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ON MULTIDIMENSIONAL BRANCHING RANDOM WALKS IN RANDOM ENVIRONMENT

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We study branching random walks in random i.i.d. environment in $\mathbb{Z}^d$, $d \geq 1$. For this model, the population size cannot decrease, and a natural definition of recurrence is introduced. We prove a dichotomy for recurrence/transience, depending only on the support of the environmental law. We give sufficient conditions for recurrence and for transience. In the recurrent case, we study the asymptotics of the tail of the distribution of the hitting times and prove a shape theorem for the set of lattice sites which are visited up to a large time.

1. Introduction and results. Branching random walks in random environment provide microscopic models for reaction–diffusion–convection phenomena in a space-inhomogeneous medium. On the other hand, much progress has been achieved in the last decade in the understanding of random walks in random environment on $\mathbb{Z}^d$, which is reviewed in Zeitouni's lecture notes [32]. It is natural to investigate such branching walks, and to relate the results to the nonbranching case. In this paper we continue the line of research of Comets, Menshikov and Popov [6] and Machado and Popov [20, 21]: each particle gives birth to at least one descendant, according to branching and jump probabilities which depend on the location, and are given by an independent identically distributed random field (environment). We stress here that the branching and the transition mechanisms are *not* supposed to be independent, and moreover, differently from [6, 20, 21], we do not suppose that the *immediate* offspring of a particle jump independently. We assume that the jumps are finite range. For an appropriate notion of recurrence and transience we prove that either the branching random walk is recurrent for almost every environment or the branching random walk is transient for almost every environment. In addition, we show that details of the distribution of the environment do not matter, but only its support. Although we could not give a complete (explicit) classification in the spirit of [6, 20, 21], this is quite interesting in view of the difficulty of the corresponding question for random walks

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in random environment. For nonreversible random walks (without branching) in random environment, very little is known on recurrence/transience. In the case of balanced environment (i.e., when the jump distribution is symmetric), Lawler proved an invariance principle [17], and the walk is recurrent for $d \leq 2$ and transient for $d \geq 3$ (Theorem 3.3.22 in [32]). For general random walks in random environment Sznitman gave sufficient conditions for the random walk to be ballistic (and, consequently, transient) in [27, 28]. These conditions, though, normally are not easily verifiable. On the other hand, for the model of the present paper, we give explicit (and easily verifiable) conditions for recurrence and transience, that, while failing to produce a complete classification, nevertheless work well in many concrete examples.

Also, we give a shape theorem for the set of visited sites in the recurrent case. In terms of random walks in random environment, this case corresponds to nestling walks (i.e., the random drift can point in all directions) as well as to nonnestling ones with strong enough branching, and the result is once again interesting in view of the (corresponding) law of large numbers for random walks [29, 34], which, so far, could allow the speed to be random in dimension $d \geq 3$ due to the lack of 0–1 law.

Some interesting problems, closely related to shape theorems, arise when studying properties of tails of the distributions of first hitting times. Here we show that branching random walks in random environment exhibit very different behaviors in dimensions $d = 1$ and $d \geq 2$ from the point of view of hitting time distribution: in the recurrent case, the annealed expectation of hitting times is always finite in $d \geq 2$, but it is not the case for the one-dimensional model. Hitting times for random walks without branching in random environment have motivated a number of papers. Tails of hitting times distributions have a variety of different behaviors for random walks in random environment in dimension $d = 1$; they have been extensively studied, both under the annealed law [7, 24] and the quenched law [13, 23]. Also, in higher dimensions, Sznitman obtains estimates for hitting times of hyperplanes [25, 26].

Many other interesting topics are left untouched in this paper. They include shape theorems for the transient case, questions related to the (global and local) size of the population, hydrodynamical equations, models with continuous space and time (where branching random walk is substituted by a branching Brownian motion and the random environment is defined appropriately), models with unbounded jumps, and so on. Also, it is a challenging problem to find the right order of decay for the tails of hitting times in dimension $d \geq 2$, in the recurrent case. Finally, since in our model each particle has at least one descendant, we do not deal at all with extinction, which seems to be a difficult issue in random environment.

An important ingredient in our paper is the notion of seeds, that is, local configurations of the environment. Some seeds can create an infinite number of particles without help from outside, potentially enforcing recurrence. So, as opposed to random walks without branching, the model of the present paper is in some sense more sensitive to the local changes in the environment. Together with the fact that more particles means more averaging, this explains why the analysis is apparently easier for the random walks with the presence of branching.

We briefly discuss different, but related, models. A multidimensional ($d \geq 3$) branching random walk for which the transition probabilities are those of the simple random walk, and the particles can branch only in some special sites (randomly placed, with a decreasing density) was considered by den Hollander, Menshikov and Popov [15], and several sufficient conditions for recurrence and transience were obtained (we mention also Volkov [31]). Dimension $d = 1$ leads to more explicit results, thanks to the order structure (see, e.g., [6]). In the case $d = 1$ with nearest-neighbor jumps, particles have to visit all intermediate locations, and, for a location-independent jump law, Greven and den Hollander [14] and Baillon, Clément, Greven and den Hollander [3] could prove some useful variational formulas. As can be seen in [21], the case where particles move on the tree has a flavor similar to $d = 1$. The case of inhomogeneous jumps with constant branching rate can be formulated as a tree-indexed random walk. In this case, a complete classification of recurrence/transience is obtained by Gantert and Müller [12], involving the branching rate and the spectral radius of the transition operator. For a constant branching rate and a random jump law on $\mathbb{Z}$ biased to the left, positivity of the velocity of the rightmost particle is studied by Devulder [8]. The occurrence of shape theorems in the branching random walk literature goes back at least to [4].

This paper is organized as follows. In the next section we define the model formally and the appropriate notions of transience and recurrence, together with their basic properties: recurrence/transience do not depend on the starting point and hold either for almost all or for almost no environments. In Section 1.2 we state the result that recurrence/transience only depend on the support of the environmental law, and we present sufficient conditions for recurrence and transience. In Section 1.3 we discuss two closely related topics: hitting times and the asymptotic behavior of the set of the sites visited up to a given time, in the recurrent case. Then, in Sections 2.1 and 2.2 we introduce two important objects which will be intensively used in the proofs of our results: induced random walks (obtained by "eliminating" the branching from the model in some reasonable way) and seeds (these are just local configurations of the environment). In Section 2.3 we simultaneously construct processes starting from all possible initial positions (that will be needed for the proof of the shape theorem). In Section 3 we prove the results related to recurrence/transience, and in Sections 4 and 5 we deal with those on hitting times and asymptotic shape correspondingly.

1.1. *Formal definitions and some basic properties of the model.* We now describe the model. Fix a finite set $\mathfrak{A} \subset \mathbb{Z}^d$ such that $\pm e_i \in \mathfrak{A}$ for all $i = 1, \ldots, d$, where e_i's are the coordinate vectors of $\mathbb{Z}^d$. Define (with $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$)

$$\mathcal{V} = \left\{ v = (v_x, x \in \mathfrak{A}) : v_x \in \mathbb{Z}_+, \sum_{x \in \mathfrak{A}} v_x \geq 1 \right\},$$

and for $v \in \mathcal{V}$ put $|v| = \sum_{x\in\mathfrak{A}} v_x$; note that $|v| \geq 1$ for all $v \in \mathcal{V}$. Furthermore, define $\mathcal{M}$ to be the set of all probability measures ω on $\mathcal{V}$:

$$\mathcal{M} = \left\{\omega = \big(\omega(v), v \in \mathcal{V}\big) : \omega(v) \geq 0 \text{ for all } v \in \mathcal{V}, \sum_{v\in\mathcal{V}} \omega(v) = 1\right\}.$$

Finally, let Q be a probability measure on $\mathcal{M}$. Now, for each $x \in \mathbb{Z}^d$ we choose a random element $\omega_x \in \mathcal{M}$ according to the measure Q, independently. The collection $\boldsymbol{\omega} = (\omega_x, x \in \mathbb{Z}^d)$ is called *the environment*. Given the environment $\boldsymbol{\omega}$, the evolution of the process is described in the following way: start with one particle at some fixed site of $\mathbb{Z}^d$. At each integer time the particles branch independently using the following mechanism: for a particle at site $x \in \mathbb{Z}^d$, a random element $v = (v_y, y \in \mathfrak{A})$ is chosen with probability $\omega_x(v)$, and then the particle is substituted by v_y particles in $x + y$ for all $y \in \mathfrak{A}$.

Note that this notion of branching random walk is more general than that of [6, 20, 21], since here we do not suppose that the *immediate* descendants of a particle jump independently (e.g., we allow situations similar to the following one [$d = 1$]: when a particle in x generates three offspring, then with probability 1 two of them go to the right and the third one goes to the left).

We denote by $\mathbb{P}, \mathbb{E}$ the probability and expectation with respect to $\boldsymbol{\omega}$ (in fact, since the environment is i.i.d., $\mathbb{P} = \bigotimes_{x\in\mathbb{Z}^d} Q_x$, where Q_x are copies of Q), and by $\mathtt{P}^x_\omega, \mathtt{E}^x_\omega$ the (so-called "quenched") probability and expectation for the process starting from x in the fixed environment $\boldsymbol{\omega}$. We use the notation $\mathbf{P}^x[\cdot] = \mathbb{E}\mathtt{P}^x_\omega[\cdot]$ for the annealed law of the branching random walk in random environment, and $\mathbf{E}^x$ for the corresponding expectation. Also, sometimes we use the symbols $\mathtt{P}_\omega, \mathtt{E}_\omega, \mathbf{P}, \mathbf{E}$ without the corresponding superscripts when it can create no confusion (e.g., when the starting point of the process is indicated elsewhere).

Throughout this paper, and without recalling it explicitly, we suppose that the two conditions below are fulfilled:

CONDITION B.

$$Q\{\omega : \text{ there exists } v \in \mathcal{V} \text{ such that } \omega(v) > 0 \text{ and } |v| \geq 2\} > 0.$$

CONDITION E.

$$Q\left\{\omega : \sum_{v:v_e\geq 1} \omega(v) > 0 \text{ for any } e \in \{\pm e_1, \dots, \pm e_d\}\right\} = 1.$$

Condition B ensures that the model cannot be reduced to random walk without branching, and Condition E is a natural ellipticity condition which ensures that the walk is really d-dimensional. In this paper, "elliptic" will mean "strictly elliptic." We will sometimes use the stronger uniform ellipticity condition:

CONDITION UE. For some $\varepsilon_0 > 0$,

$$Q\left\{\omega : \sum_{v : v_e \ge 1} \omega(v) \ge \varepsilon_0 \text{ for any } e \in \{\pm e_1, \ldots, \pm e_d\}\right\} = 1.$$

Due to Condition B, for almost all environments the population size tends to infinity, as can be seen from Lemma 2.3 below. This shows that the branching random walk is always transient as a process on $\mathbb{Z}_+^{\mathbb{Z}^d}$. So, we introduce more appropriate notions of recurrence and transience.

DEFINITION 1.1. For the particular realization of the random environment $\boldsymbol{\omega}$, the branching random walk is called recurrent if

$$\mathrm{P}_\omega^0[\text{the origin is visited infinitely often}] = 1.$$

Otherwise, the branching random walk is called transient.

By the Markov property, the recurrence is equivalent to

$$\mathrm{P}_\omega^0[\text{the origin is visited at least once}] = 1.$$

In principle, the above definition could depend on the starting point of the process and on the environment $\boldsymbol{\omega}$. However, a natural dichotomy takes place:

PROPOSITION 1.2. *We have either*:

(i) *For $\mathbb{P}$-almost all $\boldsymbol{\omega}$, the branching random walk is recurrent, in which case P_ω^x[the origin is visited infinitely often] $= 1$ for all $x \in \mathbb{Z}^d$, or*:

(ii) *For $\mathbb{P}$-almost all $\boldsymbol{\omega}$, the branching random walk is transient, in which case P_ω^x[the origin is visited infinitely often] < 1 for all $x \in \mathbb{Z}^d$.*

The next proposition refines item (ii) of Proposition 1.2:

PROPOSITION 1.3. *Let us assume that the branching random walk is transient. Then for $\mathbb{P}$-almost all $\boldsymbol{\omega}$ we have P_ω^x[the origin is visited infinitely often] $= 0$ for all $x \in \mathbb{Z}^d$.*

It is plain to construct (see, e.g., the example after the proof of Theorem 4.3 in [6]) environments $\boldsymbol{\omega}$ such that $\mathrm{P}_\omega^x[0$ is visited infinitely often$]$ is strictly between 0 and 1. The next example shows that randomness of the environment is essential for our statements (and also shows, incidentally, that there is no hope to prove Proposition 1.3 by arguments of the type "recurrence should not be sensitive to changes of the environment in finite regions").

EXAMPLE 1. Let $d = 1$, $\mathfrak{A} = \{-1, 1\}$, and consider two measures $\omega^{(1)}, \omega^{(2)}$:

(i) under $\omega^{(1)}$, with probability $2/3$ there is only one child which is located one step to the left and with probability $1/3$ there is only one child which is located one step to the right;

(ii) under $\omega^{(2)}$, with probability $1/3$ there is only one child which is located one step to the right and with probability $2/3$ there are two children one being located to the right and the other to the left.

If all sites $x < 0$ have the environment $\omega^{(1)}$ (we say they are of type 1) and all sites $x \geq 0$ are of type 2, we have $\mathtt{P}_\omega^x$[0 is visited infinitely often] is 1 for $x \geq 0$ and is less than 1 for $x < 0$. Changing the site $x = 0$ from type 2 to type 1 turns the branching random walk from recurrent to transient. This example also shows that, in general, the recurrence does depend on the environment locally. Moreover, it shows that $\mathtt{P}_\omega^0$[the origin is visited infinitely often] may be different from 0 and 1. We will see below that selecting randomly the environment in an i.i.d. fashion makes this branching random walk recurrent (for this particular example it follows, e.g., from Theorem 1.5).

Now, we begin stating the main results of this paper. As mentioned before, in Section 1.2 we formulate further results concerning transience and recurrence of the process, and in Section 1.3 we deal with questions related to (quenched and annealed) distributions of hitting times and shape theorems.

1.2. *Transience and recurrence.* It is worth keeping in mind a particular example to illustrate our results.

EXAMPLE 2. With $d = 2$ and $\mathfrak{A} = \{\pm e_1, \pm e_2\}$, consider the following v's: $v^{(1)} = \delta_{e_1}$ (with δ the Kronecker symbol), $v^{(2)} = \delta_{e_2}$, $v^{(3)} = \delta_{-e_1}$, $v^{(4)} = \delta_{-e_2}$, $v^{(5)} = \delta_{e_1} + 2\delta_{e_2} + \delta_{-e_1} + \delta_{-e_2}$, and the following $\omega^{(0)}, \omega^{(+)}, \omega^{(-)}$ defined by $(0 < a < 1)$:

$$\omega^{(0)}(v^{(1)}) = \tfrac{3}{8}, \qquad \omega^{(0)}(v^{(2)}) = \tfrac{1}{4}, \qquad \omega^{(0)}(v^{(3)}) = \tfrac{1}{8}, \qquad \omega^{(0)}(v^{(4)}) = \tfrac{1}{4},$$
$$\omega^{(+)}(v^{(1)}) = a, \qquad \omega^{(+)}(v^{(5)}) = 1 - a,$$
$$\omega^{(-)}(v^{(3)}) = \tfrac{1}{8}, \qquad \omega^{(-)}(v^{(5)}) = \tfrac{7}{8};$$

see Figure 1. Note that Conditions B and UE are satisfied.

It seems clear that the branching random walk with $Q = Q_1$ such that $Q_1(\omega^{(0)}) = \alpha = 1 - Q_1(\omega^{(-)})$ is recurrent at least for small α. In fact, it is recurrent for all $\alpha \in (0, 1)$. It seems also clear that branching random walk with $Q = Q_2$ such that $Q_2(\omega^{(0)}) = \alpha = 1 - Q_2(\omega^{(+)})$ may be recurrent or transient depending on a. The following can be obtained using Theorems 1.5 and 1.6 from this section: if $a \leq 1/2$, then the process is recurrent [since the condition (3) is fulfilled], and if $a \geq 8/9$, then the process is transient [to verify (6), use $s = e_1$ and $\lambda = 1/3$]. But it is not so clear that the behavior does not depend on α provided that $0 < \alpha < 1$; nevertheless, from Theorem 1.4 we shall see that this is the case.

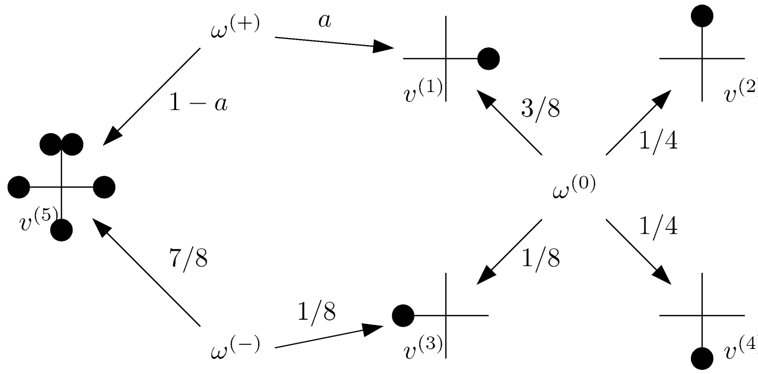

FIG. 1. *The random environment in Example 2.*

Our first result states that transience/recurrence of the process only depend on the support of the measure Q, that is, the smallest closed subset $F \subset \mathcal{M}$ such that $Q(F) = 1$. We recall that ω belongs to the support if and only if $Q(\mathcal{N}) > 0$ for all neighborhood $\mathcal{N}$ of ω in $\mathcal{M}$.

THEOREM 1.4. *Suppose that the branching random walk is recurrent* (*resp*., *transient*) *for almost all realizations of the random environment from the distribution Q. Then for any measure Q' with* $\operatorname{supp} Q \subseteq \operatorname{supp} Q'$ (*resp*., $\operatorname{supp} Q' \subseteq \operatorname{supp} Q$) *the process is recurrent* (*resp*., *transient*) *for almost all realizations of the random environment from the distribution Q'*. (*We recall that we assume that Q' satisfies Condition* E.)

The fact that recurrence and transience only depend on the support of the measure Q is not a complete surprise for this kind of model. Besides [6, 20, 21], we can mention also [10]: there in Theorem 3 it is shown that, for the branching diffusion, the intensity of "mild" Poissonian obstacles plays no role for exponential growth and local extinction.

Unlike the corresponding results of [6, 20, 21], here we did not succeed in writing down an explicit criterion for recurrence/transience. However, sufficient conditions for recurrence or transience can be obtained (they are formulated in terms of the support of Q, as they should be). To this end, for any $v \in \mathcal{V}$ and any vector $r \in \mathbb{R}^d$, define (see Figure 2)

$$D(r, v) = \max_{x \in \mathfrak{A} : v_x \geq 1} r \cdot x, \tag{1}$$

where $a \cdot b$ is the scalar product of $a, b \in \mathbb{R}^d$. Let also $\| \cdot \|$ be the Euclidean norm and $\mathbb{S}^{d-1} = \{a \in \mathbb{R}^d : \|a\| = 1\}$ be the unit sphere in $\mathbb{R}^d$.

THEOREM 1.5. *If*

$$\sup_{\omega \in \operatorname{supp} Q} \sum_{v \in \mathcal{V}} \omega(v) D(r, v) > 0 \tag{2}$$

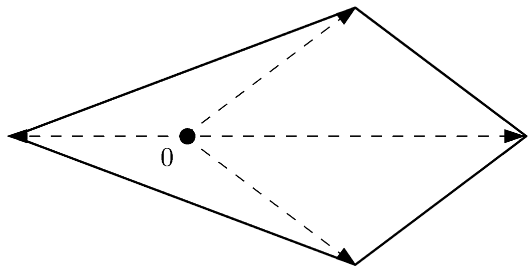

FIG. 2. *The set* $\{\sup_{\omega\in\operatorname{supp}Q}\{\sum_v \omega(v)D(r,v)\}r; r\in\mathbb{S}^{d-1}\}$ *for the branching random walk* (*the one defined by* Q_2, *with* $a<1/2$) *from Example* 2 *is the solid line*; *note that* (2) [*resp*., (3)] *means that the origin should be strictly inside* (*resp*., *outside*) *this set*.

for all $r\in\mathbb{S}^{d-1}$, *then the branching random walk is recurrent. Moreover, if*

$$\sup_{\omega\in\operatorname{supp} Q}\sum_{v\in\mathcal{V}}\omega(v)D(r,v)\ge 0 \tag{3}$$

for all $r\in\mathbb{S}^{d-1}$ *and Condition* UE *holds*, *then the branching random walk is recurrent*.

Note that (3) cannot guarantee the recurrence without Condition UE. To see this, consider the following:

EXAMPLE 3. Let $d=1$, and Q puts positive weights on $\omega^{(n)}$, $n>5$, where $\omega^{(n)}$ is described in the following way. A particle is substituted by in mean $\frac{n}{n-1}$ offspring (for definiteness, let us say that it is substituted by two offspring with probability $\frac{1}{n-1}$ and by one offspring with probability $\frac{n-2}{n-1}$); each one of the offspring goes to the left with probability $1/n$, to the right with probability $4/n$, and stays on its place with probability $1-\frac{5}{n}$, independently. In this case (3) holds, but we do not have Condition UE. Applying Theorem 1.6 below (use $\lambda=1/2$), one can see that this branching random walk is transient.

REMARK. (i) Two rather trivial sufficient conditions for recurrence are: there is $\omega\in\operatorname{supp} Q$ such that

$$\sum_{v\in\mathcal{V}}\omega(v)|v|=+\infty, \tag{4}$$

or such that

$$\sum_{v\in\mathcal{V}}\omega(v)v_0>1. \tag{5}$$

The proof is given after the proof of Proposition 1.3.

(ii) A particular case of the model considered here is the usual construction of the branching random walk, for example, [6, 20, 21]: in each x, specify the transition probabilities $\hat{P}_y^{(x)}$, $y\in\mathfrak{A}$, and branching probabilities $\hat{r}_i^{(x)}$, $i=1,2,3,\dots.$

A particle in x is first substituted by i particles with probability $\hat{r}_i^{(x)}$, then each of the offspring jumps independently to $x+y$ with probability $\hat{P}_y^{(x)}$. The pairs $((\hat{r}_i^{(x)})_{i\geq 1}, (\hat{P}_y^{(x)})_{y\in\mathfrak{A}})$ are chosen according to some i.i.d. field on $\mathbb{Z}^d$. In our notations, ω_x is a mixture of multinomial distributions on $\mathfrak{A}$:

$$\omega_x(v) = \sum_{i\geq 1} \hat{r}_i^{(x)} \operatorname{Mult}\bigl(i; \hat{P}_y^{(x)}, y \in \mathfrak{A}\bigr).$$

Note that, in this case, D defined in (1) is trivially related to the local drift of the walk by

$$\sum_{v\in\mathcal{V}} \omega_x(v) D(r, v) \geq r \cdot \sum_{y\in\mathfrak{A}} y \hat{P}_y^{(x)}.$$

The family of transition probabilities $\hat{P}_y^{(x)}$, $y \in \mathfrak{A}$, defines a random walk in i.i.d. random environment on $\mathbb{Z}^d$. The following definitions are essential in the theory of such walks [32, 33]; they are formulated here in the spirit of (1). With $\hat{Q}$ the common law of $(\hat{P}_y^{(x)})_{y\in\mathfrak{A}}$, the random walk is

- *nestling*, if for all $r \in \mathbb{S}^{d-1}$,

$$\sup_{\omega\in\operatorname{supp}\hat{Q}} r \cdot \sum_{y\in\mathfrak{A}} y \hat{P}_y > 0;$$

- *nonnestling*, if there exists $r \in \mathbb{S}^{d-1}$ such that

$$\sup_{\omega\in\operatorname{supp}\hat{Q}} r \cdot \sum_{y\in\mathfrak{A}} y \hat{P}_y < 0;$$

- *marginally nestling*, if

$$\min_{r\in\mathbb{S}^{d-1}} \sup_{\omega\in\operatorname{supp}\hat{Q}} r \cdot \sum_{y\in\mathfrak{A}} y \hat{P}_y = 0.$$

Suppose now that the random walk in random environment is nestling (or, either nestling or marginally nestling with Condition UE). Then, under Condition B, Theorem 1.5 implies that the branching random walk is recurrent, regardless of the amount of branching that is present in the model and even though the effective drift of the random walk can be arbitrarily large. This extends the observation made in this case in dimension $d = 1$, Example 1 in Section 4 of [6], to arbitrary dimension and more general branching random walks. The heuristic scenario to produce such effects remains the same: due to the nestling assumption, the medium develops atypical configurations which trap the particles at some distance from the origin; there, the branching generates an exponential number of particles, which will balance the small probability for returning to the origin. Indeed, the quenched large deviations rate function vanishes at 0 in the nestling case; see [30, 33].

Now, we turn our attention to the conditions for transience. Define for $\omega \in \mathcal{M}$, $y \in \mathfrak{A}$,

$$\mu_y^\omega = \sum_{v \in \mathcal{V}} v_y \omega(v),$$

that is, μ_y^ω is the mean number of particles sent from x to $x + y$ when the environment at x is ω. Consider the following:

CONDITION L. There exist $s \in \mathbb{S}^{d-1}$, $\lambda > 0$ such that for all $\omega \in \operatorname{supp} Q$ we have

$$\sum_{y \in \mathfrak{A}} \mu_y^\omega \lambda^{y \cdot s} \leq 1. \tag{6}$$

We note that, by continuity, Condition L is satisfied if and only if (6) holds for Q-a.e. ω.

THEOREM 1.6. *Condition* L *is sufficient for the transience of the branching random walk in random environment. Moreover, for* $\mathbb{P}$*-a.e.* ω, *with positive* P_ω^x*-probability the branching random walk will not visit the half-space* $\{y \in \mathbb{Z}^d : y \cdot s_0 \leq 0\}$*—provided that its starting point* x *is outside that half-space*, *where*

$$s_0 = \begin{cases} s, & \text{if } \lambda < 1, \\ -s, & \text{if } \lambda > 1. \end{cases}$$

(*As we will see below*, *Condition* L *cannot be satisfied with* $\lambda = 1$.) *Furthermore*, *the number of visits of the branching random walk to the half-space is a.s. finite.*

In [6, 20, 21] it was shown that, if the descendants can jump only to nearest neighbors, conditions analogous to Condition L are sufficient and *necessary* for transience in cases when the branching random walk in random environment lives on the one-dimensional lattice or on a tree. In particular, by repeating the argument of [6], it is not difficult to prove that for the present model in dimension 1 in the nearest-neighbor case, Condition L is necessary and sufficient for transience. More precisely, for this one needs, first, to prove the modified versions of Theorems 2.1 and 2.2 of [6] (with the suitable modifications in formulas (2.1) and (2.2) of [6]; note that, in the notation of [6], $r(x)P_{xy}$ is the mean offspring sent from x to y), and generalize the proof of Theorem 4.1 (one should modify the definition of $h_\xi(\lambda)$ in the beginning of Section 4 of [6]; the rest of the proof can be left practically intact). On d-dimensional lattice ($d \geq 2$) or even in dimension 1 when larger jumps are possible, the question whether Condition L is necessary for transience remains open.

1.3. *Hitting times and asymptotic shape in the recurrent case.* For the process starting from one particle at x, let us denote by $\eta_n^x(y)$ the number of particles in y at time n, and by B_n^x the set of all sites visited by the process before time n. Also, denote by $T(x, y)$ the moment of hitting $y \neq x$. For the formal definition of those quantities, see Section 2.3, although here we do not need to construct simultaneously all the branching random walks from all the possible starting points x.

First, we are going to take a closer look at the hitting times $T(0, x)$ for recurrent branching random walks. It is immediate to note that the recurrence is equivalent to $\mathbf{P}[T(0, x) < \infty \text{ for all } x] = 1$. So, for the recurrent case it is natural to ask how fast the recurrence occurs, that is, how fast (quenched and annealed) tails of the distribution of $T(0, x)$ decrease. For the (quenched) asymptotics of $\mathtt{P}_\omega[T(0, 1) > n]$ in dimension 1, we have the following result:

PROPOSITION 1.7. *Suppose that $d = 1$ and the branching random walk in random environment is recurrent. Then, for $\mathbb{P}$-almost all environments there exist $n^* = n^*(\boldsymbol{\omega})$ and $\kappa > 0$ such that*

$$\mathtt{P}_\omega[T(0, 1) > n] \leq e^{-n^\kappa} \tag{7}$$

for all $n \geq n^*$.

This result follows from a more general fact that will be proved in the course of the proof of Theorem 1.10, case $d = 1$ [see the remark just below formula (56)]. Moreover, the following example shows that, for the class of recurrent one-dimensional branching random walks in random environment, the right order of decay of $\mathtt{P}_\omega[T(0, 1) > n]$ is indeed stretched exponential.

EXAMPLE 4. We consider $d = 1$, $\mathfrak{A} = \{-1, 1\}$, and suppose that Q gives weights $1/3$ to the points $\omega^{(1)}, \omega^{(2)}, \omega^{(3)}$, which are described as follows. Fix a positive $p < 1/82$; there is no branching in $\omega^{(1)}, \omega^{(2)}$, and $\omega^{(1)}$ (resp., $\omega^{(2)}$), sends the particle to the left (resp., to the right) with probability p and to the right (resp., to the left) with probability $1 - p$. In the sites with $\omega^{(3)}$, the particle is substituted by one (resp., two) offspring with probability $2p$ (resp., $1 - 2p$); those then jump independently to the right or to the left with equal probabilities. By Theorem 1.5, this branching random walk is recurrent.

Denote $a = \ln \frac{1-p}{p}$ and Ξ the number of integers in the interval $(0, a^{-1} \ln n]$, and observe that

$$\begin{aligned}
\mathbb{P}[\omega_{-x} = \omega^{(1)}, \omega_x = \omega^{(2)} \text{ for } x \in (0, a^{-1} \ln n], \omega_0 = \omega^{(1)}]& \\
= \left(\frac{1}{3}\right)^{1+2\Xi}& \\
\geq 3^{-1-2a^{-1} \ln n}& \\
= \frac{n^{-2a^{-1} \ln 3}}{3}.&
\end{aligned}$$

Clearly, $p < 1/82$ implies that $2a^{-1}\ln 3 < 1/2$. This means that a typical environment $\boldsymbol{\omega}$ will contain a translation of the trap considered above in the box $[-n^{1/2}, 0]$, that is, there is an interval $[b-a^{-1}\ln n, b+a^{-1}\ln n] \subset [-n^{1/2}, 0]$ such that $\omega_x = \omega^{(1)}$ for $x \in [b-a^{-1}\ln n, b]$ and $\omega_x = \omega^{(2)}$ for $x \in (b, b+a^{-1}\ln n]$. For such an environment, first, the initial particle goes straight to the trap (without creating any further particles on its way) with probability at least $p^{n^{1/2}}$, and then stays there with a probability bounded away from 0 (note that the depth of the trap is $\ln n$, and this is enough to keep the particle there until time n with a good probability). This shows that $\mathrm{P}_\omega[T(0,1) > n] \geq Ce^{-n^{1/2}\ln p^{-1}}$.

One can construct other one-dimensional examples of this type; the important features are:

(i) there are ω's from $\operatorname{supp} Q$ without branching and with drifts pointing to both directions, so that traps are present;
(ii) all ω's from $\operatorname{supp} Q$ have the following property: with a positive probability the particle does not branch, that is, it is substituted by only one offspring; this permits a particle to cross a region without necessarily creating new ones.

In dimensions $d \geq 2$, finding the right order of decay of $\mathrm{P}_\omega[T(0,1) > n]$ is, in our opinion, a challenging problem. For now, we can only conjecture that it should be exponential (observe that the direct attempt to generalize the above example to $d \geq 2$ fails, since creating a trap with a logarithmic depth has higher cost). As a general fact, the annealed bound of Theorem 1.8 below is, up to a constant factor, also a quenched one by Markov inequality. This is the only rigorous result concerning the quenched asymptotics of $\mathrm{P}_\omega[T(0,1) > n]$ we can state in the case $d \geq 2$; we believe, however, that it is far from being precise.

Now, we turn our attention to the annealed distribution of hitting times.

THEOREM 1.8. *Let $d \geq 1$ and assume that the branching random walk in random environment is recurrent. For any $x_0 \in \mathbb{Z}^d$ there exists $\theta = \theta(x_0, Q)$ such that*

$$\mathbf{P}[T(0, x_0) > n] \leq \exp\{-\theta \ln^d n\} \tag{8}$$

for all n sufficiently large.

Define $\mathcal{G} \subset \mathcal{M}$ to be the set of ω's without branching, that is,

$$\mathcal{G} = \left\{\omega \in \mathcal{M} : \sum_{v \in \mathcal{V} : |v|=1} \omega(v) = 1\right\}.$$

In other words, if at a given x the environment belongs to $\mathcal{G}$, then the particles in x only jump, without creating new particles. Also, for any $\omega \in \mathcal{G}$, define the drift

$$\Delta_\omega = \sum_{x \in \mathfrak{A}} x\omega(\delta_x).$$

The following result shows that Theorem 1.8 gives in some sense the best possible bounds for the tail of the hitting time distribution that are valid for the class of recurrent branching random walk in random environment.

THEOREM 1.9. *Suppose that $Q(\mathcal{G}) > 0$ and that the origin belongs to the interior of the convex hull of $\{\Delta_\omega : \omega \in \mathcal{G} \cap \operatorname{supp} Q\}$. Then for any $x_0 \in \mathbb{Z}^d$ there exists $\theta' = \theta'(x_0, Q)$ such that*

$$\mathbf{P}[T(0, x_0) > n] \geq \exp\{-\theta' \ln^d n\} \tag{9}$$

for all n sufficiently large.

From Theorems 1.8 and 1.9 there is only a small distance to the following remarkable fact: the implication

$$\big(\mathbf{P}[T(0,x) < \infty \text{ for all } x] = 1\big) \quad \Longrightarrow \quad \big(\mathbf{E}T(0,x) < \infty \text{ for all } x\big)$$

is true for $d \geq 2$ but is false for $d = 1$. To see this, it is enough to know that the constant θ' in (9) can be less than 1 in dimension one. Consider the following example:

EXAMPLE 5. Once again, we suppose that $\mathfrak{A} = \{-1, 1\}$ and $\operatorname{supp} Q$ consists of three points $\omega^{(1)}, \omega^{(2)}, \omega^{(3)}$, with $Q(\omega^{(1)}) = \alpha_1$, $Q(\omega^{(2)}) = \alpha_2$, $Q(\omega^{(3)}) = 1 - \alpha_1 - \alpha_2$. We keep the same $\omega^{(1)}, \omega^{(2)}$ from Example 4, and let $\omega^{(3)}(\delta_1 + \delta_{-1}) = 1$. It is immediate to obtain from, for example, Theorem 1.5 that this branching random walk in random environment is recurrent. Abbreviate $a = \ln \frac{1-p}{p}$ and let

$$H = \big\{\omega_x = \omega^{(1)} \text{ for } x \in [-2a^{-1} \ln n, -a^{-1} \ln n],\\ \omega_x = \omega^{(2)} \text{ for } x \in (-a^{-1} \ln n, 0]\big\};$$

then $\mathbb{P}[H] = (\alpha_1 \alpha_2)^{a^{-1} \ln n}$. Now, on H there is a trap of depth $\ln n$ just to the left of the origin, so for such environments the quenched expectation of $T(0, 1)$ over all paths which visit the site $(-a^{-1} \ln n)$ before the site 1 is at least Cn. Indeed, with a probability bounded away from 0, the initial particle goes straight to the trap and spends time n there. Therefore, we have

$$\mathbf{E}T(0, 1) \geq \int_H \mathrm{E}_\omega T(0, 1)\, d\mathbb{P} \geq Cn\mathbb{P}[H] = Cn^{1-a^{-1} \ln(\alpha_1\alpha_2)^{-1}} \to \infty$$

when $a^{-1} \ln(\alpha_1 \alpha_2)^{-1} < 1$ [or equivalently, $p < (1 + (\alpha_1\alpha_2)^{-1})^{-1}$]. Here we could use also the same branching random walk of Example 4 (with $p < 1/10$); note, however, that in the present example we could allow sites where particles always branch.

Now, we pass to a subject closely related to hitting times, namely, we will study the set of the sites visited by time n (together with some related questions). Recall that

$$B_n^x = \{y \in \mathbb{Z}^d : \text{ there exists } m \le n \text{ such that } \eta_m^x(y) \ge 1\}.$$

Also, we define $\bar{B}_n^x$ as the set of sites that contain at least one particle at time n, and $\tilde{B}_n^x$ as the set of sites that contain at least one particle at time n and will always do in future:

$$\bar{B}_n^x = \{y \in \mathbb{Z}^d : \eta_n^x(y) \ge 1\},$$

$$\tilde{B}_n^x = \{y \in \mathbb{Z}^d : \eta_m^x(y) \ge 1 \text{ for all } m \ge n\}.$$

Evidently, $\tilde{B}_n^x \subset \bar{B}_n^x \subset B_n^x$ for all x and n.

When dealing with the shape results for $\tilde{B}_n^x$ and $\bar{B}_n^x$, we will need the following "aperiodicity" condition, where we use the standard notation $\|x\|_1 = |x^{(1)}| + \cdots + |x^{(d)}|$ for $x = (x^{(1)}, \ldots, x^{(d)}) \in \mathbb{Z}^d$:

CONDITION A. There exist $x \in \mathfrak{A}$, $v \in \mathcal{V}$ with $\|x\|_1$ even and $v_x \ge 1$ such that $Q\{\omega \in \mathcal{M} : \omega(v) > 0\} > 0$.

We refer to Condition A as the "aperiodicity condition" because, without it, the process starting from the origin would live on even sites at even times, and on odd sites at odd times.

For any set $M \subset \mathbb{Z}^d$ we define the set $\mathfrak{F}(M)$ by "filling the spaces" between the points of M:

$$\mathfrak{F}(M) = \{y + (-1/2, 1/2]^d : y \in M\} \subset \mathbb{R}^d.$$

We only deal with the recurrent case here, leaving the more delicate, transient case for further research. In the recurrent case (at least when $d \ge 2$) we are able to use the above Theorem 1.8 to control the hitting times (in particular, for proving shape theorems, it is generally important to show that the expected hitting time is finite for any site, and Theorem 1.8 takes care of that in the case $d \ge 2$).

THEOREM 1.10. *Suppose that the branching random walk in random environment is recurrent and Condition* UE *holds.*

Then there exists a deterministic compact convex set $B \subset \mathbb{R}^d$ with 0 belonging to the interior of B, such that $\mathbf{P}$*-a.s.* (*i.e., for $\mathbb{P}$-almost all $\boldsymbol{\omega}$ and* P_ω*-a.s.*), *we have for any $0 < \varepsilon < 1$*

$$(1-\varepsilon)B \subset \frac{\mathfrak{F}(B_n^0)}{n} \subset (1+\varepsilon)B$$

for all n large enough.

If, in addition, Condition A *holds, then the same result—with the same limiting shape B—holds for $\bar{B}_n^0$ and $\tilde{B}_n^0$.*

It is straightforward to note that B is a subset of the convex hull of $\mathfrak{A}$ (time being discrete, one can show by induction that B_n^0 belongs to the convex hull of $n\mathfrak{A}$); also, since $B_n^x \stackrel{\text{law}}{=} B_n^0 + x$, the same result holds for the process starting from x, for any $x \in \mathbb{Z}^d$. However, as it often happens with shape results, in general it is not easy to obtain more information about the limiting shape. Let us mention that, as opposed to the results of the previous section, the limiting shape B does not only depend on the support of Q; see the example below:

EXAMPLE 6. Let $d = 1$ and $\mathfrak{A} = \{-2, -1, 0, 1, 2\}$. Put $v^{(1)} = \delta_{-1} + \delta_0 + \delta_1$, $v^{(2)} = \delta_{-2} + \delta_{-1} + \delta_0 + \delta_1 + \delta_2$, $\omega^1 = \delta_{v^{(1)}}$, $\omega^2 = \delta_{v^{(2)}}$ and $Q(\omega^1) = 1 - Q(\omega^2) = \alpha$. Then, it is quite elementary to obtain that $B = [-(2-\alpha), 2-\alpha]$, that is, the asymptotic shape depends on Q itself, and not only on the support of Q.

Another interesting point about Theorem 1.10 is that the shape B is convex, but one finds easily examples—as the one below—where it is not strictly convex.

EXAMPLE 7. With $d = 2$ and $\mathfrak{A} = \{\pm e_1, \pm e_2\}$, consider $v^{(1)} = \delta_{e_1} + \delta_{e_2}$, $v^{(2)} = \delta_{e_1} + \delta_{e_2} + \delta_{-e_1}$, $v^{(3)} = \delta_{e_1} + \delta_{e_2} + \delta_{-e_2}$, and the following $\omega^{(1)}, \omega^{(2)}$ defined by

$$\omega^{(1)}(v^{(1)}) = \omega^{(2)}(v^{(1)}) = \omega^{(1)}(v^{(2)}) = \omega^{(2)}(v^{(3)}) = \tfrac{2}{5},$$
$$\omega^{(1)}(v^{(3)}) = \omega^{(2)}(v^{(2)}) = \tfrac{1}{5};$$

see Figure 3. Then, with $Q(\omega^{(1)}) = 1 - Q(\omega^{(2)}) = \alpha$ [with $\alpha \in (0,1)$], the branching random walk is recurrent by Theorem 1.5. For arbitrary $\alpha \in (0,1)$, $B_n^0 \cap \mathbb{Z}_+^2 = \{(x_1, x_2) : x_1, x_2 \in \mathbb{Z}_+, x_1 + x_2 \le n\}$, and so

$$B \cap \mathbb{R}_+^2 = \{(x_1, x_2) : x_1, x_2 \ge 0, x_1 + x_2 \le 1\},$$

and B has a flat edge.

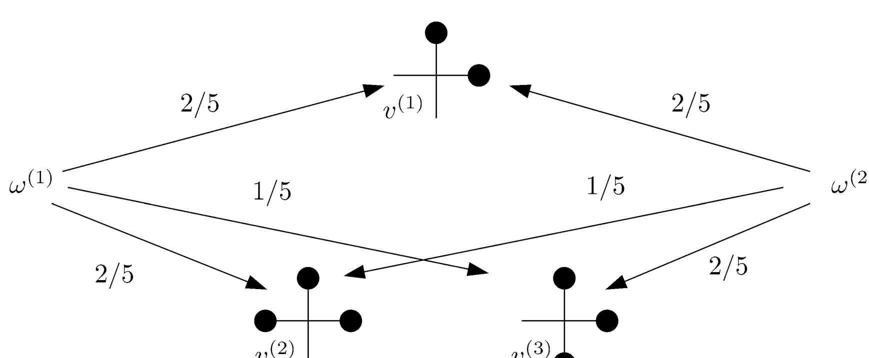

FIG. 3. *The random environment in Example 7.*

2. Some definitions and preliminary facts. First, let us introduce some more basic notation: for $x = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{Z}^d$ write

$$\|x\|_\infty = \max_{i=1,\dots,d} |x^{(i)}|.$$

Define L_0 to be the maximal jump length, that is,

$$L_0 = \max_{x \in \mathfrak{A}} \|x\|_\infty,$$

and let $\mathcal{K}_n$ be the d-dimensional cube with side of length $2n+1$:

$$\mathcal{K}_n = [-n, n]^d = \{x \in \mathbb{Z}^d : \|x\|_\infty \le n\}.$$

For $\omega \in \mathcal{M}$ and $V \subset \mathcal{V}$, sometimes we will use notation like $\omega(v \in V)$ or even $\omega(V)$ for $\sum_{v \in V} \omega(v)$.

2.1. *Induced random walks.* It is most natural to connect the branching random walk in random environment with random walks in random environment. Defining

$$\tilde{\mathcal{V}} = \{(v, \kappa) : v \in \mathcal{V}, \kappa \text{ probability measure on } \{y : v(y) \ge 1\}\},$$

we consider some probability measure $\tilde{Q}$ on $\tilde{\mathcal{V}}$ with marginal Q on $\mathcal{V}$. An i.i.d. sequence $\tilde{\boldsymbol{\omega}} = ((\omega_x, \kappa_x), x \in \mathbb{Z}^d)$ with distribution $\tilde{Q}$ defines our branching random walk as above, coupled with a random walk in random (i.i.d.) environment with transition probability

$$p_x(y) = \sum_{v \in \mathcal{V}} \omega_x(v) \kappa_x(y)$$

from x to $x + y$. In words, we pick randomly one of the children in the branching random walk. We call this walk the $\tilde{Q}$*-induced random walk* in random environment. Here are some natural choices (in the examples below κ does not depend on ω):

(i) uniform: κ is uniform on the locations $\{x \in \mathfrak{A} : v_x \ge 1\}$;

(ii) particle-uniform: $\kappa(y)$ is proportional to the number of particles sent by v to y;

(iii) r-extremal, $r \in \mathbb{S}^{d-1}$: κ is supported on the set of x's maximizing $r \cdot x$ with $x \in \mathfrak{A}$, $v_x \ge 1$.

The following proposition is a direct consequence of Theorem 1.5:

PROPOSITION 2.1. *If the branching random walk in random environment is transient, then any induced random walk is either nonnestling or marginally nestling. Moreover, if Condition* UE *holds, then any induced random walk is nonnestling.*

Note, however, that one can easily construct examples of recurrent branching random walks such that any induced random walk is nonnestling, that is, the converse for Proposition 2.1 does not hold. For completeness we give the following:

EXAMPLE 8. Let $d = 1$ and $Q = \delta_\omega$, where ω sends one particle to the left with probability $1/3$, and five particles to the right with probability $2/3$. Then, clearly, any induced random walk is nonnestling. To see that this branching random walk is recurrent it is enough to obtain by a simple computation that a mean number of grandchildren that a particle sends to the same site in two steps is strictly greater than 1 (indeed, conditioning on the first step, we see that it is $\frac{1}{3} \cdot \frac{10}{3} + \frac{2}{3} \cdot 5 \cdot \frac{1}{3} = \frac{20}{9}$).

See also Example 2 of [6].

2.2. *Seeds.* In the next definition we introduce the notion of (U, H)-*seed*, which will be frequently used throughout the paper.

DEFINITION 2.2. Fix a finite set $U \subset \mathbb{Z}^d$ containing 0, and $H_x \subset \mathcal{M}$ with $Q(H_x) > 0$ for all $x \in U$. With $H = (H_x, x \in U)$, the couple (U, H) is called a seed. We say that $\boldsymbol{\omega}$ has a (U, H)-seed at $z \in \mathbb{Z}^d$ if

$$\omega_{z+x} \in H_x \qquad \text{for all } x \in U,$$

and that $\boldsymbol{\omega}$ has a (U, H)-seed in the case $z = 0$. We call z the center of the seed.

LEMMA 2.3. *With probability* 1 *the branching random walk visits infinitely many distinct* (U, H)*-seeds* (*to visit the seed means to visit the site where the seed is centered*).

PROOF. By the ellipticity Condition E, the uniform induced random walk is elliptic, so for *every* environment it cannot stay forever in a finite subset. Take n such that $U \subset \mathcal{K}_n$, and partition the lattice $\mathbb{Z}^d$ into translates of $\mathcal{K}_n$. Since the environment is i.i.d., we can construct the induced random walk by choosing randomly the environment in a translate of K_n at the first moment when the walk enters this set. If $Q(H_x) > 0$ for all $x \in U$, then by the Borel–Cantelli lemma, infinitely many of those translates contain the desired seed, and by using Condition E, it is elementary to show that infinitely many seed centers will be visited. $\square$

As we will see, the notion of seed becomes powerful when combined with independence of the medium. Hence we give two more definitions.

DEFINITION 2.4. For a particular realization of the random environment $\boldsymbol{\omega}$, we define the branching random walk *restricted* on set $M \subset \mathbb{Z}^d$ simply by discarding all particles that step outside M, and write $\mathrm{P}_{\omega|M}, \mathrm{E}_{\omega|M}$ for corresponding probability and expectation.

DEFINITION 2.5. Let U, W be two finite subsets of $\mathbb{Z}^d$ with $0 \in W \subset U$. Let $\mathbf{p}$ be a probability distribution on $\mathbb{Z}_+$ with mean larger than 1, that is, $\mathbf{p} = (p_0, p_1, p_2, \dots)$ with $p_i \geq 0$, $\sum p_i = 1$, $\sum i p_i > 1$. A (U, H)-seed is called $(\mathbf{p}, W)$-*recurrent* if for any $\boldsymbol{\omega}$ such that $\omega_x \in H_x$, $x \in U$, we have

$$\mathrm{P}^y_{\omega|U}[W \text{ will be visited by at least } i \text{ "free" particles}] \geq \sum_{j=i}^{\infty} p_j$$

for all $i \geq 1$ and all $y \in W$. By "*free*" particles we mean that none is the descendant of another one. To shorten the notation, in the case $W = \{0\}$ we simply say that the seed is $\mathbf{p}$-recurrent.

Note that, by definition of the restricted branching random walk, the above probability depends on the environment inside U only.

The next lemma shows the relevance of $\mathbf{p}$-recurrent seeds.

LEMMA 2.6. *Suppose that there exists a (U, H)-seed that is $(\mathbf{p}, W)$-recurrent for some $\mathbf{p}$, W. Then this implies the recurrence of the branching random walk for a.e. environment $\boldsymbol{\omega}$.*

PROOF. By Lemma 2.3, an infinite number of (U, H)-seeds will be visited. Moreover, a.s. we can find a (random) sequence $z_1, z_2, z_3, \dots$ such that:

- $z_1, z_2, z_3, \dots$ are visited;
- there is a (U, H)-seed in z_i for all $i = 1, 2, 3, \dots$;
- $(z_i + U) \cap (z_j + U) = \varnothing$ for all $i \neq j$.

The point is that each of those seeds gives rise to a supercritical Galton–Watson branching process. Indeed, consider the first particle that enters z_n and set $\zeta_0^n = 1$; then let ζ_1^n be the number of free descendants of this particle that visit $z_n + W$ in the process restricted on $z_n + U$ (coupled with the original process as in Definition 2.4). By Definition 2.5, these free particles are such that the distribution of ζ_1^n dominates $\mathbf{p}$. Then, let ζ_2^n be the number of free descendants of those ζ_1^n particles in the restricted process, and so on. By construction, the process $(\zeta_k^n)_{k=0,1,2,\dots}$ dominates a supercritical Galton–Watson branching process with offspring distribution $\mathbf{p}$, which means that when the Galton–Watson process survives forever, then in the original process the set $z_n + W$ is visited infinitely often. Since the seeds centered in $z_1, z_2, z_3, \dots$ are nonoverlapping, the processes $(\zeta_\cdot^n)$, $n = 1, 2, 3, \dots$, are independent. So, almost surely at least one of the sets $z_n + W$ will be visited infinitely often, thus sending also an infinite number of particles to 0, which proves the recurrence. $\square$

PROOF OF PROPOSITION 1.2. Assume that the event $\{\mathrm{P}^0_\omega[\text{the origin is visited infinitely often}] = 1\}$ has positive $\mathbb{P}$-probability. Then, by Condition B, this event intersected with {there exists $x \in \mathbb{Z}^d : \omega_x(v : |v| \geq 2) > 0$} has also positive $\mathbb{P}$-probability. Fix $\boldsymbol{\omega}$ in the intersection, and also in the support of $\mathbb{P}$. By Condition E, the following happens with positive $\mathtt{P}_\omega^0$-probability: one particle (at least) of the branching random walk reaches this branching site x, and is then substituted by two particles (at least), each of them eventually hitting the origin. Since the number of visits to 0 is infinite, we get by the Borel–Cantelli lemma that

$$\mathtt{P}_\omega^0[\text{the origin is visited by (at least) two free particles}] = 1$$

(recall that by "free" particles, we mean that none is the descendant of the other; we do not require that they visit 0 at the same moment). Then, we can take t large enough so that

$$\mathtt{P}_\omega^0[\text{the origin is visited by (at least) two free particles before time } t] > 3/4.$$

Since the jumps are bounded, this probability is equal to

$$\mathtt{P}_{\omega|\mathcal{K}_{tL_0}}^0[\text{the origin is visited by (at least) two free particles before time } t],$$

which depends only on $\omega_x, x \in U := \mathcal{K}_{tL_0}$. By continuity, we can choose now small neighborhoods H_x of $\omega_x, x \in U$, such that

$$\mathtt{P}_{\omega'|\mathcal{K}_{tL_0}}^0[\text{the origin is visited by (at least) two free particles before time } t] > 3/4$$

for all $\boldsymbol{\omega}'$ with $\omega'_x \in H_x, x \in U$. By the support condition it holds that $Q(H_x) > 0$, and we see that the (U, H)-seed is $\mathbf{p}$-recurrent, with $\mathbf{p} = (1/4, 0, 3/4, 0, 0, \ldots)$. From Lemma 2.6, we conclude that the branching random walk is recurrent for Q-a.e. environment. Therefore the set of recurrent $\boldsymbol{\omega}'$ has probability 0 or 1.

On the other hand, it is clear by ellipticity that

$$\boldsymbol{\omega} \quad \Longleftrightarrow \quad \mathtt{P}_\omega^0[x \text{ is visited infinitely often}] = 1,$$

for all $x \in \mathbb{Z}^d$. Since the law of $\boldsymbol{\omega}$ is stationary, this means that the recurrence is also equivalent to

$$\mathtt{P}_\omega^x[0 \text{ is visited infinitely often}] = 1$$

for all $x \in \mathbb{Z}^d$. $\square$

PROOF OF PROPOSITION 1.3. Assume that with positive $\mathbb{P}$-probability,

$$\mathtt{P}_\omega^x[\text{the origin is visited infinitely often}] > 0$$

for some $x \in \mathbb{Z}^d$, and fix such an $\boldsymbol{\omega}$ in the support of $\mathbb{P}$. Then, by Condition E, the inequality holds for all $x \in \mathbb{Z}^d$, and in view of Condition B, we can assume without loss of generality that

$$\omega_x(v : |v| \geq 2) > 0. \tag{10}$$

By Condition E, x is visited infinitely often a.s. on the event $E := \{0 \text{ is visited infinitely often}\}$. Together with (10) this implies that x is visited by infinitely many free particles a.s. on the event E. With $\beta = \mathrm{P}_\omega^x[E] > 0$, fix some integers K, t such that $K\beta/2 > 1$ and

$$\mathrm{P}_\omega^x[\text{at least } K \text{ free particles visit } x \text{ before time } t] > \frac{\beta}{2}.$$

We note that this probability depends only on $\boldsymbol{\omega}$ inside $U := x + \mathcal{K}_{tL_0}$, hence it is equal to the $\mathrm{P}_{\omega|U}^x$-probability of the event under consideration. By continuity, we can choose small neighborhoods H_y of ω_y, $y \in U$, such that

$$\mathrm{P}_{\omega'|U}[\text{at least } K \text{ free particles visit } x \text{ before time } t] > \frac{\beta}{2}$$

for all $\boldsymbol{\omega}'$ with $\omega'_y \in H_y$, $y \in U$. We see that the (U, H)-seed is $(\mathbf{p}, \{x\})$-recurrent with $p_K = \beta/2 = 1 - p_0$, and has a positive $\mathbb{P}$-probability since $\boldsymbol{\omega}$ is in the support of this measure. From Lemma 2.6, we conclude that the branching random walk is recurrent, which completes the proof. $\square$

We conclude this section by proving the sufficiency (for the recurrence) of the conditions (4) and (5). In the latter case, we define $U = W = \{0\}$ and $\mathbf{p}$ by

$$\sum_{j=i}^{\infty} p_j = \inf(\omega'\{v : v_0 \ge i\}; \omega' \in \mathcal{N}), \qquad i \in \mathbb{Z}_+,$$

where $\mathcal{N}$ is a neighborhood of ω. As $\mathcal{N} \searrow \{\omega\}$, the mean of $\mathbf{p}$ increases by continuity to $\sum_v \omega(v) v_0 > 1$ in view of (5). Choosing $\mathcal{N}$ small enough so that the mean of $\mathbf{p}$ is larger than 1, the seed $(\{0\}, \mathcal{N})$ is $\mathbf{p}$-recurrent in the sense of Definition 2.5. Applying Lemma 2.6, we obtain that (5) is sufficient for the recurrence.

In the former case, we let $U = \mathfrak{A}$, $W = \{0\}$, and for $x \in \mathfrak{A} \setminus \{0\}$,

$$H_x = \left\{\omega' : \sum_{v : v_e \ge 1} \omega'(v) \ge \varepsilon \text{ for all } e \in \{\pm e_1, \dots, \pm e_d\}\right\},$$

where we fix $\varepsilon > 0$ small enough so that $Q(H_x) > 0$. Fix $a > \varepsilon^{-dL_0 - 1}$, and a distribution $\mathbf{q}$ on $\mathbb{Z}_+$ which is stochastically smaller than the distribution of $|v|$ under ω and has mean at least a. By continuity again, and in view of (4), the set

$$H_0 = \{\omega' : \text{distribution of } |v| \text{ under } \omega' \text{ is stochastically larger than } \mathbf{q}\}$$

is a neighborhood of ω. Now, if $\boldsymbol{\omega}$ has a (U, H)-seed in 0, a walker starting at 0 has a number N of offspring stochastically larger than $\mathbf{q}$, each of which is able to walk back to 0 simply by ellipticity. So, we see that, given N, the number of free visits to 0 without exiting U dominates a binomial distribution $\mathcal{B}(N, \varepsilon^{dL_0+1})$, and that, unconditionally, it dominates the mixture $\mathbf{p}$ of such binomials for $N \sim \mathbf{q}$. Since $\mathbf{p}$ has mean larger than 1, the seed (U, H) is $\mathbf{p}$-recurrent, and it has positive $\mathbb{P}$-probability. Again applying Lemma 2.6, we conclude the proof.

2.3. *Formal construction of the process and subadditivity.* Recall that, for the process starting from one particle at x_0, the variable $\eta_n^{x_0}(x)$ is the number of particles in x at time n. Given $\boldsymbol{\omega}$, for all $x \in \mathbb{Z}^d$, consider an i.i.d. family $v^{x,i}(n)$, $i = 1, 2, 3, \dots, n = 0, 1, 2, \dots,$ of random elements of $\mathcal{V}$, with $\mathrm{P}_\omega[v^{x,i}(n) = v] = \omega_x(v)$ (with a slight abuse of notation, we will still write P_ω^x for the forthcoming construction for a fixed $\boldsymbol{\omega}$, and $\mathbf{P}^x[\,\cdot\,] = \mathbb{E}\,\mathrm{P}_\omega^x[\,\cdot\,]$). Now, the idea is to construct the *collection* of branching random walks *indexed by the position of the initial particle*, using the same realization of $(v^{x,i}(n), x \in \mathbb{Z}^d, i = 1, 2, 3, \dots, n = 0, 1, 2, \dots)$.

To this end, consider first the process beginning at the origin, and put $\eta_0^0(0) = 1$, $\eta_0^0(y) = 0$ for $y \neq 0$. Inductively, define for $y \in \mathbb{Z}^d$ and $n \geq 0$ [recall that $v^{x,i}(n)$ is an element of $\mathcal{V}$, so $v_a^{x,i}(n)$ is the number of particles sent by $v^{x,i}(n)$ to $a \in \mathfrak{A}$]:

$$\eta_{n+1}^0(y) = \sum_{x\,:\,y\in\mathfrak{A}+x} \sum_{i=1}^{\eta_n^0(x)} v_{y-x}^{x,i}(n). \tag{11}$$

Define $T(0, y)$ to be the first moment when a particle enters y, provided that the process started from 0, that is,

$$T(0, y) = \inf\{n \geq 0 : \eta_n^0(y) \geq 1\},$$

and $T(0, y) = +\infty$ if there exists no such n. Now, the goal is to define η_n^z for $z \neq 0$. We distinguish two cases.

If $T(0, z) = +\infty$, we proceed as before, that is, put $\eta_0^z(z) = 1$, $\eta_0^z(y) = 0$ for $y \neq z$, and

$$\eta_{n+1}^z(y) = \sum_{x\,:\,y\in\mathfrak{A}+x} \sum_{i=1}^{\eta_n^z(x)} v_{y-x}^{x,i}(n). \tag{12}$$

When $m_0 := T(0, z) < \infty$, we put $\eta_0^z(z) = 1$, $\eta_0^z(y) = 0$ for $y \neq z$, and

$$\eta_{n+1}^z(y) = \sum_{x\,:\,y\in\mathfrak{A}+x} \sum_{i=1}^{\eta_n^z(x)} v_{y-x}^{x,i}(n + m_0). \tag{13}$$

Define also

$$T(z, y) = \inf\{n \geq 0 : \eta_n^z(y) \geq 1\}.$$

Note that the set B_n^x can now be defined by $B_n^x = \{y : T(x, y) \leq n\}$. The following lemma will be very important in the course of the proof of Theorem 1.10:

LEMMA 2.7. *For any $y, z \in \mathbb{Z}^d$ and for* all *realizations of $(v^{u,i}(n), u \in \mathbb{Z}^d$, $i = 1, 2, 3, \dots, n = 0, 1, 2, \dots)$ it holds that*

$$T(0, z) + T(z, y) \geq T(0, y). \tag{14}$$

PROOF. Inequality (14) is obvious when $T(0,z)=+\infty$, so we concentrate on the case $m_0 := T(0,z) < \infty$. In this case, by induction, it is immediate to prove that $\eta_n^z(x) \le \eta_{n+m_0}^0(x)$ for all $x \in \mathbb{Z}^d$ and all $n \ge 0$, which, in turn, shows (14). $\square$

REMARK. For the present model we failed to construct a coupling such that $T(x,y)+T(y,z) \ge T(x,z)$ holds for all $x,y,z \in \mathbb{Z}^d$. In absence of such a coupling we need to use a stronger version of the subadditive ergodic theorem, a variant of Theorem 5.3 below. An example of "branching-type" model for which such a coupling does exist can be found in [1].

3. Proofs: recurrence/transience.

3.1. *Proof of Theorem* 1.4. We need some preparations. The following lemma complements Lemma 2.6.

LEMMA 3.1. *Suppose that the branching random walk in random environment from Q is recurrent. Then there exist* $\mathbf{p}$, $m \ge 1$ *and a collection* $H = (H_z \subset \mathcal{M}, z \in \mathcal{K}_{mL_0})$ *such that* $Q(H_z) > 0$ *for all* $z \in \mathcal{K}_{mL_0}$, *and such that the* $(\mathcal{K}_{mL_0}, H)$*-seed is* $\mathbf{p}$*-recurrent.*

In fact, the reader probably has noticed that a similar result was already proved in the course of the proof of Proposition 1.2. However, for later purposes, we will construct this seed in a more explicit way (see Definition 3.2 below).

PROOF OF LEMMA 3.1. By Condition B, for some $\varepsilon > 0$ the set

$$H_0 = \{\omega : \omega(v : |v| \ge 2) \ge \varepsilon\} \cap \operatorname{supp} Q$$

has positive Q-probability. By the recurrence assumption, the set of $\boldsymbol{\omega}$ such that

$$\mathrm{P}_{\boldsymbol{\omega}}^{y}[\text{at least one particle hits } 0] = 1$$

for any $y \in \mathfrak{A}$ and such that $\omega_0 \in H_0$, has positive $\mathbb{P}$-probability. We fix $\boldsymbol{\omega}'$ in this set and also in the support of $\mathbb{P}$. Then, for any $\rho < 1$ it is possible to choose m in such a way that

$$\min_{y \in \mathfrak{A}} \mathrm{P}_{\boldsymbol{\omega}'}^{y}[\text{at least one particle hits } 0 \text{ before time } m] > \rho.$$

The probability in the above display depends only on the environment inside the cube $\mathcal{K}_{mL_0}$. By continuity we can choose neighborhoods $H_z \subset \operatorname{supp} Q$ of ω_z', $z \in \mathcal{K}_{mL_0}$, with $Q(H_z) > 0$,

$$\inf_{\boldsymbol{\omega}} \min_{y \in \mathfrak{A}} \mathrm{P}_{\boldsymbol{\omega}}^{y}[\text{at least one particle hits } 0 \text{ before time } m] > \rho,$$

where the infimum is taken over all possible environments $\boldsymbol{\omega}$ such that $\omega_z \in H_z$ for all $z \in \mathcal{K}_{mL_0}$. Due to the boundedness of jumps, for any $\boldsymbol{\omega} \in \mathcal{M}^{\mathbb{Z}^d}$ and any $y \in \mathfrak{A}$

$$\begin{aligned} &\mathrm{P}^y_\omega[\text{at least one particle hits 0 before time } m] \\ &\quad \le \mathrm{P}^y_{\omega|\mathcal{K}_{mL_0}}[\text{at least one particle hits 0}]. \end{aligned}$$

Hence, under P^0_ω, with probability ε two particles will be present at time 1 in $\mathfrak{A}$ and otherwise at least one particle, each of them having independent evolution and probability at least ρ to come back to 0 before exiting $\mathcal{K}_{mL_0}$. By an elementary computation, we see that

$$\mathrm{P}^0_{\omega|\mathcal{K}_{mL_0}}[\text{0 will be visited by at least } i \text{ free particles}] \ge \sum_{j=i}^{2} p_j$$

with

$$p_1 = (1-\varepsilon)\rho + 2\varepsilon\rho(1-\rho), \qquad p_2 = \varepsilon\rho^2, \qquad p_3 = p_4 = \cdots = 0. \tag{15}$$

It remains only to choose m large enough to assure that ρ becomes sufficiently close to 1 to guarantee that the mean $(1-\varepsilon)\rho + 2\varepsilon\rho(1-\rho) + 2\varepsilon\rho^2$ of $\mathbf{p}$ defined above is strictly larger than 1. Then, the $(\mathcal{K}_{mL_0}, H)$-seed constructed in this way is $\mathbf{p}$-recurrent, and it has a positive $\mathbb{P}$-probability. $\square$

For later purposes, it is useful to emphasize the kind of seed we constructed above.

DEFINITION 3.2. Let U, W be two finite subsets of $\mathbb{Z}^d$ such that $0 \in W$, $\mathfrak{A} + W \subset U$, and let $\varepsilon, \rho \in (0,1)$. A (U, H)-seed is called (ε, ρ, W)-good, if:

(i) for any $\omega \in H_z$ we have $\omega(v : |v| \ge 2) > \varepsilon$ for all $z \in W$;
(ii) for any $\boldsymbol{\omega}$ such that $\omega_x \in H_x$, $x \in U$, we have

$$\mathrm{P}^y_{\omega|U}[\text{at least one particle hits } W] > \rho$$

for any $y \in \mathfrak{A} + W$;
(iii) we have $(1-\varepsilon)\rho + 2\varepsilon\rho(1-\rho) + 2\varepsilon\rho^2 > 1$.

In the case $W = \{0\}$ we say that the seed is (ε, ρ)-good.

At the end of the last proof, we just showed that such a seed is $\mathbf{p}$-recurrent, in the case $W = \{0\}$. It is a simple exercise to extend the proof to the case of a general W. We state now this useful fact.

LEMMA 3.3. *Any (U, H)-seed which is (ε, ρ, W)-good is also $(\mathbf{p}, W)$-recurrent with $\mathbf{p}$ defined by* (15).

Now we finish the proof of Theorem 1.4. Observe that if a **p**-recurrent seed has positive probability under Q, then it has positive probability under Q' for any Q' such that $\operatorname{supp} Q \subset \operatorname{supp} Q'$. (One may invoke the stronger condition of Q being absolutely continuous to Q', but due to the particular form of the seed we consider here, the weaker condition of equal support is sufficient.) An application of Lemmas 2.6 and 3.1 concludes the proof.

3.2. *Proof of Theorem* 1.5.

Part 1. We start with the first statement, assuming Condition E only. For any $s \in \mathbb{S}^{d-1}$ define

$$\varphi_Q(s) = \sup_{\omega \in \operatorname{supp} Q} \sum_{v \in \mathcal{V}} \omega(v) D(s, v),$$

with D defined in (1). Since $\varphi_Q(s)$ is a continuous function of s and $\mathbb{S}^{d-1}$ is compact, (2) implies that

$$a_0 := \inf_{s \in \mathbb{S}^{d-1}} \varphi_Q(s) > 0.$$

Since $\operatorname{supp} Q$ is closed, for any s there exists $\omega^{(s)}$ such that

$$\varphi_Q(s) = \sum_{v \in \mathcal{V}} \omega^{(s)}(v) D(s, v).$$

Moreover, by continuity for any s we can find $\delta_s > 0$ and an open set $\Gamma_s \subset \mathcal{M}$ with $\omega^{(s)} \in \Gamma_s$ and $Q(\Gamma_s) > 0$ such that

$$\inf_{\substack{s' \in \mathbb{S}^{d-1}: \\ \|s - s'\| < \delta_s}} \inf_{\omega \in \Gamma_s} \sum_{v \in \mathcal{V}} \omega(v) D(s', v) > \frac{a_0}{2}, \tag{16}$$

where $\|\cdot\|$ stands for the Euclidean norm.

Since $\mathbb{S}^{d-1}$ is compact, we can choose $s_1, \dots, s_m \in \mathbb{S}^{d-1}$ that generate a finite subcovering of $\mathbb{S}^{d-1}$ by the sets $\{s' \in \mathbb{S}^{d-1} : \|s_n - s'\| < \delta_{s_n}\}$, $n = 1, \dots, m$. For each $n = 1, \dots, m$, it is possible to choose a set $U_n \subset \{s' \in \mathbb{S}^{d-1} : \|s_n - s'\| < \delta_{s_n}\}$ in such a way that $U_i \cap U_j = \varnothing$ for $i \neq j$ and $\bigcup_{i=1}^m U_i = \mathbb{S}^{d-1}$.

To prove recurrence, we construct an (ε, ρ, W)-good (A, H)-seed with a supercritical branching inside W and, in $A \setminus W$, the drift pointing toward W (and so this seed is a trap):

(i) Similarly to the proof of Theorem 1.4, we argue that there exist $\varepsilon > 0$ and $\tilde{H} \subset \operatorname{supp} Q$ such that $Q(\tilde{H}) > 0$ and $\omega(v : |v| \geq 2) \geq \varepsilon$ for any $\omega \in \tilde{H}$.

(ii) Take $W = \{y \in \mathbb{Z}^d : \|y\| \leq L_0^2 / a_0\}$, and put $H_z = \tilde{H}$ for all $z \in W$.

(iii) Choose $\rho > 0$ in such a way that condition (iii) of Definition 3.2 holds.

(iv) Denote $r_1 := L_0^2 / a_0$ and choose large enough $r_2 > r_1$ in order to guarantee that $\rho \leq \frac{r_2 - r_1 - L_0\sqrt{d}}{r_2 - r_1 + L_0\sqrt{d}}$.

(v) To complete the definition of the seed, take $A = \{y \in \mathbb{Z}^d : \|y\| \leq r_2\}$; it remains to define the environment in $A \setminus W$. It is done in the following way: if $z \in A \setminus W$, let n_0 be such that $(-z/\|z\|) \in U_{n_0}$; then put $H_z = \Gamma_{s_{n_0}}$; see Figure 4.

To prove that the seed constructed above is indeed (ε, ρ, W)-good, we construct a random walk ξ_n that is similar to example (iii) of r-extremal induced random walks from Section 2.1. Specifically, suppose that at some moment the random walk ξ_n is in site $z \neq 0$ (this does not complicate anything, since we really need the random walk to be defined only inside $A \setminus W$ and $0 \notin A \setminus W$). Generate the offspring of that particle from z according to the rules of the branching random walk; suppose that those offspring went to $z+w_1, \dots, z+w_k$. Let k_0 be the number which maximizes the quantity $(-z/\|z\|) \cdot w_l$, $l = 1, \dots, k$; then take $\xi_{n+1} = z + w_{k_0}$. Clearly, for the random walk constructed in this way,

$$\mathrm{E}_\omega\big((\xi_{n+1} - \xi_n) \cdot (-z/\|z\|) \mid \xi_n = z\big) = \sum_{v \in \mathcal{V}} \omega_z(v) D(-z/\|z\|, v). \tag{17}$$

Now, we have to bound from below the probability that the random walk ξ_n starting somewhere from $W + \mathfrak{A}$ hits W before stepping out from A. We use ideas which are classical in the Lyapunov functions approach [11]. To do that, we first

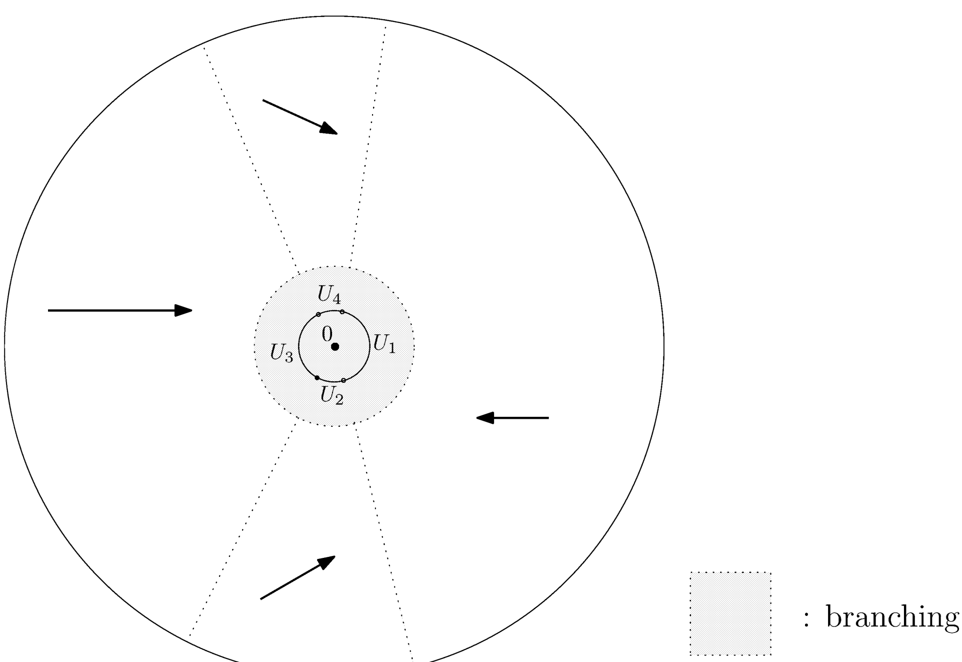

FIG. 4. *Construction of an (ε, ρ)-good seed for the branching random walk (defined by Q_2, with $a < 1/2$) of Example 2.*

recall the following elementary inequality: for any $x \geq -1$,

$$\sqrt{1+x} \leq 1 + \frac{x}{2}. \tag{18}$$

Let $p_{x,y}$ be the transition probabilities of the random walk ξ_n. Using (18), (17) and (16), we obtain

$$\begin{aligned}
&\mathrm{E}_\omega(\|\xi_{n+1}\| - \|\xi_n\| \mid \xi_n = z)\\
&\quad = \sum_{y\in\mathfrak{A}} p_{z,z+y}(\|z+y\| - \|z\|)\\
&\quad = \|z\| \sum_{y\in\mathfrak{A}} p_{z,z+y}\left(\sqrt{1 + \frac{2z\cdot y}{\|z\|^2} + \frac{\|y\|^2}{\|z\|^2}} - 1\right)\\
&\quad \leq \sum_{y\in\mathfrak{A}} p_{z,z+y}\frac{z}{\|z\|}\cdot y + \frac{L_0^2}{2\|z\|} \qquad (19)\\
&\quad \leq -\frac{a_0}{2} + \frac{L_0^2}{2\|z\|} \leq 0 \qquad (20)
\end{aligned}$$

for all $z \in A \setminus W$. Let τ be the first moment when ξ_n leaves the set $A \setminus W$; the calculation (20) shows that the process $\|\xi_{n\wedge\tau}\|$ is a (bounded) supermartingale. Denoting by $\tilde{p}$ the probability that ξ_n hits W before stepping out from A and starting the walk inside $W + \mathfrak{A}$, we obtain from the optional stopping theorem that

$$r_1 + L_0\sqrt{d} \geq \mathrm{E}_\omega\|\xi_0\| \geq \mathrm{E}_\omega\|\xi_\tau\| \geq \tilde{p}\big(r_1 - L_0\sqrt{d}\big) + (1-\tilde{p})r_2.$$

So,

$$\tilde{p} \geq \frac{r_2 - r_1 - L_0\sqrt{d}}{r_2 - r_1 + L_0\sqrt{d}} \geq \rho,$$

which shows that the (A, H)-seed constructed above is (ε, ρ, W)-good. Indeed, we have to check condition (ii) of Definition 3.2; this condition holds because of the last display and the obvious comparison between the branching random walk and the random walk ξ. With an application of Lemma 3.3 and of Lemma 2.6 we conclude the proof of the first part of Theorem 1.5.

Part 2. Now, we prove that (3) together with Condition UE implies recurrence. The basic idea is the same: we would like to construct an (A, H)-seed that is (ε, ρ, W)-good, where $W = \{y \in \mathbb{Z}^d : \|y\| \leq r_1\}$, $A = \{y \in \mathbb{Z}^d : \|y\| \leq r_2\}$, for some r_1, r_2 (to be chosen later). As above, we choose $\varepsilon > 0$ in such a way there exists $\tilde{H} \subset \operatorname{supp} Q$ such that $Q(\tilde{H}) > 0$ and $\omega(v : |v| \geq 2) \geq \varepsilon$ for any $\omega \in \tilde{H}$. Then, choose $\rho > 0$ in such a way that the condition (iii) of Definition 3.2 holds. To define the seed inside W, put $H_z = \tilde{H}$ for all $z \in W$. Now, it remains to specify r_1, r_2 and to define the seed in $A \setminus W$. To do that, we need some preparations. For any *possible* environment inside $A \setminus W$ (i.e., for any $\boldsymbol{\omega}$ such that $\omega_z \in \operatorname{supp} Q$ for all $z \in A \setminus W$) we keep the same definition of the random walk ξ_n; by Condition UE, there exists $\gamma > 0$ such that for all $x \neq 0$

$$\mathrm{E}_\omega\big((\|\xi_{n+1}\| - \|\xi_n\|)^2 \mathbf{1}_{\{\|\xi_{n+1}\| \le \|\xi_n\|\}} \mid \xi_n = x\big) \ge \gamma. \tag{21}$$

With that γ, we successively choose $\alpha > 0$ such that

$$\gamma(\alpha + 1) > L_0^2, \tag{22}$$

then $r_2 > r_1 > L_0$ with

$$\frac{(r_1 + L_0)^{-\alpha} - r_2^{-\alpha}}{(r_1 - L_0)^{-\alpha} - r_2^{-\alpha}} > \rho, \tag{23}$$

and finally, $\varepsilon' > 0$ such that

$$\varepsilon' < \frac{\gamma(\alpha+1) - L_0^2}{2r_2}. \tag{24}$$

Now we define the seed on the set $A \setminus W$ in the following way. When (3) holds, analogously to the first part of the proof of this theorem, for any $s \in \mathbb{S}^{d-1}$ we can find $\delta'_s > 0$ and an open set $\Gamma'_s \subset \mathcal{M}$ with $\omega^{(s)} \in \Gamma'_s$ and $Q(\Gamma'_s) > 0$ such that

$$\inf_{\substack{s' \in \mathbb{S}^{d-1}: \\ \|s - s'\| < \delta'_s}} \inf_{\omega \in \Gamma'_s} \sum_{v \in \mathcal{V}} \omega(v) D(s', v) > -\varepsilon'. \tag{25}$$

Similarly to what has been done before, we choose $s'_1, \dots, s'_m \in \mathbb{S}^{d-1}$ that generate a finite subcovering of $\mathbb{S}^{d-1}$ by the sets $\{s \in \mathbb{S}^{d-1} : \|s'_n - s\| < \delta_{s'_n}\}$, $n = 1, \dots, m$. For each $n = 1, \dots, m$, we choose a set $U'_n \subset \{s \in \mathbb{S}^{d-1} : \|s'_n - s\| < \delta_{s'_n}\}$ in such a way that $U'_i \cap U'_j = \varnothing$ for $i \neq j$ and $\bigcup_{i=1}^m U'_i = \mathbb{S}^{d-1}$. Now, if $z \in A \setminus W$, let n_1 be such that $(-z/\|z\|) \in U'_{n_1}$; then put $H_z = \Gamma_{s'_{n_1}}$.

To prove that the (A, H)-seed is indeed (ε, ρ, W)-good, we have to verify condition (ii) of Definition 3.2. First, it is elementary to see that the following inequality holds: for any $x \ge -1$, $\alpha > 0$

$$(1 + x)^{-\alpha} \ge 1 - \alpha x + \frac{\alpha(\alpha+1)}{2} x^2 \mathbf{1}_{\{x \le 0\}}. \tag{26}$$

Keeping the notation τ from the proof of the first part of the theorem, we are aiming to prove that $\|\xi_{n \wedge \tau}\|^{-\alpha}$ is a submartingale. Indeed, using (26), (19), (21) and (24), we obtain for $z \in A \setminus W$

$$\begin{aligned}
&\mathrm{E}_\omega(\|\xi_{n+1}\|^{-\alpha} - \|\xi_n\|^{-\alpha} \mid \xi_n = z) \\
&\quad = \|z\|^{-\alpha} \mathrm{E}_\omega\left(\left(1 + \frac{\|\xi_{n+1}\| - \|\xi_n\|}{\|\xi_n\|}\right)^{-\alpha} - 1 \middle| \xi_n = z\right) \\
&\quad \ge -\alpha \|z\|^{-\alpha-1} \mathrm{E}_\omega(\|\xi_{n+1}\| - \|\xi_n\| \mid \xi_n = z)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha(\alpha+1)}{2}\|z\|^{-\alpha-2}\mathrm{E}_\omega\big((\|\xi_{n+1}\|-\|\xi_n\|)^2\mathbf{1}_{\{\|\xi_{n+1}\|\le\|\xi_n\|\}}\mid \xi_n=z\big)\\
&\ge \alpha\|z\|^{-\alpha-1}\Big(-\varepsilon'-\frac{L_0^2}{2\|z\|}+\frac{(\alpha+1)\gamma}{2\|z\|}\Big)\\
&>0.
\end{aligned}
$$

Then, by using the optional stopping theorem again, we obtain that the probability that ξ_n hits W before stepping out from A and supposing that its starting point belongs to $W+\mathfrak{A}$ is at least

$$
\frac{(r_1+L_0)^{-\alpha}-r_2^{-\alpha}}{(r_1-L_0)^{-\alpha}-r_2^{-\alpha}},
$$

so, recalling (23), we see that condition (ii) of Definition 3.2 holds. We finish the proof of the second part of Theorem 1.5 by applying Lemma 3.3.

3.3. *Proof of Theorem* 1.6. Due to Condition B, there exists $\omega\in\operatorname{supp}Q$ such that $\sum_{y\in\mathfrak{A}}\mu_y^\omega>1$. Hence, Condition L cannot be satisfied with $\lambda=1$. Moreover, if Condition L holds for λ and s, it holds also for λ^{-1} and $(-s)$. So, we can suppose that $\lambda\in(0,1)$ without loss of generality.

Denote the half-space $Y_s=\{y\in\mathbb{Z}^d : y\cdot s\le 0\}$. Take an arbitrary starting point $z\notin Y_s$ and define

$$
F_n^z=\sum_{y\in\mathbb{Z}^d}\eta_n^z(y)\lambda^{y\cdot s}.
$$

1. Let us also modify the environment in such a way that any particle which enters Y_s neither moves nor branches anymore. By (6) it is straightforward to obtain that the process $(F_n^z, n=0,1,2,\dots)$ is a supermartingale:

$$
\begin{aligned}
\mathrm{E}_\omega^z\big(F_{n+1}^z\mid\eta_1^z(\cdot),\dots,\eta_n^z(\cdot)\big)&=\sum_{x\in Y_s}\eta_n^z(x)\lambda^{x\cdot s}+\sum_{x\notin Y_s}\eta_n^z(x)\lambda^{x\cdot s}\times\sum_{y\in\mathfrak{A}}\mu_y^{\omega_x}\lambda^{y\cdot s}\\
&\le F_n^z.
\end{aligned}
$$

Since it is also nonnegative, it converges a.s. as $n\to\infty$ to some random variable F_∞. By Fatou's lemma,

$$
\mathrm{E}_\omega^z F_\infty\le\mathrm{E}_\omega^z F_0=\lambda^{z\cdot s}<1,
$$

for $z\notin Y_s$. On the other hand, any particle stuck in Y_s contributes at least one unit to F. That shows that with positive probability the branching random walk will not enter to Y_s, so the proof of the first part of Theorem 1.6 is finished.

2. We no longer make Y_s absorbing. Note that F_n^z is still a supermartingale, and has an a.s. limit (for all $\boldsymbol{\omega}$). Let $k\ge 1$. First of all, let us show how to prove the result when Condition UE holds. Under Condition UE, each time a particle

enters the half-space Y_s it has P_ω-probability larger than ε_0^{dk} to enter $Y_s^{(k)} = \{y \in \mathbb{Z}^d : y \cdot s \le -k\}$. By the strong Markov property, an infinite number of particles will hit $Y_s^{(k)}$ a.s. on the set where the number of visits of the branching random walk to Y_s is infinite. We will then have, on this set, $\limsup_n F_n^z \ge \lambda^{-k}$ for all k (recall that $\lambda < 1$). Since F_n^z has a finite limit, this shows that the number of visits to Y_s is finite.

Now, we explain what to do when only Condition E holds. By the previous argument, it would be enough to prove that, on the event that Y_s is visited infinitely often, for any k, the set $Y_s^{(k)}$ is visited infinitely many times P_ω^z-a.s. Define $H_s^{(k)} = Y_s \setminus Y_s^{(k)}$. Suppose, without restriction of generality, that $\|s\|_\infty = 1$. For any $z \in H_s^{(k)}$ define

$$g_z^{(k)}(\boldsymbol{\omega}) = \mathrm{P}_\omega^z\big[\text{at time } k \text{ there is at least one particle in } Y_s^{(k)}\big].$$

It is elementary to obtain that:

(i) $g_u^{(k)}(\boldsymbol{\omega})$ and $g_v^{(k)}(\boldsymbol{\omega})$ are independent if $\|u - v\|_\infty > 2kL_0$, and

(ii) there exists $h_k > 0$ such that $\mathbb{P}[g_z^{(k)}(\boldsymbol{\omega}) > h_k] \ge 1/2$, uniformly in $z \in H_s^{(k)}$.

Indeed, if $\|s\|_\infty = 1$, then there exists $e \in \{\pm e_1, \dots, \pm e_d\}$ such that $e \cdot s = -1$. Then, to enter $Y_s^{(k)}$ from any point of $H_s^{(k)}$, it is enough to perform k steps in the direction e.

Now, suppose that Y_s was visited an infinite number of times, and suppose also that the number of visits to $H_s^{(k)}$ is also infinite (because otherwise, automatically, $Y_s^{(k)}$ is visited infinitely many times). Let $z_1, z_2, z_3, \dots$ be the locations of those visits. Using (i) and (ii), we can extract an infinite subsequence $i_1 < i_2 < i_3 < \cdots$ such that $g_{z_{i_j}}^{(k)}(\boldsymbol{\omega}) > h_k$, for all j. Similarly to the previous argument, we obtain that in this case $Y_s^{(k)}$ will be visited infinitely often, which leads to a contradiction with the existence of a finite limit for F_n^z.

4. Proofs of Theorems 1.8 and 1.9.

PROOF OF THEOREM 1.8. Roughly, the idea is as follows: by recurrence we know that there are **p**-recurrent [in fact, even (ε, ρ)-good] seeds, each of them supporting a supercritical Galton–Watson process (i.e., if we consider the branching random walk restricted on such a seed, it dominates in some sense a supercritical Galton–Watson process). To prove (8) it suffices essentially to control the time to reach a large enough quantity of these seeds.

By Lemma 3.1, there exist $n_0, \varepsilon, \rho > 0$ and a collection $H = (H_z \subset \mathcal{M}, z \in \mathcal{K}_{n_0})$ having positive $\mathbb{P}$-probability, such that the $(\mathcal{K}_{n_0}, H)$-seed is (ε, ρ)-good (in the proof of Lemma 3.1, we indeed constructed such a seed). Moreover, it is straightforward to see that there exists t_0 such that

$$\mathbb{P}\Big[\text{the } (\mathcal{K}_{n_0}, H)\text{-seed is } (\varepsilon,\rho)\text{-good and for any } y \in \mathfrak{A} \\ \mathrm{P}^y_{\omega|\mathcal{K}_{n_0}}[\text{at least one particle hits 0 before time } t_0] > \rho\Big] > 0 \tag{27}$$

(recall that $\mathrm{P}^y_{\omega|\mathcal{K}_{n_0}}$ corresponds to the branching random walk starting from y and restricted on $\mathcal{K}_{n_0}$; cf. Definition 2.4). For any $\boldsymbol{\omega}$, define the random subset $S_{\boldsymbol{\omega}}$ of the lattice with spacing $2n_0+1$:

$$S_{\boldsymbol{\omega}} = \{z \in (2n_0+1)\mathbb{Z}^d : z \text{ is the center of } (\mathcal{K}_{n_0}, H)\text{-seed} \\ \text{which is } (\varepsilon,\rho)\text{-good and satisfies (27)}\}.$$

We need to consider two cases separately: $d \geq 2$ and $d = 1$.

Case $d \geq 2$. Consider the event

$$M_n = \{\forall y \in \mathcal{K}_{L_0 n \ln^{-1} n}\ \exists z \in S_{\boldsymbol{\omega}} : \|y - z\|_\infty \leq \alpha \ln n\};$$

the (small enough) constant α will be chosen later. We will use the bound

$$\mathbf{P}[T(0,x_0) > n] \leq \sup_{\boldsymbol{\omega}\in M_n} \mathrm{P}^0_{\omega}[T(0,x_0) > n] + \mathbb{P}[M_n^c]. \tag{28}$$

Let us begin by estimating the second term in the right-hand side of (28). We have

$$\mathbb{P}[M_n^c] = \mathbb{P}[\exists y \in \mathcal{K}_{L_0 n \ln^{-1} n} : (y + \mathcal{K}_{\alpha \ln n}) \cap S_{\boldsymbol{\omega}} = \varnothing] \\ \leq |\mathcal{K}_{L_0 n \ln^{-1} n}| \mathbb{P}[\mathcal{K}_{\alpha \ln n} \cap S_{\boldsymbol{\omega}} = \varnothing]. \tag{29}$$

The point is that the events $\{x \in S_{\boldsymbol{\omega}}\}$ and $\{y \in S_{\boldsymbol{\omega}}\}$ are independent for any $x, y \in (2n_0+1)\mathbb{Z}^d$, $x \neq y$. Denoting the left-hand side of (27) by $p_0 = \mathbb{P}[0 \in S_{\boldsymbol{\omega}}] > 0$, we obtain

$$\mathbb{P}[\mathcal{K}_{\alpha \ln n} \cap S_{\boldsymbol{\omega}} = \varnothing] \leq (1-p_0)^{\alpha^d \ln^d n/(2n_0+1)^d},$$

so, from (29),

$$\mathbb{P}[M_n^c] \leq L_0^d n^d \ln^{-d} n \exp\left\{-\frac{\alpha^d \ln(1-p_0)^{-1}}{(2n_0+1)^d} \ln^d n\right\} \\ \leq \exp\{-C_1 \ln^d n\} \tag{30}$$

for some $C_1 > 0$ and for all n large enough.

Now, we estimate the first term in the right-hand side of (28). Let ξ_n be the uniform induced random walk in random environment; compare example (i) in Section 2.1. By Condition UE, this random walk will be uniformly elliptic as well, in the sense that for any $x \in \mathbb{Z}^d$ and any $\omega \in \operatorname{supp} Q$

$$\mathrm{P}^x_{\omega}[\xi_1 = x + e] \geq \varepsilon_1 > 0 \tag{31}$$

for all $e \in \{\pm e_i, i = 1, \dots, d\}$ with a new constant $\varepsilon_1 = \varepsilon_0(2L_0 + 1)^{-1}$. By (31), we have that for arbitrary $\boldsymbol{\omega} \in M_n$ and any m with $0 \le m \le n \ln^{-1} n - d\alpha \ln n$,

$$\mathrm{P}_{\omega}^{0}\big[\{\xi_m, \dots, \xi_{m+d\alpha \ln n}\} \cap S_{\boldsymbol{\omega}} \neq \varnothing | \xi_0, \dots, \xi_{m-1}\big] \ge \varepsilon_1^{d\alpha \ln n}. \tag{32}$$

Define

$$\tau = \inf\left\{m : \sum_{i=0}^{m} \mathbf{1}_{\{\xi_i \in S_{\omega}\}} \ge \ln^d n\right\}, \tag{33}$$

that is, τ is the moment when the random walk ξ hits the set $S_{\boldsymbol{\omega}}$ for the $\lceil \ln^d n \rceil$th time. Also, let us recall Chernoff's bound for the binomial distribution: if S_k is a binomial $\mathscr{B}(n, p)$ random variable, for any k and a with $0 < a < p < 1$, we have

$$\mathbf{P}\left[\frac{S_k}{k} \le a\right] \le \exp\{-kU(a, p)\}, \tag{34}$$

where

$$U(a, p) = a \ln \frac{a}{p} + (1 - a) \ln \frac{1 - a}{1 - p} > 0.$$

Now, divide the time interval $[0, n \ln^{-1} n]$ into $(d\alpha)^{-1} n \ln^{-2} n$ subintervals of length $d\alpha \ln n$. Fix the constant α in such a way that $d\alpha \ln \varepsilon_1^{-1} < 1/2$. Use the Markov property for ξ under P_{ω}^{0}, the inequality (32), and (34) with $p = \varepsilon_1^{d\alpha \ln n}$, $k = (d\alpha)^{-1} n \ln^{-2} n$, $a = d\alpha n^{-1} \ln^{2+d} n$ [and an elementary computation shows that then $U(a, p)$ is of order $n^{-d\alpha \ln \varepsilon_1^{-1}}$] to obtain that for some $C_2, C_3 > 0$

$$\begin{aligned}
\mathrm{P}_{\omega}^{0}[\tau \le n/3] &\ge \mathrm{P}_{\omega}^{0}[\tau \le n \ln^{-1} n] \\
&\ge \mathrm{P}_{\omega}^{0}\left[\sum_{i=1}^{k} \mathbf{1}_{\{\xi_j ; (i-1) d\alpha \ln n < j \le i\alpha d \ln n\} \bigcap S_{\omega} \neq \varnothing} \ge ka\right] \\
&\ge 1 - \exp\{-C_2 (d\alpha)^{-1} n^{1 - d\alpha \ln \varepsilon_1^{-1}} \ln^{-2} n\} \\
&\ge 1 - \exp\{-C_3 n^{1/2}\}
\end{aligned} \tag{35}$$

for any $\boldsymbol{\omega} \in M_n$ (supposing also that n is large enough so that $a < p$).

Now, we show that each time the random walk ξ passes through the points of $S_{\boldsymbol{\omega}}$ it gives rise to a supercritical Galton–Watson process, and that on the set $\{\tau \le n/3\}$, about $\ln^d n$ such independent Galton–Watson processes will be started before time $n/3$. Indeed, analogously to the proof of Lemma 3.3, if we have a particle in the center of the seed, its direct offspring in this Galton–Watson process are those descendants (in the branching random walk restricted on the seed) that pass through the center not later than t_0. (Actually, we must take this Galton–Watson process independent of the random walk ξ, so when ξ passes through the seed, we cannot use the corresponding particle in the Galton–Watson process. This, however, does not spoil anything, because with uniformly positive probability another particle will be generated somewhere in the set $x+\mathfrak{A}$—with x the center—at that moment, so it can be used to start the Galton–Watson process.) By construction, this Galton–Watson process is "uniformly" supercritical, so there exists $p_1>0$ such that with probability at least p_1 in the $[n/3t_0]$th generation of the process the number of particles will be at least $C_4\alpha_1^n$, for some $C_4>0$, $\alpha_1>1$. So, since the real time between the generations is at most t_0, this means that for any $x\in S_{\boldsymbol{\omega}}$

$$\mathtt{P}_{\omega}^{x}[\text{the seed in } x \text{ generates at least } C_4\alpha_1^n \text{ free particles}\\ \text{before time } n/3]>p_1. \tag{36}$$

Now, by (36) we have

$$\begin{aligned}&\mathtt{P}_{\omega}^{0}[\text{at least one seed generates at least } C_4\alpha_1^n \text{ free particles}\\ &\qquad\qquad \text{before time } (2n/3)\mid \tau<n/3]\\ &\quad\geq 1-(1-p_1)^{\ln^d n}.\end{aligned} \tag{37}$$

Consider those $C_4\alpha_1^n$ free particles. By Condition UE, any descendants of each one will hit x_0 by the time $2L_0dn\ln^{-1}n<n/3$ with probability at least $\varepsilon_0^{2L_0dn\ln^{-1}n}$, so at least one particle will hit x_0 with probability at least

$$1-(1-\varepsilon_0^{2L_0dn\ln^{-1}n})^{C_4\alpha_1^n}\geq 1-\exp\{-C_4\exp\{n[\ln\alpha_1-2L_0d\ln\varepsilon_0^{-1}\ln^{-1}n]\}\}$$

where the quantity in the brackets is positive for large enough n. Taking into account (35) and (37), we then obtain that for any $\boldsymbol{\omega}\in M_n$

$$\mathtt{P}_{\omega}^{0}[T(0,x_0)>n]\leq e^{-C_5\ln^d n} \tag{38}$$

for some $C_5>0$ and all n large enough. We plug now (30) and (38) into (28) to conclude the proof of Theorem 1.8 in the case $d\geq 2$.

Case $d=1$. Now, we prove Theorem 1.8 in dimension 1. For $d=1$ the above approach fails, because if α is small, then (30) will not work, if α is large, then we would have problems with (35), and it is not always possible to find a value of α such that both inequalities would work.

First, we do the proof assuming that $L_0=1$, that is, $\mathfrak{A}$ is either $\{-1,1\}$ or $\{-1,0,1\}$. Analogously to the proof for higher dimensions, if we prove that, on the set of environments of $\mathbb{P}$-probability at least $1-e^{-C_1\ln n}$, the initial particle hits at least *const*$\cdot\ln n$ many good seeds from $S_{\boldsymbol{\omega}}$, we are done. To this end, note that, on the time interval of length $\frac{\ln n}{2\ln\varepsilon_0^{-1}}$ a single particle (even if it does not generate any offspring) covers a space interval of the same length with probability at least

$$\varepsilon_0^{(\ln n)/(2\ln\varepsilon_0^{-1})}=n^{-1/2}.$$

So, by time $n \ln^{-1} n$, with large (of at least stretched exponential order) probability there is an interval of length $\frac{\ln n}{2 \ln \varepsilon_0^{-1}}$ containing 0 such that all sites from there are visited.

Analogously to the proof for higher dimensions, consider the set

$$M_n^{(1)} = \Big\{ \text{the number of good seeds from } S_\omega \text{ in all the intervals} \\ \text{of length } \frac{\ln n}{2 \ln \varepsilon_0^{-1}} \text{ containing 0 is at least } C_2 \ln n \Big\},$$

which corresponds to the set of "good" environments. Since S_ω has a positive density, we can choose small enough C_2 in such a way that

$$\mathbb{P}\big[(M_n^{(1)})^c\big] \le e^{-C_3 \ln n}$$

for some C_3. Now, on $M_n^{(1)}$, with probability at least $1 - e^{-C_4 n^{C_5}}$ by time $n \ln^{-1} n$ at least $C_2 \ln n$ good seeds from S_ω will be visited. The rest of the proof is completely analogous to the proof for $d \ge 2$.

Let us explain how to proceed in the case of a general L_0. In the above argument the fact $L_0 = 1$ was used only for the following purpose: if we know that a particle crossed a (space) interval, then we are sure that all the good seeds that might be there were visited. For a general L_0, instead of (ε, ρ)-good seeds of S_ω, use (ε, ρ, W)-good seeds with $W = \{0, 1, \dots, L_0 - 1\}$, so that particles cannot jump over the translates of this W. [Indeed, it is clear that a recurrent branching random walk generates (ε, ρ, W)-good seeds for any finite W.] So, proof of Theorem 1.8 is concluded. $\square$

PROOF OF THEOREM 1.9. The method of the proof is very similar to the construction of Example 5. Roughly speaking, for given n and x, we create a (rather improbable) environment that has a trap near the origin; for such an environment with a good probability the event $\{T(0,x) > n\}$ occurs.

Now, let us work out the details. Rather than doing the proof for $T(0,x)$ with a general $x \in \mathbb{Z}^d$, we use $x = e_1$, the general case being completely analogous. Suppose that the origin belongs to the interior of the convex hull of $\{\Delta_\omega : \omega \in \mathcal{G} \cap \operatorname{supp} Q\}$. Then, analogously to the proof of Theorem 1.5 (see Section 3.2), one can split the sphere $\mathbb{S}^{d-1}$ into a finite number (say, m_0) of nonintersecting subsets $\hat{U}_1, \dots, \hat{U}_{m_0}$ and find a finite collection $\hat{\Gamma}_1, \dots, \hat{\Gamma}_{m_0} \subset \mathcal{G}$ having the following properties: for all $i = 1, \dots, m_0$,

(i) there exists $p_1 > 0$ such that $Q(\hat{\Gamma}_i) > p_1$,

(ii) there exists $a_1 > 0$ such that for any $z \in \hat{U}_i$ and any $\omega \in \hat{\Gamma}_i$ we have $z \cdot \Delta_\omega < -a_1$.

Take $A = \{y \in \mathbb{Z}^d : \|y\| \le u \ln n\}$, where u is a (large) constant to be chosen later. Consider the (A, H)-seed with $H_x, x \in A$ defined as follows. First, put $H_0 = \mathcal{G}$; for $x \neq 0$, let i_0 be such that $\frac{x}{\|x\|} \in \hat{U}_{i_0}$ (note that i_0 is uniquely defined), then put $H_x = \hat{\Gamma}_{i_0}$. Clearly

$$\mathbb{P}[\text{there is } (A, H)\text{-seed in } y] \ge p_1^{(2u)^d \ln^d n}. \tag{39}$$

Note that for any possible environment inside the (A, H)-seed there is no branching. This means that the process restricted on A is a random walk (without branching), which will be denoted by ξ_n.

Analogously to (20), we can prove that there exist $a_2 > 0$, $C_1 \ge 0$ such that

$$\mathrm{E}_\omega(\|\xi_{n+1}\| - \|\xi_n\| \mid \xi_n = z) < -a_2 \tag{40}$$

for all $z \in A \setminus \{y : \|y\| \le C_1\}$, provided there is an (A, H)-seed in 0. Let $\tilde{\tau}$ be the hitting time of the set $(\mathbb{Z}^d \setminus A) \cup \{y : \|y\| \le C_1\}$ by ξ_n. Next, we prove that, when $a_3 > 0$ is small enough, the process $e^{a_3\|\xi_{n\wedge\tilde{\tau}}\|}$ is a supermartingale. Indeed, first, note that there exist $C_2, C_3 > 0$ such that

$$e^x < 1 + x + C_2 x^2 \tag{41}$$

when $|x| < C_3$. We can choose a_3 small enough so that $|\|\xi_{n+1}\| - \|\xi_n\|| < C_3/a_3$ a.s. From (41) we obtain

$$\begin{aligned}\mathrm{E}_\omega\big(e^{a_3\|\xi_{n+1}\|} - e^{a_3\|\xi_n\|} \mid \xi_n = z\big) &= e^{a_3\|z\|}\mathrm{E}_\omega\big(e^{a_3(\|\xi_{n+1}\| - \|\xi_n\|)} - 1 \mid \xi_n = z\big)\\ &\le e^{a_3\|z\|}(-a_3 a_2 + C_2 a_3^2 L_0^2)\\ &< 0\end{aligned}$$

if a_3 is small enough, so $e^{a_3\|\xi_{n\wedge\tilde{\tau}}\|}$ is indeed a supermartingale.

Now, we need to make two observations concerning the exit probabilities. First, consider any y such that $C_1 + 1 \le \|y\| < C_1 + 2$. If $\hat{p}_1$ is the probability that, starting from y, the random walk ξ_n hits the set $\mathbb{Z}^d \setminus A$ before the set $\{y : \|y\| \le C_1\}$, then it is straightforward to obtain from the optional stopping theorem that

$$e^{a_3(C_1+2)} \ge \mathrm{E}_\omega e^{a_3\|\xi_0\|} \ge \mathrm{E}_\omega e^{a_3\|\xi_{\tilde{\tau}}\|} \ge \hat{p}_1 e^{a_3 u \ln n},$$

so

$$\hat{p}_1 \le \frac{e^{a_3(C_1+2)}}{n^{a_3 u}}. \tag{42}$$

Second, suppose now that the random walk ξ_n starts from a point y with $\|y\| = u \ln n$ (i.e., on the boundary of A). Analogously, using the optional stopping theorem, one can show that, with probability bounded away from 0, the random walk hits the set $\{y : \|y\| \le C_1\}$ before stepping out of A. Now, suppose that $u > a_3^{-1}$, and that there is an (A, H)-seed centered at $(-u \ln n)e_1$ (i.e., touching the origin; cf. Figure 5). Using the previous observation together with (42), one

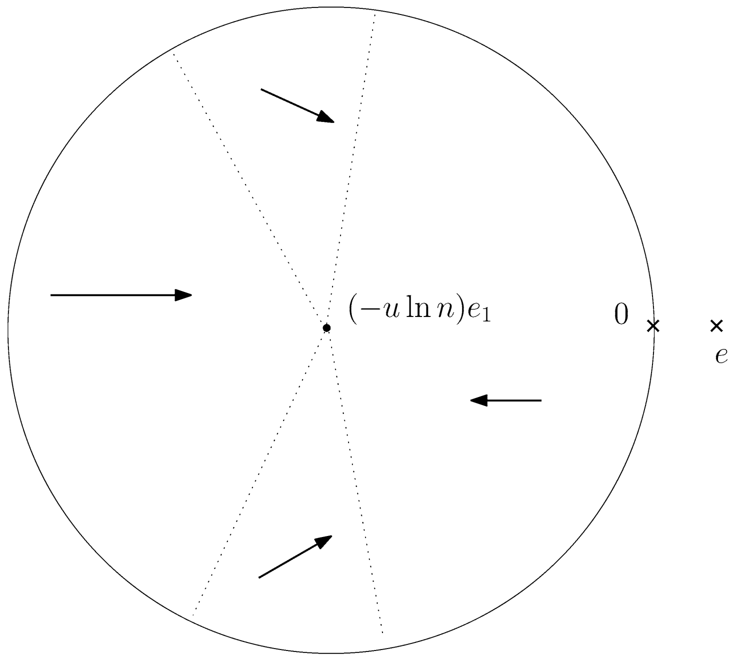

FIG. 5. *Construction of a trap.*

can obtain that, with probability bounded away from 0, the particle will go to the set $(-u \ln n)e_1 + A$ and will stay there until time n [without generating any other particles, since there is no branching in the sites of (A, H)-seed]. So, by (39),

$$\mathbf{P}[T(0, e_1) > n] \geq e^{-C_4 \ln^d n},$$

thus completing the proof of Theorem 1.9. □

5. Proof of Theorem 1.10. We prove this theorem separately for two cases: $d \geq 2$ and $d = 1$. There are, essentially, two reasons for splitting the proof into these two cases. First, as usual, in dimension 1 we have to care about only one (well, in fact, two) directions of growth, while for $d \geq 2$ there are infinitely many possible directions. So, one may think that the proof for $d = 1$ should be easy when compared to the proof for $d \geq 2$. For the majority of growth models this is indeed true, but not for the model of the present paper. This comes from Theorems 1.8, 1.9 and Example 5: recurrence implies that the annealed expectation of the hitting time is finite only for $d \geq 2$, but not for $d = 1$.

5.1. *Case $d \geq 2$.* First, we need to show that the sets of interest grow at least linearly. Recall the notation $\mathcal{K}_n = [-n, n]^d$ and Condition A given before the theorem.

LEMMA 5.1. *Suppose that $d \geq 2$ and the branching random walk in random environment is recurrent and Condition* UE *holds. Then*:

(i) *There exist $\delta_0, \theta_0 > 0$ such that*

$$\mathbf{P}[\mathcal{K}_{\delta_0 n} \subset B_n^0] \geq 1 - \exp\{-\theta_0 \ln^d n\} \tag{43}$$

for all n sufficiently large.

(ii) *Suppose, in addition, that Condition* A *holds. Then there exist $\delta_1, \theta_1 > 0$ such that*

$$\mathbf{P}[\mathcal{K}_{\delta_1 n} \subset \tilde{B}_n^0] \geq 1 - \exp\{-\theta_1 \ln^d n\} \tag{44}$$

for all n sufficiently large.

PROOF. We will use the notation from the proof of Theorem 1.8.

Step 1. Let us prove part (ii) first. To do that, we need to examine in more detail the supercritical Galton–Watson process arising in seeds centered in the points of S_ω. Specifically, we need more information about how (conditioned on survival) the particles of that process are distributed in time. As we have seen before, in that Galton–Watson process a particle has one offspring with probability $(1-\varepsilon)\rho + 2\varepsilon\rho(1-\rho)$, two offspring with probability $\varepsilon\rho^2$ and zero offspring with the remaining probability, and the parameters ε, ρ are such that $(1-\varepsilon)\rho + 2\varepsilon\rho(1-\rho) + 2\varepsilon\rho^2 > 1$, so the process is uniformly supercritical. Moreover, the real time interval between the particle and its offspring is not larger than t_0; however, the exact distribution of this time interval is unknown. So, let us suppose that if a particle has one offspring, then it reappears in the center of the seed after k time units with probability q_k, $k = 1, \dots, t_0$, and if a particle has two offspring, then they reenter the center after i and j time units with probability $q_{i,j}$, $i, j = 1, \dots, t_0$, $i \leq j$ (we have $\sum_k q_k = 1$ and $\sum_{i,j} q_{i,j} = 1$).

Supposing for a moment that the Galton–Watson process starts from one particle at time 0, denote by $\zeta(n)$ the number of particles of that process at time n. We are going to prove that, conditioned on survival, $\sum_{i=nt_0}^{(n+1)t_0} \zeta(i)$ grows rapidly in n. To do so, we construct two processes $Z_n^i, \hat{Z}_n^i$, $n = 0, 1, 2, \dots, i = 0, \dots, t_0 - 1$. We start by defining $\hat{Z}_0^i = 0$ for all i, $Z_0^i = 0$ for $i = 1, \dots, t_0 - 1$, and $Z_0^0 = 1$. Inductively, suppose that the processes $Z, \hat{Z}$ are constructed up to n. Suppose, for example, that $Z_n^{i_0} = a > 0$; this means that there are a particles of Z in the center of the seed at time $i_0 + nt_0$. For each of those a particles, do the following:

- let it generate its offspring according to the rules of the Galton–Watson process; those offspring reenter the center either during the time interval $[nt_0, (n+1)t_0)$, or during $[(n+1)t_0, (n+2)t_0)$;
- for those offspring that appeared in the center of the seed during the interval $[nt_0, (n+1)t_0)$, repeat the above step.

Doing that, we obtain a cloud of free particles (again, in the sense that one cannot be descendant of another) in the interval $[(n+1)t_0, (n+2)t_0)$. Fix a parameter $h>0$ and declare each of those particles to be of type 1 with probability $1-h$ and to be of type 2 with probability h, independently. Repeat the same procedure for all $i_0 \in \{0,\dots,t_0-1\}$ (note that the particles from $\hat{Z}$ are not used in this construction). Then, define Z^i_{n+1} to be the number of type 1 particles at the moment $i+(n+1)t_0$, and $\hat{Z}^i_{n+1}$ to be the number of type 2 particles at the same moment, $i=0,\dots,t_0-1$. Then, $Z_n=(Z^0_n,\dots,Z^{t_0-1}_n)$ is a multitype branching process with t_0 types. Furthermore, it is straightforward to see that if h is small enough, then the mean number of particles (of all types) generated by a particle from Z_n is greater than 1, so that process is supercritical (this follows from, e.g., Theorem 2 of Section 3 of Chapter V of [2], noting also that if, for a nonnegative matrix, the sum of entries is strictly greater than 1 for each row, then the maximum eigenvalue of that matrix is strictly greater than 1). That is, with positive probability the size of nth generation of Z grows exponentially in n. From that it is quite elementary to obtain that there exist constants $\gamma_2, p_2>0$, $\alpha_2>1$ (depending on q_i's and $q_{i,j}$'s) such that $|\hat{Z}_n|>\gamma_2\alpha_2^n$ for all n with probability at least p_2, where $|\hat{Z}_n|=\hat{Z}^0_n+\cdots+\hat{Z}^{t_0-1}_n$. In fact, we think that with some more effort one should be able to prove that these constants can be chosen uniformly in q_i's and $q_{i,j}$'s; however, it is easier to proceed as follows. Clearly, there are $\gamma_3, p_3>0$, $\alpha_3>1$ (*not* depending on q_i's and $q_{i,j}$'s) such that

$$\mathbb{P}\big[\mathtt{P}^0_\omega[|\hat{Z}_n|>\gamma_3\alpha_3^n]\geq p_3, \text{ for all } n \,|\, 0\in S_{\boldsymbol{\omega}}\big]\geq \tfrac{1}{2}. \tag{45}$$

Now, we recall the aperiodicity Condition A. Essentially, it says that the density of the aperiodic sites is positive, where by "aperiodic site" we mean the following: for a given $\boldsymbol{\omega}$, $x\in\mathbb{Z}^d$ is an aperiodic site if there exists y such that $\|x-y\|_1$ is even, and a particle in x sends at least one offspring to y with a positive $\mathtt{P}_\omega$-probability. We need Condition A here because without the aperiodic sites the process would live on even sites at even moments of time and on odd sites at odd moments of time.

For any $z\in S_{\boldsymbol{\omega}}$ and $\hat{Z}$ the process defined above starting from z, define the event

$$E_z=\{|\hat{Z}_n|>\gamma_3\alpha_3^n, \text{ for all } n\}.$$

Define

$$M'_n=\{\forall y\in\mathcal{K}_{L_0 n\ln^{-1}n}\ \exists z\in S_{\boldsymbol{\omega}}:\mathtt{P}^z_\omega[E_z]\geq p_3, \|y-z\|_\infty\leq\alpha\ln n,$$
$$\text{and there is an aperiodic site } x_1 \text{ such that } x_1+\mathfrak{A}\subset\mathcal{K}_{L_0 n\ln^{-1}n}\}.$$

Due to (45), an estimate similar to (30) holds for $\mathbb{P}[(M'_n)^c]$. As in the proof of Theorem 1.8 between (30) and (35), one can prove that, with overwhelming $\mathtt{P}^0_\omega$-probability when $\boldsymbol{\omega}\in M'_n$, before time $n\ln^{-1}n$ the random walk ξ_n will meet a seed (centered, say, in z_0) where an "explosion" (i.e., the event E_{z_0}) happens.

Suppose that t_1 is the moment when the Galton–Watson process in z_0 starts; we have $t_1 \le n \ln^{-1} n$.

Now, take an arbitrary $m \ge n$ and suppose that $\boldsymbol{\omega} \in M'_n$. Supposing that n is large enough and δ_1 is small enough, there exists k_0 such that

$$\big[t_1 + k_0 t_0, t_1 + (k_0+1)t_0\big) \subset [m - 4\delta_1 n, m - 3\delta_1 n]$$

and $k_0 \ge \frac{m}{2t_0}$. Then, since the event E_{z_0} occurs, there exists $t_2 \in \{0, \dots, t_0 - 1\}$ such that $\hat{Z}_{k_0}^{t_2} \ge \gamma_3 \alpha_3^{k_0}/t_0$; that is, there are at least $\gamma_3\alpha_3^{k_0}/t_0$ "unused particles" (they were not used in the construction of the branching processes, so we do not have any information about their future) at z_0 at the moment $t_1 + k_0 t_0 + t_2$. Take any $x_0 \in \mathcal{K}_{\delta_1 n}$ and suppose, for definiteness, that $\|x_0 - z_0\|_1$ is odd. Denote $\hat{t} = m - (t_1 + k_0 t_0 + t_2)$ (notice that $3\delta_1 n \le \hat{t} \le 4\delta_1 n$) and consider two cases:

Case 1: $\hat{t}$ is odd.

Then, by Condition UE, any particle in z_0 will send a descendant to x_0 in time *exactly* $\hat{t}$ with probability at least $\varepsilon_0^{\hat{t}}$.

Case 2: $\hat{t}$ is even.

Here we will have to use the fact that on M'_n there exists an aperiodic site somewhere in $\mathcal{K}_{L_0 n \ln^{-1} n}$. That is, when going from z_0 to x_0 in time $\hat{t}$, on the way we pass through the aperiodic site, and this happens with probability at least $C_6 \varepsilon_0^{\hat{t}-\ell}$.

So, in both cases we see that a particle in z_0 will send a descendant to x_0 in time $\hat{t}$ with probability at least $C_7 \varepsilon_0^{\hat{t}}$ for some $C_7 > 0$. Recall that we dispose of at least $\gamma_3 \alpha_3^{k_0}/t_0$ independent particles in z_0, so the probability that at least one particle will be in x_0 at time m is at least

$$\begin{aligned}1 - (1 - C_7 \varepsilon_0^{\hat{t}})^{\gamma_3 \alpha_3^{k_0}/t_0} &\ge 1 - (1 - C_7 \varepsilon_0^{4\delta_1 n})^{\gamma_3 t_0^{-1} \alpha_3^{m/(2t_0)}} \\ &\ge 1 - \exp\biggl\{-\frac{C_8 \gamma_3}{t_0} \exp\biggl[\frac{\ln \alpha_3}{2t_0} m - 4\delta_1 n \ln \varepsilon_0^{-1}\biggr]\biggr\}.\end{aligned}$$

Choosing δ_1 small enough, it is straightforward to complete the proof of part (ii).

Step 2. As for part (i), it can be proved analogously to part (ii), by writing

$$\{T(0,x) \le n\} \supset \big(\{T(0,x) = n\} \cup \{T(0,x) = n-1\}\big)$$

and noting that for handling of one of the events in the right-hand side of the above display Condition A is unnecessary. The proof of Lemma 5.1 is completed. □

Consider any $x_0 \in \mathbb{Z}^d \setminus \{0\}$, and define a family of random variables

$$Y^{x_0}(m,n) = T(mx_0, nx_0), \qquad 0 \le m < n.$$

Let us point out that the sequence of random variables $(Y^{x_0}(n-1,n), n = 1, 2, 3, \dots)$ is in general *not* stationary (although they are of course identically distributed). To see this, note that, conditioned on $\boldsymbol{\omega}$, the random variables $T(0, x_0)$

and $T(x_0, 2x_0)$ are independent [because (recall the construction of Section 2.3), given $T(0, x_0) = r$, the random variable $T(x_0, 2x_0)$ depends on $\upsilon^{x,i}(n)$ for $n \geq r$], while $T(x_0, 2x_0)$ and $T(2x_0, 3x_0)$ need not be so. Nevertheless, we will prove that the above sequence satisfies the strong law of large numbers:

LEMMA 5.2. *Denote $\beta_{x_0} = \mathbf{E}Y^{x_0}(0,1)$. Then for any $\varepsilon > 0$ there exists $\theta_2 = \theta_2(\varepsilon)$ such that*

$$\mathbf{P}\left[\left|\frac{1}{n}\sum_{i=1}^{n} Y^{x_0}(i-1,i) - \beta_{x_0}\right| > \varepsilon\right] \leq \exp\{-\theta_2 \ln^d n\} \tag{46}$$

for all n. In particular,

$$\frac{1}{n}\sum_{i=1}^{n} Y^{x_0}(i-1,i) \longrightarrow \beta_{x_0} \qquad \mathbf{P}\text{-a.s. and in } L^p,\ p \geq 1. \tag{47}$$

PROOF. Abbreviate $Y_i := Y^{x_0}(i-1,i)$ and introduce the events $G_i = \{Y_i < \sqrt{n}/(2L_0)\}$, $i = 1, \dots, n$. Suppose for simplicity that $\sqrt{n}$ is integer, the general case can be treated analogously. Define the events

$$F = \left\{\left|\frac{1}{n}\sum_{i=1}^{n} Y_i - \beta_{x_0}\right| > \varepsilon\right\},$$

$$F_i = \left\{\left|\frac{1}{\sqrt{n}}\sum_{j=1}^{\sqrt{n}} Y_{i+(j-1)\sqrt{n}} - \beta_{x_0}\right| > \varepsilon\right\},$$

$i = 1, \dots, \sqrt{n}$; since

$$\frac{1}{n}\sum_{i=1}^{n} Y_i = \frac{1}{\sqrt{n}}\sum_{i=1}^{\sqrt{n}}\sum_{j=1}^{\sqrt{n}} Y_{i+(j-1)\sqrt{n}},$$

we can write

$$F \subset \bigcup_{i=1}^{\sqrt{n}} F_i. \tag{48}$$

Now, to bound from above the probability of a single event F_i, we write

$$\begin{aligned}
\mathbf{P}[F_i] &\leq \mathbf{P}\left[\left|\frac{1}{\sqrt{n}}\sum_{j=1}^{\sqrt{n}} Y_{i+(j-1)\sqrt{n}} \mathbf{1}_{G_{i+(j-1)\sqrt{n}}} - \beta_{x_0}\right| > \varepsilon\right] \\
&\qquad + \mathbf{P}\bigl[\text{there exists } j \leq \sqrt{n} \text{ such that } G^c_{i+(j-1)\sqrt{n}} \text{ occurs}\bigr] \\
&=: I_1 + I_2.
\end{aligned}$$

By Theorem 1.8, with some $C_9 > 0$

$$I_2 \le \sqrt{n}\mathbf{P}[G_1^c] \le \sqrt{n}\exp\{-C_9 \ln^d n\}. \tag{49}$$

To bound the term I_1, we note first that it is elementary to obtain from Theorem 1.8 that, for some C_{10},

$$\beta_{x_0} - \mathbf{E}Y_1\mathbf{1}_{G_1} \le \exp\{-C_{10} \ln^d n\}. \tag{50}$$

The key point here is that the random variables $Y_{i+(j_1-1)\sqrt{n}}\mathbf{1}_{G_{i+(j_1-1)\sqrt{n}}}$ and $Y_{i+(j_2-1)\sqrt{n}}\mathbf{1}_{G_{i+(j_2-1)\sqrt{n}}}$ are independent when $j_1 \neq j_2$. Indeed, on the event

$$\left\{\max\{T((n_1-1)x_0, n_1x_0), T((n_2-1)x_0, n_2x_0)\} < \frac{|n_2-n_1|}{2L_0}\right\},$$

the random variables $T((n_1-1)x_0, n_1x_0)$ and $T((n_2-1)x_0, n_2x_0)$ are functions of $v^{x,i}(n)$'s where the superscript x belongs to nonintersecting subsets of $\mathbb{Z}^d$. Therefore, having in mind (50) and Theorem 1.8, to bound the term I_1 from above we can use some large deviation result for the sums of i.i.d. random variables without exponential moments (use, e.g., Corollary 1.11 from [22] with $x = \frac{\varepsilon\sqrt{n}}{2}$, $y = n^{1/4}$, noting also that, since the number of terms here is $\sqrt{n}$, the quantity B_n^2 of [22] is $O(\sqrt{n})$) to obtain that

$$I_1 < \exp\{-C_{11} \ln^d n\}. \tag{51}$$

Using (49), (51) and (48), we conclude the proof of (46). Since (47) follows from (46) immediately for $p = 1$, the proof of Lemma 5.2 is finished in this case. To extend it to a general p, it suffices to note that for all $p' \ge 1$,

$$\left(\frac{1}{n}\sum_{i=1}^n Y^{x_0}(i-1,i)\right)^{p'} \le \frac{1}{n}\sum_{i=1}^n (Y^{x_0}(i-1,i))^{p'},$$

which has a finite expectation. $\square$

To proceed with the proof of Theorem 1.10, we state the result of [18], which is an improved version of Kingman's subadditive ergodic theorem [16].

THEOREM 5.3. *Suppose that $\{Y(m,n)\}$ is a collection of positive random variables indexed by integers satisfying $0 \le m < n$ such that*:

(i) $Y(0,n) \le Y(0,m) + Y(m,n)$ *for all* $0 \le m < n$ (*subadditivity*);

(ii) *the joint distribution of* $\{Y(m+1, m+k+1), k \ge 1\}$ *is the same as that of* $\{Y(m, m+k), k \ge 1\}$ *for each* $m \ge 0$;

(iii) *for each* $k \ge 1$ *the sequence of random variables* $\{Y(nk, (n+1)k), n \ge 0\}$ *is a stationary ergodic process*;

(iv) *the expectation of* $Y(0,1)$ *is finite.*

Then

$$\lim_{n\to\infty} \frac{Y(0,n)}{n} \to \gamma \qquad a.s.,$$

where

$$\gamma = \inf_{n\geq 0} \frac{\mathbf{E}Y(0,n)}{n}.$$

Similarly to the proof of a number of other shape results, our original intention was to apply Theorem 5.3 to the family $(Y^{x_0}(m,n), 0 \leq m < n)$. Indeed, assumption (i) of Theorem 5.3 holds due to Lemma 2.7, from the construction of the random variables $T(\cdot,\cdot)$ it is elementary to observe that assumption (ii) holds as well, and assumption (iv) follows from Theorem 1.8. However, as we observed just before Lemma 5.2, the sequence of random variables in (iii) need not be stationary (even though it has good mixing properties and the random variables there are equally distributed). So, we take a slightly different route: consider the proof of Theorem 5.3 (here we use the proof of Theorem 2.6 of Chapter VI of [19]), and follow its steps carefully. One sees that assumption (iii) (which is assumption (b) in Theorem 2.6 of Chapter VI of [19]) is used only in (2.11) and between the displays (2.14) and (2.15) of Chapter VI of [19] to prove that a certain sequence converges a.s. and in L^1 to its mean. So, we can state the following extension of the standard subadditive ergodic theorem.

THEOREM 5.4. *Theorem 5.3 remains valid if assumption* (iii) *is substituted by the following one*:

(iii)′ *for all $k \geq 1$ there exists a constant $\gamma^{(k)}$ such that the sequence of random variables*

$$\left(\frac{1}{n+1}\sum_{j=0}^{n} Y\big(jk,(j+1)k\big)\right)_{n\geq 0}$$

converges almost surely and in L^1 to $\gamma^{(k)}$.

Now, let us notice that, in our situation, condition (iii)′ holds due to Lemma 5.2 [note also that $Y^{x_0}(mk,nk) = Y^{kx_0}(m,n)$].

From the above argument we conclude that for any $x \in \mathbb{Z}^d \setminus \{0\}$ there exists a number $\mu(x)$ (depending also on Q) such that

$$\frac{T(0,nx)}{n} \longrightarrow \mu(x) \qquad \mathbf{P}\text{-a.s., } n\to\infty. \tag{52}$$

From this point on, the proof of the shape result for B_n^0 becomes completely standard, so we only briefly outline the main steps and refer to, for example, [1, 5, 9] for details:

- It is easy to obtain that for any $x \in \mathbb{Z}^d$, $a \in \mathbb{Z}_+$, we have $\mu(ax) = a\mu(x)$.
- Using that, $\mu(x)$ is first extended on $x \in \mathbb{R}^d$ with rational coordinates (if $x \in \mathbb{Q}^d$ and $ax \in \mathbb{Z}^d$, with $a \in \mathbb{Z}_+$, then $\mu(x) := \frac{\mu(ax)}{a}$), and then, using subadditivity and part (i) of Lemma 5.1, to the whole $\mathbb{R}^d$.
- The limiting shape B is then identified by $B = \{x \in \mathbb{R}^d : \mu(x) \le 1\}$ [note that B is convex since the subadditivity property $\mu(x+y) \le \mu(x) + \mu(y)$ is preserved; however, B need not be symmetric, since generally $\mu(x)$ need not be equal to $\mu(-x)$].
- To complete the proof (for B_n^0), cover B and a sufficiently large annulus of B by balls of radius δ', where δ' is sufficiently small, and then use part (i) of Lemma 5.1.

To complete the proof of Theorem 1.10 (for dimension $d \ge 2$), we recall the relation $\tilde{B}_n^x \subset \bar{B}_n^x \subset B_n^x$, so all we need to prove is that for any $\varepsilon > 0$, $(1-\varepsilon)B \subset \mathfrak{F}(\tilde{B}_n^0)$ for all n large enough. This follows easily from part (ii) of Lemma 5.1 and the corresponding shape result for B_n^0.

5.2. *Case* $d = 1$. As noticed in the beginning of Section 5, here we cannot guarantee that $\mathbf{E}T(0,1) < \infty$ (although it may be so), so we need to develop a different approach. On the other hand, still a number of the steps of the proof for $d = 1$ will be quite analogous to the corresponding steps of the proof for $d \ge 2$; in such cases we will prefer to refer to the case $d \ge 2$ rather than writing down a similar argument once again.

The main idea of the proof of Theorem 1.10 in the case $d = 1$ is the following. From the proofs of Theorems 1.8 and 1.9 we saw that, while usually $\mathtt{P}_\omega[T(0,1) > n]$ is well behaved [and, in particular, $\mathtt{E}_\omega T(0,1) < \infty$], there are some "exceptional" environments that may cause $\mathbf{E}T(0,1) = \infty$ in dimension 1 (see Example 5). So, if the environment is "atypical," instead of starting with one particle, we start with a number of particles depending on the environment (and the more atypical is the environment, the larger is that number).

For the sake of simplicity, we suppose now that the maximal jump L_0 is equal to 1; afterward we explain how to deal with general L_0.

Keeping the notation $S_{\boldsymbol{\omega}}$ from the proof of Theorem 1.8, we note that the set $S_{\boldsymbol{\omega}}$ has positive density in $\mathbb{Z}$, so there exists (small enough) γ_1 such that an interval of length k contains at least $\gamma_1 k$ good seeds from $S_{\boldsymbol{\omega}}$ with $\mathbb{P}$-probability at least $1 - e^{-C_1 k}$. Let us say that an interval is *nice*, if (k being its length) it contains at least $\gamma_1 k$ good seeds from $S_{\boldsymbol{\omega}}$.

Fix $r < C_1$ (e.g., $r := C_1/2$) and define

$$h^x(\boldsymbol{\omega}) = \min\{m : \text{ all the intervals of length } k \ge m \text{ intersecting with } x + [-e^{rk}, e^{rk}] \text{ are nice}\};$$

with this choice of r it is elementary to obtain that there exists $C_2 > 0$ such that

$$\mathbb{P}[h^0(\boldsymbol{\omega}) = n] \le e^{-C_2 n}. \tag{53}$$

Now, suppose that, instead of starting with one particle, the process starts with $e^{Kh^0(\boldsymbol{\omega})}$ particles in 0, where K is a (large) constant to be chosen later. For $\ell \geq 1$, define

$$\tilde{T}(0,\ell) = \min\{n \geq 0 : \eta_n^0(\ell) \geq e^{Kh^\ell(\boldsymbol{\omega})}\},$$

that is, $\tilde{T}(0,\ell)$ is the first moment when we have at least $e^{Kh^\ell(\boldsymbol{\omega})}$ particles in ℓ. Now, our goal is to prove that if K is large enough, then $\mathbf{E}\tilde{T}(0,1) < \infty$. Denote $Z = h^0(\boldsymbol{\omega}) \vee h^1(\boldsymbol{\omega})$, and write

$$\begin{aligned} \mathbf{E}\tilde{T}(0,1) &= \mathbb{E}\mathrm{E}_\omega \tilde{T}(0,1) \\ &\leq \sum_{m=1}^{\infty} \Bigl(\sup_{\boldsymbol{\omega}: Z=m} \mathrm{E}_\omega \tilde{T}(0,1)\Bigr)\mathbb{P}[Z=m]. \end{aligned} \tag{54}$$

Let us obtain an upper bound on the supremum in the right-hand side of (54). Fix $m \geq 1$ and let us consider an environment $\boldsymbol{\omega}$ such that $Z = m$ [so that $h^0(\boldsymbol{\omega}) \leq m$]. First, we prove the estimate (55) below, in the following way:

(i) Consider the time interval $[0, \theta_0 m]$, where $\theta_0 = \frac{K}{2\ln \varepsilon_0^{-1}}$. Each particle that is initially in the origin (even if it does not generate new offspring) will cover the box $[0, \theta_0 m] \subset \mathbb{Z}$ by time $\theta_0 m$ (simply by going always one unit to the right), with probability at least $\varepsilon_0^{\theta_0 m}$ (ε_0 is from Condition UE). Recall that initially we had $e^{Kh^0(\boldsymbol{\omega})}$ particles in 0; since $K > \theta_0 \ln \varepsilon_0^{-1}$, there exists C_3 such that, with probability at least $1 - e^{-C_3 m}$ all the sites of the box $[0, \theta_0 m]$ will be visited by time $\theta_0 m$.

(ii) By definition of the quantity $h^0(\boldsymbol{\omega})$, the box $[0, \theta_0 m]$ contains at least $\theta_0 \gamma_1 m$ good seeds from $S_{\boldsymbol{\omega}}$ (here we suppose that $\theta_0 > 1$, i.e., $K > 2\ln \varepsilon_0^{-1}$). Since all of them were visited, there will be an explosion in at least one of these seeds with probability at least $1 - e^{-C_4 \theta_0 m}$.

(iii) Now, we only have to wait $C_5 m$ (where C_5 is a [large] constant depending on K) time units more to be able to guarantee that at least $e^{Kh^1(\boldsymbol{\omega})}$ particles will simultaneously be in the site 1 at some moment from the time interval $[\theta_0 m, \theta_0 m + C_5 m]$ (to see that, use an argument of the type "if the number of visits to the site 1 during the time interval of length n_1 was at least n_2, then at some moment at least n_2/n_1 particles were simultaneously in that site"). So, finally one can obtain that there exist C_6, θ_1 (depending on K) such that

$$\mathrm{P}_\omega[\tilde{T}(0,1) > C_6 m] < e^{-\theta_1 m}, \tag{55}$$

and the crucial point is that θ_1 can be made arbitrarily large by enlarging K. So, choose K in such a way that $\theta_1 > 5\ln \varepsilon_0^{-1}$.

Next, the goal is to obtain an upper estimate on $\mathrm{P}_\omega[\tilde{T}(0,1) > n]$ which does not depend on K. Specifically, we are going to prove that, for some positive constants

C_7, C_8, we have, on $\{\boldsymbol{\omega} : h^0(\boldsymbol{\omega}) \vee h^1(\boldsymbol{\omega}) = m\}$ and for large enough m,

$$\mathbb{P}_\omega[\tilde{T}(0,1) > n] \le e^{-C_7 n^{C_8}} \tag{56}$$

for all $n \ge e^{5m \ln \varepsilon_0^{-1}}$.

REMARK. To obtain the estimate (56), we will use only one initial particle in 0; so, the same estimate will be valid for $T(0,1)$, thus giving us the proof of Proposition 1.7.

Now, to prove (56), we proceed in the following way.

(i) Consider one particle starting from the origin. During any time interval of length $\frac{\ln n}{5 \ln \varepsilon_0^{-1}}$ it will cover a space interval of the same length (by going to the right on each step) with probability at least

$$\varepsilon_0^{(\ln n)/(5 \ln \varepsilon_0^{-1})} = n^{-1/5}$$

(note that there is a similar argument in the proof of Theorem 1.8 for the case $d = 1$). So, in time $n^{1/4}$ a single particle will cover an interval of that length with probability at least $1 - e^{-C_9 n^{1/20}}$ (note that these estimates do not depend on $\boldsymbol{\omega}$).

(ii) Abbreviate $r' = \frac{r}{5 \ln \varepsilon_0^{-1}}$ [r is from the definition of $h^x(\boldsymbol{\omega})$]. If $n \ge e^{5m \ln \varepsilon_0^{-1}}$, then all the intervals of length $\frac{\ln n}{5 \ln \varepsilon_0^{-1}}$ intersecting with the interval $[-n^{r'}, n^{r'}]$ are nice (so, in particular, they contain at least one good seed from $S_{\boldsymbol{\omega}}$) on $\{\boldsymbol{\omega} : h^0(\boldsymbol{\omega}) \vee h^1(\boldsymbol{\omega}) = m\}$.

(iii) Consider the time interval $[0, n^{1/2}]$. One of the following two alternatives will happen:

(iii.a) Either some of the particles from the cloud of the offspring of the initial particle will go out of the interval $[-n^{r'}, n^{r'}]$, or

(iii.b) all the offspring of the initial particle will stay in the interval $[-n^{r'}, n^{r'}]$ up to time $n^{1/2}$.

In the case (iii.a), at least $\gamma_1 n^{r'}$ good seeds from $S_{\boldsymbol{\omega}}$ will be visited. In the case (iii.b), argue as follows: we have $n^{1/4}$ time subintervals of length $n^{1/4}$; during each one a good seed will be visited with overwhelming probability. So, with probability greater than $1 - n^{1/4} e^{-C_9 n^{1/20}}$ the number of visits to good seeds will be at least $n^{1/4}$ (and all of these good seeds are in the interval $[-n^{1/2}, n^{1/2}]$).

(iv) Thus, in any case, by time $n^{1/2}$ there will be a polynomial number of visits to good seeds. So, with overwhelming probability one of them will explode and produce enough particles to guarantee that there are at least $e^{C_{10} n}$ particles which were created at distance no more than $n^{1/2}$ from 0 before time $n/2$. Then, it is elementary to obtain that, with overwhelming probability, we will have at least

$e^{C_{11}n}$ particles in site 1, for some $C_{11} > 0$. Since $n \geq e^{5m \ln \varepsilon_0^{-1}}$, this will be enough to make the event $\{\tilde{T}(0,1) \leq n\}$ occur when $C_{11}e^{5m \ln \varepsilon_0^{-1}} > Km$, and the last inequality holds for all, except possibly finitely many, values of m.

Now, we finish the proof of the fact that $\mathbf{E}\tilde{T}(0,1) < \infty$. Write

$$\mathrm{E}_\omega \tilde{T}(0,1) = \sum_{n=0}^{\infty} \mathrm{P}_\omega[\tilde{T}(0,1) > n],$$

and use (56) to bound $\mathrm{P}_\omega[\tilde{T}(0,1) > n]$ for $n \geq e^{5m \ln \varepsilon_0^{-1}}$, and (55) for $n \in [C_6 m, e^{5m \ln \varepsilon_0^{-1}})$. Since $\theta_1 > 5 \ln \varepsilon_0^{-1}$, we obtain that $\mathrm{E}_\omega \tilde{T}(0,1) < C_{12}m + C_{13}$ on $\{\boldsymbol{\omega} : h^0(\boldsymbol{\omega}) \vee h^1(\boldsymbol{\omega}) = m\}$ for m large enough, so using (54) and (53), we conclude the proof of the fact that $\mathbf{E}\tilde{T}(0,1) < \infty$.

The rest of the proof (for $L_0 = 1$) is straightforward. First, we define variables $\tilde{T}(k,m)$, $1 \leq k < m$, repeating the construction of Section 2.3, with the following modifications: the process initiating in k starts from $e^{Kh^k(\boldsymbol{\omega})}$ particles, at the moment [with respect to $\upsilon^{x,i}(n)$] $\tilde{T}(0,k)$ [instead of $T(0,k)$]. Then, it is elementary to see that we still have the subadditivity relation $\tilde{T}(0,m) \leq \tilde{T}(0,k) + \tilde{T}(k,m)$. There is again a problem with the absence of the stationarity for the sequence $\tilde{T}(0,1), \tilde{T}(1,2), \tilde{T}(2,3), \ldots$; this problem can be dealt with in exactly the same way as in Section 5.1.

So, the above arguments show that $\frac{\tilde{T}(0,n)}{n}$ converges to a limit as $n \to \infty$, which immediately implies the shape theorem in dimension 1 (we do not need the analogue of Lemma 5.1 here) for the model starting with $e^{Kh^0(\boldsymbol{\omega})}$ particles from 0.

We now complete the proof of Theorem 1.10 in the case $d = 1$ (and, for now, $L_0 = 1$): it is elementary to obtain that, for a recurrent branching random walk in random environment starting with one particle, for $\mathbb{P}$-almost all $\boldsymbol{\omega}$'s, at some (random) time we will have at least $e^{Kh^0(\boldsymbol{\omega})}$ particles in the origin. Now, it remains only to erase all other particles and apply the above reasoning.

To treat the case of a general $L_0 \geq 1$, we apply the same reasoning as in the proof of Theorem 1.8 for $d = 1$ [viz. instead of (ε, ρ)-good seeds, we consider (ε, ρ, W)-good seeds with $W = \{0, 1, \ldots, L_0 - 1\}$, so that a particle cannot overjump W].

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