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A DEFINITION AND SOME CHARACTERISTIC PROPERTIES OF PSEUDO-STOPPING TIMES

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Dedicated to David Williams

Recently, Williams [*Bull. London Math. Soc.* **34** (2002) 610–612] gave an explicit example of a random time ρ associated with Brownian motion such that ρ is not a stopping time but $\mathbb{E}M_\rho = \mathbb{E}M_0$ for every bounded martingale M. The aim of this paper is to characterize such random times, which we call pseudo-stopping times, and to construct further examples, using techniques of progressive enlargements of filtrations.

1. Introduction. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, and $\rho : (\Omega, \mathcal{F}) \to (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ be a random time. We recall that the space $\mathcal{H}^1$ is the Banach space of (càdlàg) $(\mathcal{F}_t)$-martingales (M_t) such that

$$\|M\|_{\mathcal{H}^1} = \mathbb{E}\Big[\sup_{t\geq 0} |M_t|\Big] < \infty.$$

DEFINITION 1. We say that ρ is a $(\mathcal{F}_t)$-pseudo-stopping time if, for every $(\mathcal{F}_t)$-martingale (M_t) in $\mathcal{H}^1$, we have

$$\mathbb{E}M_\rho = \mathbb{E}M_0. \tag{1.1}$$

REMARK 1. It is equivalent to assume that (1.1) holds for bounded martingales, since these are dense in $\mathcal{H}^1$.

We indicate immediately that a class of pseudo-stopping times with respect to a filtration $(\mathcal{F}_t)$, which are not in general $(\mathcal{F}_t)$-stopping times, may be obtained by considering stopping times with respect to a larger filtration $(\mathcal{G}_t)$ such that $(\mathcal{F}_t)$ is immersed in $(\mathcal{G}_t)$, that is, every $(\mathcal{F}_t)$-martingale is a $(\mathcal{G}_t)$-martingale. This situation is described in [3] and refered to there as the (H) hypothesis. We shall discuss this situation in more detail in Section 3. For now, we give a well-known example: let

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$B_t = (B_t^1, \dots, B_t^d)$ be a d-dimensional Brownian motion, and $R_t = |B_t|$, $t \geq 0$, its radial part; it is well known that

$$(\mathcal{R}_t \equiv \sigma\{R_s, s \leq t\}, t \geq 0),$$

the natural filtration of R, is immersed in $(\mathcal{B}_t \equiv \sigma\{B_s, s \leq t\}, t \geq 0)$, the natural filtration of B. Thus, an example of $(\mathcal{R}_t)$-pseudo-stopping time is

$$T_a^{(1)} = \inf\{t, B_t^1 > a\}.$$

Recently, Williams [20] showed that, with respect to the filtration $(\mathcal{F}_t)$ generated by a one-dimensional Brownian motion $(B_t)_{t\geq 0}$, there exist pseudo-stopping times ρ which are not $(\mathcal{F}_t)$-stopping times. Williams' example is the following: let

$$T_1 = \inf\{t : B_t = 1\} \quad \text{and} \quad \sigma = \sup\{t < T_1 : B_t = 0\};$$

then

$$\rho = \sup\{s < \sigma : B_s = S_s\} \qquad \text{where } S_s = \sup_{u \leq s} B_u$$

is a $(\mathcal{F}_t)$-pseudo-stopping time. This paper has two main aims:

- to understand better the nature of pseudo-stopping times;
- to construct further examples of pseudo-stopping times.

In Section 2 with the help of the theory of progressive enlargements of filtrations, we give some equivalent properties for ρ to be a pseudo-stopping time. We also comment there on the difference between (1.1) and the property

$$\mathbb{E}[M_\infty | \mathcal{F}_\rho] = M_\rho \tag{1.2}$$

for every uniformly integrable $(\mathcal{F}_t)$-martingale (M_t), which was shown by Knight and Maisonneuve [12] to be equivalent to ρ being a $(\mathcal{F}_t)$-stopping time.

In Section 3 we give some other examples of pseudo-stopping times. We associate with the end L of a given $(\mathcal{F}_t)$ predictable set Γ, that is,

$$L = \sup\{t : (t, \omega) \in \Gamma\},$$

a pseudo-stopping time $\rho < L$ in a manner which generalizes Williams' example. We also link the pseudo-stopping times with randomized stopping times.

In Section 4 we give a discrete time analogue of the Williams random time ρ. This approach is based on the analogue of Williams' path decomposition proposed by Le Gall for the standard random walk [13].

2. Some characteristic properties of pseudo-stopping times.

2.1. *Basic facts about progressive enlargements.* We recall here some basic results about the progressive enlargement of a filtration $(\mathcal{F}_t)$ by a random time ρ. All these results may be found in [4, 9, 11, 17, 21].

We enlarge the initial filtration $(\mathcal{F}_t)$ with the process $(\rho \wedge t)_{t\geq 0}$, so that the new enlarged filtration $(\mathcal{F}_t^\rho)_{t\geq 0}$ is the smallest filtration containing $(\mathcal{F}_t)$ and making ρ a stopping time. A few processes will play a crucial role in our discussion:

- the $(\mathcal{F}_t)$-supermartingale

$$Z_t^\rho = \mathbb{P}[\rho > t | \mathcal{F}_t] \tag{2.1}$$

  chosen to be càdlàg, associated to ρ by Azéma (see [9] for detailed references);
- the $(\mathcal{F}_t)$-dual optional and predictable projections of the process $\mathbf{1}_{\{\rho\leq t\}}$, denoted, respectively, by A_t^ρ and a_t^ρ;
- the càdlàg martingale

$$\mu_t^\rho = \mathbb{E}[A_\infty^\rho | \mathcal{F}_t] = A_t^\rho + Z_t^\rho,$$

  which is in BMO$(\mathcal{F}_t)$ (see [4] or [21]). We recall that the space of BMO martingales (see [6] for more details and references) is the Banach space of (càdlàg) square integrable $(\mathcal{F}_t)$-martingales (Y_t) which satisfy

$$\|Y\|_{\mathrm{BMO}}^2 = \mathrm{essup}_T \mathbb{E}[(Y_\infty - Y_{T-})^2 | \mathcal{F}_T] < \infty,$$

  where T ranges over all $(\mathcal{F}_t)$-stopping times.

We also consider the Doob–Meyer decomposition of (2.1):

$$Z_t^\rho = m_t^\rho - a_t^\rho.$$

If ρ avoids any $(\mathcal{F}_t)$-stopping time, that is, to say $P[\rho = T > 0] = 0$ for any stopping time T, then $A_t^\rho = a_t^\rho$ is continuous.

Finally, we recall that every $(\mathcal{F}_t)$-local martingale (M_t), stopped at ρ, is a $(\mathcal{F}_t^\rho)$-semimartingale, with canonical decomposition:

$$M_{t\wedge\rho} = \widetilde{M}_t + \int_0^{t\wedge\rho} \frac{d\langle M, \mu^\rho\rangle_s}{Z_{s-}^\rho}, \tag{2.2}$$

where $(\widetilde{M}_t)$ is an $(\mathcal{F}_t^\rho)$-local martingale.

REMARK 2. We also recall that, in a filtration $(\mathcal{F}_t)$ where all martingales are continuous, $A_t^\rho = a_t^\rho$ since optional processes are predictable (see [18], Chapter IV).

2.2. *A characterization of pseudo-stopping times.* We now discuss some characteristic properties of pseudo-stopping times. We assume throughout that $\mathbb{P}[\rho = \infty] = 0$.

THEOREM 1. *The following four properties are equivalent*:

(1) *ρ is a $(\mathcal{F}_t)$-pseudo-stopping time*, *that is*, (1.1) *is satisfied*;
(2) $\mu_t^\rho \equiv 1$, *a.s.*,
(3) $A_\infty^\rho \equiv 1$, *a.s.*,
(4) *every $(\mathcal{F}_t)$-local martingale (M_t) satisfies*

$$(M_{t\wedge\rho})_{t\geq 0} \text{ is a local } (\mathcal{F}_t^\rho)\text{-martingale}.$$

If, *furthermore*, *all $(\mathcal{F}_t)$-martingales are continuous*, *then each of the preceding properties is equivalent to*
(5)

$$(Z_t^\rho)_{t\geq 0} \text{ is a decreasing } (\mathcal{F}_t) \text{ predictable process}.$$

PROOF. (1) $\Rightarrow$ (2). For every square integrable $(\mathcal{F}_t)$-martingale (M_t), we have

$$\mathbb{E}[M_\rho] = \mathbb{E}\left[\int_0^\infty M_s\, dA_s^\rho\right] = \mathbb{E}[M_\infty A_\infty^\rho] = \mathbb{E}[M_\infty \mu_\infty^\rho].$$

Since $\mathbb{E}M_\rho = \mathbb{E}M_0 = \mathbb{E}M_\infty$, we have

$$\mathbb{E}[M_\infty] = \mathbb{E}[M_\infty A_\infty^\rho] = \mathbb{E}[M_\infty \mu_\infty^\rho].$$

Consequently, $\mu_\infty^\rho \equiv 1$, a.s., hence, $\mu_t^\rho \equiv 1$, a.s., which is equivalent to $A_\infty^\rho \equiv 1$, a.s. Hence, (2) and (3) are equivalent.

(2) $\Rightarrow$ (4). This is a consequence of the decomposition formula (2.2).

(4) $\Rightarrow$ (1). It suffices to consider any $\mathcal{H}^1$-martingale (M_t), which, assuming (4), satisfies $(M_{t\wedge\rho})_{t\geq 0}$ is a martingale in the enlarged filtration $(\mathcal{F}_t^\rho)$. Then, as a consequence of the optional stopping theorem applied in $(\mathcal{F}_t^\rho)$ at time ρ, we get

$$\mathbb{E}[M_\rho] = \mathbb{E}[M_0],$$

hence, ρ is a pseudo-stopping time.

Finally, in the case where all $(\mathcal{F}_t)$-martingales are continuous, we show the following:

(a) (2) $\Rightarrow$ (5). If ρ is a pseudo-stopping time, then Z_t^ρ decomposes as

$$Z_t^\rho = 1 - A_t^\rho.$$

As all $(\mathcal{F}_t)$-martingales are continuous, optional processes are, in fact, predictable, and so (Z_t^ρ) is a predictable decreasing process.

(b) (5) $\Rightarrow$ (2). Conversely, if (Z_t^ρ) is a predictable decreasing process, then, from the unicity in the Doob–Meyer decomposition, the martingale part μ_t^ρ is constant, that is, $\mu_t^\rho \equiv 1$, a.s. Thus, (2) is satisfied. □

In the next proposition, we deal with uniformly integrable martingales (M_t) instead of martingales in $\mathcal{H}^1$ (or $\mathcal{H}^2, \dots$).

PROPOSITION 1. *The following properties are equivalent*:

(1) *ρ is a $(\mathcal{F}_t)$-pseudo-stopping time*;
(2) *for every uniformly integrable martingale*,

$$\mathbb{E}[|M_\rho|] \le \mathbb{E}[|M_\infty|].$$

REMARK 3. In fact, we shall further show in the next proof that, for ρ a pseudo-stopping time and for (M_t) any uniformly integrable martingale,

$$\mathbb{E}[|M_\rho|] < \infty \quad \text{and} \quad \mathbb{E}[M_\rho] = \mathbb{E}[M_\infty].$$

PROOF OF PROPOSITION 1. (1) $\Rightarrow$ (2). If (M_t) is uniformly integrable, it may be decomposed as

$$M_t = M_t^{(+)} - M_t^{(-)}, \tag{2.3}$$

where

$$M_t^{(+)} = \mathbb{E}[M_\infty^+ | \mathcal{F}_t] \quad \text{and} \quad M_t^{(-)} = \mathbb{E}[M_\infty^- | \mathcal{F}_t].$$

[Note that $M_\infty^\pm$ indicate the positive and negative parts of M_∞, whereas $(M_t^{(\pm)})$ are the martingales with terminal values $M_\infty^\pm$.] Thus, to prove (2), it suffices to prove

$$\mathbb{E}[M_\rho] = \mathbb{E}[M_\infty],$$

under the further assumption that $M \ge 0$. In this latter case, we have $M_t = \mathbb{E}[M_\infty | \mathcal{F}_t]$, with $M_\infty \ge 0$. Now let

$$M_t^{(n)} = \mathbb{E}[(M_\infty \wedge n) | \mathcal{F}_t].$$

$(M_t^{(n)})$ is a bounded martingale, hence, we have

$$\mathbb{E}[M_\infty^{(n)}] = \mathbb{E}[M_\rho^{(n)}].$$

Doob's maximal inequality yields

$$\mathbb{P}\Big[\sup_{t \ge 0} (M_t - M_t^{(n)}) > \varepsilon\Big] \le \frac{1}{\varepsilon} \mathbb{E}[M_\infty - M_\infty^{(n)}],$$

so that $(M_\rho^{(n)})$ converges to (M_ρ) in probability; but the sequence $(M_\rho^{(n)})$ is increasing, so it, in fact, converges almost surely. Hence, the monotone convergence theorem yields

$$\mathbb{E}[M_\infty] = \mathbb{E}[M_\rho].$$

Finally, going back to (2.3) in the general case, we obtain

$$\begin{aligned}\mathbb{E}[|M_\rho|] &\le \mathbb{E}\big[M_\rho^{(+)} + M_\rho^{(-)}\big]\\ &= \mathbb{E}[M_\infty^+ + M_\infty^-]\\ &= \mathbb{E}[|M_\infty|].\end{aligned}$$

Hence, (2) holds. Further, we may now write

$$\begin{aligned}\mathbb{E}[M_\rho] &= \mathbb{E}\big[M_\rho^{(+)} - M_\rho^{(-)}\big]\\ &= \mathbb{E}[M_\infty^+ - M_\infty^-]\\ &= \mathbb{E}[M_\infty].\end{aligned}$$

(2) $\Rightarrow$ (1). We need only apply property (2) to any martingale (M_t) taking values in [0, 1]. Thus,

$$\mathbb{E}[M_\rho] \le \mathbb{E}[M_\infty],$$
$$\mathbb{E}[1 - M_\rho] \le \mathbb{E}[1 - M_\infty].$$

But, since the sums on both sides add up to 1, we must have

$$\mathbb{E}[M_\rho] = \mathbb{E}[M_\infty].$$

Hence, ρ is a $(\mathcal{F}_t)$-pseudo-stopping time. $\square$

As an application of the theorem, we can check that in Williams' example, his time ρ associated with a Brownian motion is a pseudo-stopping time. Indeed, the dual predictable (= optional) projection A_t^ρ of $\mathbf{1}_{\{\rho \le t\}}$ is $\max_{s \le t \wedge T_1} B_s$ [19, 20] and $A_\infty^\rho \equiv 1$.

2.3. *Around the result of Knight and Maisonneuve.* We now comment on the statement of the third property in Theorem 1.

For the properties of the different sigma fields $\mathcal{F}_\rho$, $\mathcal{F}_{\rho+}$, $\mathcal{F}_{\rho-}$, associated with a general random time ρ, the reader can consult [19] or [21]. Here, we just recall their definitions:

DEFINITION 2. Three classical σ-fields associated with a filtration $(\mathcal{F}_t)$ and any random time ρ are the following:

$$\mathcal{F}_{\rho+} = \sigma\{z_\rho, (z_t) \text{ any } (\mathcal{F}_t) \text{ progressively measurable process}\};$$
$$\mathcal{F}_{\rho} = \sigma\{z_\rho, (z_t) \text{ any } (\mathcal{F}_t) \text{ optional process}\};$$
$$\mathcal{F}_{\rho-} = \sigma\{z_\rho, (z_t) \text{ any } (\mathcal{F}_t) \text{ predictable process}\}.$$

The result of Knight and Maisonneuve which was recalled in the Introduction may be stated as follows:

THEOREM 2. *If for all uniformly integrable* $(\mathcal{F}_t)$*-martingales* (M_t), *one has*

$$\mathbb{E}[M_\infty|\mathcal{F}_\rho] = M_\rho \qquad \text{on } \{\rho < \infty\},$$

then ρ *is a* $(\mathcal{F}_t)$*-stopping time* (*the converse is Doob's optional stopping theorem*).

Refining slightly the argument in [12], we obtain the following:

THEOREM 3. *If for all bounded* $(\mathcal{F}_t)$*-martingales* (M_t), *one has*

$$\mathbb{E}[M_\infty|\sigma\{M_\rho, \rho\}] = M_\rho \qquad \text{on } \{\rho < \infty\},$$

then ρ *is a* $(\mathcal{F}_t)$*-stopping time.*

PROOF. For $t \geq 0$, we have

$$\mathbb{E}[M_\infty \mathbf{1}_{(\rho \leq t)}] = \mathbb{E}[M_\rho \mathbf{1}_{(\rho \leq t)}] = \mathbb{E}\left[\int_0^t M_s \, dA_s^\rho\right] = \mathbb{E}[M_\infty A_t^\rho].$$

Comparing the two extreme terms, we get

$$\mathbf{1}_{(\rho \leq t)} = A_t^\rho,$$

that is, ρ is a $(\mathcal{F}_t)$-stopping time. □

An interesting open question in view of what has been proved for pseudo-stopping times is whether $\mathbb{E}[M_\infty|M_\rho] = M_\rho$, on $\{\rho < \infty\}$ is equivalent to ρ being a stopping time.

To illustrate the result of Knight and Maisonneuve, we show explicitly how, in the framework of Williams' example, M_ρ and $\mathbb{E}[M_\infty|\mathcal{F}_\rho]$ differ, for

$$M_t = \exp\left(\lambda B_{t \wedge T_1} - \frac{\lambda^2}{2}(t \wedge T_1)\right), \qquad \lambda > 0.$$

We write

$$\begin{aligned} M_\infty &= \exp\left(\lambda - \frac{\lambda^2}{2} T_1\right) \\ &= \exp(\lambda) \exp\left(-\frac{\lambda^2}{2}(\rho + (\sigma - \rho) + (T_1 - \sigma))\right). \end{aligned} \tag{2.4}$$

We now recall Williams' path decomposition results for $(B_u)_{u \leq T_1}$ on the intervals $(0, \rho)$, (ρ, σ), (σ, T_1):

- $(B_{\sigma+u})_{u \leq T_1 - \sigma}$ is a BES(3) process, independent of $\mathcal{F}_\sigma$; hence, we have

$$\mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}(T_1 - \sigma)\right)\Big|\mathcal{F}_\sigma\right] = \mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}(T_1 - \sigma)\right)\right] = \frac{\lambda}{\sinh(\lambda)}.$$

- S_ρ, where $S_s = \sup_{u \leq s} B_u$, is uniformly distributed on $(0, 1)$;

- Conditionally on $S_\rho = h$, the processes $(B_u)_{u \le \rho}$ and $(B_{\sigma-u})_{u \le \sigma-\rho}$ are two independent Brownian motions considered up to their first hitting time of h. Consequently, we have

$$\mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}(\sigma - \rho)\right) \Big| \mathcal{F}_\rho\right] = \exp(-\lambda S_\rho).$$

Plugging this information in (2.4), we obtain

$$\mathbb{E}[M_\infty | \mathcal{F}_\rho] = \exp\left(\lambda(1 - B_\rho) - \frac{\lambda^2}{2}\rho\right)\left(\frac{\lambda}{\sinh(\lambda)}\right),$$

while

$$M_\rho = \exp\left(\lambda B_\rho - \frac{\lambda^2}{2}\rho\right) \tag{2.5}$$

and these two quantities are obviously different.

2.4. *Further properties of pseudo-stopping times.* Besides the assumption that ρ is a $(\mathcal{F}_t)$-pseudo-stopping time, we also make the hypothesis that ρ avoids all $(\mathcal{F}_t)$-stopping times. We saw that, in this case,

$$a_t^\rho = A_t^\rho = 1 - Z_t^\rho$$

is continuous.

For simplicity, we shall write (Z_u) instead of (Z_u^ρ).

PROPOSITION 2. *Under the previous hypotheses, for all uniformly integrable $(\mathcal{F}_t)$-martingales (M_t), and all bounded Borel measurable functions f, one has*

$$\begin{aligned}\mathbb{E}[M_\rho f(Z_\rho)] &= \mathbb{E}[M_0] \int_0^1 f(x)\,dx \\ &= \mathbb{E}[M_\rho] \int_0^1 f(x)\,dx.\end{aligned}$$

REMARK 4. On the other hand, it is not true that

$$\mathbb{E}[M_\infty f(Z_\rho)] = \mathbb{E}[M_\rho f(Z_\rho)], \tag{2.6}$$

for every bounded Borel function f. Indeed, from Proposition 2, the right-hand side of (2.6) is equal to

$$\mathbb{E}\left[M_\infty \int_0^1 f(x)\,dx\right].$$

Thus, our hypothesis (2.6) would imply the absurd equality between $f(Z_\rho)$ and $\int_0^1 f(x)\,dx$.

PROOF OF PROPOSITION 2. Under our assumptions, we have

$$\begin{aligned}\mathbb{E}[M_\rho f(Z_\rho)] &= \mathbb{E}\left[\int_0^\infty M_u f(Z_u)\, dA_u^\rho\right]\\ &= \mathbb{E}\left[\int_0^\infty M_u f(1 - A_u^\rho)\, dA_u^\rho\right]\\ &= \mathbb{E}\left[M_\infty \int_0^\infty f(1 - A_u^\rho)\, dA_u^\rho\right]\\ &= \mathbb{E}\left[M_\infty \int_0^1 f(1-x)\, dx\right]\\ &= \mathbb{E}\left[M_\infty \int_0^1 f(x)\, dx\right]. \qquad \square\end{aligned}$$

Taking $M_t \equiv 1$, we find that (Z_ρ) is uniformly distributed on (0, 1), which is already known [11, 21] since (recalling that Z_u is decreasing)

$$Z_\rho = \inf_{u \le \rho} Z_u.$$

In fact, we have a stronger result: under all changes of probability on $\mathcal{F}_\rho$, of the form

$$d\mathbb{Q} = M_\rho\, d\mathbb{P},$$

where (M_t) is a positive uniformly integrable $(\mathcal{F}_t)$-martingale such that $\mathbb{E}[M_0] = 1$, the law of Z_ρ (is unchanged and) is uniform.

COROLLARY 1. *Under the assumptions of Proposition* 2, *we have*

$$\mathbb{E}[M_\rho | Z_\rho] = \mathbb{E}[M_\rho] = \mathbb{E}[M_0].$$

On the other hand, the quantity $\mathbb{E}[M_\infty | Z_\rho]$ *is not easy to evaluate, as is seen with Williams' example, and is different from* $\mathbb{E}[M_\rho|Z_\rho]$. *Indeed, in this framework and with the already used notation,*

$$\mathbb{E}[M_\infty | Z_\rho] = \exp(\lambda)\mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2} T_1\right)\Big| B_\rho\right].$$

Decomposing again T_1 *as* $T_1 = \rho + (\sigma - \rho) + (T_1 - \sigma)$, *and using Williams path decomposition, we obtain*

$$\begin{aligned}\mathbb{E}[M_\infty | Z_\rho] &= \exp(\lambda)\left(\frac{\lambda}{\sinh(\lambda)}\right)\exp(-\lambda B_\rho)\mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}\rho\right)\Big| B_\rho\right]\\ &= \left(\frac{2\lambda}{1 - \exp(-2\lambda)}\right)\exp(-2\lambda B_\rho).\end{aligned}$$

COROLLARY 2. *The family* $\{M_\rho; M$ *uniformly integrable* $(\mathcal{F}_t)$*-martingale*$\}$ *is not dense in* $L^1(\mathcal{F}_\rho)$.

This negative result led us to look for some representation of the generic element of $L^1(\mathcal{F}_\rho)$ in terms of $(\mathcal{F}_t)$-martingales taken at time ρ on one hand, and the variable Z_ρ, on the other hand.

PROPOSITION 3. (i) *Let* $K:[0,1]\times\mathbb{R}_+\times\Omega\to\mathbb{R}_+$ *be a* $\mathcal{B}_{[0,1]}\otimes\mathcal{P}(\mathcal{F}_\bullet)$ *measurable process, where* $\mathcal{P}(\mathcal{F}_\bullet)$ *denotes the* $(\mathcal{F}_t)$ *predictable* σ*-field on* $\mathbb{R}_+\times\Omega$. *Then*

$$\mathbb{E}[K(1-Z_\rho,\rho)]=\mathbb{E}\left[\int_0^1 dy K(y,\alpha_y)\right], \tag{2.7}$$

where

$$\alpha_y=\inf\{u:A_u^\rho>y\}.$$

(ii) *Let* $(H_u, u\geq 0)$ *be a bounded predictable process. Define a measurable family* $(M_t^y)_{t\geq 0}$ *of martingales through their terminal values*

$$M_\infty^y=H_{\alpha_y}.$$

Then

$$H_\rho=M_\rho^{1-Z_\rho}\qquad a.s.$$

PROOF. (i) This follows from the monotone class theorem, once we have shown

$$\mathbb{E}[f(1-Z_\rho)H_\rho]=\mathbb{E}\left[\int_0^1 dy f(y)H_{\alpha_y}\right] \tag{2.8}$$

for every bounded predictable process H and every Borel bounded function f. But, this identity follows from the fact that $1-Z_\rho=A_\rho$; and so

$$\begin{aligned}\mathbb{E}[f(A_\rho)H_\rho]&=\mathbb{E}\left[\int_0^\infty dA_u f(A_u)H_u\right]\\&=\mathbb{E}\left[\int_0^1 dy f(y)H_{\alpha_y}\right].\end{aligned}$$

We shall prove the second statement by showing that, for every bounded (k_u) predictable process,

$$\mathbb{E}[k_\rho H_\rho]=\mathbb{E}[k_\rho M_\rho^{1-Z_\rho}].$$

From (2.7), we deduce

$$\begin{aligned}\mathbb{E}[k_\rho M_\rho^{1-Z_\rho}] &= \mathbb{E}\left[\int_0^1 dy M_{\alpha_y}^y k_{\alpha_y}\right]\\ &\overset{\text{(a)}}{=} \int_0^1 dy \mathbb{E}[M_\infty^y k_{\alpha_y}]\\ &\overset{\text{(b)}}{=} \int_0^1 dy \mathbb{E}[H_{\alpha_y} k_{\alpha_y}]\\ &\overset{\text{(c)}}{=} \mathbb{E}[k_\rho H_\rho]\end{aligned}$$

[(a) follows from the optional stopping theorem for (M_t^y); (b) follows from the definition of M_∞^y; (c) is another consequence of (2.7)]. Comparing the extreme terms in the above, we get

$$H_\rho = M_\rho^{1-Z_\rho}. \qquad \square$$

3. Some systematic constructions and some examples of pseudo-stopping times.

3.1. *First constructions.* Here we discuss some combinations of several pseudo-stopping times which yield a pseudo-stopping time. Here is a first easy result:

PROPOSITION 4. *Let ρ be a $(\mathcal{F}_t)$-pseudo-stopping time and let τ be a $(\mathcal{F}_t^\rho)$-stopping time. Then $\rho \wedge \tau$ is a $(\mathcal{F}_t)$-pseudo-stopping time.*

PROOF. Let M be any uniformly integrable $(\mathcal{F}_t)$-martingale. We know that $M_{t\wedge\rho}$ is a uniformly integrable martingale in the enlarged filtration $(\mathcal{F}_t^\rho)$ and ρ is a stopping time in this filtration. If τ is also a $(\mathcal{F}_t^\rho)$-stopping time, then so is $\rho \wedge \tau$. Hence, $\mathbb{E}M_{\rho\wedge\tau} = \mathbb{E}M_0$. $\square$

EXAMPLE 1. Let ρ be as in Williams' example. Let $0 < a < 1$, and $T_a = \inf\{t > 0 : B_t = a\}$. Then

$$\rho_a = \rho \wedge T_a, \qquad 0 < a < 1,$$

is an increasing family of pseudo-stopping times.

REMARK 5. As a further comment about Proposition 4, we remark that pseudo-stopping times do not inherit all the "nice" properties of stopping times. As an example, a pseudo-stopping time of a given filtration does not remain, in general, a pseudo-stopping time in a larger filtration, whereas a stopping time does. Indeed, keep the same notation as in Section 2.3 and look at the pseudo-stopping time ρ in the larger filtration $(\mathcal{F}_t^{\sigma})$. Using the computations we have already done in Section 2.3 and the projections formula (see [4], page 186), we get

$$\mathbb{P}[\rho > t | \mathcal{F}_t^{\sigma}] = \frac{1 - \max_{s \le t \wedge T_1} B_s}{1 - B^+_{t \wedge T_1}},$$

which is not decreasing. In fact, any end of predictable set that avoids stopping times is not a pseudo-stopping time. We shall see it in the next subsection.

3.2. *A generalization of Williams' example.* To keep the discussion as simple as possible, we assume that we are working with an original filtration $(\mathcal{F}_t)$ such that:

- All $(\mathcal{F}_t)$-martingales are continuous [e.g., $(\mathcal{F}_t)$ is the Brownian filtration].
- Moreover, we consider L, the end of a $(\mathcal{F}_t)$ predictable set, such that for every $(\mathcal{F}_t)$-stopping time T, $\mathbb{P}[L = T] = 0$.

Under these two conditions, the supermartingale $Z_t = P[L > t | \mathcal{F}_t]$ associated with L is a.s. continuous, and satisfies $Z_L = 1$. Then we let

$$\rho = \sup\left\{ t < L : Z_t = \inf_{u \le L} Z_u \right\}.$$

The following holds:

PROPOSITION 5. (i) $I_L = \inf_{u \le L} Z_u$ *is uniformly distributed on* $[0, 1]$; (*see* [21]).

(ii) *The supermartingale* $Z_t^{\rho} = P[\rho > t | \mathcal{F}_t]$ *associated with* ρ *is given by*

$$Z_t^{\rho} = \inf_{u \le t} Z_u.$$

As a consequence, ρ *is a* $(\mathcal{F}_t)$*-pseudo-stopping time.*

PROOF. (i) Let

$$T_b = \inf\{t, Z_t \le b\}, \qquad 0 < b < 1,$$

then

$$\mathbb{P}[I_L \le b] = \mathbb{P}[T_b < L] = \mathbb{E}[Z_{T_b}] = b.$$

(ii) Note that, for every $(\mathcal{F}_t)$-stopping time T, we have

$$\{T < \rho\} = \{T' < L\},$$

where

$$T' = \inf\left\{ t > T, Z_t \le \inf_{s \le T} Z_s \right\}.$$

Consequently, we have

$$\mathbb{E}[Z_T^\rho] = \mathbb{P}[T < \rho] = \mathbb{P}[T' < L] = \mathbb{E}[Z_{T'}] = \mathbb{E}\Big[\inf_{u \le T} Z_u\Big],$$

which yields

$$\mathbb{E}\big[Z_T^\rho \mathbf{1}_{\{T<\infty\}}\big] = \mathbb{E}\Big[\inf_{u \le T} Z_u \mathbf{1}_{\{T<\infty\}}\Big],$$

since (Z_u^ρ) and (Z_u) converge to 0 as $u \to \infty$. We now deduce the desired result from the optional section theorem. $\square$

In the literature about enlargements of filtrations ([9, 11, 21], etc.), a number of explicit computations of supermartingales associated to various L's have been given. We shall use some of these computations to produce some examples of pseudo-stopping times, with the help of the proposition.

(1) First let us check again that we recover the example of Williams from the proposition. With the notation of the Introduction ($L = \sigma$), it is not hard to see that (see [19])

$$Z_t = 1 - B^+_{t \wedge T_1}.$$

Hence,

$$\rho = \sup\{s < \sigma : B_s = S_s\}.$$

(2) Consider $(R_t)_{t \ge 0}$ a three-dimensional Bessel process, starting from zero, its filtration $(\mathcal{F}_t)$, and

$$L = L_1 = \sup\{t : R_t = 1\}.$$

Then

$$\rho = \sup\Big\{t < L : R_t = \sup_{u \le L} R_u\Big\} \tag{3.1}$$

is a $(\mathcal{F}_t)$-pseudo-stopping time. This follows from the fact that

$$Z_t^L = 1 \wedge \frac{1}{R_t},$$

hence, (3.1) is equivalent to

$$\rho = \sup\Big\{t < L : Z_t^L = \inf_{u \le L} Z_u^L\Big\},$$

and from the above proposition,

$$Z_t^\rho = 1 \wedge \bigg(\frac{1}{\sup_{u \le t} R_u}\bigg).$$

We can generalize further this example by noticing that, for $n > 2$, we have for $(R_t)_{t\geq 0}$ a BES(n), $Z_t^L = 1 \wedge (\frac{1}{R_t})^{n-2}$. More generally, let us consider a transient diffusion (X_t). Let s be a scale function such that $s(-\infty) = 0$ and $s(x) > 0$. Let

$$L_a = \sup\{t; X_t = a\},$$

the last passage time at level a. We have (see [16])

$$Z_t^{L_a} = 1 \wedge \frac{s(X_t)}{s(a)}.$$

Thus,

$$\rho_a = \sup\Big\{t < L_a : s(X_t) = \inf_{u \leq L_a} s(X_u)\Big\}$$

is a pseudo-stopping time in the natural filtration of (X_t). For example, consider the case of a Brownian motion with a negative drift:

$$X_t \equiv x + \mu t + \sigma B_t, \qquad \mu < 0.$$

In this case, the scale function is

$$s(x) = \exp\left(-\frac{2\mu x}{\sigma^2}\right).$$

Hence,

$$\rho_a = \sup\Big\{t < L_a : \mu t + \sigma B_t = \inf_{u \leq L_a} (\mu u + \sigma B_u)\Big\}$$

is a pseudo-stopping time in the natural filtration of (B_t).

(3) Consider $(B_u)_{u\geq 0}$ a one-dimensional Brownian motion, $(\mathcal{F}_t)$ its filtration, and

$$g_t = \sup\{s < t : B_s = 0\},$$

then

$$\rho_t = \sup\left\{s < g_t : \frac{|B_s|}{\sqrt{t-s}} = \sup_{u < g_t} \frac{|B_u|}{\sqrt{t-u}}\right\} \tag{3.2}$$

is a $\mathcal{F}_t$-pseudo-stopping time. Again, this follows from the fact that ρ_t is, in fact, defined from $g_t(= L)$, as in the framework preceding the proposition, since

$$Z_u^{g_t} \equiv \Phi\left(\frac{|B_u|}{\sqrt{t-u}}\right),$$

with $\Phi(x) = \mathbb{P}(|N| \geq x)$, where N is a standard Gaussian variable.

(4) We can reinterpret the previous example via a deterministic time-change. We remark that we can write

$$\frac{B_u}{\sqrt{1-u}} = Y_{\log(1/(1-u))},$$

where $(Y_s)_{s\geq 0}$ is an Ornstein–Uhlenbeck process satisfying

$$Y_s = \beta_s + \tfrac{1}{2}\int_0^s du Y_u.$$

We then deduce form Example 3 that

$$\rho' = \sup\left\{s < L_0' : |Y_s| = \sup_{u\leq L_0'} |Y_u|\right\}$$

is a $(\mathcal{F}_t')$-pseudo-stopping time, where

$$L_0' \equiv \log\left(\frac{1}{1-g_1}\right) = \sup\{s > 0, Y_s = 0\}$$

and $(\mathcal{F}_t^{'})$ is the natural filtration of (Y_t).

As for Williams' example, none of these pseudo-stopping times remains a pseudo-stopping time in the larger filtration $(\mathcal{F}_t^L)$. This is a consequence of a result of Azéma [1].

PROPOSITION 6. *Let L be the end of a predictable set such that $\mathbb{P}[L = T] = 0$. Then L is not a pseudo-stopping time.*

PROOF. From a result of Azéma [1], as $A_t^L = a_t^L$ is continuous, the law of A_∞^L is the exponential law of parameter 1, while for pseudo-stopping times, the law of A_∞^L is δ_1, the Dirac mass at one. Hence, L cannot be a pseudo-stopping time. □

3.3. *Further examples.* In this section we shall link pseudo-stopping times with other random times that appear in the literature. In particular, we will see that the random times allowing the (H) hypothesis (see [7]) to hold are special cases of pseudo-stopping times.

3.3.1. *The hypothesis* (H). First, we give the following obvious result:

PROPOSITION 7. *If ρ is a random time that is independent from $\mathcal{F}_\infty$, then it is a pseudo-stopping time.*

EXAMPLE 2. If ρ is an exponential time of parameter λ that is independent from $\mathcal{F}_\infty$, then it is a pseudo-stopping time.

EXAMPLE 3. Another example is given by what Williams [20] calls a "silly" time:

$$\rho = \frac{1}{1 + |B_2 - B_1|},$$

which is independent from $\mathcal{F}_1$.

Now suppose that our probability space supports a uniform random variable Θ on $(0, 1)$ that is independent of the sigma field $\mathcal{F}_\infty$. Assume we are given an $(\mathcal{F}_t)$-adapted increasing and continuous process satisfying $A_0 = 0$ and $A_\infty = 1$. Let us consider the random time defined by

$$\rho = \inf\{t; A_t > \Theta\}.$$

It is not difficult to check that

$$\mathbb{P}[\rho > t | \mathcal{F}_t] = 1 - A_t. \tag{3.3}$$

Hence, we can state the following:

PROPOSITION 8. *Let (A_t) be a nonincreasing, continuous and adapted process such that*

$$A_0 = 1,$$
$$A_\infty = 0.$$

Then, if our probability space supports a uniform random variable Θ on $(0, 1)$ that is independent of the sigma field $\mathcal{F}_\infty$, there always exists a pseudo-stopping time ρ such that $Z_t^\rho = A_t$, for $t \geq 0$.

We have thus constructed a pseudo-stopping time associated with a given continuous process (A_t). This construction is well known, see [8] for more details and references.

But the pseudo-stopping times that are constructed in the way of (3.3) enjoy the following noticeable property [5, 8]:

$$\mathbb{P}[\rho > t | \mathcal{F}_t] = \mathbb{P}[\rho > t | \mathcal{F}_\infty]. \tag{3.4}$$

Random times with this property are often used in the literature on default modeling (see [7, 8]) and were studied in [3, 5]. There are several equivalent formulations for (3.4). Before we mention them, let us notice that any random time satisfying (3.4) is a pseudo-stopping time. In fact, we have a stronger result: every $(\mathcal{F}_t)$-martingale is an $(\mathcal{F}_t^\rho)$-martingale (see [5]). Thus, the fourth statement in Theorem 1 is satisfied.

Now let us consider the (H) hypothesis in our framework of progressive enlargement with a random time ρ: every $(\mathcal{F}_t)$-square integrable martingale is an $(\mathcal{F}_t^\rho)$-square integrable martingale. This hypothesis was studied, in a general framework, by Dellacherie and Meyer [5] and Brémaud and Yor [3]. It is equivalent to one of the following hypothesis (see [7] for more references):

(1) $\forall t$, the σ-algebras $\mathcal{F}_\infty$ and $\mathcal{F}_t^\rho$ are conditionally independent given $\mathcal{F}_t$.
(2) For all bounded $\mathcal{F}_\infty$-measurable random variables $\mathbf{F}$ and all bounded $\mathcal{F}_t^\rho$-measurable random variables $\mathbf{G}_t$, we have

$$\mathbb{E}[\mathbf{F}\mathbf{G}_t|\mathcal{F}_t] = \mathbb{E}[\mathbf{F}|\mathcal{F}_t]\mathbb{E}[\mathbf{G}_t|\mathcal{F}_t].$$

(3) For all bounded $\mathcal{F}_t^\rho$-measurable random variables $\mathbf{G}_t$,

$$\mathbb{E}[\mathbf{G}_t|\mathcal{F}_\infty] = \mathbb{E}[\mathbf{G}_t|\mathcal{F}_t].$$

(4) For all bounded $\mathcal{F}_\infty$-measurable random variables $\mathbf{F}$,

$$\mathbb{E}[\mathbf{F}|\mathcal{F}_t^\rho] = \mathbb{E}[\mathbf{F}|\mathcal{F}_t].$$

(5) For all $s \le t$,

$$\mathbb{P}[\rho \le s|\mathcal{F}_t] = \mathbb{P}[\rho \le s|\mathcal{F}_\infty].$$

Thus, pseudo-stopping times may be considered as a generalized or a weakened form of the (H) hypothesis, since then local martingales in the initial filtration remain local martingales in the enlarged one up to time ρ. Moreover, for most of the examples we have considered, such as Williams', (3.4) is not satisfied.

3.3.2. *Randomized stopping times and Föllmer measures.* Now we give a relation between pseudo-stopping times and randomized stopping times as presented in [15]. First we give some definitions. We always consider a given probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$.

DEFINITION 3. A randomized random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability measure μ on $([0,\infty] \times \Omega, \mathcal{B}([0,\infty]) \otimes \mathcal{F})$ such that its projection on Ω is equal to $\mathbb{P}$.

For example, let ρ be a random time; then μ_ρ defined by

$$\mu_\rho(X) = \mathbb{E}[X_\rho],$$

for all bounded measurable processes (X_t), is a randomized random variable.

We know from a result of Föllmer (see [6]) that there exists an increasing càdlàg process (A_t) such that $A_0 = 0$ and

$$\mu(X) = \mathbb{E}\left[\int_0^\infty X_s \, dA_s\right],$$

for all nonnegative process (X_t). The fact that the projection on Ω is equal to $\mathbb{P}$ means that $A_\infty = 1$, a.s.

DEFINITION 4. If the process (A_t) associated with μ on $([0,\infty] \times \Omega, \mathcal{B}([0,\infty]) \otimes \mathcal{F})$ is adapted, then we say that μ is a randomized stopping time.

By considering the new space $\overline{\Omega} = [0,1] \times \Omega$ endowed with the σ-fields $\overline{\mathcal{F}} = \mathcal{B}([0,1]) \otimes \mathcal{F}$, $\overline{\mathcal{F}}_t = \mathcal{B}([0,1]) \otimes \mathcal{F}_t$ (augmented in the usual way) and the probability measure $\overline{\mathbb{P}} = \lambda \otimes \mathbb{P}$, it is possible to show that, for every randomized stopping time μ, there exists a stopping time ρ in this new filtered space such that

$$\mu(X) = \overline{\mathbb{E}}[X_\rho],$$

for all bounded measurable process (X_t) on $([0,\infty] \times \Omega, \mathcal{B}([0,\infty]) \otimes \mathcal{F})$. We take the convention that a random variable H on Ω can be considered as the random variable on $\overline{\Omega}: (u,\omega) \to H(\omega)$. Conversely, to every stopping time of $\overline{\mathcal{F}}_t$, there corresponds a randomized stopping time.

This construction is always carried on the enlarged space $\overline{\Omega}$. The third statement in Theorem 1 allows us to use pseudo-stopping times to construct randomized stopping times without enlarging the initial space.

PROPOSITION 9. *Let ρ be a pseudo-stopping time and A_t^ρ the $(\mathcal{F}_t)$ dual optional projection of the process $\mathbf{1}_{\{\rho \le t\}}$. Then the Föellmer measure μ associated with A_t^ρ is a randomized stopping time. Moreover, for every bounded or non-negative $(\mathcal{F}_t)$ optional process (X_t)*

$$\mu(X) = \mathbb{E}[X_\rho].$$

3.3.3. *Randomized stopping times and families of stopping times.*

PROPOSITION 10. *Let $(T_u)_{u \ge 0}$ be a family of $(\mathcal{F}_t)$-stopping times and S a positive random variable, independent of $(\mathcal{F}_\infty)$. Then*

$$\rho = T_S$$

is a $(\mathcal{F}_t)$-pseudo-stopping time.

PROOF. Let (M_t) be a bounded $(\mathcal{F}_t)$-martingale;

$$\mathbb{E}\big[M_{T_S}\big] = \mathbb{E}\big[\mathbb{E}[M_{T_S}|S]\big] = \mathbb{E}[M_0]. \qquad \square$$

The previous proposition shows that any independently time changed family of stopping times is a pseudo-stopping time. In fact, this proposition admits a converse: every pseudo-stopping time is, in law, a time changed family of stopping times. More precisely:

PROPOSITION 11. *Let ρ be a $(\mathcal{F}_t)$-pseudo-stopping time, which avoids all $(\mathcal{F}_t)$-stopping times, and $Z_t = \mathbb{P}[\rho > t | \mathcal{F}_t]$ its associated supermartingale. Set*

$$\alpha_u \equiv \inf\{t \ge 0, (1 - Z_t) > u\}, \qquad 0 \le u \le 1,$$

the right-continuous generalized inverse of the increasing continuous process $(1 - Z_t)$. Then $(\alpha_u)_{0 \le u \le 1}$ is a family of $(\mathcal{F}_t)$-stopping times and

$$\rho \overset{law}{=} \alpha_U,$$

where U is a random variable with uniform law, independent of $\mathcal{F}_\infty$.

PROOF. The fact α_u is a stopping time, for all u, follows from

$$\{\alpha_u \leq t\} = \{u \leq (1 - Z_t)\} \qquad \forall t \geq 0.$$

From (2.8), we also have

$$\mathbb{E}[g(\rho)] = \mathbb{E}\left[\int_0^1 g(\alpha_u)\,du\right],$$

for any bounded Borel function g. This establishes: $\rho \overset{\text{law}}{=} \alpha_U$. $\square$

4. A discrete analogue: the coin-tossing case. Let $(X_n)_{n\geq 1}$ be the standard random walk with Bernoulli increments. In his paper [13], Le Gall proved an analogue of Williams' path decomposition for (X_n). To fix ideas, we shall consider the canonical space $\Omega = \mathbb{Z}^N$ endowed with the product σ-field. (X_n) will be the coordinate process and $(\mathbb{P}_x)_{x\in\mathbb{Z}}$ the family of probability laws which make (X_n) the standard random walk with Bernoulli increments. We also denote by $(\mathbb{Q}_x)_{x\in N}$ the unique family of probability measures such that $(X_n, \mathbb{Q}_x)$ is a Markov chain with transition probabilities:

$$\mathbb{Q}_0[X_1 = 1] = 1$$

$$\text{if } x \geq 1 \qquad \mathbb{Q}_x[X_1 = x+1] = \frac{1}{2}\left(1 + \frac{1}{x}\right),$$

$$\mathbb{Q}_x[X_1 = x-1] = \frac{1}{2}\left(1 - \frac{1}{x}\right).$$

Now let $p \geq 1$ and define

$$\sigma_p = \inf\{k; X_k = p\},$$

$$\eta = \sup\{k \leq \sigma_p : X_k = 0\},$$

$$m = \sup\{X_k, k \leq \eta\},$$

$$\gamma = \inf\{k \geq 0; X_k = m\}.$$

Then under $\mathbb{P}_0$ the following hold:

(1) The processes $(X_k)_{0\leq k\leq\eta}$ and $(X_{\eta+k})_{0\leq k\leq\sigma_p-\eta}$ are independent, with the second being distributed as $(X_k)_{0\leq k\leq\sigma_p}$ under $\mathbb{Q}_0$;
(2) m is uniformly distributed on $\{0, 1, \dots, p-1\}$;
(3) Conditionally on $\{m = j\}$, the processes $(X_k)_{0\leq k\leq\gamma}$ and $(X_{\eta-k})_{0\leq k\leq\eta-\gamma}$ are independent, the first being distributed as $(X_k)_{0\leq k\leq\sigma_j}$ under $\mathbb{P}_0$, and the second as $(X_k)_{0\leq k\leq\sigma_{j+1}-1}$ under $\mathbb{Q}_0$.

PROPOSITION 12. *If $(M_n)_{n\in\mathbb{N}}$ is a bounded martingale, then*

$$\mathbb{E}_0[M_\gamma] = \mathbb{E}_0[M_0],$$

and, thus, γ is a pseudo-stopping time.

PROOF. We have

$$M_n = f_n(X_1, X_2, \ldots, X_n),$$

for a sequence of bounded measurable functions f_n depending only on n variables. Thus, for any bounded function f,

$$\mathbb{E}_0[M_\gamma f(m)] = \mathbb{E}_0\big[\mathbb{E}_0[M_\gamma | m] f(m)\big].$$

But

$$\begin{aligned}\mathbb{E}_0[M_\gamma | m = j] &= \mathbb{E}_0\big[f_{\sigma_j}(X_1, X_2, \ldots, X_{\sigma_j})\big]\\ &= \mathbb{E}_0\big[M_{\sigma_j}\big] = \mathbb{E}_0[M_0].\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\mathbb{E}_0[M_\gamma f(m)] &= \mathbb{E}_0[M_\gamma]\mathbb{E}_0[f(m)]\\ &= \mathbb{E}_0[M_\infty]\mathbb{E}_0[f(m)].\end{aligned}$$ □

REMARK 6. Again (as in the continuous time setting), note that, in general,

$$\mathbb{E}_0[M_\infty | \mathcal{F}_\gamma] \neq M_\gamma.$$

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