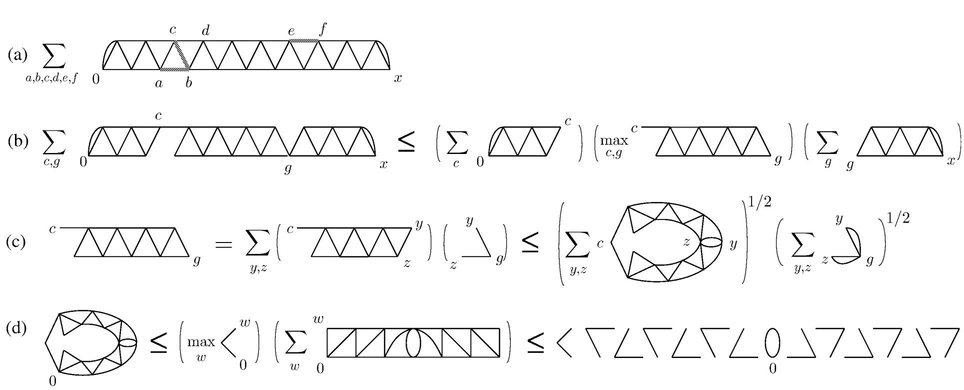


FIG. 3. (a) *A typical lace diagram for self-avoiding walk and long segments*. (b) *Diagram of* (a) *after extracting three long segments*, *and how to bound it by decomposing into three factors*. (c) *How to use the Schwarz inequality*: *vertices* c, g *are fixed*. (d) *How to decompose the first factor of* (c). *Vertices of degree one are not summed over.*

over all the possible choices of the long segments. As shown in the proof of Lemma 1.8, the number of choices of long segments is bounded by N^2. Using our assumption on the decay of G, we thus have

$$
\begin{aligned}
\Pi^{(N)}(x) &\le cN^2 \times \lambda^{N-3} \times (G_{x,N})^3 \le cN^2\lambda^{N-3}\left(\frac{\beta}{(|x|/N)^{\alpha}}\right)^3 \\
&= cN^{2+3\alpha}\lambda^{N-3}\frac{\beta^3}{|x|^{3\alpha}}
\end{aligned}
\tag{3.10}
$$

with a finite constant c. Summing this over $N \ge 3$ (the sum converges as long as $\lambda < 1$), and noting that the lowest order ($N = 2$) is bounded by $G(x)^3 \le \beta^3/|x|^{3\alpha}$ proves the lemma.

3.4. *Proof of Lemma* 1.8 *for percolation.* This is proven along the same line as for self-avoiding walk, but we encounter more complicated percolation diagrams [8]. Although we have to consider general N-loop diagrams, details are explained by using 4-loop diagrams as examples. General cases will be extrapolated rather easily.

Diagrams of $\Pi^{(4)}$ look like those of Figure 4(a), plus 14 others. [In general, there are 2^N diagrams for $\Pi^{(N)}$.] Dealing with the right diagram (and 14 others) is easier, and we only explain how to deal with the left one.

Before going into details we explain about a special feature of percolation diagrams. In percolation diagrams, we encounter $x \Vdash\!\!\!-\!\!\!- y$, which represents $2dp(D * G)(y - x)$ [8]. This is almost the same as $G(y - x)$ for large $|y - x|$, because

$$
\begin{aligned}
2dp(D * G)(y - x) &= 2dp \sum_{z\,:\,|z-x|=1} \frac{1}{2d} G(y - z) \\
&\le (1 + c_4\lambda) \sum_{z\,:\,|z-x|=1} \frac{1}{2d} G(y - z).
\end{aligned}
\tag{3.11}
$$

Some care is needed when $|y - x| = 1$ can happen. For example, the rightmost factor of Figure 4(d) is

$$
\begin{aligned}
2dp &\sum_{|u|=1} \frac{1}{2d}\big(G^{(\gamma)} * G * G\big)(f - u) \\
&= p \sum_{|u|=1} \big(G * G^{(\gamma)} * G\big)(f - u) \\
&= p \sum_{|u|=1} \big\{T^{(0,\gamma)}(f - u) + \delta_{\gamma,0}\delta_{f-u,0}\big\}.
\end{aligned}
\tag{3.12}
$$

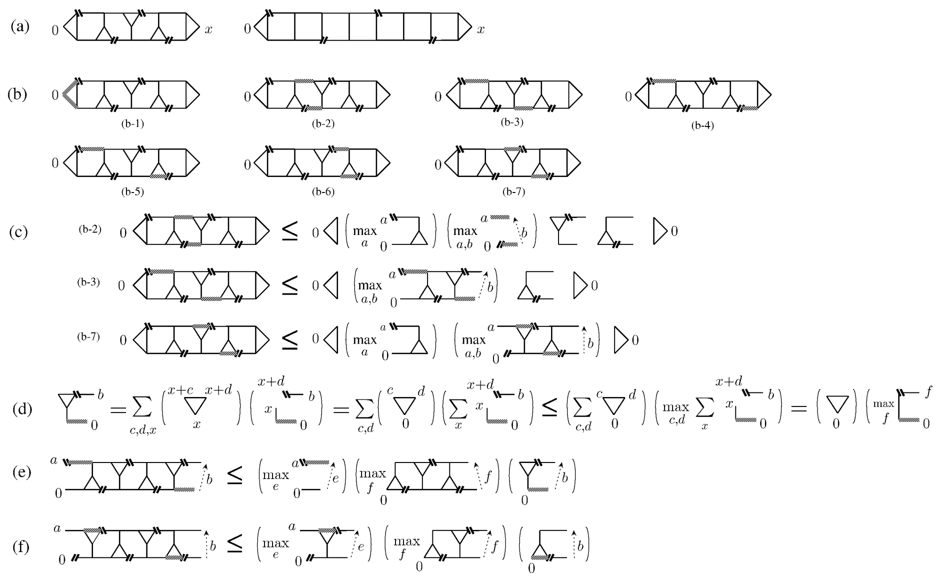


FIG. 4. (a) *Typical four loop diagrams of* $\Pi^{(4)}(x)$ *for percolation*. (b) *Possible choices of* "*long*" *segments*, *indicated by shadowed thick lines*. (c) *How to decompose some cases of* (b) *into simple components*. *Note that the left and right factors can be further decomposed into* (*open*) *triangles*, *and produce powers of* λ. (d) *How to decompose a factor appearing in* (c) *into a triangle and a weighted triangle*, $\bar{T}^{(0,\gamma)}$. *The second equality follows from translation invariance*. *The rightmost factor is not exactly equal to*, *but is bounded by a constant multiple of*, $\bar{T}^{(0,\gamma)}$ *as explained around* (3.13). (e) *How to decompose the middle factor of* (b-3) *into basic components*. (f) *How to decompose the middle factor of* (b-7) *into basic components*. *The leftmost diagram is new and is bounded by a constant multiple of* $\bar{H}^{(\beta)}$.

When $|f| \neq 1$, the above is bounded by $2dp\bar{T}^{(0,\gamma)} \leq (1 + c_4\lambda)\bar{T}^{(0,\gamma)}$. But when $|f| = 1$, $f - u$ can be zero for one u. In this case we get

$$\begin{aligned} &2dp \sum_{|u|=1} \frac{1}{2d}\big(G^{(\gamma)} * G * G\big)(f-u) \\ &\quad \leq p\big[(2d-1)\bar{T}^{(0,\gamma)} + \big(1 + \bar{T}^{(0,\gamma)}\big)\big] \\ &\quad \leq (1 + c_4\lambda)\bigg[\bar{T}^{(0,\gamma)} + \frac{1}{2d}\bigg]. \end{aligned} \tag{3.13}$$

We can thus conclude that the rightmost factor of Figure 4(d) is bounded by (3.13).

Step 1. *Distributing the weight* $|x|^{\beta+\gamma}$. To deal with $|x|^{\beta+\gamma}$, we first note that there are *two* (upper and lower) disjoint paths which connect 0 and x. Out of each line, we pick up the longest segment, as we did for self-avoiding walk. Because there are at most $(2N + 1)$ segments for each of the upper and lower lines of a N-loop diagram, these "long" segments are not shorter than $|x|/(2N + 1)$. Various choices of these elements are illustrated as Figure 4(b), where long segments are indicated by thick shadowed lines. Suppose for concreteness that $|x|^\beta$ is on the upper line, and $|x|^\gamma$ is on the lower line.

Step 2. *Decomposition of the diagram.* Next we control the sum over all vertices of the diagram. This procedure is illustrated in Figure 4(c). We can peel off (open) triangles from left and right, leaving $|x|^\beta$-, $|x|^\gamma$-weighted parts in the middle.

For (b-2), the middle factor is nothing but $W^{(\beta,\gamma)}$, which is assumed to be finite, and we are done.

The case (b-3) is explained in Figure 4(e). (We have increased the number of loops in the middle, to illustrate more general N-loop diagrams.) As shown, we can peel off $W^{(\beta,0)}$ from the left, decompose the middle part into triangles, and are left with the right factor. The right factor itself is decomposed as in Figure 4(d), and is bounded by the product of a triangle and $T^{(0,\gamma)}$.

The case (b-7) is more complicated. Decomposing as before, we encounter the leftmost component of Figure 4(f). [Other parts can be decomposed into triangles and $T^{(0,\gamma)}$, and are controlled well.] This is nothing but $H^{(\beta)}(a,b)$ of (1.17), and is finite by the assumption of the lemma.

Step 3. *Summary.* Proceeding this way, we see all the cases of weighted N-loop Π diagrams are bounded above by

$$(2N+1)^{\beta+\gamma}\times\left[\bar{W}^{(\beta,\gamma)}\text{ or }\bar{W}^{(\beta,0)}\bar{T}^{(0,\gamma)}\text{ or }\bar{H}^{(\beta)}\bar{T}^{(0,\gamma)}\right]\times(\text{triangles}). \tag{3.14}$$

The diagram consists of N nontrivial loops, and at most two of them are used as $\bar{W}^{(\beta,\gamma)}$, $\bar{W}^{(\beta,0)}$, $\bar{H}^{(\beta)}$ and/or $\bar{T}^{(0,\gamma)}$. So there are at least $(N-2)$ open triangles, each of which is bounded by $2dp(\lambda+\frac{1}{2d})$ ($\le 2\lambda$ for sufficiently large d and small λ). In addition, we have small triangles which are bounded by $(1+\lambda)$ [8]. So the triangles contribute $(c\lambda)^{N-2}$ with some c.

The number of choices of "long" segments are bounded by $(2N+1)^2$, because there are at most $(2N+1)$ segments for upper and lower lines. Also, there are 2^N diagrams for $\Pi^{(N)}$. The N-loop contribution is thus bounded by

$$c2^N\times N^2\times(2N+1)^{\beta+\gamma}\times(c'\lambda)^{N-2}=cN^{2+\beta+\gamma}(2c'\lambda)^{N-2}. \tag{3.15}$$

Summing this over $N\ge 2$ and taking care of $N=0,1$ separately proves (3.5) and the lemma. (The cases of $N=0,1$ are rather simple, and the details are omitted.)

3.5. *Proof of Lemma* 1.7 *for percolation.* The basic idea is the same as for the self-avoiding walk. We extract *two* (cf. *three* for self-avoiding walk) factors of long G from the upper and lower lines connecting 0 and x, and bound the rest by $(c\lambda)^{N-3}$.

As for self-avoiding walk, we first pick "long" segments. We have a segment of length $\ge |x|/(2N+1)$ on the upper and lower sides of the diagram connecting 0 and x. These long segments can be any lines which lie on the upper and lower sides of the diagram, so the total number of choices are bounded by $(2N+1)^2$. Several different cases are shown in Figure 5(b), where the shaded thick lines represent these long segments. These cases are grouped into two.

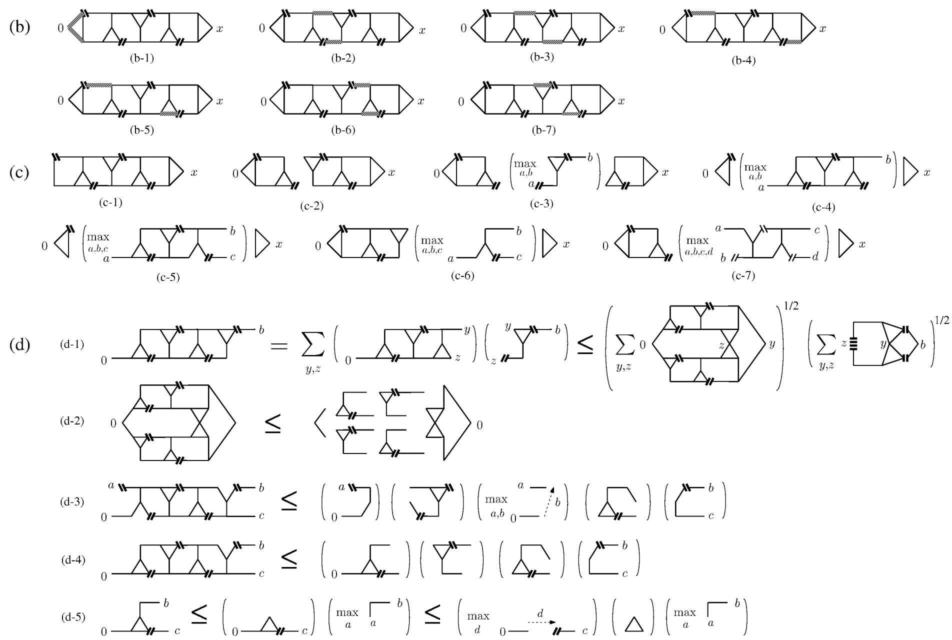


FIG. 5. (b) *Several cases of* "*long*" *lines for the diagram of Figure* 4(a) *on the left*. (c) *Diagrams of* (b), *after extracting two long lines*, *and how to decompose them into smaller components*. (d) *Typical nasty diagrams of case* 2. (d-1) *is bounded by the Schwarz inequality*, *and the result is further decomposed as shown in* (d-2). (d-3) *is decomposed into triangles*, *squares*, *and two* G'*s*. (d-4) *is decomposed into triangles*, *squares*, *and a factor of* (d-5), *which is further decomposed into a triangle and bubbles*.

Case 1. This is when (i) we have these "long" segments on two lines on a rectangle (or triangle) facing each other, like Figure 5(b-1) and (b-2), or (ii) we have long lines on adjacent rectangles, like Figure 5(b-3). In either case, we just bound the diagram by extracting two factors of

$$G_{x,N} := \sup_{y\,:\,|y|\ge|x|/(2N+1)} G(y) \le \frac{\beta(2N+2)^\alpha}{|x|^\alpha}, \tag{3.16}$$

where the second inequality follows from our assumption (1.45) of the lemma on $G(x)$.

The effect of extracting these G's is nothing but erasing these two lines in the diagram, so the case (b-2) is bounded by (c-2), after extracting two factors of $G_{x,N}$. The remaining components in (c-2) as well as in (c-1) are easily bounded in terms of triangles, because 0 and x are now fixed.

The case (b-3) is similar. By peeling off from left and right, we get (c-3). Now the factor in the middle is easily seen to be bounded by two triangles (just extract the small triangle first).

To summarize, case 1 can be bounded by $G_{x,N}^2$ times convergent diagrams, which are bounded by some powers of triangles. By counting the number of nontrivial loops, we see that these triangles are bounded by a $O((c\lambda)^{N-2})$, just as in the proof of Lemma 1.8.

Case 2. There remain more complicated cases, but the basic idea is the same. As shown in Figure 5(c-4) through Figure 5(c-7), we decompose into the component in the middle which is hard to deal with, and the left and the right components which can be easily decomposed into triangles. We now concentrate on the component in the middle.

There are essentially three kinds of these, which are shown in Figure 5(d), as (d-1), (d-3) and (d-4). The case (d-3) has two lines at each end of the diagram, while the case (d-1) has only one line. The case (d-4) is a kind of "cross-term" of these two.

This is where we have to impose the restriction $d > 8$ even if we assume $\lambda \ll 1$. It is natural that the lemma holds for $d > 6$, but currently we cannot control the middle factor in $6 < d \leq 8$. In $d > 8$, we can bound the middle factor by decomposing it into open triangles and a square $\bar{S}^{(0)}$, as shown in Figure 5(d). The infrared bound (1.25) guarantees that $\bar{S}^{(0)}$ is finite in $d > 8$.

The factor of Figure 5(d-1) is bounded by the Schwarz inequality as has been done for self-avoiding walk diagrams, as shown in the right-hand side of Figure 5(d-1). The resulting components are further bounded by $\bar{B}, \bar{T}$ and $\bar{S}$ as shown in Figure 5(d-2). The factor of Figure 5(d-3) is bounded as shown, in terms of open triangles, squares, and two G's. The factor of Figure 5(d-4) is decomposed as shown, and its first factor is further decomposed as in (d-5).

In all these cases, we can collect at least $(N-3)$ factors of $c\lambda$ and two factors of $G_{x,N}$ for each diagram. Multiplying by the number of different choices of "long" segments [which is $O(N^2)$], and summing over N proves the lemma.

3.6. *Proof of Lemma* 1.8 *for lattice trees and animals.* The proof proceeds along the same line as for self-avoiding walk and percolation, and we will be brief.

Typical diagrams for $\Pi^{(7)}(x)$ of lattice trees are given in Figure 6(a). In general, diagrams for $\Pi^{(N)}(x)$ consist of N small squares, with an extra vertex on each inner square. These inner extra vertices (and x itself) can appear on either (upper and lower) side of the diagram, and there are 2^{N-1} diagrams for $\Pi^{(N)}(x)$.

As in the percolation diagrams, there are two (upper and lower) disjoint paths which connect 0 and x. Out of each line, we pick the longest segment, as we did for self-avoiding walk. Because there are at most $2N$ segments for each of the upper and lower lines of a N-loop diagram, these "long" segments are not shorter than $|x|/(2N)$. Several choices of these segments are illustrated in Figure 6(b), where long segments are represented by thick shadowed lines. Suppose for concreteness that $|x|^{\beta}$ is on the upper line, and $|x|^{\gamma}$ is on the lower line.

Next we control the sum over all vertices of the diagram. This procedure is illustrated in Figure 6(c). We peel off $S^{(0)}(a)$, $S^{(\gamma)}(a)$, or $T^{(\beta,\gamma)}(a)$ from right to left. For (b-1), we peel off open squares from the right, and the remaining leftmost

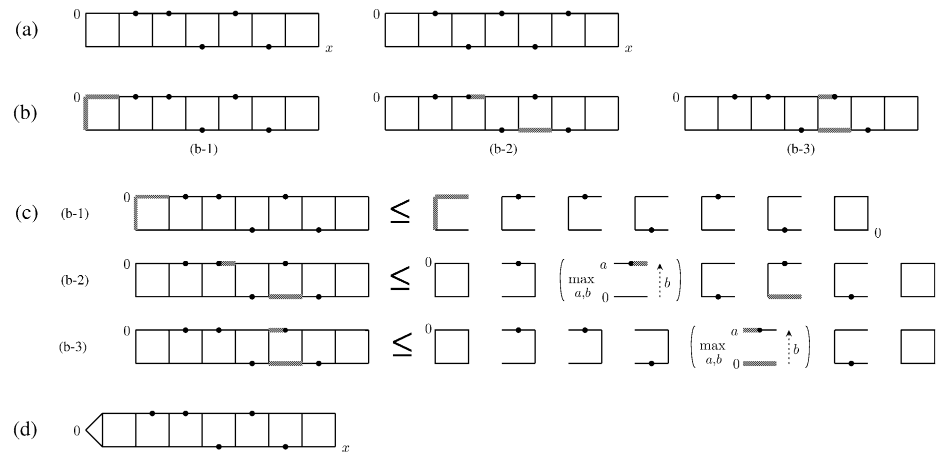


FIG. 6. (a) *Two examples of diagrams of* $\Pi^{(7)}(x)$ *for lattice trees*. (b) *Possible choices of "long" segments, represented by shadowed thick lines, for the left diagram of* (a). (c) *How to decompose three cases of* (b) *into basic components*. (d) *A typical diagram of* $\Pi^{(7)}(x)$ *for lattice animals. The only difference between lattice trees and animals is that we have an extra triangle on the left* (*at* 0).

factor is bounded by $\bar{T}^{(\beta,\gamma)}$. For (b-2), we proceed similarly, but encounter $\bar{S}^{(\gamma)}$ and $\bar{T}^{(\beta,0)}$ in the process. For (b-3), we encounter $\bar{T}^{(\beta,\gamma)}$.

Proceeding this way, we see that all the cases of weighted N-loop Π diagrams are bounded above by

$$(2N)^{\beta+\gamma} \times \big[\bar{T}^{(\beta,\gamma)} \text{ or } \bar{T}^{(\beta,0)}\bar{S}^{(\gamma)}\big] \times (\text{squares}). \tag{3.17}$$

The diagram consists of N nontrivial loops, and at most two of them are used as $\bar{T}^{(\beta,\gamma)}$, $\bar{T}^{(\beta,0)}$, and/or $\bar{S}^{(\gamma)}$. So there are at least $(N-2)$ open squares, each of which is bounded by λ.

The number of choices of "long" segments are bounded by $(2N)^2$, because there are at most $2N$ segments for upper and lower lines. Also, there are 2^N diagrams for $\Pi^{(N)}$. The N-loop contribution is thus bounded by

$$c2^N \times (2N)^2 \times (2N)^{\beta+\gamma} \times (\lambda)^{N-2} = cN^{2+\beta+\gamma}(2\lambda)^{N-2}. \tag{3.18}$$

Summing this over $N \geq 2$ and considering $N = 1$ separately proves (3.5) and the lemma for lattice trees.

Diagrams for lattice animals are almost the same as those for lattice trees, except that there is an extra triangle at 0, as shown in Figure 6(d). (Diagrams in Figure 6(d) incorporate an improvement achieved in [7] over the analysis in [9].) These can be handled in the same way as for lattice trees.

3.7. *Proof of Lemma* 1.7 *for lattice trees and animals.* The basic idea is again the same as for self-avoiding walk and percolation, and we will be brief. We illustrate for a typical example of the left of Figure 6(a). As for percolation, we extract

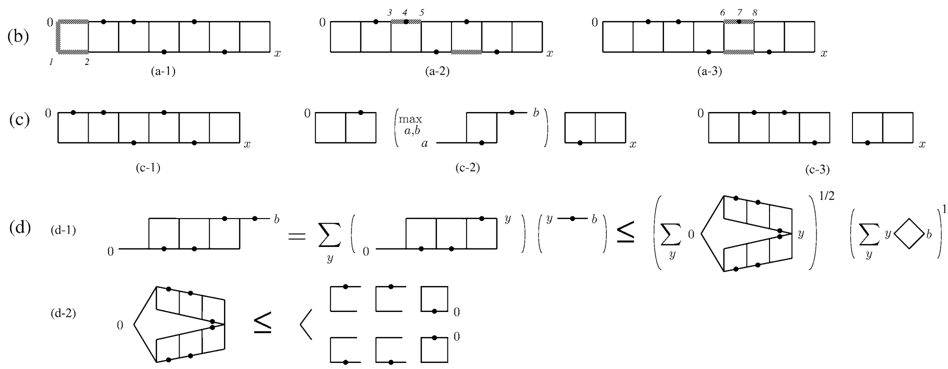


FIG. 7. (b) *Several cases of* "*long*" *segments for a lattice tree diagram. Vertices* 1 *through* 8 *are summed over*; *they are here just for the explanation in the main text*. (c) *Diagrams of* (b), *after extracting two long segments*. (d) *How to decompose the middle factor* (*with more loops*) *of* (c-2).

two factors of long segments from the upper and lower lines connecting 0 and x, and bound the rest by $(c\lambda)^{N-3}$. However, we now consider the convolution $G * G$ appearing in the diagram as one segment. Three typical choices of long segments are shown in Figure 7(b). Here each of 0-1-2, 3-4-5, and 6-7-8 is considered to be a single segment.

Because of our modified definition of line segments, there are N segments for upper and lower lines connecting 0 and x. Therefore, each long segment is at least as long as $|x|/N$, and there are N^2 choices of these long segments.

We now extract contributions of "long" segments from upper and lower lines. Because of our modified definition of line segments, factors extracted will be either $G_{x,N} := \sup_{|y|\geq|x|/N} G(y)$ (as before), or

$$\sup_{|y|\geq|x|/N} (G*G)(y) \leq c\beta^2\left(\frac{N}{|x|}\right)^{2\alpha-d} \qquad (\text{if } 0 < 2\alpha - d < d) \tag{3.19}$$

(which is new), where the inequality comes from our assumption (1.45) and a basic property of convolution, Lemma B.1 (a). Contributions from two long segments are thus bounded by

$$c\beta^2\left(\frac{N}{|x|}\right)^{2\alpha} \vee c\beta^4\left(\frac{N}{|x|}\right)^{4\alpha-2d}. \tag{3.20}$$

Examples of remaining factors are shown in Figure 7(c). Of these, (c-1) and (c-3) are easily decomposed into (open) squares, and pose no problem. The middle factor of (c-2) is more complicated, like several middle factors of Figure 5(c) for percolation.

For this (and similar middle factors) we use the Schwarz inequality as we did for percolation. Concretely (we increased the number of loops to illustrate more complicated typical cases), we proceed as in Figure 7(d). We first use the Schwarz inequality to get two diagrams on the right of (d-1). The second factor is bounded by $\bar{S}^{(0)} + 1$. The first factor is bounded by decomposing it into open squares and pentagons $\bar{P}$, as shown in (d-2). Existing bound (1.25) guarantees that $\bar{P}$ is finite in $d > 10$.

In all these cases, we can collect at least $(N-3)$ factors of $c\lambda$ and a factor of (3.20) for each diagram. Multiplying by the number of different choices of "long" segments (which is N^2) and the number of N-loop diagrams (which is 2^N), and summing over N proves the lemma.

The proof for lattice animals proceeds similarly and is omitted.

4. Proof of Lemma 1.9. In this section, we prove Lemma 1.9. The basic idea of the proof is simple, but technical details can be complicated (especially for noninteger exponents). Therefore, we first explain the framework of the proof in Section 4.1, and give details in later sections.

4.1. *Framework of the proof of Lemma 1.9.*

4.1.1. *Reduction of the proof to certain integrability conditions.* Our goal is to prove that $G^{(\alpha)}(a)$, $W^{(\beta,\gamma)}(a), \dots, H^{(\beta)}(a,b)$ are finite uniformly in $a, b \in \mathbb{Z}^d$. However, it is cumbersome to deal with $|x|^\alpha$-weighted quantities, especially when α is not an even integer. We thus define $(j, l = 1, 2, \dots, d)$

$$G_j^{(\alpha)}(a) := |a_j|^\alpha G(a), \qquad W_{jl}^{(\beta,\gamma)}(a) := \bigl(G_j^{(\beta)} * G_l^{(\gamma)}\bigr)(a), \tag{4.1}$$

$$\begin{aligned} T_{jl}^{(\beta,\gamma)}(a) &:= \bigl(G_j^{(\beta)} * G_l^{(\gamma)} * G\bigr)(a), \\ S_j^{(\gamma)}(a) &:= \bigl(G_j^{(\gamma)} * G * G * G\bigr)(a), \end{aligned} \tag{4.2}$$

and similarly $H_j^{(\beta)}(a,b)$. In view of an elementary inequality

$$\begin{aligned} |x|^\alpha &= \left(\sum_{j=1}^d x_j^2\right)^{\alpha/2} \le c_\alpha \sum_{j=1}^d |x_j|^\alpha \qquad \text{with} \\ c_\alpha &= \begin{cases} d^{\alpha-1}, & (\alpha \ge 1), \\ d^\alpha, & (0 < \alpha < 1) \end{cases} \end{aligned} \tag{4.3}$$

it suffices to prove that $G_j^{(\alpha)}(a)$, $W_{jl}^{(\beta,\gamma)}(a), \dots, H_j^{(\beta)}(a,b)$ are finite uniformly in $a, b \in \mathbb{Z}^d$.

Now, these quantities are represented in Fourier space as

$$G_j^{(\alpha)}(a) = \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ika} \hat{G}_j^{(\alpha)}(k), \tag{4.4}$$

$$W_{jl}^{(\beta,\gamma)}(a) = \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ika} \hat{G}_j^{(\beta)}(k) \hat{G}_l^{(\gamma)}(k), \tag{4.5}$$

$$T_{jl}^{(\beta,\gamma)}(a) = \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ika} \hat{G}_j^{(\beta)}(k) \hat{G}_l^{(\gamma)}(k) \hat{G}(k), \tag{4.6}$$

$$S_j^{(\gamma)}(a) = \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ika} \hat{G}_j^{(\gamma)}(k) \hat{G}(k)^3, \tag{4.7}$$

$$\begin{aligned} H_j^{(\beta)}(a,b) = \iiint_{[-\pi,\pi]^{3d}} & \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} e^{i(ka+lb)} \\ & \times \hat{G}_j^{(\beta)}(p) \hat{G}(k)^2 \hat{G}(l)^2 \\ & \times \hat{G}(p-k) \hat{G}(p+l) \hat{G}(k+l). \end{aligned} \tag{4.8}$$

Therefore, if we have a good control over $\hat{G}_j^{(\alpha)}(k)$, $\hat{G}_j^{(\beta)}(k), \dots,$ so that we can prove integrability of $\hat{G}_j^{(\alpha)}$, $\hat{G}_j^{(\beta)} \hat{G}_l^{(\gamma)}, \dots,$ we are done.

4.1.2. *Proof of Lemma 1.9 for even integer exponents.* The above scenario works perfectly when α, β, γ are even integers, because when n is a positive integer ($\partial_j := \frac{\partial}{\partial k_j}$),

$$\begin{aligned} G_j^{(2n)}(x) &= |x_j|^{2n} G(x) = (-1)^n \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} (\partial_j)^{2n} \hat{G}(k) \\ &\Longrightarrow \quad \hat{G}_j^{(2n)}(k) = (-1)^n (\partial_j)^{2n} \hat{G}(k), \end{aligned} \tag{4.9}$$

and a good bound on the derivative is given by the following lemma.

LEMMA 4.1. *Suppose we have*

$$\sum_x |x|^M |\Pi(x)| < \infty \tag{4.10}$$

for a positive integer M. Then, $\hat{G}(k)$ of (1.18)–(1.22) satisfies, for all $1 \le m \le M$ and $j = 1, 2, \dots, d$

$$\left| \frac{\partial^m}{\partial k_j^m} \hat{G}(k) \right| \le \frac{c}{|k|^{2+m}} \tag{4.11}$$

with a possibly m-dependent constant c.

PROOF OF LEMMA 1.9 WHEN α, β, γ ARE EVEN INTEGERS, ASSUMING LEMMA 4.1. Lemma 1.9 for even integer exponents can now be proved, by counting powers of k and checking integrability. When α (resp. β, γ) is an even positive integer which satisfies $\alpha \le \phi$ (resp. $\beta \le \lfloor \phi \rfloor$, $\gamma \le \lfloor \phi \rfloor$), the assumption of the lemma (1.53) guarantees that (4.10) holds with $M = \alpha$ (resp. β, γ). This allows us to use (4.9) and (4.11) to get $|\hat{G}_j^{(m)}(k)| \le c|k|^{-2-m}$ ($m = \alpha$, or β, or γ). This implies

$$G_j^{(\alpha)}(a) \le \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \left|\hat{G}_j^{(\alpha)}(k)\right| \le \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \frac{c}{|k|^{2+\alpha}}, \tag{4.12}$$

which is finite for $2 + \alpha < d$. Similarly, by (4.4) and (4.6),

$$\begin{aligned} W_{jl}^{(\beta,\gamma)}(a) &\le \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \frac{c}{|k|^{2+\beta}|k|^{2+\gamma}}, \\ S_j^{(\gamma)}(a) &\le \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \frac{c}{|k|^{2+\gamma}|k|^6}. \end{aligned} \tag{4.13}$$

The first integral is finite if $4 + \beta + \gamma < d$. The second integral is finite if $2 + \gamma + 6 < d$. $T_{jl}^{(\beta,\gamma)}(a)$ is handled in exactly the same way.

$H_j^{(\beta)}(a, b)$ requires more care. Using (4.8), we have

$$\begin{aligned} H_j^{(\beta)}(a,b) \le \iiint_{[-\pi,\pi]^{3d}} & \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} \\ & \times \frac{c}{|p|^{2+\beta}|k|^4|l|^4|p-k|^2|p+l|^2|k+l|^2}. \end{aligned} \tag{4.14}$$

This $3d$-dimensional integral is seen to be finite by elementary power counting. In short, these integrals are finite, as long as singularities at the origin are integrable when some (or all) integral variables are sent to zero simultaneously (see [21, 22] for details). In our case, this is satisfied if $2 + \beta < d$, $2 + \beta + 4 + 2 < 2d$, $2 + \beta + 14 < 3d$. These conditions are satisfied when $d > 6$ and $4 + \beta < d$. □

4.1.3. *Proof of Lemma 1.9 for noninteger exponents*, $\alpha < \lfloor \phi \rfloor$. When α, β, γ are not even integers, the Fourier transform of $G_j^{(\alpha)}(x) = |x_j|^\alpha G(x)$ is not a simple derivative of $\hat{G}(k)$. [When α is an odd integer, Fourier transform of $(x_j)^\alpha G(x)$ is given by a simple derivative; this is not true for $|x_j|^\alpha G(x)$.] The answer is given in terms of *fractional derivatives*, which is explained in Section 4.3. As a result, we get:

LEMMA 4.2. *Suppose we have*

$$\sum_x |x|^M |\Pi(x)| < \infty \tag{4.15}$$

for a positive integer M. Then G of (1.18)–(1.22) *satisfies, for any integer $n \in [1, M \wedge (d-2)]$ and for $0 < \varepsilon < 1$,*

$$|x_1|^{n-\varepsilon} G(x) = G_1^{(n-\varepsilon)}(x) = \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ikx} \hat{G}_1^{(n-\varepsilon)}(k) \tag{4.16}$$

with

$$\left|\hat{G}_1^{(n-\varepsilon)}(k_1,\vec{k})\right| \le \frac{c}{|k_1|^{1-\varepsilon}|\vec{k}|^n|k|}, \tag{4.17}$$

where c may depend on ε. Here $k=(k_1,\vec{k})$ and $|k|:=(|k_1|^2+|\vec{k}|^2)^{1/2}$.

PROOF OF LEMMA 1.9 FOR NONINTEGER EXPONENTS, ASSUMING LEMMA 4.2. Thanks to Lemma 4.2, we have the bound (4.17) for $1\le n\le \lfloor\phi\rfloor \wedge (d-2)$. We estimate our quantities of interest one by one, using the above bound.

We start from $\bar{G}^{(\alpha)}$ when $\alpha<\lfloor\phi\rfloor$. In this case, $1\le n\le\lfloor\phi\rfloor$ is satisfied if we write $\alpha=n-\varepsilon$ with $\varepsilon\in(0,1)$. Therefore, the bound (4.17) allows us to conclude

$$G_1^{(n-\varepsilon)}(a) \le \int_{[-\pi,\pi]^d}\frac{d^dk}{(2\pi)^d}\left|\hat{G}_1^{(n-\varepsilon)}(k)\right| \le \int_{[-\pi,\pi]^d}\frac{d^dk}{(2\pi)^d}\frac{c}{|k_1|^{1-\varepsilon}|\vec{k}|^n|k|}. \tag{4.18}$$

Dividing the integration region according to $|k_1|>|\vec{k}|$ or not, we can easily see that the above integral is finite as long as $1-\varepsilon<1$, $n<d-1$, and $2+n-\varepsilon<d$. (These conditions are equivalent to $2+\alpha<d$.) This proves the lemma for $\bar{G}^{(\alpha)}$, for $\alpha<\lfloor\phi\rfloor\wedge(d-2)$. We need a separate argument to deal with $\bar{G}^{(\alpha)}$ for $\alpha>\lfloor\phi\rfloor$, to which we will come back later.

Controlling $\bar{S}^{(\gamma)}$ and $\bar{H}^{(\beta)}$ is similar. By $\mathbb{Z}^d$-symmetry, it suffices to show that $\bar{S}_1^{(\gamma)}$ and $\bar{H}_1^{(\beta)}$ are finite. $1\le n\le\lfloor\phi\rfloor$ is satisfied for noninteger γ satisfying $\gamma\le\lfloor\phi\rfloor$, if we write $\gamma=n-\varepsilon$ with $\varepsilon\in(0,1)$. Using (4.17) and the Fourier representation (4.6), we have

$$\begin{aligned} S_1^{(\gamma)}(a) &\le \int_{[-\pi,\pi]^d}\frac{d^dk}{(2\pi)^d}\left|\hat{G}_1^{(n-\varepsilon)}(k)\hat{G}(k)^3\right| \\ &\le \int_{[-\pi,\pi]^d}\frac{d^dk}{(2\pi)^d}\frac{c}{|k_1|^{1-\varepsilon}|\vec{k}|^n|k|}\times\frac{1}{|k|^6}. \end{aligned} \tag{4.19}$$

The integral on the right is finite as long as $n<d-1$ and $n+8-\varepsilon<d$, or equivalently, $\gamma+8<d$. Similarly, writing $\beta=m-\delta$, using (4.17) and the Fourier representation (4.8), we have

$$\begin{aligned} H_1^{(\beta)}(a,b) \le \iiint_{[-\pi,\pi]^{3d}} &\frac{d^dk}{(2\pi)^d}\frac{d^dl}{(2\pi)^d}\frac{d^dp}{(2\pi)^d}\frac{c}{|p_1|^{1-\delta}|\vec{p}|^m|p|} \\ &\times\frac{c}{|k|^4|l|^4|p-k|^2|p+l|^2|k+l|^2}. \end{aligned} \tag{4.20}$$

This integral is finite as long as $d>6$, $m-\delta+4<d$, and $m<d-1$. This condition is satisfied if $\beta<d-4$ and $d>6$. These prove Lemma 1.9 for $\bar{S}^{(\gamma)}$ and $\bar{H}^{(\beta)}$.

Next we move on to $W^{(\beta,\gamma)}$. By $\mathbb{Z}^d$-symmetry, it suffices to show that $W_{11}^{(\beta,\gamma)}$ and $W_{12}^{(\beta,\gamma)}$ are finite. Writing $\beta = m - \delta$ and $\gamma = n - \varepsilon$, using (4.17) and the Fourier representation (4.4),

$$W_{11}^{(\beta,\gamma)}(a) \le \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \frac{c}{|k_1|^{1-\delta}|\vec{k}|^m|k|} \times \frac{c}{|k_1|^{1-\varepsilon}|\vec{k}|^n|k|}. \tag{4.21}$$

This integral is finite as long as $n + m < d - 1$, $2 - \varepsilon - \delta < 1$, and $(n + m + 2) + (2 - \varepsilon - \delta) < d$. These conditions are equivalent to $\beta + \gamma < d - 4$ and $\beta + \gamma - (\lfloor\beta\rfloor + \lfloor\gamma\rfloor) < 1$.

$W_{12}^{(\beta,\gamma)}$ is similar. By $\mathbb{Z}^d$-symmetry, $\hat{G}_2^{(\gamma)}$ obeys the same bound as $\hat{G}_1^{(\gamma)}$, if we interchange k_1 and k_2. Writing $\boldsymbol{k}$ for $k_3, k_4, \dots, k_d$, and using the Fourier representation (4.4), we have

$$\begin{aligned} W_{12}^{(\beta,\gamma)}(a) \le \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} & \frac{c}{|k_1|^{1-\delta}(|k_2|^2 + |\boldsymbol{k}|^2)^{m/2}|k|} \\ & \times \frac{c}{|k_2|^{1-\varepsilon}(|k_1|^2 + |\boldsymbol{k}|^2)^{n/2}|k|}. \end{aligned} \tag{4.22}$$

This integral is seen to be finite if $\beta + \gamma < d - 4$, by exhausting six cases depending on the lengths of $|k_1|$, $|k_2|$, and $|\boldsymbol{k}|$. We have thus proved Lemma 1.9 for $\bar{W}^{(\beta,\gamma)}$. Proof for $\bar{T}^{(\beta,\gamma)}$ proceeds in exactly the same way and is omitted. □

4.1.4. *Proof of Lemma 1.9 for noninteger $\alpha > \lfloor\phi\rfloor$.* Finally, we control $\bar{G}^{(\alpha)}$ for $\alpha > \lfloor\phi\rfloor$. This is the most complicated of all the cases relevant for Lemma 1.9. In this case, n exceeds $\lfloor\phi\rfloor$ if we write $\alpha = n - \varepsilon$. Thus the assumption of the lemma is not sufficient to guarantee the finiteness of $\partial_1^n \hat{G}(k)$, and we cannot rely on the bound (4.17), which has been so useful in previous cases.

To overcome this difficulty, we proceed as follows. Instead of directly controlling the Fourier transform of $G_1^{(n-\varepsilon)}(a)$, we treat this quantity by considering it as a product of $|a_1|^{1-\varepsilon}$ and $(a_1)^{n-1}G(a)$. When n is even, this product has a different sign from $|a_1|^{n-\varepsilon}$, but this suffices for our purpose.

The Fourier transform of $(a_1)^{n-1}G(a)$ is given by $(i\partial_1)^{n-1}\hat{G}(k)$. Using the explicit differentiation formula [i.e., (4.31) in the proof of Lemma 4.1], we see that terms in $(i\partial_1)^{n-1}\hat{G}(k)$ can be grouped into two: (1) terms with $(n-1)$ derivatives on a single function [i.e., terms with $\partial_1^{n-1}\hat{J}(k)$ or $\partial_1^{n-1}\hat{g}(k)$], and (2) terms which contain lower order derivatives of $\hat{J}$ and $\hat{g}$. We call the first group $\hat{P}(k)$, and the second $\hat{Q}(k)$. Explicitly,

$$\begin{aligned} \hat{P}(k) &= \frac{\hat{g}(k)(i\partial_1)^{n-1}\hat{J}(k)}{\{1 - \hat{J}(k)\}^2} + \frac{(i\partial_1)^{n-1}\hat{g}(k)}{1 - \hat{J}(k)}, \\ \hat{Q}(k) &= (i\partial_1)^{n-1}\hat{G}(k) - \hat{P}(k). \end{aligned} \tag{4.23}$$

We denote their inverse Fourier transforms by $P(a)$ and $Q(a)$, so that

$$G(a) = P(a) + Q(a) \quad \text{and} \tag{4.24}$$
$$|a_1|^{1-\varepsilon} G(a) = |a_1|^{1-\varepsilon} P(a) + |a_1|^{1-\varepsilon} Q(a).$$

Our task is to show that two quantities on the right are finite uniformly in a.

We begin with $|a_1|^{1-\varepsilon} P(a)$. We introduce $\hat{P}_1(k)$, $\hat{P}_2(k)$, $\hat{\psi}_1$ and $\hat{\psi}_2$ as

$$\hat{P}_1(k) := \frac{\hat{g}(k)(i\partial_1)^{n-1}\hat{J}(k)}{\{1-\hat{J}(k)\}^2} := \hat{\psi}_1(k)(i\partial_1)^{n-1}\hat{J}(k), \tag{4.25}$$
$$\hat{P}_2(k) := \frac{(i\partial_1)^{n-1}\hat{g}(k)}{1-\hat{J}(k)} := \hat{\psi}_2(k)(i\partial_1)^{n-1}\hat{g}(k).$$

We only consider P_1, because dealing with P_2 is similar and easier.

Consider P_1 in x-space, which reads

$$P_1(a) = \sum_y (y_1)^{n-1} J(y)\psi_1(a-y). \tag{4.26}$$

Multiply both sides by $|a_1|^{1-\varepsilon}$, and on the right-hand side use $|a_1|^{1-\varepsilon} \le c(|a_1 - y_1|^{1-\varepsilon} + |y_1|^{1-\varepsilon})$ with some constant c. As a result, we get two terms:

$$\begin{aligned} |a_1|^{1-\varepsilon}|P_1(a)| &\le c\sum_y |y_1|^{n-\varepsilon}|J(y)| \times |\psi_1(a-y)| \\ &\quad + c\sum_y |y_1|^{n-1}|J(y)| \times |a_1-y_1|^{1-\varepsilon}|\psi_1(a-y)| \\ &\le c\left[\sum_y |y_1|^{n-\varepsilon}|J(y)|\right]\left[\sup_x |\psi_1(x)|\right] \\ &\quad + c\left[\sum_y |y_1|^{n-1}|J(y)|\right]\left[\sup_x |x_1|^{1-\varepsilon}|\psi_1(x)|\right]. \end{aligned} \tag{4.27}$$

Our task is to show that the four factors are all finite.

First $\sum_y |y_1|^{n-\varepsilon}|J(y)|$ is finite, because of our assumption (recall $n-\varepsilon = \alpha \le \phi$). This also shows that $\sum_y |y_1|^{n-1}|J(y)|$ is finite.

To prove that $\sup_x |x_1|^{1-\varepsilon}|\psi_1(x)|$ is finite, we use the following fact which is proved in Section 4.4 using fractional derivatives:

$$|x_1|^{1-\varepsilon}\psi_1(x) = \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ikx}\hat{\psi}_1^{(1-\varepsilon)}(k) \qquad \text{with} \tag{4.28}$$
$$\left|\hat{\psi}_1^{(1-\varepsilon)}(k)\right| \le \frac{c}{|k_1|^{1-\varepsilon}|\vec{k}|^3|k|}.$$

$\hat{\psi}_1^{(1-\varepsilon)}(k)$ is integrable if $d > 5 - \varepsilon$, and $\sup_x |x_1|^{1-\varepsilon}|\psi_1(x)|$ and $\sup_x |\psi_1(x)|$ are finite [$\psi_1(0)$ is easily seen to be finite in $d > 4$ by (4.74)]. We have therefore shown that (4.27) is finite uniformly in a.

We now turn to $|a_1|^{1-\varepsilon} Q(a) = Q_1^{(1-\varepsilon)}(a)$. We will prove in Section 4.4:

$$|x_1|^{1-\varepsilon} Q_1(x) = \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ikx} \hat{Q}_1^{(1-\varepsilon)}(k) \qquad \text{with} \\ \big|\hat{Q}_1^{(1-\varepsilon)}(k)\big| \le \frac{c}{|k_1|^{1-\varepsilon}|\vec{k}|^n |k|}. \tag{4.29}$$

$\hat{Q}_1^{(1-\varepsilon)}(k)$ is integrable in k, and thus $|a_1|^{1-\varepsilon} Q(a)$ is finite, as long as $2 + n - \varepsilon < d$, or $\alpha + 2 < d$.

We have thus shown that both $|a|^{1-\varepsilon} P(a)$ and $|a|^{1-\varepsilon} Q(a)$ are finite uniformly in a, and the proof is complete.

In the following, we prove Lemma 4.1, Lemma 4.2, (4.28), and (4.29), one by one.

4.2. *Proof of Lemma* 4.1. By $\mathbb{Z}^d$-symmetry, it suffices to prove (4.11) for $j = 1$, and we abbreviate ∂^m for $\frac{\partial^m}{\partial k_1^m}$. We first note that (4.10) implies

$$|\partial^m \hat{g}(k)|, |\partial^m \hat{J}(k)| < \infty \qquad \text{for all } m \le M. \tag{4.30}$$

By explicit differentiation, we see from (1.18) that

$$\partial^m \hat{G}(k) = \partial^m \left(\frac{\hat{g}(k)}{1 - \hat{J}(k)} \right) = \sum_{p=0}^{m} \binom{m}{p} (\partial^{m-p} \hat{g}(k)) \partial^p \left(\frac{1}{1 - \hat{J}(k)} \right) \\ = \partial^m \hat{g}(k) + \sum_{p=1}^{m} \binom{m}{p} (\partial^{m-p} \hat{g}(k)) \sum_{q=2}^{p+1} \sum_{\vec{r}} C_{\vec{r}} \frac{\prod_{\ell \ge 1} [\partial^\ell \hat{J}(k)]^{r_\ell}}{\{1 - \hat{J}(k)\}^q}. \tag{4.31}$$

In the above, $\vec{r} = \{r_\ell\}_{\ell \ge 1}$ is a vector of nonnegative integers, $C_{\vec{r}}$ is a coefficient which depends on $\vec{r}$. The vector $\vec{r}$ satisfies

$$\sum_{\ell \ge 1} \ell r_\ell = p, \qquad \sum_{\ell \ge 1} r_\ell = q - 1, \tag{4.32a}$$

$$r_\ell = 0 \qquad \text{for } \ell + q \ge p + 3. \tag{4.32b}$$

Because $\partial^\ell \hat{J}(k)$ are all finite (for relevant values of ℓ) and because $\hat{J}(k)$ is even in k_j, we have for *odd* ℓ satisfying $\ell + 1 \le M$

$$|\partial^\ell \hat{J}(k)| \le \sup_k |\partial^{\ell+1} \hat{J}(k)||k| = c|k|. \tag{4.33}$$

By (4.32a),

$$\sum_{\ell \ge 2} (\ell - 1) r_\ell = \sum_{\ell \ge 1} \ell r_\ell - \sum_{\ell \ge 1} r_\ell = p - q + 1 \tag{4.34}$$

and we have

$$r_1 = \sum_{\ell \ge 1} r_\ell - \sum_{\ell \ge 2} r_\ell \ge \sum_{\ell \ge 1} r_\ell - \sum_{\ell \ge 2} (\ell - 1) r_\ell \\ = q - 1 - (p - q + 1) = 2q - p - 2. \tag{4.35}$$

We now combine (4.33) and (4.35), dividing into two cases:

(i) For q sufficiently large such that $2q > p + 2$, we have $r_1 > 0$ by (4.35). We have at least r_1 powers of $|k|$ in the numerator, and the terms in (4.31) with $2q > p + 2$ are bounded as

$$|\text{case (i) of (4.31)}| \le c \frac{|k|^{r_1}}{|k|^{2q}} = c|k|^{2q-p-2-2q} = c|k|^{-p-2}. \tag{4.36}$$

(ii) For $2q \le p + 2$, it may happen that there is no first derivative in the numerator. But we at least know that the numerator is finite. We simply bound these terms as

$$|\text{case (ii) of (4.31)}| \le \frac{c}{|k|^{2q}} \le \frac{c}{|k|^{p+2}}. \tag{4.37}$$

Combining these two cases, we get (for $m \le M$)

$$|\partial^m \hat{G}(k)| \le c \sum_{p=0}^{m} [|k|^{-p-2} + |k|^{-p-2}] \le \frac{c}{|k|^{m+2}}. \tag{4.38}$$

This proves (4.11). □

4.3. *Fourier analysis of fractional powers.* One way to prove that a given function $f(x)$ decays at least as fast as $|x|^{-n}$ when $|x| \to \infty$, where n is a positive integer, is to show that the nth-derivative of its Fourier transform $\hat{f}(k)$ is integrable. However, there are cases where the nth-derivative is not integrable, whereas suitably defined $(n - \varepsilon)$th-derivative is, for some $0 < \varepsilon < 1$. We then expect that $f(x)$ decays at least as fast as $|x|^{-(n-\varepsilon)}$. In this subsection, we summarize results which will be useful in such cases. The subject is closely related to fractional derivatives and can be considered as a special case of *Weyl fractional derivatives* ([24], Section 19) if we consider $f(x)$ as the "Fourier coefficient" of $\hat{f}(k)$.

In this subsection, $f(x) : \mathbb{Z}^d \to \mathbb{R}$ always denotes a $\mathbb{Z}^d$-symmetric function, which is represented as

$$f(x) = \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ikx} \hat{f}(k) \qquad \text{with } \hat{f}(k) \in L^1([-\pi, \pi]^d) \tag{4.39}$$

and $\hat{f}(k)$ is periodic in each k_j $(k = 1, 2, \dots, d)$ with period 2π. We treat the first component k_1 of k differently from $k_2, k_3, \dots, k_d$, and write $k = (k_1, \vec{k})$. Also, we write $\partial_1 \hat{f}(k)$ for the partial derivative with respect to the first argument of $\hat{f}$. We define

$$\operatorname{sgn} x_1 = \begin{cases} 1, & (x_1 > 0), \\ 0, & (x_1 = 0), \\ -1, & (x_1 < 0), \end{cases} \tag{4.40}$$

and ($\alpha \in \mathbb{R}$)

$$f_1^{(\alpha)}(x) := |x_1|^\alpha I[x_1 \neq 0] f(x), \qquad f_1'^{(\alpha)}(x) := |x_1|^\alpha (\operatorname{sgn} x_1) f(x). \tag{4.41}$$

Note that the prime on $f_1'^{(\alpha)}(x)$ does *not* mean a derivative. We introduce for $0 < \varepsilon < 1$

$$\begin{aligned} L_{o,\varepsilon}(p_1) &:= \frac{1}{2\pi i \Gamma(\varepsilon)} \int_0^\infty dt\, t^{\varepsilon-1} \frac{\sin p_1}{\cosh t - \cos p_1}, \\ L_{e,\varepsilon}(p_1) &:= \frac{1}{2\pi \Gamma(\varepsilon)} \int_0^\infty dt\, t^{\varepsilon-1} \frac{\cos p_1 - e^{-t}}{\cosh t - \cos p_1}. \end{aligned} \tag{4.42}$$

These are Fourier transforms of $|x_1|^{-\varepsilon}(\operatorname{sgn} x_1)$ and $|x_1|^{-\varepsilon} I[x_1 \neq 0]$ respectively, in the sense that

$$\begin{aligned} |x_1|^{-\varepsilon}(\operatorname{sgn} x_1) &= \int_{-\pi}^{\pi} dp_1 e^{ip_1 x_1} L_{o,\varepsilon}(p_1), \\ |x_1|^{-\varepsilon} I[x_1 \neq 0] &= \int_{-\pi}^{\pi} dp_1 e^{ip_1 x_1} L_{e,\varepsilon}(p_1) \end{aligned} \tag{4.43}$$

hold. These identities can be proved, for example, by interchanging the order of t and p_1 integrations and using residue calculus.

We begin with the following proposition which represents $f_1^{(-\varepsilon)}(x)$ and $f_1'^{(-\varepsilon)}(x)$ in terms of Fourier transforms for $0 < \varepsilon < 1$. The proposition looks almost obvious in view of (4.43); it is a special case of a well-known fact that the Fourier transform of $f(x)g(x)$ is given by $\hat{f} * \hat{g}$.

PROPOSITION 4.3. *Suppose $f(x)$ is represented by* (4.39), *and define for* $0 < \varepsilon < 1$

$$\begin{aligned} \hat{f}_1^{(-\varepsilon)}(k) &:= \int_{-\pi}^{\pi} L_{e,\varepsilon}(p_1) \hat{f}(k_1 - p_1, \vec{k})\, dp_1, \\ \hat{f}'^{(-\varepsilon)}_1(k) &:= \int_{-\pi}^{\pi} L_{o,\varepsilon}(p_1) \hat{f}(k_1 - p_1, \vec{k})\, dp_1. \end{aligned} \tag{4.44}$$

Then we have

$$\begin{aligned} f_1^{(-\varepsilon)}(x) &= \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ikx} \hat{f}_1^{(-\varepsilon)}(k), \\ f_1'^{(-\varepsilon)}(x) &= \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ikx} \hat{f}_1'^{(-\varepsilon)}(k). \end{aligned} \tag{4.45}$$

For $f_1^{(\alpha)}(x)$ with $\alpha > 0$, which is of our main interest, several representations with differing conditions of applicability can be obtained. Of these, the following will be useful for our analysis.

PROPOSITION 4.4. *Suppose $f(x)$ is represented by (4.39). Let m be a positive integer, $0 < \varepsilon < 1$, and assume $(\partial_1)^m \hat{f}(k)$ is integrable in k_1 for each $\vec{k} \neq \vec{0}$. Define*

$$\hat{f}_1^{(m-\varepsilon)}(k_1, \vec{k}) := \int_{-\pi}^{\pi} L_{*,\varepsilon}(p_1)\{(i\partial_1)^m \hat{f}(k_1 - p_1, \vec{k})\}\, dp_1, \tag{4.46}$$

where $L_{*,\varepsilon} = L_{o,\varepsilon}$ if m is odd, and $L_{*,\varepsilon} = L_{e,\varepsilon}$ if m is even. Then,

$$f_1^{(m-\varepsilon)}(x) = \lim_{\delta\downarrow 0} \int_{\substack{[-\pi,\pi]^{d-1} \\ |\vec{k}|>\delta}} \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} e^{ikx} \hat{f}_1^{(m-\varepsilon)}(k). \tag{4.47}$$

If we further assume $\hat{f}_1^{(m-\varepsilon)} \in L^1([-\pi, \pi]^d)$, then

$$f_1^{(m-\varepsilon)}(x) = \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ikx} \hat{f}_1^{(m-\varepsilon)}(k). \tag{4.48}$$

As for the integral kernels, we have:

PROPOSITION 4.5. *Fix $0 < \varepsilon < 1$. $L_{o,\varepsilon}(p_1)$ is pure imaginary, odd in p_1, and satisfies for $p_1 \in [-\pi, \pi]$*

$$|L_{o,\varepsilon}(p_1)| \le \tfrac{1}{2}|p_1|^{\varepsilon-1}, \qquad |\partial L_{o,\varepsilon}(p_1)| \le |p_1|^{\varepsilon-2}. \tag{4.49}$$

$L_{e,\varepsilon}(p_1)$ is real-valued, even in p_1, and satisfies for $p_1 \in [-\pi, \pi]$

$$-\frac{\log 2}{\pi} \le L_{e,\varepsilon}(p_1) \le \frac{|p_1|^{\varepsilon-1}}{\pi(1-\varepsilon)}, \qquad |\partial L_{e,\varepsilon}(p_1)| \le \frac{|p_1|^{\varepsilon-2}}{\pi}. \tag{4.50}$$

The above bounds on the derivatives of $L_{o,\varepsilon}$ and $L_{e,\varepsilon}$ are not optimal, in the sense that the coefficients on the right-hand side can be multiplied by $1 - \varepsilon$. However, the current bounds suffice for our purpose.

We in the following briefly prove these propositions, in this order. Because Proposition 4.5 can be proved independently of the rest, we use its result (especially the integrability of $L_{*,\varepsilon}$) in the proofs of Propositions 4.3 and 4.4.

SKETCH OF THE PROOF OF PROPOSITION 4.3. The proof is almost identical for $f_1^{(-\varepsilon)}$ and $f_1'^{(-\varepsilon)}$, and we only treat $f_1^{(-\varepsilon)}$. Note first that $\hat{f}_1^{(-\varepsilon)}$ is well defined and is integrable, thanks to the integrability of $L_{e,\varepsilon}$ and $\hat{f}$. We calculate (4.45) using the definition of $L_{e,\varepsilon}$. Starting from

$$\int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ikx} \hat{f}_1^{(-\varepsilon)}(k) = \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ikx} \int_{-\pi}^{\pi} dp_1 L_{e,\varepsilon}(p_1) \hat{f}(k_1 - p_1, \vec{k}),$$

we write the d-dimensional integral as an iterated integral (guaranteed by the integrability of $L_{e,\varepsilon}$ and $\hat{f}$) and then change variables from k_1, p_1 to $k_1' = k_1 - p_1, p_1' = p_1$. By the periodicity of $\hat{f}$, we get

$$\begin{aligned}
&\int_{[-\pi,\pi]^{d-1}} \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} e^{i\vec{k}\vec{x}} \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} e^{ik_1x_1} \int_{-\pi}^{\pi} dp_1 L_{e,\varepsilon}(p_1) \hat{f}(k_1 - p_1, \vec{k}) \\
&\quad = \int_{[-\pi,\pi]^{d-1}} \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} e^{i\vec{k}\vec{x}} \int_{-\pi}^{\pi} \frac{dk_1'}{2\pi} \\
&\qquad \times \int_{-\pi}^{\pi} dp_1' e^{i(k_1'+p_1')x_1} L_{e,\varepsilon}(p_1') \hat{f}(k_1', \vec{k}) \\
&\quad = \int_{[-\pi,\pi]^{d}} \frac{d^{d}k}{(2\pi)^{d}} e^{ikx} \hat{f}(k_1, \vec{k}) \int_{-\pi}^{\pi} dp_1 e^{ip_1x_1} L_{e,\varepsilon}(p_1) \\
&\quad = f(x) \int_{-\pi}^{\pi} dp_1 e^{ip_1x_1} L_{e,\varepsilon}(p_1).
\end{aligned} \tag{4.51}$$

Now the last integral is $|x_1|^{-\varepsilon} I[x_1 \neq 0]$ by (4.43). □

SKETCH OF THE PROOF OF PROPOSITION 4.4. We deal with even m only—odd m can be treated in the same way. Because $L_{e,\varepsilon}$ is integrable and $(i\partial_1)^m \hat{f}(k)$ is integrable in k_1 for $\vec{k} \neq \vec{0}$, we can interchange the p_1-integral and $(i\partial_1)^m$ to get

$$\begin{aligned}
\hat{f}_1^{(m-\varepsilon)}(k_1, \vec{k}) &= (i\partial_1)^m \int_{-\pi}^{\pi} L_{e,\varepsilon}(p_1) \hat{f}(k_1 - p_1, \vec{k})\, dp_1 \\
&= (i\partial_1)^m \hat{f}_1^{(-\varepsilon)}(k_1, \vec{k})
\end{aligned} \tag{4.52}$$

for $\vec{k} \neq \vec{0}$. Using this, the right-hand side of (4.47) can be calculated as

$$\begin{aligned}
&\lim_{\delta \downarrow 0} \int_{\substack{[-\pi,\pi]^{d-1} \\ |\vec{k}|>\delta}} \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} e^{ikx} \hat{f}_1^{(m-\varepsilon)}(k) \\
&\quad = \lim_{\delta \downarrow 0} \int_{\substack{[-\pi,\pi]^{d-1} \\ |\vec{k}|>\delta}} \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} e^{ikx} \{(i\partial_1)^m \hat{f}_1^{(-\varepsilon)}(k_1, \vec{k})\} \\
&\quad = (x_1)^m \lim_{\delta \downarrow 0} \int_{\substack{[-\pi,\pi]^{d-1} \\ |\vec{k}|>\delta}} \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} e^{ikx} \hat{f}_1^{(-\varepsilon)}(k_1, \vec{k}) \\
&\quad = (x_1)^m \int_{[-\pi,\pi]^{d}} \frac{d^{d}k}{(2\pi)^{d}} e^{ikx} \hat{f}_1^{(-\varepsilon)}(k_1, \vec{k}).
\end{aligned} \tag{4.53}$$

Here the second equality follows from integration by parts with respect to k_1, and the last equality follows because $\hat{f}_1^{(-\varepsilon)}$ is integrable in k. In view of Proposition 4.3, the integral on the far right is nothing but $f_1^{(-\varepsilon)}(x)$ for even m, and we get (4.47). Now (4.48) follows trivially. □

SKETCH OF THE PROOF OF PROPOSITION 4.5. That $iL_{o,\varepsilon}(p_1)$ is odd in p_1 and is positive for $p_1 > 0$ is easily seen from its integral representation (4.42). The integral can be performed exactly by residue calculus, and we get for $0 < p_1 < \pi$

$$\begin{aligned} &iL_{o,\varepsilon}(p_1) \\ &\quad = \frac{\csc((\pi/2)\varepsilon)}{2\Gamma(\varepsilon)}\left[p_1^{\varepsilon-1} - \sum_{n=1}^{\infty}\{(2\pi n - p_1)^{\varepsilon-1} - (2\pi n + p_1)^{\varepsilon-1}\}\right]. \end{aligned} \tag{4.54}$$

As the summand is positive, we immediately get for $0 < p_1 < \pi$

$$\begin{aligned} &0 \le iL_{o,\varepsilon}(p_1) \le \frac{\csc((\pi/2)\varepsilon)}{2\Gamma(\varepsilon)} p_1^{\varepsilon-1} = C_{o,\varepsilon} p_1^{\varepsilon-1} \\ &\qquad \text{with } C_{o,\varepsilon} := \frac{\csc((\pi/2)\varepsilon)}{2\Gamma(\varepsilon)}. \end{aligned} \tag{4.55}$$

The coefficient $C_{o,\varepsilon}$ is increasing in ε for $0 < \varepsilon \le 1$ and is bounded by its value at $\varepsilon = 1$: $C_{o,\varepsilon} \le 1/2$. We thus get the first half of (4.49).

Differentiating (4.54), we get

$$\begin{aligned} &i\partial L_{o,\varepsilon}(p_1) \\ &\quad = -C_{o,\varepsilon}(1-\varepsilon)\left[p_1^{\varepsilon-2} + \sum_{n=1}^{\infty}\{(2\pi n - p_1)^{\varepsilon-2} + (2\pi n + p_1)^{\varepsilon-2}\}\right]. \end{aligned} \tag{4.56}$$

This is even in p_1 and is obviously negative. To get its lower bound, we bound the summation by its value at $p_1 = \pi$ (because the summand is increasing in $|p_1|$), to get

$$\begin{aligned} &\sum_{n=1}^{\infty}\{(2\pi n - p_1)^{\varepsilon-2} + (2\pi n + p_1)^{\varepsilon-2}\} \\ &\quad \le \sum_{n=1}^{\infty}\{(2\pi n - \pi)^{\varepsilon-2} + (2\pi n + \pi)^{\varepsilon-2}\} \\ &\quad = \pi^{\varepsilon-2}\left[1 + 2\sum_{n=1}^{\infty}(2n+1)^{\varepsilon-2}\right]. \end{aligned} \tag{4.57}$$

Because $(2x+1)^{\varepsilon-2}$ is convex, we can bound the sum as

$$\sum_{n=1}^{\infty}(2n+1)^{\varepsilon-2} \le \int_{1/2}^{\infty}(2x+1)^{\varepsilon-2}\,dx = \frac{2^{\varepsilon-1}}{2(1-\varepsilon)} = \frac{2^{\varepsilon-2}}{1-\varepsilon}. \tag{4.58}$$

As a result, we get

$$
\begin{aligned}
-i\partial L_{o,\varepsilon}(p_1) &\le C_{o,\varepsilon}(1-\varepsilon)p_1^{\varepsilon-2} + C_{o,\varepsilon}\pi^{\varepsilon-2}(1-\varepsilon+2^{\varepsilon-1}) \\
&\le \frac{1-\varepsilon}{2}p_1^{\varepsilon-2} + \frac{1}{2\pi}.
\end{aligned}
\tag{4.59}
$$

In the last step above, we used $C_{o,\varepsilon} \le 1/2$ and the fact that $\pi^{\varepsilon-2}(1-\varepsilon+2^{\varepsilon-1})$ is increasing in ε and is bounded by its value at $\varepsilon = 1$, that is, by $1/\pi$.

As for $L_{e,\varepsilon}(p_1)$, we start from an integral representation of its derivative:

$$
\begin{aligned}
\partial L_{e,\varepsilon}(p_1) &= -\frac{\sin p_1}{2\pi\Gamma(\varepsilon)}\int_0^\infty dt\,\frac{t^{\varepsilon-1}\sinh t}{(\cosh t - \cos p_1)^2} \\
&= -\frac{\sin p_1}{2\pi\Gamma(\varepsilon)}\lim_{\delta\to 0}\biggl[\frac{\delta^{\varepsilon-1}}{\cosh\delta - \cos p_1} \\
&\qquad + (\varepsilon-1)\int_\delta^\infty dt\,\frac{t^{\varepsilon-2}}{\cosh t - \cos p_1}\biggr],
\end{aligned}
\tag{4.60}
$$

where in the second step we integrated by parts. The first line of (4.60) shows that $\partial L_{e,\varepsilon}(p_1)$ is negative. The integral on the second line can be performed by residue calculus, and leads to the following representation:

$$
\begin{aligned}
\partial L_{e,\varepsilon}(p_1) = -C_{e,\varepsilon}\biggl[p_1^{\varepsilon-2} - \sum_{n=1}^\infty\{&(2\pi n - p_1)^{\varepsilon-2} \\
&- (2\pi n + p_1)^{\varepsilon-2}\}\biggr],
\end{aligned}
\tag{4.61}
$$

with $C_{e,\varepsilon} = (1-\varepsilon)\sec(\frac{\pi}{2}\varepsilon)/\{2\Gamma(\varepsilon)\}$. The coefficient $C_{e,\varepsilon}$ is increasing in ε for $0 < \varepsilon < 1$ and is bounded by its limiting value at $\varepsilon = 1^-$: $C_{e,\varepsilon} \le 1/\pi$. Because the summand in (4.61) is positive, we get

$$
\partial L_{e,\varepsilon}(p_1) \ge -C_{e,\varepsilon}p_1^{\varepsilon-2} \ge -\frac{1}{\pi}p_1^{\varepsilon-2}.
\tag{4.62}
$$

Finally, we turn to $L_{e,\varepsilon}(p_1)$. Because $\partial L_{e,\varepsilon}(p_1)$ is negative, we can get a lower bound on $L_{e,\varepsilon}(p_1)$ as

$$
\begin{aligned}
L_{e,\varepsilon}(p_1) \ge L_{e,\varepsilon}(\pi) &= -\frac{1}{\pi\Gamma(\varepsilon)}\int_0^\infty \frac{t^{\varepsilon-1}}{e^t+1}\,dt \\
&= \frac{1}{\pi}\sum_{n=1}^\infty \frac{(-1)^n}{n^\varepsilon} \ge -\frac{\log 2}{\pi},
\end{aligned}
\tag{4.63}
$$

where in the last step we bounded the sum by its value at $\varepsilon = 1^-$. To get an upper bound, we integrate (4.61) from p_1 to π, to get

$$L_{e,\varepsilon}(p_1) = \frac{C_{e,\varepsilon}}{1-\varepsilon}\Bigg[p_1^{\varepsilon-1} - \pi^{\varepsilon-1} + \sum_{n=1}^{\infty}\{(2\pi n - p_1)^{\varepsilon-1} + (2\pi n + p_1)^{\varepsilon-1} - (2\pi n - \pi)^{\varepsilon-1} - (2\pi n + \pi)^{\varepsilon-1}\}\Bigg] + L_{e,\varepsilon}(\pi). \tag{4.64}$$

The sum in the above is negative because $(2\pi n - p)^{\varepsilon-1} + (2\pi n + p)^{\varepsilon-1}$ is increasing in p. Because $L_{e,\varepsilon}(\pi) \le 0$ as is seen in (4.63), we get an upper bound

$$L_{e,\varepsilon}(p_1) \le \frac{C_{e,\varepsilon}}{1-\varepsilon}(p_1^{\varepsilon-1} - \pi^{\varepsilon-1}) \le \frac{C_{e,\varepsilon}}{1-\varepsilon}p_1^{\varepsilon-1} \le \frac{1}{\pi(1-\varepsilon)}p_1^{\varepsilon-1}. \tag{4.65}$$ □

4.4. *Bounds on the Fourier transform of "fractionally weighted" two-point functions and related quantities.* In this subsection, we make use of the representation obtained in the previous subsection and prove bounds on the Fourier transform of $G_j^{(\alpha)}(x)$, Lemma 4.2, and two bounds (4.28) and (4.29). We start from the following simple lemma concerning one-dimensional convolution, whose proof is postponed to the end of this section.

LEMMA 4.6. *Fix $0 < \varepsilon < 1$ and $\rho > 1$, and suppose*

$$|f(k_1, \vec{k})| \le \frac{1}{|k|^\rho}, \qquad |\partial_1 f(k_1, \vec{k})| \le \frac{1}{|k|^{\rho+1}} \tag{4.66}$$

and

$$|g(p_1)| \le |p_1|^{\varepsilon-1}, \qquad |\partial_1 g(p_1)| \le |p_1|^{\varepsilon-2}. \tag{4.67}$$

Then the 1*-dimensional convolution*

$$(\partial_1 f * g)(k_1, \vec{k}) = \int_{-\pi}^{\pi} g(p_1)\partial_1 f(k_1 - p_1, \vec{k})\, dp_1 \tag{4.68}$$

obeys

$$|(\partial_1 f * g)(k_1, \vec{k})| \le c \times \begin{Bmatrix} |k_1|^{\varepsilon-1}|\vec{k}|^{-\rho} & (\textit{for } |k_1| \le |\vec{k}|) \\ |k_1|^{\varepsilon-2}|\vec{k}|^{1-\rho} & (\textit{for } |k_1| \ge |\vec{k}|) \end{Bmatrix} \approx \frac{c}{|k_1|^{1-\varepsilon}|\vec{k}|^{\rho-1}|k|} \tag{4.69}$$

with a possibly ε-dependent constant c.

PROOF OF LEMMA 4.2. We consider only positive k_1; bounds on negative k_1 follow by $\mathbb{Z}^d$-symmetry. We start from the following expression for the Fourier transform suggested by (4.46):

$$\hat{G}_1^{(n-\varepsilon)}(k) = \int_{-\pi}^{\pi} L_{*,\varepsilon}(p_1)\{(i\partial_1)^n \hat{G}(k_1 - p_1, \vec{k})\}\, dp_1, \tag{4.70}$$

where $L_{*,\varepsilon} = L_{o,\varepsilon}$ if n is odd, and $L_{*,\varepsilon} = L_{e,\varepsilon}$ if n is even. [We will soon check that the above $\hat{G}_1^{(n-\varepsilon)}(k)$ in fact satisfies (4.16).] Lemma 4.1 gives

$$|\partial_1^n \hat{G}(k)| \le c|k|^{-2-n} \qquad \text{for } n \le M, \tag{4.71}$$

and Proposition 4.5 gives

$$|L_{*,\varepsilon}(p)| \le c|p|^{\varepsilon-1}, \qquad |\partial L_{*,\varepsilon}(p)| \le c|p|^{\varepsilon-2}. \tag{4.72}$$

We combine these and estimate (4.70) by the following lemma, by setting $f(k) = \partial_1^{n-1}\hat{G}$ and $\rho = n+1$. (We can apply the lemma, because $\rho > 1$ thanks to $n \ge 1$). The result turns out to be (4.17).

The proof is complete if we show that $\hat{G}_1^{(n-\varepsilon)}(k)$ of (4.70) does satisfy (4.16) for $1 \le n \le M \wedge (d-2)$. For this, note that (4.71) guarantees the integrability of $\partial_1^n \hat{G}(k)$ in k_1 for fixed $\vec{k} \ne \vec{0}$. Also, (4.17) means $\hat{G}_1^{(n-\varepsilon)}(k)$ is integrable in k, for n under consideration. Therefore Proposition 4.4 can be applied and (4.48) holds for G, which is nothing but (4.16). □

PROOF OF (4.28). As we did for $G_1^{(n-\varepsilon)}$, we start from the following expression for the Fourier transform suggested by (4.46):

$$\hat{\psi}_1^{(1-\varepsilon)}(k) = \int_{-\pi}^{\pi} L_{o,\varepsilon}(k_1 - p_1)\partial_1 \hat{\psi}_1(p_1, \vec{k})\, dp_1. \tag{4.73}$$

By $\hat{J}_{p_c}(0) = 1$, the estimate (1.24), and the fact that $\hat{J}(k)$ is even in k_1, it is easily seen that $\hat{\psi}_1(k)$ obeys the bound

$$|\hat{\psi}_1(k)| \le c|k|^{-4}, \qquad |\partial_1 \hat{\psi}_1(k)| \le c|k|^{-5}. \tag{4.74}$$

So Lemma 4.6 implies the desired bound,

$$|\hat{\psi}_1^{(1-\varepsilon)}(k)| \le \frac{c}{|k_1|^{1-\varepsilon}|\vec{k}|^3|k|}. \tag{4.75}$$

The proof is complete if we show that the inverse Fourier transform of $\hat{\psi}_1^{(1-\varepsilon)}(k)$ is equal to $|x_1|^{1-\varepsilon}\psi_1(x)$, but this can be done in exactly the same way as for $\hat{G}^{(n-\varepsilon)}$. □

PROOF OF (4.29). The proof for $|a_1|^{1-\varepsilon}Q(a) = Q_1^{(1-\varepsilon)}(a)$ is similar. We start from

$$\hat{Q}_1^{(1-\varepsilon)}(k) = \int_{-\pi}^{\pi} L_{o,\varepsilon}(k_1 - p_1)\partial_1 Q(p_1, \vec{k})\, dp_1. \tag{4.76}$$

We first note that the total number of differentiations appearing in each term of $\hat{Q}(k)$ is $n-1$, and that of $\partial_1 \hat{Q}(k)$ is n. Second, we note that all the derivatives appearing in the expression of $\partial_1 \hat{Q}(k)$ are finite; this is because the highest order of differentiation in $\hat{Q}(k)$ is $n-2$, and thus the highest order of differentiation in $\partial_1 \hat{Q}(k)$ is $n-1 \leq \lfloor \phi \rfloor$. So the expression for $\partial_1 \hat{Q}(k)$ now reads [cf. (4.31)]:

$$\partial_1 \hat{Q}(k) = \sum_{p=1}^{n} \binom{n}{p} \left(\partial_1^{n-p} \hat{g}(k)\right) \sum_{q=2}^{p+1} \sum_{\vec{r}} C_{\vec{r}} \frac{\prod_{\ell \geq 1} [\partial_1^{\ell} \hat{J}(k)]^{r_\ell}}{\{1-\hat{J}(k)\}^q}, \tag{4.77}$$

but $C_{\vec{r}} = 0$ if $p = n$ and $q = 2$ (i.e., only $\ell < n$ is allowed in the numerator). Arguing as in the proof of Lemma 4.1 and counting powers of $|k|$ of each term, we get

$$|\hat{Q}(k)| \leq c|k|^{-(1+n)}, \qquad |\partial_1 \hat{Q}(k)| \leq c|k|^{-(2+n)}. \tag{4.78}$$

Therefore, estimating (4.76) using Lemma 4.6 leads to

$$\left|\hat{Q}_1^{(1-\varepsilon)}(k)\right| \leq \frac{c}{|k_1|^{1-\varepsilon}|\vec{k}|^n |k|}. \tag{4.79}$$

Finally, we can show that the inverse Fourier transform of the above $\hat{Q}_1^{(1-\varepsilon)}(k)$ is in fact $|x_1|^{1-\varepsilon} Q(x)$, just as we did for $\hat{G}_1^{(n-\varepsilon)}(k)$. □

PROOF OF LEMMA 4.6. We first rewrite (4.68), dividing the integration region and integrating by parts as follows. To simplify notation, we write $a := k_1$ and $b := |\vec{k}|$:

$$\begin{aligned}(\partial_1 f * g)(k) &= \int_{|p_1|<a/2} g(a-p_1)\partial_1 f(p_1,\vec{k})\,dp_1 \\ &\quad + \int_{a/2<|p_1|<\pi} g(a-p_1)\partial_1 f(p_1,\vec{k})\,dp_1 \\ &= [g(a-p_1) f(p_1,\vec{k})]_{-a/2}^{a/2} \\ &\quad + \int_{|p_1|<a/2} \partial_1 g(a-p_1) f(p_1,\vec{k})\,dp_1 \\ &\quad + \int_{a/2<|p_1|<\pi} g(a-p_1)\partial_1 f(p_1,\vec{k})\,dp_1.\end{aligned} \tag{4.80}$$

The integration by parts was done only for the interval $[-a/2, a/2]$—this is justified because there is no singularity of $g(p_1)$ in this interval. We estimate these terms one by one.

The first and second terms are simple. For the first term, we have

$$\begin{aligned}|[g(a-p_1) f(p_1,\vec{k})]_{-a/2}^{a/2}| &\leq \left|g\left(\frac{a}{2}\right) f\left(\frac{a}{2},\vec{k}\right)\right| + \left|g\left(\frac{3a}{2}\right) f\left(-\frac{a}{2},\vec{k}\right)\right| \\ &\leq c a^{\varepsilon-1} \times (a^2+b^2)^{-\rho/2}.\end{aligned} \tag{4.81}$$

For the second term, we have

$$\left|\int_{|p_1|<a/2} \partial g(a-p_1) f(p_1,\vec{k})\,dp_1\right| \tag{4.82}$$
$$\begin{aligned}
&\le c\int_{|p_1|<a/2} |a-p_1|^{\varepsilon-2}(p_1^2+b^2)^{-\rho/2}\,dp_1\\
&\le ca^{\varepsilon-2}\int_{-a/2}^{a/2}(p_1^2+b^2)^{-\rho/2}\,dp_1\\
&\le ca^{\varepsilon-2}\times(a\wedge b)b^{-\rho} = ca^{\varepsilon-1}b^{1-\rho}(a\vee b)^{-1}.
\end{aligned}$$

The third term is bounded and divided as

$$\begin{aligned}
&\left|\int_{a/2<|p_1|<\pi} g(a-p_1)\partial_1 f(p_1,\vec{k})\,dp_1\right|\\
&\quad\le c\int_{a/2<|p_1|<\pi} |a-p_1|^{\varepsilon-1}(p_1^2+b^2)^{-(\rho+1)/2}\,dp_1\\
&\quad=\int_{-\pi}^{-a/2}(\cdots)\,dp_1+\int_{a/2}^{3a/2}(\cdots)\,dp_1+\int_{3a/2}^{\pi}(\cdots)\,dp_1\\
&\quad:=(I)+(II)+(III).
\end{aligned}\tag{4.83}$$

In (I) and (III), we have $|a-p_1|\ge|p_1|/3$ and $|a-p_1|^{\varepsilon-1}\le 3^{1-\varepsilon}p_1^{\varepsilon-1}$. Therefore, we can bound them as

$$(I)+(III)\le c\int_{a/2}^{\infty}(p_1^2+b^2)^{-(\rho+1)/2}p_1^{\varepsilon-1}\,dp_1. \tag{4.84}$$

This integral can be bounded in two ways. First, by neglecting b^2 in the integrand,

$$\begin{aligned}
(I)+(III)&\le c\int_{a/2}^{\infty}(p_1^2+b^2)^{-(\rho+1)/2}p_1^{\varepsilon-1}\,dp_1\\
&\le\int_{a/2}^{\infty}p_1^{\varepsilon-1-\rho-1}\,dp_1=ca^{\varepsilon-\rho-1}.
\end{aligned}\tag{4.85}$$

Also, extending the integration region to $p_1\ge 0$ and changing the variable to $q_1 = p_1/b$,

$$\begin{aligned}
(I)+(III)&\le c\int_{a/2}^{\infty}(p_1^2+b^2)^{-(\rho+1)/2}p_1^{\varepsilon-1}\,dp_1\\
&\le b^{\varepsilon-\rho-1}\int_0^{\infty}(1+q_1^2)^{-(\rho+1)/2}q_1^{\varepsilon-1}\,dq_1.
\end{aligned}\tag{4.86}$$

For $0<\varepsilon<\rho+1$, the last integral is finite. We have thus shown $(I)+(III)\le$

$c(a \vee b)^{\varepsilon-\rho-1}$. Now in (II), $p_1^2 + b^2$ is of the same order as $a^2 + b^2$. Thus

$$(II) \le c(a^2+b^2)^{-(\rho+1)/2} \int_{a/2}^{3a/2} |a-p_1|^{\varepsilon-1}\, dp_1 \\ = c(a^2+b^2)^{-(\rho+1)/2} a^{\varepsilon}. \tag{4.87}$$

We can thus conclude

$$(I) + (II) + (III) \le c(a \vee b)^{\varepsilon-\rho-1} + c(a^2+b^2)^{-(\rho+1)/2} a^{\varepsilon} \\ \le c(a \vee b)^{\varepsilon-\rho-1}. \tag{4.88}$$

Combining (4.81), (4.82) and (4.88), we get

$$|(\partial_1 f * g)(k_1, \vec{k})| \\ \le c[a^{\varepsilon-1}(a \vee b)^{-\rho} + a^{\varepsilon-1} b^{1-\rho}(a \vee b)^{-1} + (a \vee b)^{\varepsilon-\rho-1}] \\ \le c \begin{cases} a^{\varepsilon-1} b^{-\rho} & (a < b), \\ a^{\varepsilon-2} b^{1-\rho} & (a \ge b), \end{cases} \tag{4.89}$$

which proves the lemma [note that $(a \vee b) \le |k| \le 2(a \vee b)$]. □

APPENDIX A: QUANTITIES AT $p = p_c$

Proposition 1.3 is a slightly improved version of corresponding results obtained in previous works: [10, 11] (SAW), [8] (percolation), and [9] (LTLA). It is slightly improved, in the sense that original works mainly dealt with quantities for $p < p_c$ (although all the estimates were uniform in p). More precisely, estimates (1.24) and (1.26)–(1.28), the Fourier representation (1.18), and the bound (1.25) are proved for $p < p_c$; the critical point p_c is characterized by

$$\lim_{p \uparrow p_c} \hat{J}_p(0) = 1 \tag{A.1}$$

instead of (1.30). We in this Appendix show how to extend these results to $p = p_c$, so that we have Proposition 1.3.

1. We first explain how to extend estimates (1.24) and (1.26)–(1.28) to $p = p_c$. Note that $G_p(x)$ is left continuous and increasing in p; this is because $G_p(x)$ can be realized as an increasing limit (finite sum/volume approximation) of a function which is continuous and increasing in p. [In fact $G_p(x)$ for percolation is continuous in p for all p ([4], page 203), although we do not need this fact.] The left continuity of $G_p(x)$ in p and the dominated convergence theorem establish (1.26)–(1.28) at $p = p_c$. Diagrammatic bounds of the lace expansion and the dominated convergence theorem now guarantee absolute convergence of the sums over x and n defining $\hat{\Pi}_p(k)$ at $p = p_c$. [We are not using continuity of $\hat{\Pi}_p(k)$ here; we just bound each term of $\hat{\Pi}_{p_c}(k)$ in terms of quantities appearing in (1.26)–(1.28) at $p = p_c$.] Therefore (1.24) holds even at $p = p_c$.

2. Moreover, (1.23) is now proved, where $h^{(n)}(x)$ is obtained by bounding diagrams for $\Pi_{p_c}^{(n)}(x)$ in terms of (products of) critical two-point function G_{p_c}. Finally, both equations of (1.10) hold even at $p = p_c$ for $f(x) = \Pi_{p_c}(x), g_{p_c}(x), J_{p_c}(x)$.

3. We will, later in this Appendix, show that $\hat{\Pi}_p(k)$ and $\hat{g}_p(k)$ are left continuous at $p = p_c$. Equation (1.30) now follows from (A.1) by the left-continuity.

4. Finally, we deal with (1.18) and (1.25) at $p = p_c$. This is rather subtle, because $G_{p_c}(x)$ is not summable as Theorem 1.1 implies. To make sense of (1.18), let $p \uparrow p_c$ on both sides of (1.18). By the left continuity of $G_p(x)$ in p, the left-hand side of (1.18) goes to $G_{p_c}(x)$. On the right-hand side, the integrand is integrable in k uniformly in $p < p_c$, thanks to the infrared bound (1.25) (recall that we are considering $d > 2$).

Therefore, by the dominated convergence theorem and the left-continuity of $\hat{\Pi}_p(k)$ and $\hat{g}_p(k)$ stated above,

$$
\begin{aligned}
G_{p_c}(x) = \lim_{p\uparrow p_c} G_p(x) &= \lim_{p\uparrow p_c} \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ikx} \frac{\hat{g}_p(k)}{1-\hat{J}_p(k)} \\
&= \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ikx} \left(\lim_{p\uparrow p_c} \frac{\hat{g}_p(k)}{1-\hat{J}_p(k)} \right) \qquad \text{(A.2)} \\
&= \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{ikx} \frac{\hat{g}_{p_c}(k)}{1-\hat{J}_{p_c}(k)}.
\end{aligned}
$$

Thus, we still have the Fourier representation (1.18) and the infrared bound (1.25) at $p = p_c$, with the understanding that $\hat{G}_{p_c}(k)$ is defined by the integrand of the right-hand side of (A.2).

5. Our remaining task is to prove the left-continuity of $\hat{J}_p(k)$ and $\hat{g}_p(k)$ at $p = p_c$. The proof is based on the following lemma:

LEMMA A.1. *Consider SAW, percolation, or LTLA for which Proposition 1.3 holds.* $\Pi_p^{(n)}(x)$ *is continuous in* p *for* $p < p_c$, *and is left-continuous at* $p = p_c$.

PROOF THAT $\hat{\Pi}_p(k)$ IS CONTINUOUS IN p FOR $p \le p_c$, ASSUMING LEMMA A.1. Equation (1.23) implies the double sum $\sum_x \sum_n (-1)^n \Pi_p^{(n)}(x) e^{-ikx}$ is absolutely and uniformly convergent for $p \le p_c$, and Lemma A.1 claims the summand is continuous in p for $p < p_c$ and is left-continuous at $p = p_c$. Because the uniform convergent limit of a continuous function is continuous, $\hat{\Pi}_p(k)$ is continuous in p for $p < p_c$, and is left-continuous at $p = p_c$. $\hat{J}_p(k)$ and $\hat{g}_p(k)$ are now left-continuous at $p = p_c$, as is easily seen from their definition. □

In the rest of this Appendix, we prove Lemma A.1 above.

A.1. Proof of Lemma A.1 for SAW and LTLA. For self-avoiding walk and lattice trees/animals, Lemma A.1 follows rather easily. $\Pi_p^{(n)}(x)$ of these models are, by definition, power series in p with positive coefficients: $\Pi_p^{(n)}(x) = \sum_{m=0}^{\infty} a_m(n,x)p^m$. Equation (1.23) guarantees that the radius of convergence of this series is at least p_c, and that the series converges absolutely at $p = p_c$. The finite sum $f_M(p) := \sum_{m=0}^{M} a_m(n,x)p^m$ is of course continuous in p. Therefore, $\Pi_p^{(n)}(x)$, being a uniformly convergent limit of a continuous function $f_M(p)$, is thus continuous in p for $p < p_c$ and is left-continuous at $p = p_c$.

A.2. Proof of Lemma A.1 for percolation. Proving Lemma A.1 for percolation is more subtle, because finite volume approximation to $\Pi_p^{(n)}(x)$ does not seem to be either increasing or decreasing in p. To overcome this difficulty, we decompose $\Pi_p(x)$ further, and express it in terms of increasing/decreasing events.

Step 1. We begin by recalling the definitions of $\Pi_p^{(n)}(x)$. Because the following description is very brief, the reader is advised to consult [8, 12, 26] for details.

- Given a set of sites $A \in \mathbb{Z}^d$ and a bond configuration, two sites x and y are *connected in* A if there is a path of occupied bonds from x to y having all of its sites in A, or if $x = y \in A$. The set of all sites which are connected to x is denoted by $C(x)$. [This $C(x)$ has nothing to do with $C(x)$ of Theorem 1.4.]
- Given a set of sites $A \in \mathbb{Z}^d$ and a bond configuration, two sites x and y are *connected through* A if they are connected but they are *not* connected in $\mathbb{Z}^d \backslash A$.
- Given a bond configuration and a bond $\{u, v\}$, we define $\tilde{C}^{\{u,v\}}(x)$ to be the set of sites which remain connected to x in the new configuration obtained by setting $\{u, v\}$ to be vacant.

For $x, y \in \mathbb{Z}^d$ and $A \subset \mathbb{Z}^d$, let $E_0(x, y)$ be the event that x and y are doubly connected, and let $E_2(x, y; A)$ be the event that x is connected to y through A and there is no pivotal bond for the connection from x to y whose first endpoint is connected to x through A. We now define

$$\Pi_p^{(0)}(x) := \mathbb{E}[I[E_0(0, x)]] \tag{A.3}$$

and for $n \geq 1$

$$\Pi_p^{(n)}(x) := \sum_{(y_1,y_1')} p \sum_{(y_2,y_2')} p \cdots \sum_{(y_n,y_n')} p \Pi_p^{(n)}(x; y_1, y_1', y_2, y_2', \dots, y_n, y_n'). \tag{A.4}$$

Here the sums are over all directed pairs of nearest neighbor sites, and

$$\begin{aligned} &\Pi_p^{(n)}(x; y_1, y_1', \dots, y_n, y_n') \\ &\quad := \mathbb{E}_0\big[I_0\mathbb{E}_1\big[I_1\mathbb{E}_2\big[I_2 \cdots \mathbb{E}_{n-1}\big[I_{n-1}\mathbb{E}_n[I_n]\big]\cdots\big]\big]\big], \end{aligned} \tag{A.5}$$

where we abbreviated $I_0 = I[E_0(0, y_1)]$, $I_j = I[E(y_j', y_{j+1}; \tilde{C}_{j-1})]$ with $\tilde{C}_{j-1} = \tilde{C}^{\{y_j,y_j'\}}(y_{j-1}')$ and $y_{n+1} = x$, and $\mathbb{E}_j$'s represent nested expectations.

Step 2. We in the following prove that $\Pi_p^{(0)}(x)$ and $\Pi_p^{(n)}(x; y_1, y_1', \dots, y_n, y_n')$ are continuous in p for $p < p_c$ and are left continuous at $p = p_c$. Once this is done, continuity of $\Pi_p^{(n)}(x)$ follows easily as follows. In the proof of Proposition 1.3, one proves that there is a function $h^{(n)}(x; y_1, y_1', \dots, y_n, y_n')$ which satisfies

$$\Pi_p^{(n)}(x; y_1, y_1', \dots, y_n, y_n') \le h^{(n)}(x; y_1, y_1', \dots, y_n, y_n') \qquad (p \le p_c), \tag{A.6}$$

and

$$\sum_{(y_1, y_1')} p \sum_{(y_2, y_2')} p \dots \sum_{(y_n, y_n')} p h^{(n)}(x; y_1, y_1', \dots, y_n, y_n') < \infty. \tag{A.7}$$

This bound guaranties that the sum over (y_j, y_j') $(j = 1, 2, \dots, n)$ in (A.4) converges uniformly and absolutely for $p \le p_c$. This implies the limit $\Pi_p^{(n)}(x)$ is continuous for $p < p_c$ and is left continuous at $p = p_c$.

Our task has thus been reduced to proving continuity of $\Pi_p^{(0)}(x)$ and $\Pi_p^{(n)}(x; y_1, y_1', \dots, y_n, y_n')$. We consider $\Pi_p^{(n)}(x; y_1, y_1', \dots, y_n, y_n')$ only, because $\Pi_p^{(0)}(x)$ is (much) easier.

Step 3. To prove continuity of $\Pi_p^{(n)}(x; y_1, y_1', \dots, y_n, y_n')$, we express it in terms of increasing/decreasing events. For $A \subset \mathbb{Z}^d$, we define:

- $F_1(x, y; A)$: the event that x and y are connected in $\mathbb{Z}^d \setminus A$.
- $F_2(x, y; A)$: the event that x and y are connected, and x and z are connected in $\mathbb{Z}^d \setminus A$, where z is the first endpoint of the last pivotal bond for the connection from x to y. (When there is no pivotal bond for the connection from x to y, F_2 is simply the event that x and y are connected.)

Note that $F_1(x, y; A) \subset F_2(x, y; A)$ and $E_2(x, y; A) = F_2(x, y; A) \setminus F_1(x, y; A)$. So, abbreviating $I_j^{(\varepsilon)} = I[F_\varepsilon(y_j', y_{j+1}; \tilde{C}_{j-1})]$ (for $\varepsilon = 1, 2$), we have $I_j = I_j^{(2)} - I_j^{(1)}$. Using this, we can decompose as

$$\Pi_p^{(n)}(x; y_1, y_1', \dots, y_n, y_n') = \sum_{\vec{\varepsilon}} (-1)^{\sum_{j=1}^n \varepsilon_j} \Pi_p^{(n, \vec{\varepsilon})}(x; y_1, y_1', \dots, y_n, y_n'), \tag{A.8}$$

where $\vec{\varepsilon}$ stands for $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, the sum over $\vec{\varepsilon}$ runs over all choices of $\varepsilon_j = 1, 2$, and

$$\begin{aligned} &\Pi_p^{(n, \vec{\varepsilon})}(x; y_1, y_1', \dots, y_n, y_n') \\ &\qquad := \mathbb{E}_0\big[I_0 \mathbb{E}_1\big[I_1^{(\varepsilon_1)} \mathbb{E}_2\big[I_2^{(\varepsilon_2)} \cdots \mathbb{E}_{n-1}\big[I_{n-1}^{(\varepsilon_{n-1})} \mathbb{E}_n\big[I_n^{(\varepsilon_n)}\big]\big] \cdots\big]\big]\big]. \end{aligned} \tag{A.9}$$

We in the following prove that $\Pi_p^{(n, \vec{\varepsilon})}$ is continuous in p for every choice of $\varepsilon_j = 1, 2$ $(1 \le j \le n)$. Continuity of $\Pi_p^{(n)}$ in p immediately follows from this.

Step 4. Continuity of $\Pi_p^{(n, \vec{\varepsilon})}$ is proved by considering its finite volume approximations which are increasing/decreasing in the volume. Let Λ be a finite set of sites in $\mathbb{Z}^d$, and write $\partial\Lambda$ for its boundary sites. We define for $A \subset \Lambda$ and for $\varepsilon = 1, 2$:

- $\tilde{E}_{0,\Lambda}(x, y) := E_0(x, y) \cap \{C(x) \cap \partial\Lambda = \varnothing\}$,
- $\tilde{\tilde{E}}_{0,\Lambda}(x, y) := E_0(x, y) \cup \{C(x) \cap \partial\Lambda \neq \varnothing\}$,
- $\tilde{F}_{\varepsilon,\Lambda}(x, y; A) := F_\varepsilon(x, y; A) \cap \{C(x) \cap \partial\Lambda = \varnothing\}$,
- $\tilde{\tilde{F}}_{\varepsilon,\Lambda}(x, y; A) := F_\varepsilon(x, y; A) \cup \{C(x) \cap \partial\Lambda \neq \varnothing\}$.

Now define $\tilde{\Pi}^{(n,\vec{\varepsilon})}_{p,\Lambda}$ by replacing E_0 and F_ε by $\tilde{E}_{0,\Lambda}$ and $\tilde{F}_{\varepsilon,\Lambda}$ in the definition of $\Pi^{(n,\vec{\varepsilon})}_p$; also define $\tilde{\tilde{\Pi}}^{(n,\vec{\varepsilon})}_{p,\Lambda}$ by replacing E_0 and F_ε by $\tilde{\tilde{E}}_{0,\Lambda}$ and $\tilde{\tilde{F}}_{\varepsilon,\Lambda}$.

$\tilde{\Pi}^{(n,\vec{\varepsilon})}_{p,\Lambda}$ and $\tilde{\tilde{\Pi}}^{(n,\vec{\varepsilon})}_{p,\Lambda}$ converge to $\Pi^{(n,\vec{\varepsilon})}_p$ in the infinite volume limit as long as the percolation density is zero (which has been proven to be the case for $p \leq p_c$ in high dimensions). Thanks to their definition, $\tilde{\Pi}^{(n,\vec{\varepsilon})}_{p,\Lambda}$ is increasing in the volume, while $\tilde{\tilde{\Pi}}^{(n,\vec{\varepsilon})}_{p,\Lambda}$ is decreasing in the volume. Moreover, these functions are continuous in p for $p \leq p_c$; this is because events with tildes and double tildes are essentially finite volume events.

The (increasing) infinite volume limit of a continuous function $\tilde{\Pi}^{(n,\vec{\varepsilon})}_{p,\Lambda}$ is lower semicontinuous in p for $p \leq p_c$. The (decreasing) infinite volume limit of a continuous function $\tilde{\tilde{\Pi}}^{(n,\vec{\varepsilon})}_{p,\Lambda}$ is upper semicontinuous in p for $p \leq p_c$. Their common limit, $\Pi^{(n,\vec{\varepsilon})}_p$, is thus continuous in p for $p < p_c$ and is left-continuous at $p = p_c$.

APPENDIX B: BASIC PROPERTIES OF CONVOLUTION

LEMMA B.1. (i) *Let f, g be functions on $\mathbb{Z}^d$ which satisfy*

$$|f(x)| \leq \frac{1}{|||x|||^\alpha}, \qquad |g(x)| \leq \frac{1}{|||x|||^\beta}, \tag{B.1}$$

with $\alpha, \beta > 0$. Then there exists a constant C depending on α, β, d such that

$$|(f * g)(x)| \leq \begin{cases} C|||x|||^{-(\alpha\wedge\beta)}, & (\alpha > d \text{ or } \beta > d), \\ C|||x|||^{-(\alpha+\beta-d)}, & (\alpha, \beta < d \text{ and } \alpha + \beta > d). \end{cases} \tag{B.2}$$

(ii) *Let $d > 2$ and let f, g be $\mathbb{Z}^d$-symmetric functions on $\mathbb{Z}^d$, which satisfy*

$$f(x) = \frac{A}{|||x|||^{d-2}} + O\left(\frac{B}{|||x|||^{d-2+\rho}}\right), \qquad |g(x)| \leq \frac{C}{|||x|||^{d+\rho}} \tag{B.3}$$

with positive A, B, C and $0 < \rho < 2$. Then

$$(f * g)(x) = \frac{A\sum_y g(y)}{|||x|||^{d-2}} + O\left(\frac{C(A + B)}{|||x|||^{d-2+\rho}}\right), \tag{B.4}$$

where the constants in the error term depend on d and ρ.

(iii) *Let f, g be functions on $\mathbb{Z}^d$ which satisfy*

$$|f(x)| \le \frac{1}{|||x|||^\alpha}, \qquad \sum_x |g(x)| = K, \qquad |g(x)| \le \frac{K}{|x|^d} \tag{B.5}$$

with positive K and $0 < \alpha < d$. Then, there exists a constant C depending on α, d such that

$$|(f * g)(x)| \le \frac{CK}{|x|^\alpha}. \tag{B.6}$$

(iv) *Let f, g be functions on $\mathbb{Z}^d$ which satisfy*

$$\begin{aligned} f(x) &\sim \frac{A}{|x|^\alpha}, \qquad (|x| \uparrow \infty), \\ \sum_x |g(x)| &= K, \qquad |g(x)| \le \frac{K}{|x|^d} \end{aligned} \tag{B.7}$$

with positive A, K and $0 < \alpha < d$. Then,

$$(f * g)(x) \sim \frac{A \sum_y g(y)}{|x|^\alpha} \qquad (|x| \uparrow \infty). \tag{B.8}$$

PROOF. Parts (i) and (ii) are Proposition 1.7 of [7], and their proofs are omitted. We now prove (iii) and (iv). The case $x = 0$ is trivial, so we only consider $x \neq 0$.

(iii) Divide the sum defining $f * g$ into two, $(f * g)(x) = T_1 + T_2$, with

$$T_1 := \sum_{y : |y| < |x|/2} f(y) g(x - y), \qquad T_2 := \sum_{y : |y| \ge |x|/2} f(y) g(x - y). \tag{B.9}$$

For T_1, note that $|x - y| \ge |x|/2$. Therefore,

$$\begin{aligned} |T_1| &\le \sum_{y : |y| < |x|/2} \frac{1}{|||y|||^\alpha} \frac{K}{|x - y|^d} \\ &\le \frac{K}{(|x|/2)^d} \sum_{y : |y| < |x|/2} \frac{1}{|||y|||^\alpha} \le C' K 2^d |x|^{-\alpha} \end{aligned} \tag{B.10}$$

with some constant C'. On the other hand, using $|y| \ge |x|/2$ for T_2, we have

$$\begin{aligned} |T_2| &\le \sum_{y : |y| \ge |x|/2} \frac{1}{|||y|||^\alpha} |g(x - y)| \\ &\le \frac{1}{(|x|/2)^\alpha} \sum_y |g(x - y)| \le 2^d |x|^{-\alpha} \times K. \end{aligned} \tag{B.11}$$

Combining the above two proves (iii).

(iv) Fix $0 < \varepsilon \ll 1$, and divide the sum defining $f * g$ into three parts, $(f * g)(x) = S_1 + S_2 + S_3$, with

$$S_1 := \sum_{y:\,|y|<\varepsilon|x|} f(x-y)g(y),$$
$$S_2 := \sum_{y:\,|x-y|<\varepsilon|x|} f(x-y)g(y), \tag{B.12}$$
$$S_3 := (f*g)(x) - S_1 - S_2.$$

In the following, we prove

$$S_1 = \frac{A}{|x|^\alpha}\sum_y g(y) + o(|x|^{-\alpha}) + \varepsilon O(|x|^{-\alpha}),$$
$$S_2 = \varepsilon^{d-\alpha} O(|x|^{-\alpha}), \qquad S_3 = \varepsilon^{-\alpha} o(|x|^{-\alpha}), \tag{B.13}$$

where $O(|x|^{-\alpha})$ does not, but $o(|x|^{-\alpha})$ may, depend on ε. These give, for fixed $0 < \varepsilon \ll 1$,

$$\limsup_{|x|\to\infty} |x|^\alpha (f*g)(x) \le A \sum_y g(y) + [C'\varepsilon + C''\varepsilon^{d-\alpha}], \tag{B.14}$$

$$\liminf_{|x|\to\infty} |x|^\alpha (f*g)(x) \ge A \sum_y g(y) - [C'\varepsilon + C''\varepsilon^{d-\alpha}] \tag{B.15}$$

with some C', C''. Letting $\varepsilon \downarrow 0$ yields $\lim_{|x|\to\infty} |x|^\alpha (f*g)(x) = A\sum_y g(y)$ for $\alpha < d$, and proves the lemma. In the following, we prove (B.13).

We first note that the asymptotic condition (B.7) implies

$$\forall \varepsilon > 0\ \exists M > 0 \text{ s.t.} \qquad \frac{A-\varepsilon}{|x|^\alpha} \le f(x) \le \frac{A+\varepsilon}{|x|^\alpha} \qquad \text{for } |x| \ge M. \tag{B.16}$$

We also note that by taking M' large, we can have a uniform bound for all $x \in \mathbb{Z}^d$:

$$|f(x)| \le \frac{M'}{|||x|||^\alpha}. \tag{B.17}$$

We fix $0 < \varepsilon \ll 1$, and choose M, M' as above. We only consider sufficiently large $|x|$ depending on M, M' and ε.

Dealing with S_1. We further divide S_1 as

$$S_1 = \sum_{y:\,|y|<\varepsilon|x|} \frac{A}{|x|^\alpha} g(y) + \sum_{y:\,|y|<\varepsilon|x|} \left[\frac{A}{|x|^\alpha} - \frac{A}{|x-y|^\alpha}\right] g(y)$$
$$+ \sum_{y:\,|y|<\varepsilon|x|} \left[f(x-y) - \frac{A}{|x-y|^\alpha}\right] g(y) =: S_{11} + S_{12} + S_{13}. \tag{B.18}$$

For S_{11}, we have

$$
\begin{aligned}
S_{11} &:= \sum_{y:\,|y|<\varepsilon|x|} \frac{A}{|x|^\alpha} g(y) = \left[\sum_{y\in\mathbb{Z}^d} g(y)\right]\frac{A}{|x|^\alpha} - \left[\sum_{|y|\geq\varepsilon|x|} g(y)\right]\frac{A}{|x|^\alpha} \\
&= \frac{A}{|x|^\alpha}\sum_y g(y) + o(|x|^{-\alpha}),
\end{aligned}
\tag{B.19}
$$

where we used the fact that $\sum_{|y|\geq\varepsilon|x|}|g(y)|$ goes to zero as $|x|\to\infty$, because $\sum_y |g(y)|<\infty$. Here $o(|x|^{-\alpha})$ may depend on ε.

For S_{12}, note that $|x-y|^{-\alpha} = (1+O(\varepsilon))|x|^{-\alpha}$ for $|y|\leq\varepsilon|x|$. So, for $0<\varepsilon\ll 1$,

$$
\begin{aligned}
|S_{12}| &\leq \sum_{y:\,|y|<\varepsilon|x|} \left|\frac{A}{|x|^\alpha} - \frac{A}{|x-y|^\alpha}\right| |g(y)| \\
&\leq O(\varepsilon)\frac{A}{|x|^\alpha}\sum_{y:\,|y|<\varepsilon|x|} |g(y)| = \varepsilon O(|x|^{-\alpha}).
\end{aligned}
\tag{B.20}
$$

For S_{13}, we use (B.16) to conclude

$$
\begin{aligned}
|S_{13}| &:= \left|\sum_{y:\,|y|<\varepsilon|x|}\left[f(x-y) - \frac{A}{|x-y|^\alpha}\right]g(y)\right| \\
&\leq \sum_{y:\,|y|<\varepsilon|x|}\frac{\varepsilon}{|x-y|^\alpha}|g(y)| \\
&\leq \frac{\varepsilon}{(1-\varepsilon)^\alpha|x|^\alpha}\sum_{y:\,|y|<\varepsilon|x|}|g(y)| \leq \frac{\varepsilon}{(1-\varepsilon)^\alpha|x|^\alpha}\times K.
\end{aligned}
\tag{B.21}
$$

As a result, we have

$$
S_1(x) = \frac{A}{|x|^\alpha}\sum_y g(y) + o(|x|^{-\alpha}) + \varepsilon O(|x|^{-\alpha}).
\tag{B.22}
$$

Dealing with S_2. Note first that $|y|\geq(1-\varepsilon)|x|$ if $|x-y|<\varepsilon|x|$. So using (B.17) to bound f and the pointwise bound on g of (B.7), we get

$$
\begin{aligned}
|S_2(x)| &:= \left|\sum_{y:\,|x-y|<\varepsilon|x|} f(x-y)g(y)\right| \\
&\leq \left(\sum_{y:\,|x-y|<\varepsilon|x|}|f(x-y)|\right)\times\left(\sup_{y:\,|x-y|<\varepsilon|x|}|g(y)|\right) \\
&\leq \left(\sum_{y:\,|x-y|<\varepsilon|x|}\frac{M'}{|||x-y|||^\alpha}\right)\times\frac{C'}{|x|^d} = \varepsilon^{d-\alpha}O(|x|^{-\alpha}).
\end{aligned}
\tag{B.23}
$$

Dealing with S_3. We use (B.16) (we only consider those x's satisfying $\varepsilon|x| \ge M$) and calculate as

$$
\begin{aligned}
|S_3(x)| &:= \Bigg| \sum_{\substack{y : |x-y| \ge \varepsilon|x| \\ |y| \ge \varepsilon|x|}} f(x-y) g(y) \Bigg| \le \sum_{\substack{y : |x-y| \ge \varepsilon|x| \\ |y| \ge \varepsilon|x|}} \frac{A+\varepsilon}{|x-y|^{\alpha}} |g(y)| \\
&\le \sum_{\substack{y : |x-y| \ge \varepsilon|x| \\ |y| \ge \varepsilon|x|}} \frac{A+\varepsilon}{(\varepsilon|x|)^{\alpha}} |g(y)| \qquad \text{(B.24)} \\
&\le \frac{A+\varepsilon}{\varepsilon^{\alpha}|x|^{\alpha}} \times \sum_{|y| \ge \varepsilon|x|} |g(y)| = \varepsilon^{-\alpha} o(|x|^{-\alpha}).
\end{aligned}
$$

The last equality follows again because $|g(y)|$ is summable, and $o(|x|^{-\alpha})$ may depend on ε. □

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