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STOCHASTIC PROCESSES IN RANDOM GRAPHS

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We study the asymptotics of large, moderate and normal deviations for the connected components of the sparse random graph by *the method of stochastic processes*. We obtain the logarithmic asymptotics of large deviations of the joint distribution of the number of connected components, of the sizes of the giant components and of the numbers of the excess edges of the giant components. For the supercritical case, we obtain the asymptotics of normal deviations and the logarithmic asymptotics of large and moderate deviations of the joint distribution of the number of components, of the size of the largest component and of the number of the excess edges of the largest component. For the critical case, we obtain the logarithmic asymptotics of moderate deviations of the joint distribution of the sizes of connected components and of the numbers of the excess edges. Some related asymptotics are also established. The proofs of the large and moderate deviation asymptotics employ methods of idempotent probability theory. As a byproduct of the results, we provide some additional insight into the nature of phase transitions in sparse random graphs.

1. Introduction. The random graph $\mathcal{G}(n, p)$ is defined as a nondirected graph on n vertices where every two vertices are independently connected by an edge with probability p. The graph is said to be sparse if $p = c/n$ for $c > 0$ and n large. Properties of sparse random graphs have been studied at length and major developments have been summarized in the recent monographs by Bollobas (2001), Janson, Łuczak and Ruciński (2000) and Kolchin (1999). The focus of this paper is on the asymptotics as $n \to \infty$ of the sizes of the giant connected components, that is, components of order n in size, of $\mathcal{G}(n, c_n/n)$, where $c_n \to c > 0$. It is known that for $c > 1$, with probability tending to 1 as $n \to \infty$, there exists a unique giant component of $\mathcal{G}(n, c/n)$, which is asymptotically βn in size, where $\beta \in (0, 1)$ is the positive root to the equation $1 - \beta = \exp(-\beta c)$, the rest of the components being of sizes not greater than of order $\log n$. For $c < 1$, with probability tending to 1, there are no connected components of sizes greater than of order $\log n$, while for $c = 1$ the size of the largest component is of order $n^{2/3}$. Our primary objective is to evaluate the probabilities that there

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exist several giant connected components. As to be expected, these probabilities are exponentially small in n, so we study the decay rates and state our results in the form of the large deviation principle (LDP). In addition, influenced by the papers of Stepanov (1970b) and Aldous (1997), we concern ourselves with the large deviation asymptotics of the number of the connected components and of the numbers of the excess edges of the connected components. Thus, the main result is an LDP for the joint distribution of the normalized number of the connected components of $\mathcal{G}(n, c_n/n)$, of the normalized sizes of the connected components and of the normalized numbers of the excess edges. Projecting yields LDPs for the sizes and for the number of the connected components. Stepanov (1970b) and later Bollobás, Grimmett and Janson (1996), analyzing a more general setting, have obtained the logarithmic asymptotics of the moment generating function of the number of the connected components of $\mathcal{G}(n, c/n)$. If $c \leq 2$, the latter asymptotics also yield the LDP for the number of components, as Bollobás, Grimmett and Janson (1996) demonstrate, but not for arbitrary $c > 0$. This anomaly is caused by a phase transition occurring at $c = 2$ discovered by Stepanov (1970b), which results, as we show, in the action functional becoming nonconvex as c passes through the value of 2. Moreover, the phase transition turns out to consist in a giant component breaking up.

Another group of results presented in the paper has to do with the properties of the largest connected component. We establish normal deviation, moderate deviation and large deviation asymptotics for the joint distribution of the size of the largest connected component, of the number of its excess edges and of the number of the connected components. In related work, O'Connell (1998) proves an LDP for the size of the largest connected component of $\mathcal{G}(n, c/n)$ and Stepanov (1970a, 1972) obtains central limit theorems for the size of the largest component and the number of components; different proofs of the central limit theorem for the size of the largest component are given in Pittel (1990) and Barraez, Boucheron and Fernandez de la Vega (2000), the latter authors also provide estimates of the rate of convergence. Our third group of results concerns the critical random graph when $c = 1$. We complement the result of Aldous (1997) on the convergence in distribution of the suitably normalized sizes and numbers of the excess edges of the connected components with moderate deviation asymptotics for these random variables.

Our analysis employs a surprising (to us) connection to queueing theory. The results outlined above are derived as consequences of the asymptotic properties of a "master" stochastic process, which captures the partitioning of the random graph into connected components and builds on an earlier construction of a similar sort, see Janson, Łuczak and Ruciński (2000). This stochastic process is intimately related to the waiting-time process (or the queue-length process) in a certain time- and state-dependent queueing system and the connected components correspond to the busy cycles of the system. We capitalize on this connection by invoking our intuition for the behavior of queues as well as some standard queueing theory tools such as properties of the Skorohod reflection mapping. Thus, at first we apply the methods of the asymptotic theory of stochastic processes, namely, the methods of weak and large deviation convergence, in order to establish asymptotics of the master process and then translate them into the properties of the connected components of the random graph. In the context of the random graph theory, the present paper can thus be considered as developing the approach pioneered by Aldous (1997) of deriving asymptotic properties of random graphs as consequences of asymptotics of associated stochastic processes. On the technical side, we extensively use the observation also made by Aldous (1997) that the connected components can be identified with the excursions of a certain stochastic process. Yet, the specific construction in this paper is different from the one of Aldous (1997). It is actually much the same as the one of Barraez, Boucheron and Fernandez de la Vega (2000), as we learned after the paper had been submitted, except for an important distinction, which we discuss below.

There are also other interesting technical aspects of the proofs, which concern all three types of asymptotics: large deviations, moderate deviations and normal deviations. The proof of the LDP for the master process relies on the results of the large deviation theory of semimartingales [Puhalskii (2001)], which seem to be called for since the action functional is “non-Markovian” and “non-time homogeneous.” The cumulant that characterizes the action functional is not nondegenerate, which is known to present certain difficulties for establishing the LDP. In the standard approach the problem reveals itself when the large deviation lower bound is proved and is usually tackled via a perturbation argument: an extra term is added to the process under study so that the perturbed process has a nondegenerate cumulant and then a limit is taken in the lower bound for the perturbed process as the perturbation term tends to zero, see Liptser (1996), de Acosta (2000) and Liptser, Spokoiny and Veretennikov (2002). Our approach to proving the LDP replaces establishing the upper and lower bounds with the requirement that the limiting maxingale problem has a unique solution. The degeneracy of the cumulant presents a problem here too. We cope with it via a perturbation argument as well the important difference being that the perturbation is applied to the limit idempotent process that specifies the maxingale problem rather than to the pre-limiting stochastic processes. This change of the object has important methodological advantages. First, the proof of the LDP is simplified as compared with the case where the perturbation is introduced at the pre-limiting stage. Second, once the perturbation argument has been carried out for a given cumulant, one can use it to prove LDPs for a range of stochastic processes that produce the same cumulant in the limit. We expand on these ideas in Puhalskii (2004). The actual implementation of the perturbation approach for the setting in the paper relies on the techniques of idempotent probability theory, Puhalskii (2001), and also draws on time-change arguments in Ethier and Kurtz [(1986), Chapter 6], thus, applying probabilistic ideas to an idempotent probability setting. Idempotent probability theory techniques are also instrumental in the proofs of the moderate-deviation asymptotics. These proofs are modeled on the preceding proofs of the normal-deviation asymptotics and to a large degree replicate them by replacing limit stochastic processes with their idempotent counterparts.

An interesting feature of the proof of the normal deviation asymptotics for the largest component is that it provides an instance of convergence in distribution of stochastic processes "with unmatched jumps in the limit process" [Whitt (2002)], that is, though the jumps of the pre-limiting processes vanish, the limit process is discontinuous, moreover, it is not right-continuous with left-hand limits. We thus do not have convergence in distribution in the Skorohod topology and have to use some ad-hoc techniques to obtain the needed conclusions. As it is explained in Whitt (2002), convergence with unmatched jumps often occurs in the study of diffusion approximation of time dependent queues, so it is not surprising (but is amusing) to see it here. Incidentally, we are faced with a similar situation in the proof of the moderate-deviation asymptotics when no LDP for the Skorohod topology is available and the corresponding limit theorem can be viewed as an example of large deviation convergence in distribution of stochastic processes with unmatched jumps in the limit idempotent process.

We now outline the structure of the paper. In Section 2 we define the underlying stochastic processes, derive queue-like equations for them, state the results on the properties of the connected components, and comment on them. Section 3 contains technical preliminaries. Section 4 is concerned with proving the LDP for the basic processes. In Section 5 the LDPs for the connected components are proved. Section 6 contains proofs of the normal and moderate deviation asymptotics for the largest component. Section 7 considers critical random graphs. The Appendix provides an overview of the notions and facts of idempotent probability theory invoked in the proofs.

2. The model equations and main results. We model the formation of the sparse random graph on n vertices with edge probability $p_n = c_n/n$ via stochastic processes $V^n = (V_i^n,\ i = 0, 1, \dots, n)$ and $E^n = (E_i^n,\ i = 0, 1, \dots, n)$. At time 0 the processes are at 0. At time 1 an arbitrary vertex of the graph is picked and is connected by edges to the other vertices independently with probability p_n. We say that this vertex has been first *generated* and then *saturated*. The vertices, to which it has been connected, are called *generated*. The value of V_1^n is defined as the number of vertices in the resulting connected component, that is, the number of the generated vertices at time 1; $E_1^n = 0$. At time 2 we pick one of the generated, nonsaturated vertices, if any, and saturate it by connecting it independently with probability p_n to the vertices that either have not been generated yet or have been generated but not saturated. If there are no generated, nonsaturated vertices, we pick an arbitrary nongenerated vertex, declare it generated and saturate it by attempting to connect it to the nongenerated vertices, thus, generating those of these vertices connection to which is established. We denote as V_2^n the total number of vertices generated at times 1 and 2 and we denote as E_2^n the number of edges connecting the vertex that was saturated at time 2 with the other vertices generated at time 1, if any. We proceed in this fashion by saturating one vertex per unit of time until time n. Thus, at time i a generated, nonsaturated vertex is picked and is connected by edges with probability p_n to the nonsaturated vertices, both generated and not yet generated; if there are no generated, nonsaturated vertices available, then an arbitrary nongenerated vertex is chosen, is declared generated and is then saturated. The increment $V_i^n - V_{i-1}^n$ is defined as the number of vertices generated at i, the increment $E_i^n - E_{i-1}^n$ is defined as the number of edges drawn at i between the vertex being saturated and the vertices generated by i. Thus, $V_i^n - V_{i-1}^n$ equals either the number of new vertices joined to a connected component at time i if $V_{i-1}^n > i-1$ or it is the number of vertices that start a new component at i if $V_{i-1}^n = i-1$. Accordingly, the increment $E_i^n - E_{i-1}^n$ either equals the number of excess edges in a connected component appeared at time i, that is, the edges in excess of those that are necessary to maintain connectedness, or $E_i^n - E_{i-1}^n = 0$. Since during this process every two vertices independently attempt connection with probability p_n exactly once, the resulting configuration of edges at time n has the same distribution as the one in the random graph $\mathcal{G}(n, p_n)$. In fact, the sizes of the connected components of $\mathcal{G}(n, p_n)$ can be recovered from the process V^n as time-spans between successive moments when V_i^n is equal to i. The numbers of the excess edges in the connected components are equal to the increments of the process E^n over such time periods. In addition, the number of times when V_i^n is equal to $i \in \{1, 2, \dots, n\}$ equals the number of the connected components of $\mathcal{G}(n, p_n)$. We now turn this description into equations.

Since at time i there are V_i^n generated vertices, the evolution of V^n is given by the following recursion:

$$\begin{aligned} V_i^n &= \left(V_{i-1}^n + \sum_{j=1}^{n-V_{i-1}^n} \xi_{ij}^n\right)\mathbf{1}(V_{i-1}^n > i-1) \\ &\quad + \left(i + \sum_{j=1}^{n-i} \xi_{ij}^n\right)\mathbf{1}(V_{i-1}^n = i-1), \qquad i = 1, 2, \dots, n,\ V_0^n = 0, \end{aligned} \tag{2.1}$$

where the ξ_{ij}^n, $i \in \mathbb{N}$, $j \in \mathbb{N}$, $n \in \mathbb{N}$, are mutually independent Bernoulli random variables with $\mathbf{P}(\xi_{ij}^n = 1) = p_n$ and $\mathbf{1}(\Gamma)$ is the indicator function of an event Γ that equals 1 on Γ and 0 outside of Γ. Let Q_i^n denote the number of nonsaturated, generated vertices at time i. Since $Q_i^n = V_i^n - i$, (2.1) implies that

$$\begin{aligned} Q_i^n &= \left(Q_{i-1}^n + \sum_{j=1}^{n-Q_{i-1}^n-(i-1)} \xi_{ij}^n - 1\right)\mathbf{1}(Q_{i-1}^n > 0) \\ &\quad + \sum_{j=1}^{n-i} \xi_{ij}^n \mathbf{1}(Q_{i-1}^n = 0), \qquad i = 1, 2, \dots, n,\ Q_0^n = 0. \end{aligned} \tag{2.2}$$

The evolution of the process E^n is governed by the recursion

$$E_i^n = E_{i-1}^n + \sum_{j=1}^{Q_{i-1}^n - 1} \zeta_{ij}^n, \qquad i = 1, 2, \dots, n,\ E_0^n = 0, \tag{2.3}$$

where the ζ_{ij}^n, $i \in \mathbb{N}$, $j \in \mathbb{N}$, $n \in \mathbb{N}$, are mutually independent Bernoulli random variables with $\mathbf{P}(\zeta_{ij}^n = 1) = p_n$, which are independent of the ξ_{ij}^n, and sums are assumed to be equal to 0 if the upper summation index is less than the lower one.

We use for the analysis of (2.2) the following insight. Let us introduce a related process $Q'^n = (Q'^n_i, i = 0, 1, \dots, n)$ by

$$Q'^n_i = \left(Q'^n_{i-1} + \sum_{j=1}^{n - Q'^n_{i-1} - i} \xi_{ij}^n - 1 \right)^+, \qquad i = 1, 2, \dots, n,\ Q'^n_0 = 0, \tag{2.4}$$

where $a^+ = \max(a, 0)$. We note that Q'^n_i is the waiting time of the ith request, where $i = 0, 1, \dots, n-1$, in the queueing system that starts empty, has $\sum_{j=1}^{n - Q'^n_i - (i+1)} \xi_{i+1,j}^n$ as the ith request's service time and 1 as the interarrival times. (Alternatively, Q'^n_i can be considered as the queue length at time i for the discrete-time queueing system that serves one request per unit time, the number of arrivals in $[i, i+1]$ being equal to $\sum_{j=1}^{n - Q'^n_i - (i+1)} \xi_{i+1,j}^n$.) It is seen that $Q'^n_i = (Q_i^n - 1)^+$, so the asymptotic properties of the process $Q^n = (Q_i^n,\ i = 0, 1, \dots, n)$ multiplied by a vanishing constant are the same as those of the process $Q'^n = (Q'^n_i,\ i = 0, 1, \dots, n)$. In addition, connected components of the random graph correspond to busy cycles of this queueing system, that is, the excursions of Q'^n. Thus, a possible way to study the random graph is through the process Q'^n. This approach is, in effect, pursued by Barraez, Boucheron and Fernandez de la Vega (2000) who study what in our notation is the process $(Q'^n_i + 1,\ i = 0, 1, \dots, n)$. It is, however, inconvenient for our purposes because $Q'^n_i = 0$ not only when $Q_i^n = 0$, but also when $Q_i^n = 1$, so the queueing system may have more busy cycles than there are connected components. For this reason, we choose to work with Q^n directly. Yet, the queueing theory connection serves us as a guide. Let us recall that the solution of (2.4) is given by $Q'^n = \mathcal{R}(\widetilde{S}^n)$, where the process $\widetilde{S}^n = (\widetilde{S}_i^n,$ $i = 0, 1, \dots, n)$ with $\widetilde{S}_0^n = 0$ defined by $\widetilde{S}_i^n = \sum_{k=1}^i \sum_{j=1}^{n - Q'^n_{k-1} - k} \xi_{kj}^n - i$, and $\mathcal{R}$ is the Skorohod reflection operator: $\mathcal{R}(\mathbf{x})_t = \mathbf{x}_t - \inf_{s \in [0,t]} \mathbf{x}_s \wedge 0$ for $\mathbf{x} = (\mathbf{x}_t,$ $t \in \mathbb{R}_+)$, where $\wedge$ denotes the minimum. We find it productive to express Q^n as a reflection too. The idea is to sacrifice the Markovian character of recursion (2.2) for the nice properties of the reflection mapping.

A manipulation of (2.2) yields the following equality:

$$Q_i^n = S_i^n + \varepsilon_i^n + \Phi_i^n, \qquad i = 0, 1, \dots, n, \tag{2.5}$$

where

$$S_i^n = \sum_{k=1}^{i} \left(\sum_{j=1}^{n-Q_{k-1}^n-(k-1)} \xi_{kj}^n - 1 \right), \tag{2.6}$$

$$\varepsilon_i^n = \mathbf{1}(Q_i^n > 0) - \sum_{k=1}^{i} \xi_{k,n-k+1}^n \mathbf{1}(Q_{k-1}^n = 0), \tag{2.7}$$

$$\Phi_i^n = \sum_{k=1}^{i} \mathbf{1}(Q_k^n = 0). \tag{2.8}$$

For the sequel it is useful to note that Φ_n^n equals the number of the connected components of $\mathcal{G}(n, p_n)$.

Denoting as $\lfloor x \rfloor$ the integer part of $x \in \mathbb{R}_+$, we introduce continuous-time processes $\overline{Q}^n = (\overline{Q}_t^n,\ t \in [0,1])$, $\overline{S}^n = (\overline{S}_t^n,\ t \in [0,1])$, $\overline{\Phi}^n = (\overline{\Phi}_t^n,\ t \in [0,1])$, and $\overline{E}^n = (\overline{E}_t^n,\ t \in [0,1])$ by the respective equalities $\overline{Q}_t^n = Q_{\lfloor nt \rfloor}^n / n$, $\overline{S}_t^n = S_{\lfloor nt \rfloor}^n / n$, $\overline{\Phi}_t^n = \Phi_{\lfloor nt \rfloor}^n / n$, and $\overline{E}_t^n = E_{\lfloor nt \rfloor}^n / n$. By (2.8) $\overline{\Phi}_t^n = \int_0^t \mathbf{1}(\overline{Q}_s^n = 0)\, d\overline{\Phi}_s^n$, so, by (2.5) the pair $(\overline{Q}^n, \overline{\Phi}^n)$ solves the Skorohod problem in $\mathbb{R}$ for $\overline{S}^n + \bar{\varepsilon}^n$, consequently,

$$\overline{Q}^n = \mathcal{R}(\overline{S}^n + \bar{\varepsilon}^n), \tag{2.9}$$

$$\overline{\Phi}^n = \mathcal{T}(\overline{S}^n + \bar{\varepsilon}^n), \tag{2.10}$$

where $\mathcal{T}(\mathbf{x})_t = -\inf_{s \in [0,t]} \mathbf{x}_s \wedge 0$ for $\mathbf{x} = (\mathbf{x}_t, t \in \mathbb{R}_+)$ and $\bar{\varepsilon}^n = (\bar{\varepsilon}_t^n,\ t \in [0,1])$ is defined by

$$\bar{\varepsilon}_t^n = \frac{\varepsilon_{\lfloor nt \rfloor}^n}{n}. \tag{2.11}$$

Equation (2.3) yields the representation

$$\overline{E}_t^n = \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=1}^{Q_{i-1}^n - 1} \zeta_{ij}^n, \qquad t \in [0,1]. \tag{2.12}$$

Equations (2.6), (2.9), (2.10) and (2.12) play a central part in establishing the main results of the paper. In some more detail, the processes $\bar{\varepsilon}^n$ prove to be inconsequential and may be disregarded (see Lemma 3.1), so (2.6), (2.12) and (2.9) enable us to obtain functional limit theorems for the processes $(\overline{S}^n, \overline{E}^n)$, which on making another use of (2.9) and (2.10) yield the asymptotics of the connected components (we note that the latter step does not reduce to a mere application of the continuous mapping principle). Before embarking on this programme, we state and discuss the results.

We will say that a sequence $\mathbf{P}_n$, $n \in \mathbb{N}$, of probability measures on the Borel σ-algebra of a metric space Υ (or a sequence of random elements X_n, $n \in \mathbb{N}$, with values in Υ and distributions $\mathbf{P}_n$) obeys the large deviation principle (LDP) for scale k_n, where $k_n \to \infty$ as $n \to \infty$, with action functional $I : \Upsilon \to [0, \infty]$ if the sets $\{\upsilon \in \Upsilon : I(\upsilon) \le a\}$ are compact for all $a \in \mathbb{R}_+$,

$$\limsup_{n\to\infty} \frac{1}{k_n} \log \mathbf{P}_n(F) \le -\inf_{\upsilon\in F} I(\upsilon) \qquad \text{for all closed sets } F \subset \Upsilon$$

and

$$\liminf_{n\to\infty} \frac{1}{k_n} \log \mathbf{P}_n(G) \ge -\inf_{\upsilon\in G} I(\upsilon) \qquad \text{for all open sets } G \subset \Upsilon.$$

Let for $u \in [0, 1]$, $\rho \in \mathbb{R}_+$ and $c > 0$,

$$K_\rho(u) = u \log \frac{\rho u}{1 - e^{-\rho u}} - \frac{\rho u^2}{2}, \tag{2.13}$$

$$L_c(u) = (1 - u) \log(1 - u) + (c - \log c)u - \frac{cu^2}{2}, \tag{2.14}$$

where we adopt the conventions $0/0 = 1$ and $0 \cdot \infty = 0$. We also denote $a \vee b = \max(a, b)$, $\pi(x) = x \log x - x + 1$, $x \in \mathbb{R}_+$, and assume that $\pi(\infty) \cdot 0 = \infty$.

Let $\mathbb{S}$ denote the subset of $\mathbb{R}_+^{\mathbb{N}}$ of sequences $\mathbf{u} = (u_1, u_2, \ldots)$ such that $\sum_{i=1}^\infty u_i < \infty$. Given a convex function $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\chi(0) = 0$, $\chi(x) > 0$ for $x > 0$, and $\chi(x)/x \to 0$ as $x \to 0$, we endow $\mathbb{S}$ with an Orlicz space topology that is generated by a Luxembourg metric $d_\chi(\mathbf{u}, \mathbf{u}') = \inf\{b \in \mathbb{R}_+ : \sum_{i=1}^\infty \chi(|u_i - u_i'|/b) \le 1\}$, where $\mathbf{u} = (u_1, u_2, \ldots)$ and $\mathbf{u}' = (u_1', u_2', \ldots)$ [cf., e.g., Krasnosel'skii and Rutickii (1961) and Bennett and Sharpley (1988)]. Let also $\mathbb{S}_1$ denote the subspace of $\mathbb{S}$ of nonincreasing sequences $\mathbf{u} = (u_1, u_2, \ldots)$ with $\sum_{i=1}^\infty u_i \le 1$. It is endowed with induced topology which is equivalent to the product topology.

Let $(U_1^n, U_2^n, \ldots)$ be the sequence of the sizes of the connected components of the random graph $\mathcal{G}(n, c_n/n)$ arranged in descending order appended with zeros to make it infinite, $(R_1^n, R_2^n, \ldots)$ be the sequence of the corresponding numbers of the excess edges appended with zeros, and α^n be the number of the connected components of $\mathcal{G}(n, c_n/n)$. We define $\overline{U}^n = (U_1^n/n, U_2^n/n, \ldots)$ and $\overline{R}^n = (R_1^n/n, R_2^n/n, \ldots)$, and consider $(\alpha^n/n, \overline{U}^n, \overline{R}^n)$ as a random element of $[0, 1] \times \mathbb{S}_1 \times \mathbb{S}$, which is assumed to be equipped with product topology.

THEOREM 2.1. *Let $c_n \to c > 0$ as $n \to \infty$. Then the sequence $(\alpha^n/n, \overline{U}^n, \overline{R}^n)$, $n \in \mathbb{N}$, obeys the LDP in $[0, 1] \times \mathbb{S}_1 \times \mathbb{S}$ for scale n with action functional $I_c^{\alpha,U,R}$ defined for $a \in [0, 1]$, $\mathbf{u} = (u_1, u_2, \ldots) \in \mathbb{S}_1$ and $\mathbf{r} = (r_1, r_2, \ldots) \in \mathbb{S}$ by*

$$\begin{aligned} I_c^{\alpha,U,R}&(a, \mathbf{u}, \mathbf{r}) \\ &= \sum_{i=1}^\infty \sup_{\rho\in\mathbb{R}_+} \left(K_\rho(u_i) + r_i \log \frac{\rho}{c}\right) + L_c\left((1 - 2a) \vee \sum_{i=1}^\infty u_i\right) \\ &\quad + \frac{c}{2}\left(1 - (1 - 2a) \vee \sum_{i=1}^\infty u_i\right)^2 \pi\left(\frac{2(1 - a - (1 - 2a) \vee \sum_{i=1}^\infty u_i)}{c(1 - (1 - 2a) \vee \sum_{i=1}^\infty u_i)^2}\right) \end{aligned}$$

if $\sum_{i=1}^{\infty} u_i \leq 1 - a$ *and* $I_c^{\alpha,U,R}(a, \mathbf{u}, \mathbf{r}) = \infty$ *otherwise.*

As a consequence, we obtain some marginal LDPs.

COROLLARY 2.1. *Let* $c_n \to c > 0$ *as* $n \to \infty$. *Then the sequences* $(\overline{U}^n, \overline{R}^n)$, $n \in \mathbb{N}$, *and* $(\alpha^n/n, \overline{U}^n)$, $n \in \mathbb{N}$, *obey the LDPs for scale* n *in the respective spaces* $\mathbb{S}_1 \times \mathbb{S}$ *and* $[0, 1] \times \mathbb{S}_1$ *with respective action functionals* $I_c^{U,R}$ *and* $I_c^{\alpha,U}$, *defined for* $\mathbf{u} = (u_1, u_2, \dots) \in \mathbb{S}_1$, $\mathbf{r} = (r_1, r_2, \dots) \in \mathbb{S}$ *and* $a \in [0, 1]$ *by*

$$I_c^{U,R}(\mathbf{u}, \mathbf{r}) = \sum_{i=1}^{\infty} \sup_{\rho \in \mathbb{R}_+} \left(K_\rho(u_i) + r_i \log \frac{\rho}{c} \right) + L_c \left(\left(1 - \frac{1}{c} \right) \vee \sum_{i=1}^{\infty} u_i \right)$$

and

$$I_c^{\alpha,U}(a, \mathbf{u}) = \sum_{i=1}^{\infty} K_c(u_i) + L_c \left((1 - 2a) \vee \sum_{i=1}^{\infty} u_i \right)$$
$$+ \frac{c}{2} \left(1 - (1 - 2a) \vee \sum_{i=1}^{\infty} u_i \right)^2 \pi \left(\frac{2(1 - a - (1 - 2a) \vee \sum_{i=1}^{\infty} u_i)}{c(1 - (1 - 2a) \vee \sum_{i=1}^{\infty} u_i)^2} \right)$$

if $\sum_{i=1}^{\infty} u_i \leq 1 - a$ *and* $I_c^{\alpha,U}(a, \mathbf{u}) = \infty$ *otherwise.*

COROLLARY 2.2. *Let* $c_n \to c > 0$ *as* $n \to \infty$. *Then the sequences* α^n/n, $n \in \mathbb{N}$, *and* $\overline{U}^n/n$, $n \in \mathbb{N}$, *obey the LDPs in the respective spaces* $[0, 1]$ *and* $\mathbb{S}_1$ *for scale* n *with the respective action functionals*

$$I_c^{\alpha}(a) = \inf_{\tau \in [(1-2a)^+, 1-a]} \left(K_c(\tau) + L_c(\tau) + \frac{c(1 - \tau)^2}{2} \pi \left(\frac{2(1 - a - \tau)}{c(1 - \tau)^2} \right) \right)$$

and

$$I_c^{U}(\mathbf{u}) = \sum_{i=1}^{\infty} K_c(u_i) + L_c \left(\left(1 - \frac{1}{c} \right) \vee \sum_{i=1}^{\infty} u_i \right).$$

The next corollary clarifies the structure of the most probable configurations of the giant components. Let, given $\delta > 0$, $m \in \mathbb{N}$, and $u_i \in (0, 1]$, $i = 1, 2, \dots, m$ with $\sum_{i=1}^{m} u_i \leq 1$, $A_\delta^n(u_1, \dots, u_m)$ denote the event that there exist m connected components of $\mathcal{G}(n, c_n/n)$, whose respective sizes are between $n(u_i - \delta)$ and $n(u_i + \delta)$ for $i = 1, 2, \dots, m$. For $\varepsilon > 0$, we define event $\widetilde{A}_{\delta,\varepsilon}^n(u_1, \dots, u_m)$ as follows. Let $r_i^* = cu_i^2/(1 - \exp(-cu_i)) - cu_i^2/2 - u_i$. If $\sum_{i=1}^{m} u_i \geq 1 - 1/c$, then $\widetilde{A}_{\delta,\varepsilon}^n(u_1, \dots, u_m)$ equals the intersection of $A_\delta^n(u_1, \dots, u_m)$, the event that the numbers of the excess edges of the m components are within the respective intervals $(n(r_i^* - \varepsilon), n(r_i^* + \varepsilon))$, and the event that any other connected component is of size less than $n\varepsilon$. If $\sum_{i=1}^{m} u_i < 1 - 1/c$, then $\widetilde{A}_{\delta,\varepsilon}^n(u_1, \dots, u_m)$ equals the intersection of $A_\delta^n(u_1, \dots, u_m)$, the event that there exists another connected component whose size is in the interval $(n(u^*-\varepsilon), n(u^*+\varepsilon))$, where $u^*/(1-\exp(-cu^*)) = 1-\sum_{i=1}^m u_i$, the event that the numbers of the excess edges of these $m+1$ components are within the respective intervals $(n(r_i^*-\varepsilon), n(r_i^*+\varepsilon))$ for $i=1,2,\dots,m$ and $(n(r^*-\varepsilon), n(r^*+\varepsilon))$, where $r^* = cu^{*2}/(1-\exp(-cu^*)) - cu^{*2}/2 - u^*$, and the event that any other connected component is of size less than $n\varepsilon$.

COROLLARY 2.3. *Let $c_n \to c > 0$ as $n\to\infty$. Then*

$$\begin{aligned}
\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\log \mathbf{P}\big(A_\delta^n(u_1,\dots,u_m)\big)
&= \lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{n}\log \mathbf{P}\big(A_\delta^n(u_1,\dots,u_m)\big)\\
&= \lim_{\substack{\delta\to 0\\ \varepsilon\to 0}}\limsup_{n\to\infty}\frac{1}{n}\log \mathbf{P}\big(\widetilde{A}_{\delta,\varepsilon}^n(u_1,\dots,u_m)\big)\\
&= \lim_{\substack{\delta\to 0\\ \varepsilon\to 0}}\liminf_{n\to\infty}\frac{1}{n}\log \mathbf{P}\big(\widetilde{A}_{\delta,\varepsilon}^n(u_1,\dots,u_m)\big)\\
&= -\left(\sum_{i=1}^m K_c(u_i) + L_c\left(\sum_{i=1}^m u_i\right)\right)
\end{aligned}$$

and

$$\lim_{\delta\to 0}\liminf_{n\to\infty}\mathbf{P}\big(\widetilde{A}_{\delta,\varepsilon}^n(u_1,\dots,u_m)\,|\,A_\delta^n(u_1,\dots,u_m)\big) = 1.$$

Let β^n denote the size of the largest connected component of $\mathcal{G}(n, c_n/n)$ and γ^n denote the number of its excess edges. We state results on the asymptotics of $(\alpha^n/n, \beta^n/n, \gamma^n/n)$.

COROLLARY 2.4. *Let $c_n \to c > 0$ as $n\to\infty$. Then the following hold*:

1. *The sequence $(\alpha^n/n, \beta^n/n, \gamma^n/n)$, $n\in\mathbb{N}$, obeys the LDP in $[0,1]^2\times\mathbb{R}_+$ for scale n with action functional defined by*

$$I_c^{\alpha,\beta,\gamma}(a,0,0) = L_c\big((1-2a)^+\big) + \frac{c}{2}\big(1-(1-2a)^+\big)^2\pi\left(\frac{2(1-a-(1-2a)^+)}{c(1-(1-2a)^+)^2}\right),$$

$$\begin{aligned}
I_c^{\alpha,\beta,\gamma}(a,u,r) = \sup_{\rho\in\mathbb{R}_+}&\left(K_\rho(u) + r\log\frac{\rho}{c}\right) - K_c(u)\\
&+ \inf_{\tau\in[(1-2a)\vee u,1-a]}\left(\left\lfloor\frac{\tau}{u}\right\rfloor K_c(u) + K_c\left(\tau - u\left\lfloor\frac{\tau}{u}\right\rfloor\right)\right.\\
&\qquad\left. + L_c(\tau) + \frac{c}{2}(1-\tau)^2\pi\left(\frac{2(1-a-\tau)}{c(1-\tau)^2}\right)\right)
\end{aligned}$$

if $u \in (0, 1-a]$ *and* $I_c^{\alpha,\beta,\gamma}(a, u, r) = \infty$ *otherwise.*

2. *The sequence* $(\beta^n/n, \gamma^n/n)$, $n \in \mathbb{N}$, *obeys the LDP in* $[0,1] \times \mathbb{R}_+$ *for scale* n *with action functional* $I_c^{\beta,\gamma}$ *defined by* $I_c^{\beta,\gamma}(0,0) = L_c((1-1/c)^+)$, $I_c^{\beta,\gamma}(0,r) = \infty$ *if* $r > 0$,

$$I_c^{\beta,\gamma}(u,r) = \sup_{\rho \in \mathbb{R}_+}\left(K_\rho(u) + r\log\frac{\rho}{c}\right) + \left(\left\lfloor \frac{1}{u}\left(1-\frac{1}{c}\right)\right\rfloor - 1\right)K_c(u) + K_c(\hat{u}\wedge u) + L_c\left(\left\lfloor \frac{1}{u}\left(1-\frac{1}{c}\right)\right\rfloor u + \hat{u}\wedge u\right)$$

if $u \in (0, (1-1/c)^+)$, *where* $\hat{u} \in [0,1]$ *satisfies the equality* $\hat{u}/(1-\exp(-c\hat{u})) = 1 - \lfloor (1-1/c)/u \rfloor u$, *and by* $I_c^{\beta,\gamma}(u,r) = \sup_{\rho\in\mathbb{R}_+}(K_\rho(u) + r\log(\rho/c)) + L_c(u)$ *if* $u \ge (1-1/c)^+$.

3. *The sequence* β^n/n, $n \in \mathbb{N}$, *obeys the LDP in* $[0,1]$ *for scale* n *with action functional* I_c^β *defined as follows*: $I_c^\beta(0) = L_c((1-1/c)^+)$,

$$I_c^\beta(u) = \left\lfloor \frac{1}{u}\left(1-\frac{1}{c}\right)\right\rfloor K_c(u) + K_c(\hat{u}\wedge u) + L_c\left(\left\lfloor \frac{1}{u}\left(1-\frac{1}{c}\right)\right\rfloor u + \hat{u}\wedge u\right)$$

if $u \in (0, (1-1/c)^+)$ *and* $I_c^\beta(u) = K_c(u) + L_c(u)$ *if* $u \ge (1-1/c)^+$.

The next theorem considers normal deviations of $(\alpha^n, \beta^n, \gamma^n)$. We recall that $\beta \in (0,1)$ is defined as the positive solution of the equation $1 - \beta = \exp(-\beta c)$ if $c > 1$. For $c \le 1$, we define $\beta = 0$. Let also $\alpha = 1 - \beta - c(1-\beta)^2/2$ and $\gamma = (c-1)\beta - c\beta^2/2$.

THEOREM 2.2. *Let* $\sqrt{n}(c_n - c) \to \theta \in \mathbb{R}$ *as* $n \to \infty$, *where* $c > 0$. *Then the following hold*.

1. *The sequence* $\sqrt{n}(\alpha^n/n - \alpha)$, $n \in \mathbb{N}$, *converges in distribution in* $\mathbb{R}$ *as* $n \to \infty$ *to a Gaussian random variable* $\tilde{\alpha}$ *with* $\mathbf{E}\tilde{\alpha} = -\theta(1-\beta^2)/2$ *and* $\operatorname{Var}\tilde{\alpha} = \beta(1-\beta) + c(1-\beta)^2/2$.

2. *If, in addition,* $c > 1$, *then the sequence* $(\sqrt{n}(\alpha^n/n - \alpha), \sqrt{n}(\beta^n/n - \beta), \sqrt{n}(\gamma^n/n - \gamma))$, $n \in \mathbb{N}$, *converges in distribution in* $\mathbb{R}^3$ *as* $n \to \infty$ *to a Gaussian random variable* $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ *with* $\mathbf{E}\tilde{\beta} = \theta\beta(1-\beta)/(1-c(1-\beta))$, $\mathbf{E}\tilde{\gamma} = \theta\beta^2/2$, $\operatorname{Var}\tilde{\beta} = \beta(1-\beta)/(1-c(1-\beta))^2$, $\operatorname{Var}\tilde{\gamma} = \beta(1-\beta) + c\beta(3\beta/2 - 1)$, $\operatorname{Cov}(\tilde{\alpha},\tilde{\beta}) = -\beta(1-\beta)/(1-c(1-\beta))$, $\operatorname{Cov}(\tilde{\alpha},\tilde{\gamma}) = -\beta(1-\beta)(c-1)$ *and* $\operatorname{Cov}(\tilde{\beta},\tilde{\gamma}) = \beta(1-\beta)(c-1)/(1-c(1-\beta))$.

We now state a moderate deviation asymptotics result for $(\alpha^n, \beta^n, \gamma^n)$. We assume as given a real-valued sequence b_n, $n \in \mathbb{N}$, such that $b_n \to \infty$ and $b_n/\sqrt{n} \to 0$ as $n \to \infty$. Let y^T denote the transpose of $y \in \mathbb{R}^3$.

THEOREM 2.3. *Let* $(\sqrt{n}/b_n)(c_n - c) \to \hat{\theta} \in \mathbb{R}$ *as* $n \to \infty$, *where* $c > 0$. *Then the following hold*.

1. *The sequence* $(\sqrt{n}/b_n)(\alpha^n/n-\alpha)$, $n\in\mathbb{N}$, *obeys the LDP in* $\mathbb{R}$ *for scale* b_n^2 *with action functional* $(x-\mu_\alpha)^2/(2\sigma_\alpha^2)$, $x\in\mathbb{R}$, *where* $\mu_\alpha=-\hat{\theta}(1-\beta^2)/2$ *and* $\sigma_\alpha^2=\beta(1-\beta)+c(1-\beta)^2/2$.

2. *If, in addition,* $c>1$, *then the sequence* $((\sqrt{n}/b_n)(\alpha^n/n-\alpha),(\sqrt{n}/b_n)\times(\beta^n/n-\beta),(\sqrt{n}/b_n)(\gamma^n/n-\gamma))$, $n\in\mathbb{N}$, *obeys the LDP in* $\mathbb{R}^3$ *for scale* b_n^2 *with action functional* $(y-\mu)^T\Sigma^{-1}(y-\mu)/2$, $y\in\mathbb{R}^3$, *where* $\mu=(\mu_\alpha,\mu_\beta,\mu_\gamma)^T$ *and* $\Sigma=\begin{pmatrix}\sigma_\alpha^2 & \sigma_{\alpha\beta} & \sigma_{\alpha\gamma}\\ \sigma_{\alpha\beta} & \sigma_\beta^2 & \sigma_{\beta\gamma}\\ \sigma_{\alpha\gamma} & \sigma_{\beta\gamma} & \sigma_\gamma^2\end{pmatrix}$ *are given by* $\mu_\beta=\hat{\theta}\beta(1-\beta)/(1-c(1-\beta))$, $\mu_\gamma=\hat{\theta}\beta^2/2$, $\sigma_\beta^2=\beta(1-\beta)/(1-c(1-\beta))^2$, $\sigma_\gamma^2=\beta(1-\beta)+c\beta(3\beta/2-1)$, $\sigma_{\alpha\beta}=-\beta(1-\beta)/(1-c(1-\beta))$, $\sigma_{\alpha\gamma}=-\beta(1-\beta)(c-1)$ *and* $\sigma_{\beta\gamma}=\beta(1-\beta)(c-1)/(1-c(1-\beta))$.

The list of results is concluded with the critical-graph case. We recall that excursions of a nonnegative function $\mathbf{x}=(\mathbf{x}_t,\ t\in\mathbb{R}_+)$ are defined as intervals $[s_i,t_i]$, where $s_i<t_i$, such that $\mathbf{x}_{s_i}=\mathbf{x}_{t_i}=0$ and $\mathbf{x}_p>0$ for $p\in(s,t)$, t_i-s_i is called the excursion's length; continuous functions have at most countably many excursions. Let, given $\tilde{\theta}\in\mathbb{R}$, process $\widetilde{X}=(\widetilde{X}_t,\ t\in\mathbb{R}_+)$ be defined as the Skorohod reflection of the process $(W_t+\tilde{\theta}t-t^2/2,\ t\in\mathbb{R}_+)$, where $W=(W_t,\ t\in\mathbb{R}_+)$ is a Wiener process. In the next theorem, $\widetilde{U}=(\widetilde{U}_1,\widetilde{U}_2,\dots)$ is the sequence of the excursion lengths of $\widetilde{X}$ arranged in descending order and $\widetilde{R}=(\widetilde{R}_1,\widetilde{R}_2,\dots)$ is the sequence of the increments of the process $(N_{\int_0^t\widetilde{X}_s\,ds},\ t\in\mathbb{R}_+)$ over these excursions, where $(N_t,\ t\in\mathbb{R}_+)$ is a Poisson process. Let $\mathbb{S}$ denote the subspace of $\mathbb{R}_+^{\mathbb{N}}$ of nonincreasing sequences $\mathbf{u}=(u_1,u_2,\dots)$ equipped with induced topology. The sequence b_n is defined as in Theorem 2.3.

THEOREM 2.4. 1. *Let* $n^{1/3}(c_n-1)\to\tilde{\theta}\in\mathbb{R}$ *as* $n\to\infty$. *Then the sequences* $\widetilde{U}^n=(U_1^n/n^{2/3},U_2^n/n^{2/3},\dots)$ *and* $\widetilde{R}^n=(R_1^n/n^{2/3},R_2^n/n^{2/3},\dots)$ *jointly converge in distribution in* $\mathbb{S}\times\mathbb{R}_+^{\mathbb{N}}$ *to the respective sequences* $\widetilde{U}=(\widetilde{U}_1,\widetilde{U}_2,\dots)$ *and* $\widetilde{R}=(\widetilde{R}_1,\widetilde{R}_2,\dots)$. *If, moreover,* $\sqrt{n}(c_n-1)\to\theta\in\mathbb{R}$ *as* $n\to\infty$ (*so* $\tilde{\theta}=0$), *then the* $(\widetilde{U}^n,\widetilde{R}^n)$ *are asymptotically independent of* $\sqrt{n}(\alpha^n/n-1/2)$ *so that* $(\sqrt{n}(\alpha^n/n-1/2),\widetilde{U}^n,\widetilde{R}^n)$ *jointly converge in distribution in* $\mathbb{R}\times\mathbb{S}\times\mathbb{R}_+^{\mathbb{N}}$ *to* $(\tilde{\alpha},\widetilde{U},\widetilde{R})$, *where* $(\widetilde{U},\widetilde{R})$ *correspond to* $\tilde{\theta}=0$, $\tilde{\alpha}$ *is independent of* $(\widetilde{U},\widetilde{R})$ *and is Gaussian with* $\mathbf{E}\tilde{\alpha}=-\theta/2$ *and* $\operatorname{Var}\tilde{\alpha}=1/2$.

2. *Let* $(n^{1/3}/b_n^{2/3})(c_n-1)\to\breve{\theta}\in\mathbb{R}$ *as* $n\to\infty$. *Then the sequences* $\breve{U}^n=(U_1^n/(nb_n)^{2/3},U_2^n/(nb_n)^{2/3},\dots)$ *and* $\breve{R}^n=(R_1^n/(nb_n)^{2/3},R_2^n/(nb_n)^{2/3},\dots)$ *jointly obey the LDP in* $\mathbb{S}\times\mathbb{R}_+^{\mathbb{N}}$ *for scale* b_n^2 *with action functional*

$$\breve{I}_{\breve{\theta}}^{U,R}(\mathbf{u},\mathbf{r})=-\frac{1}{24}\sum_{i=1}^\infty u_i^3+\frac{1}{6}\left(\sum_{i=1}^\infty u_i-\breve{\theta}\right)^3\vee 0+\frac{\breve{\theta}^3}{6}+\frac{1}{24}\sum_{i=1}^\infty u_i^3\pi\left(\frac{12r_i}{u_i^3}\right)$$

if $\sum_{i=1}^\infty u_i<\infty$ *and* $r_i=0$ *when* $u_i=0$, *and* $\breve{I}_{\breve{\theta}}^{U,R}(\mathbf{u},\mathbf{r})=\infty$ *otherwise, where*

$\mathbf{u} = (u_1, u_2, \dots)$ *and* $\mathbf{r} = (r_1, r_2, \dots)$. *If, moreover,* $(\sqrt{n}/b_n)(c_n - 1) \to \hat{\theta}$ *as* $n \to \infty$ (*so* $\breve{\theta} = 0$), *then the* $((\sqrt{n}/b_n)(\alpha^n/n - 1/2), \breve{U}^n, \breve{R}^n)$ *obey the LDP in* $\mathbb{R} \times \breve{\mathbb{S}} \times \mathbb{R}_+^{\mathbb{N}}$ *with action functional*

$$\breve{I}_{\theta}^{\alpha,U,R}(a, \mathbf{u}, \mathbf{r}) = \left(a + \frac{\hat{\theta}}{2}\right)^2 + \breve{I}_0^{U,R}(\mathbf{u}, \mathbf{r}).$$

COROLLARY 2.5. *Let* $(n^{1/3}/b_n^{2/3})(c_n - 1) \to \breve{\theta} \in \mathbb{R}$ *as* $n \to \infty$. *Then the following hold*:

1. *The sequence* $\breve{U}^n$, $n \in \mathbb{N}$, *obeys the LDP in* $\breve{\mathbb{S}}$ *for scale* b_n^2 *with action functional*

$$\breve{I}_{\breve{\theta}}^{U}(\mathbf{u}) = -\frac{1}{24}\sum_{i=1}^{\infty} u_i^3 + \frac{1}{6}\left(\sum_{i=1}^{\infty} u_i - \breve{\theta}\right)^3 \vee 0 + \frac{\breve{\theta}^3}{6}$$

if $\sum_{i=1}^{\infty} u_i < \infty$ *and* $\breve{I}_{\breve{\theta}}^{U}(\mathbf{u}) = \infty$ *otherwise.*

2. *The sequence* $\beta^n/(nb_n)^{2/3}, n \in \mathbb{N}$, *obeys the LDP in* $\mathbb{R}_+$ *for scale* b_n^2 *with action functional* $\breve{I}_{\breve{\theta}}^{\beta}$ *given by* $\breve{I}_{\breve{\theta}}^{\beta}(0) = \breve{\theta}^3 \vee 0/6$,

$$\breve{I}_{\breve{\theta}}^{\beta}(u) = -\left\lfloor \frac{\breve{\theta}}{u} \right\rfloor \frac{u^3}{24} - \frac{1}{24}\left(\left(2\left(\breve{\theta} - \left\lfloor \frac{\breve{\theta}}{u} \right\rfloor u\right)\right) \wedge u\right)^3$$
$$+ \frac{1}{6}\left(\left\lfloor \frac{\breve{\theta}}{u} \right\rfloor u + \left(2\left(\breve{\theta} - \left\lfloor \frac{\breve{\theta}}{u} \right\rfloor u\right)\right) \wedge u - \breve{\theta}\right)^3 + \frac{\breve{\theta}^3}{6}$$

if $u \in (0, \breve{\theta}^+)$ *and*

$$\breve{I}_{\breve{\theta}}^{\beta}(u) = -\frac{u^3}{24} + \frac{(u - \breve{\theta})^3}{6} + \frac{\breve{\theta}^3}{6}$$

if $u \geq \breve{\theta}^+$.

We now comment on the results and relate them to earlier ones. Equation (2.4) in a slightly different form appears in Barraez, Boucheron and Fernandez de la Vega (2000). The sequences $\overline{U}^n$ and $\overline{R}^n$ have been suggested by the form of the results of Aldous (1997). Corollary 2.3 implies, in particular, that provided there exist m components asymptotically nu_i in size, where $\sum_{i=1}^{m} u_i < 1 - 1/c$, then with probability close to 1, there exists another giant component. This can be explained by noting that the number of vertices outside of the m components is asymptotically equal to $n(1 - \sum_{i=1}^{m} u_i)$, so the "effective" expected degree of an outside node is $c(1 - \sum_{i=1}^{m} u_i) > 1$, which means there is enough potential for another giant component.

Part 3 of Corollary 2.4 is due to O'Connell (1998), who provides an alternative, elegant form of the action functional for $c > 1$ and $u > 0$: $I_c^{\beta}(u) = kK_c(u) + L_c(ku)$ for $u \in [x_k, x_{k-1}]$, $k \in \mathbb{N}$, where $x_0 = 1$ and the x_k, $k \in \mathbb{N}$, are the solutions of the equations $x_k/(1-\exp(-cx_k)) = 1 - kx_k$. [Note that the expression for the action functional in Theorem 3.1 of O'Connell (1998) has a misprint.] O'Connell (1998) also noted that the action functional I_c^β is not convex. The advantage of the form of I_c^β used in Corollary 2.4 is that it is suggestive of the structure of the most probable configuration with the largest component asymptotically nu in size: if $u \ge 1 - 1/c$, then the component of size nu is the only giant component, while if $u < 1 - 1/c$, then there are $\lfloor (1-1/c)/u \rfloor$ components, whose sizes are asymptotically nu, and one component asymptotically $n(\hat{u} \wedge u)$ in size. [A similar remark has been made by O'Connell (1998).] This conjecture is confirmed by the proof of Corollary 2.4. In addition, the number of components in an optimal configuration is asymptotically equal to $n(1-u-c(1-u)^2/2)$ if $u \ge 1-1/c$ and $n(1-\hat{\tau}-c(1-\hat{\tau})^2/2)$ if $u < 1-1/c$, where $\hat{\tau} = \lfloor (1-1/c)/u \rfloor u + \hat{u} \wedge u$.

Corollary 2.5 leads to similar conclusions. The action functional $\breve{I}_{\breve{\theta}}^\beta(u)$ can be written for $\breve{\theta} > 0$ and $u \in (0, 2\breve{\theta}]$ as $\breve{I}_{\breve{\theta}}^\beta(u) = -ku^3/24 + (ku - \breve{\theta})^3/6 + \breve{\theta}^3/6$, where $k \in \mathbb{N}$ is such that $u \in [\breve{\theta}/(k+1/2), \breve{\theta}/(k-1/2)]$. It is not convex for $\breve{\theta} > 0$ either. Figure 1 shows the action functional for $\breve{\theta} = 2$. [Note that the form of the curve is the same for all $\breve{\theta} > 0$ since $\breve{I}_{x\breve{\theta}}^\beta(xu) = x^3 \breve{I}_{\breve{\theta}}^\beta(u)$ for $x > 0$.] Interestingly, the graph of $\breve{I}_{\breve{\theta}}^\beta$ is reminiscent of the one of I_c^β given in O'Connell's paper (1998), which we reproduce in Figure 2 for comparison's sake. For $\breve{\theta} > 0$, the most probable configuration with the largest component asymptotically $(nb_n)^{2/3}u$ in size consists of only one such component if $u \ge \breve{\theta}$ and has $\lfloor \breve{\theta}/u \rfloor$ components asymptotically $(nb_n)^{2/3}u$ in size along with one component asymptotically $(nb_n)^{2/3}((2(\breve{\theta} - \lfloor \breve{\theta}/u \rfloor u)) \wedge u)$ in size if $u < \breve{\theta}$. Since the action functional $\breve{I}_{\breve{\theta}}^\beta(u)$ equals zero at the only point $u = 2\breve{\theta}^+$, the $\beta^n/(nb_n)^{2/3}$ converge in probability to $2\breve{\theta}^+$ as $n \to \infty$, which is consistent with the asymptotics $\beta/(c-1) \to 2$ as $c \downarrow 1$. There is also an analogue for the critical graph of Corollary 2.3 on the most probable "conditional" configurations. In particular, given there exists a component asymptotically $(nb_n)^{2/3}u$ in size, with probability tending to 1, it has asymptotically $(nb_n)^{2/3}u^3/12$ excess edges, and if $u < \breve{\theta}$, also with probability tending to 1, there exists another component asymptotically $(nb_n)^{2/3}2(\breve{\theta} - u)$ in size.

The first assertion of part 1 of Theorem 2.4 is due to Aldous (1997), who establishes the convergence of the sizes of the connected components for the stronger ℓ_2 topology. Our proof uses similar ideas. Part 2 of Theorem 2.4 can also be expressed as a statement on a certain type of convergence of excursions. Let idempotent process $\breve{X} = (\breve{X}_t,\, t \in \mathbb{R}_+)$ be defined as the reflection of the idempotent process $(\breve{W}_t + \breve{\theta}t - t^2/2,\, t \in \mathbb{R}_+)$, where $\breve{W} = (\breve{W}_t,\, t \in \mathbb{R}_+)$ is an idempotent Wiener process, and let $(\breve{N}_t,\, t \in \mathbb{R}_+)$ be an idempotent Poisson process independent of $\breve{W}$ [for the notions of idempotent probability the reader is referred either to Puhalskii (2001) or the Appendix]. Let $(\breve{U}_1, \breve{U}_2, \dots)$ be the sequence of

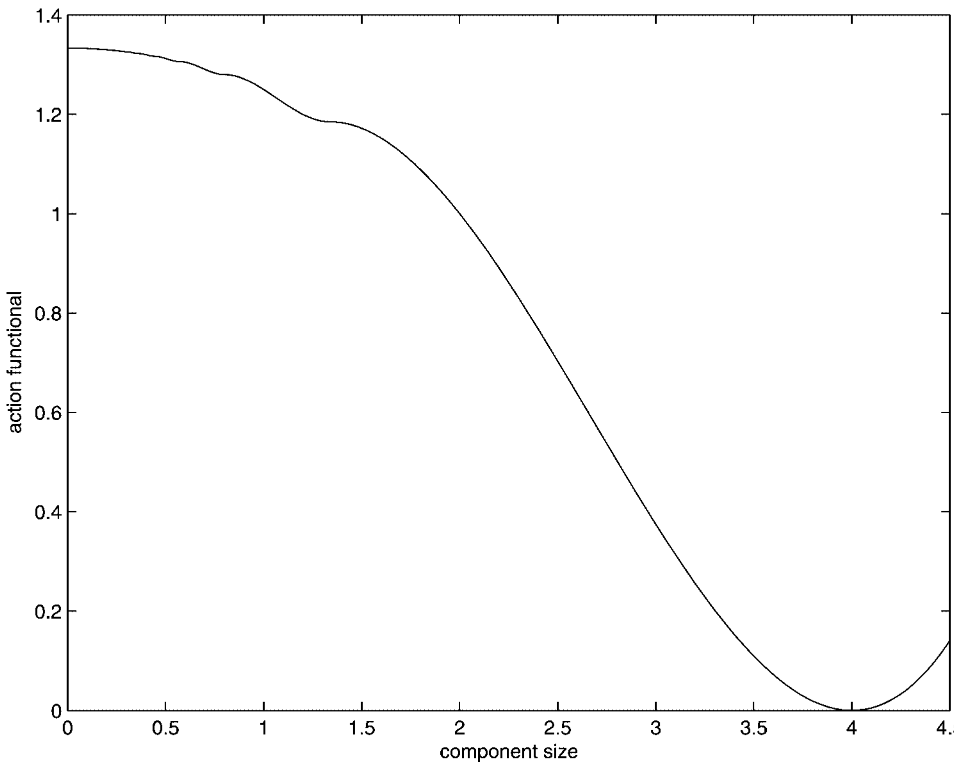

FIG. 1. *Moderate deviations of the size of the largest component of the critical graph* ($\breve{\theta} = 2$).

the excursion lengths of $\breve{X}$ arranged in descending order and $(\breve{R}_1, \breve{R}_2, \dots)$ be the sequence of the increments of $(\breve{N}_{\int_0^t \breve{X}_p\,dp}, t \in \mathbb{R}_+)$ over these excursions. Then the sequences $(U_1^n/(nb_n)^{2/3}, U_2^n/(nb_n)^{2/3}, \dots)$ and $(R_1^n/(nb_n)^{2/3}, R_2^n/(nb_n)^{2/3}, \dots)$ jointly large deviation converge in distribution in $\breve{\mathbb{S}} \times \mathbb{R}_+^{\mathbb{N}}$ at rate b_n^2 to the respective sequences $(\breve{U}_1, \breve{U}_2, \dots)$ and $(\breve{R}_1, \breve{R}_2, \dots)$ as $n \to \infty$. (The definition of large deviation convergence is recalled in Section 3.) Thus, the actual assertion combines statements on large deviation convergence and on the idempotent distribution of the limit. The LDP for $(\overline{U}^n, \overline{R}^n)$ of Corollary 2.1 admits a similar reformulation.

Part 1 of Theorem 2.2 for the case where $c_n = c$ and, accordingly, $\theta = 0$ is due to Stepanov (1970a, 1972). Part 2 of Theorem 2.2 complements the results of Stepanov (1970a, 1972) [see also Pittel (1990) and Barraez, Boucheron and Fernandez de la Vega (2000)] by allowing for $\theta \neq 0$, incorporating γ^n and indicating the covariance of $\tilde{\alpha}$ and $\tilde{\beta}$. As to be expected, the latter two random variables are negatively correlated. Parts 1 and 2 of Theorem 2.3, equivalently, state that the $(\sqrt{n}/b_n)(\alpha^n/n - \alpha)$ and $((\sqrt{n}/b_n)(\alpha^n/n - \alpha), (\sqrt{n}/b_n)(\beta^n/n - \beta), (\sqrt{n}/b_n)(\gamma^n/n - \gamma))$ large deviation converge at rate b_n^2 as $n \to \infty$ to Gaussian idempotent variables with respective parameters $(\mu_\alpha, \sigma_\alpha^2)$ and (μ, Σ). This

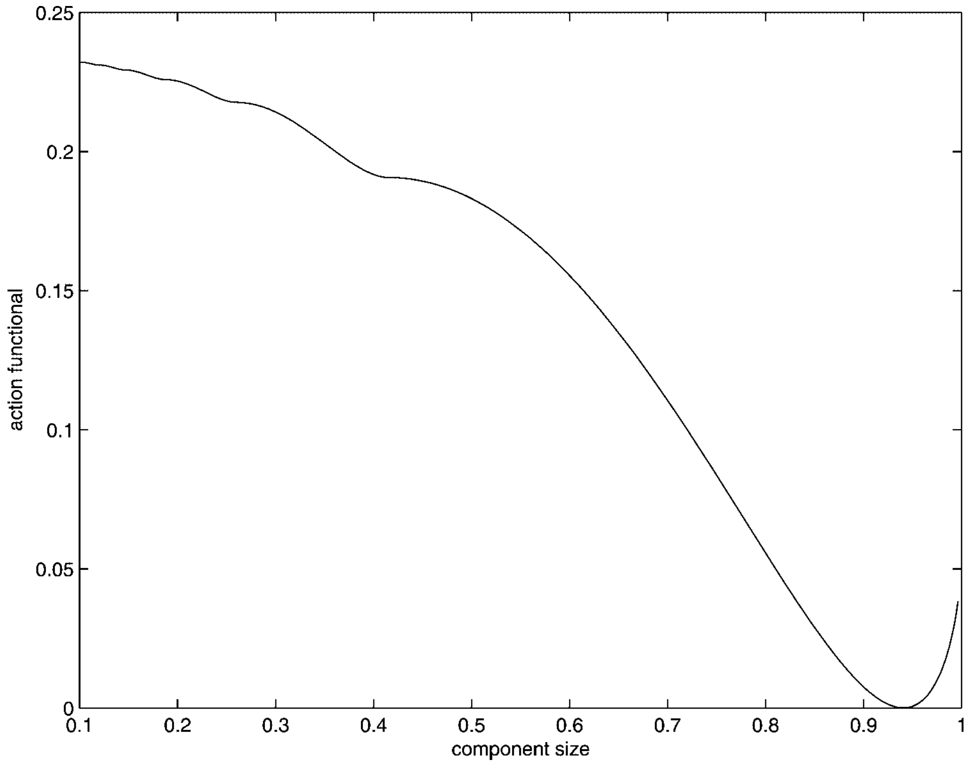

FIG. 2. *Large deviations of the size of the largest component of* $\mathcal{G}(n, 3/n)$ [*O'Connell* (1998)].

formulation not only emphasizes analogy with Theorem 2.2, but is instrumental in the proof below.

We now consider implications of the LDP for α^n/n of Corollary 2.2, which provide some revealing insights. The derivative with respect to τ of the function in the infimum on the right-hand side of the expression for I_c^α equals

$$2\left(1-\frac{a}{1-\tau}\right)-\frac{c\tau}{e^{c\tau}-1}-\log\left(2\left(1-\frac{a}{1-\tau}\right)\right)+\log\frac{c\tau}{e^{c\tau}-1}.$$

Since $\tau \ge 1-2a$, the derivative is nonnegative if and only if $a \ge (1-\tau)(1-c\tau/(2(e^{c\tau}-1)))$. The function on the right-hand side of the latter inequality, as a function of $\tau \in [0,1]$, is concave, is decreasing if $c \le 2$ and is first increasing and then decreasing if $c > 2$. Let $a^* \in [1/2, 1]$ denote the maximum of this function on $[0,1]$. For $a \in [0, a^*]$, the equation

$$a = (1-\tau)\left(1-\frac{1}{2}\frac{c\tau}{e^{c\tau}-1}\right) \tag{2.15}$$

has one root if either $a < 1/2$ or $a = a^*$ and has two roots otherwise. Let $\tau^*(a) \in [0,1]$, where $a \in [0, a^*]$, denote the greatest root of (2.15). Then the infimum on the right-hand side of the expression for I_c^α is attained at $\tau = \tau^*(a)$ if $a \in [0, 1/2]$, at $\tau = 0$ if $a \in [a^*, 1]$ and either at $\tau = \tau^*(a)$ or $\tau = 0$ if $a \in (1/2, a^*)$. Accordingly, the optimal configuration has either one giant component asymptotically $n\tau^*(a)$ in size or no giant components. We can, therefore, write

$$I_c^\alpha(a) = K_c(\tau^*(a)) + L_c(\tau^*(a)) + \frac{c(1-\tau^*(a))^2}{2}\pi\left(\frac{2(1-a-\tau^*(a))}{c(1-\tau^*(a))^2}\right) \tag{2.16}$$

if $a \in [0, 1/2]$,

$$I_c^\alpha(a) = \left(\frac{c}{2}\pi\left(\frac{2(1-a)}{c}\right)\right) \wedge \left(K_c(\tau^*(a)) + L_c(\tau^*(a)) + \frac{c(1-\tau^*(a))^2}{2}\pi\left(\frac{2(1-a-\tau^*(a))}{c(1-\tau^*(a))^2}\right)\right) \tag{2.17}$$

if $a \in (1/2, a^*)$, and

$$I_c^\alpha(a) = \frac{c}{2}\pi\left(\frac{2(1-a)}{c}\right) \tag{2.18}$$

if $a \in [a^*, 1]$. If $c \le 2$, the action functional is, in fact, given by (2.16) and (2.18) since $a^* = 1/2$. It is seen to be convex and differentiable in a. If $c > 2$, then $a^* > 1/2$, the difference between the first and the second functions in the minimum on the right-hand side of (2.17) is positive for $a = 1/2$, is decreasing in a for $a > 1/2$, and there exists a unique $\hat{a} \in (1/2, a^*)$ where these two functions are equal. Thus, for $c > 2$,

$$I_c^\alpha(a) = \begin{cases} K_c(\tau^*(a)) + L_c(\tau^*(a)) \\ \quad + \dfrac{c(1-\tau^*(a))^2}{2}\pi\left(\dfrac{2(1-a-\tau^*(a))}{c(1-\tau^*(a))^2}\right), & \text{if } a \in [0, \hat{a}], \\ \dfrac{c}{2}\pi\left(\dfrac{2(1-a)}{c}\right), & \text{if } a \in [\hat{a}, 1]. \end{cases}$$

For $c > 2$, the function $I_c^\alpha(a)$ is strictly convex to the right of $\hat{a}$ and is strictly concave in a neighborhood to its left. As a matter of fact, there exists $\tilde{a} \in (0, 1/2)$ such that $I_c^\alpha(a)$ is strictly convex for $a < \tilde{a}$ (and $a > \hat{a}$), and is strictly concave for $\tilde{a} < a < \hat{a}$. The value of $\tilde{a}$ is given by (2.15) for $\tau = \tilde{\tau}$, where $\tilde{\tau}$ solves the equation $\exp(-c\tau) - 1 + c\tau = c\tau^2$. In addition, $\tilde{a} \downarrow 0$ and $\hat{a} \uparrow 1$ as $c \to \infty$ (in fact, $\tilde{a} < 2/c$ for $c > 2$), so the concavity region grows as c does. Figure 3 shows the action functionals for various values of c and Figure 4 shows the regions of convexity and concavity for I_c^α.

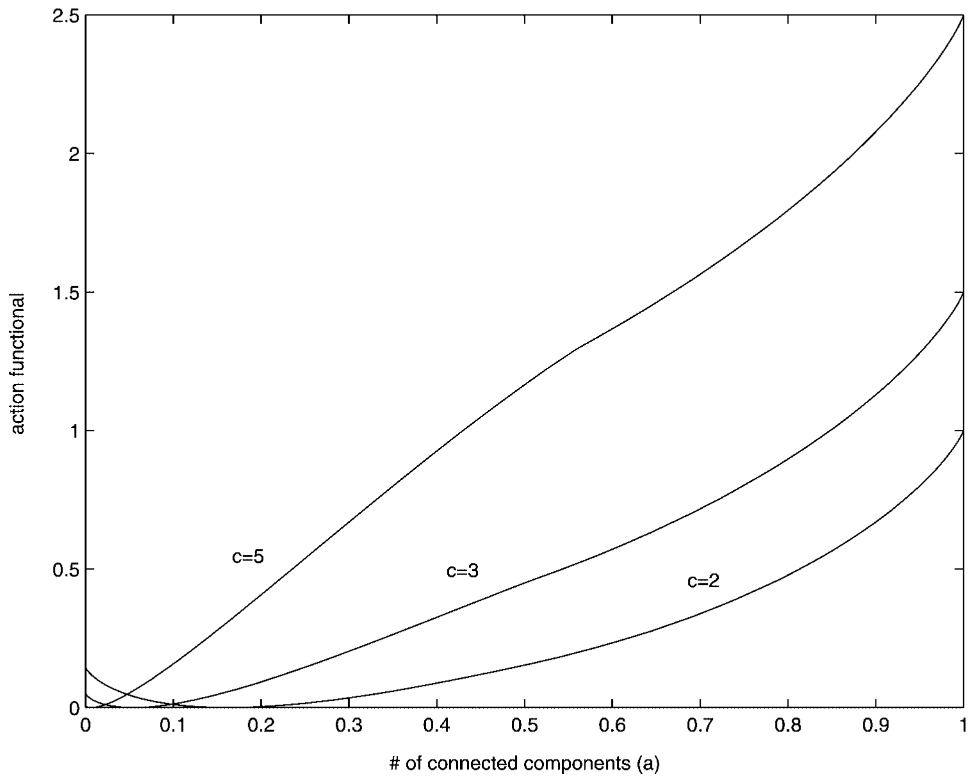

FIG. 3. *LDP for the number of connected components.*

Another distinguishing feature of point $\hat{a}$ is that at it, the left derivative of $I_c^\alpha(a)$ is greater than the right one, $I_c^\alpha(a)$ being differentiable in a elsewhere. It is, moreover, a point of phase transition: for $a < \hat{a}$, the most probable configuration has one giant component asymptotically $n\tau^*(a)$ in size, while for $a > \hat{a}$, it is optimal to have no giant components. Hence, for $c > 2$, we have the following structure of the random graph with a given number of components of order na: for small values of a, with probability close to 1, there is one giant component asymptotically $n\tau^*(a)$ in size and many (actually asymptotically na) small components of sizes not greater than of order $o(n)$ (it can be conjectured their sizes are of order $\log n$ or less); as a increases, more small components split off from the giant component and the size of the giant component decreases gradually; however, at $a = \hat{a}$ the giant component breaks up in that its size drastically reduces from being of order $n\tau^*(\hat{a})$ to being of order $o(n)$, and for $a > \hat{a}$, only small components remain which disintegrate further as a increases. If $c \leq 2$, then as a increases, the giant component, which is asymptotically $n\tau^*(a)$ in size, gradually decreases in size and disappears at $a = 1/2$, so no drastic changes occur. We thus shed new light on the observation by Stepanov (1970b) [see also Bollobás, Grimmett and Janson (1996)] of $c = 2$ being a critical point.

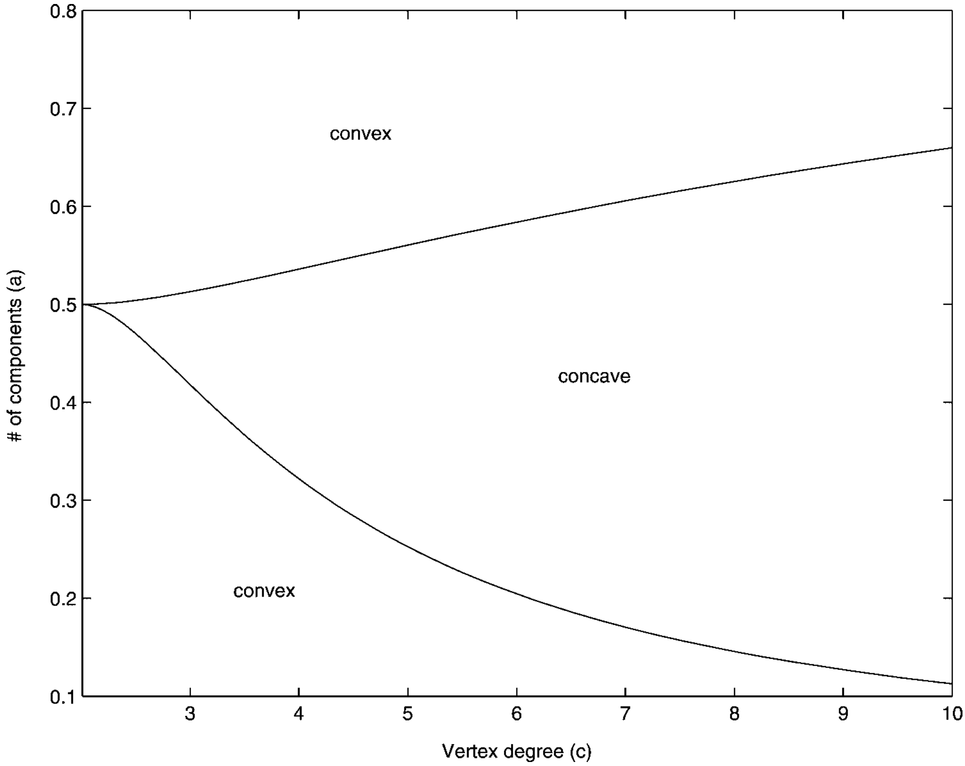

FIG. 4. *Convexity–concavity regions for* I_c^α.

There is another connection between our results and those of Stepanov (1970b), as well as of Bollobás, Grimmett and Janson (1996), to which we alluded in the Introduction. The above observation has been made by Stepanov (1970b) on the basis of the asymptotics

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbf{E} e^{\lambda \alpha^n} = S_c(\lambda), \qquad \lambda \in \mathbb{R}, \tag{2.19}$$

where

$$\begin{aligned} S_c(\lambda) = \sup_{\tau \in [(1-e^\lambda/c)^+,1]} \biggl(&\lambda(1-\tau) + \frac{c}{2}(1-\tau)^2 e^{-\lambda} - (1-\tau)\log(1-\tau) \\ &- \frac{c}{2}(1-\tau^2) - \tau\log\tau + \tau\log(1-e^{-c\tau})\biggr), \end{aligned} \tag{2.20}$$

and a subsequent analysis of the function $S_c(\lambda)$. We are able to reproduce (2.19) by using the LDP for α^n/n and Varadhan's lemma; see, for example, Dembo and Zeitouni (1998). Moreover, since I_c^α is strictly convex for $c \le 2$, it is possible to derive the LDP for α^n/n of Corollary 2.2 from limit (2.19) via Gärtner's theorem, see Gärtner (1977) or Freidlin and Wentzell (1998), so that $I_c(\alpha)$ is given by the Legendre–Fenchel transform of $S_c(\lambda)$. This has been done actually by Bollobás, Grimmett and Janson (1996), who obtain asymptotics (2.19) independently of Stepanov (1970b) and, in effect, provide a solution to the optimization problem (2.20), though they do not find the form of I_c^α in Corollary 2.2. However, for $c > 2$, Gärtner's theorem is not applicable because of "the onset of concavity" described above. The Legendre–Fenchel transform of $S_c(\lambda)$, being the convex hull of $I_c(\alpha)$, no longer coincides with $I_c(\alpha)$, which causes Bollobás, Grimmett and Janson's (1996) stopping short of obtaining the above LDP.

3. Technical preliminaries. In this section we collect pieces of terminology and notation used throughout the paper, recall some results on weak convergence and large deviation asymptotics pertinent to the developments below, and provide a number of auxiliary lemmas.

We denote by $\mathbb{D}_C([a,b],\mathbb{R}^d)$, where $d \in \mathbb{N}$, the space of right-continuous with left-hand limits $\mathbb{R}^d$-valued functions on an interval $[a,b]$ equipped with uniform metric and Borel σ-algebra. Space $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$ is defined as the space of $\mathbb{R}^d$-valued right-continuous with left-hand limits functions on $\mathbb{R}_+$ equipped with the Skorohod topology and Borel σ-algebra. Spaces $\mathbb{C}([0,1],\mathbb{R}^d)$ and $\mathbb{C}(\mathbb{R}_+,\mathbb{R}^d)$ are the subspaces of the respective spaces $\mathbb{D}_C([0,1],\mathbb{R}^d)$ and $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$ consisting of continuous functions with induced topologies. Elements of these spaces are mostly denoted by boldface lower-case Roman letters, for example, $\mathbf{x} = (\mathbf{x}_t,\ t \in [a,b])$; $\mathbf{x}_{t-}$ denotes the left-hand limit at t; $\dot{\mathbf{x}}_t$ denotes the Radon–Nykodim derivative at t with respect to Lebesgue measure of an absolutely continuous $\mathbf{x}$. We denote by p_1 the projection $(\mathbf{x}_t,\ t \in \mathbb{R}_+) \to (\mathbf{x}_t,\ t \in [0,1])$ from $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$ to $\mathbb{D}_C([0,1],\mathbb{R}^d)$ and note that it is continuous at $\mathbf{x} \in \mathbb{C}(\mathbb{R}_+,\mathbb{R}^d)$. Maps $\mathcal{R}$ and $\mathcal{T}$ from $\mathbb{D}(\mathbb{R}_+,\mathbb{R})$ to $\mathbb{D}(\mathbb{R}_+,\mathbb{R})$ are defined by $\mathcal{R}(\mathbf{x})_t = \mathbf{x}_t - \inf_{s\in[0,t]} \mathbf{x}_s \wedge 0$ and $\mathcal{T}(\mathbf{x})_t = -\inf_{s\in[0,t]} \mathbf{x}_s \wedge 0$. If $\mathbf{x}_0 \geq 0$, then the functions $\mathbf{y} = \mathcal{R}(\mathbf{x})$ and $\phi = \mathcal{T}(\mathbf{x})$ can be equivalently defined as a solution to a Skorohod problem in that $\mathbf{y} = \mathbf{x} + \phi$, $\mathbf{y}_t \geq 0$, ϕ is nondecreasing with $\phi_0 = 0$ and $\phi_t = \int_0^t \mathbf{1}(\mathbf{y}_s = 0)\, d\phi_s$, $t \in \mathbb{R}_+$. Unless specified otherwise, "almost everywhere (a.e.)" refers to Lebesgue measure and product topological spaces are equipped with product topologies; besides, $\inf_\varnothing$ is understood as ∞ and $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra on $\mathbb{R}$.

We assume that all the random objects we consider are defined on a complete probability space $(\Omega,\mathcal{F},\mathbf{P})$, the expectation of a random variable ξ is denoted as $\mathbf{E}\xi$. For a sequence of $\mathbb{R}^d$-valued random variables ξ_n, $n \in \mathbb{N}$, and a sequence of real numbers $k_n \to \infty$, we write $\xi_n \overset{\mathbf{P}^{1/k_n}}{\to} 0$ and say that the ξ_n tend to zero super-exponentially in probability at rate k_n if $\lim_{n\to\infty} \mathbf{P}(|\xi_n| > \varepsilon)^{1/k_n} = 0$ for arbitrary $\varepsilon > 0$. We also let $\overset{\mathbf{P}}{\to}$ denote convergence in probability, $\overset{d}{\to}$ denote convergence in distribution in the associated metric space, and $\underset{k_n}{\overset{ld}{\to}}$ denote large deviation (LD) convergence in distribution at rate k_n. To recall the definition of the latter [see, e.g., Puhalskii (2001)], we say that a $[0,1]$-valued function $\mathbf{\Pi}$, defined on the power set of a metric space Υ, is a deviability on Υ if $\mathbf{\Pi}(\Gamma) = \sup_{\upsilon\in\Gamma}\exp(-I(\upsilon))$, $\Gamma \subset \Upsilon$, where I is an action functional on Υ, that is, a $[0,\infty]$-valued function on Υ such that the sets $\{\upsilon \in \Upsilon : I(\upsilon) \le a\}$ are compact for $a \in \mathbb{R}_+$. We say that a sequence $\mathbf{P}_n$, $n \in \mathbb{N}$, of probability measures on the Borel σ-algebra of Υ LD converges at rate k_n to a deviability $\mathbf{\Pi}$ on Υ if $\lim_{n\to\infty}(\int_\Upsilon f(\upsilon)^{k_n}\, d\mathbf{P}_n(\upsilon))^{1/k_n} = \sup_{\upsilon\in\Upsilon} f(\upsilon)\mathbf{\Pi}(\{\upsilon\})$ for every continuous bounded $\mathbb{R}_+$-valued function f on Υ. Equivalently, the sequence $\mathbf{P}_n$, $n \in \mathbb{N}$, LD converges at rate k_n to $\mathbf{\Pi}$ if it obeys the LDP with action functional I for scale k_n. We recall that if the sequence $\mathbf{P}_n$ is exponentially tight of order k_n, that is, for every $\varepsilon > 0$, there exists a compact $K \subset \Upsilon$ such that $\limsup_{n\to\infty}\mathbf{P}_n(\Upsilon \setminus K)^{1/k_n} < \varepsilon$, then it is LD relatively sequentially compact, that is, for every subsequence $\mathbf{P}_{n'}$ there exists a subsubsequence $\mathbf{P}_{n''}$ that LD converges at rate $k_{n''}$ to some deviability; every such deviability is called an LD accumulation point of the $\mathbf{P}_n$. We also say that a sequence of random variables X_n, $n \in \mathbb{N}$, with values in Υ LD converges in distribution at rate k_n to a Luzin idempotent variable X with values in Υ if the sequence of laws of the X_n LD converges at rate k_n to the idempotent distribution of X.

Let $H^n = (H^n_t, t \in \mathbb{R}_+)$, $n \in \mathbb{N}$, be a sequence of $\mathbb{R}^d$-valued stochastic processes having right-continuous, with left-hand limits, paths. The sequence H^n is said to be $\mathbb{C}$-tight if the sequence of the distributions of the H^n on $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$ is tight for weak convergence of probability measures on $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$, with its every accumulation point being the law of a continuous process. The following limits provide necessary and sufficient conditions for $\mathbb{C}$-tightness:

$$\lim_{B\to\infty}\limsup_{n\to\infty}\mathbf{P}(|H^n_0| > B) = 0,$$

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbf{P}\left(\sup_{s,t\in[0,T]:\,|s-t|\le\delta}|H^n_t - H^n_s| > \varepsilon\right) = 0, \qquad T \in \mathbb{R}_+,\ \varepsilon > 0.$$

The sequence H^n is said to be $\mathbb{C}$-exponentially tight of order k_n if the sequence of the distributions of the H^n is exponentially tight of order k_n as a sequence of probability measures on $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$ and its every LD accumulation point $\mathbf{\Pi}$ is such that $\mathbf{\Pi}(\mathbf{x}) = 0$ for every $\mathbf{x} \in \mathbb{D}(\mathbb{R}_+,\mathbb{R}^d) \setminus \mathbb{C}(\mathbb{R}_+,\mathbb{R}^d)$. The sequence of laws of the H^n is $\mathbb{C}$-exponentially tight of order k_n if and only if

$$\lim_{B\to\infty}\limsup_{n\to\infty}\mathbf{P}(|H^n_0| > B)^{1/k_n} = 0,$$

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbf{P}\left(\sup_{s,t\in[0,T]:\,|s-t|\le\delta}|H^n_t - H^n_s| > \varepsilon\right)^{1/k_n} = 0, \qquad T \in \mathbb{R}_+,\ \varepsilon > 0.$$

We denote by ξ^n_{ij} and ζ^n_{ij}, where $i \in \mathbb{N}$, $j \in \mathbb{N}$, $n \in \mathbb{N}$, i.i.d. Bernoulli random variables on $(\Omega,\mathcal{F},\mathbf{P})$ with $\mathbf{P}(\xi^n_{ij} = 1) = c_n/n$ and define $\mathcal{F}^n_t$, $t \in \mathbb{R}_+$, as the σ-algebras generated by the ξ^n_{ij} and ζ^n_{ij} for $i = 1, 2, \dots, \lfloor n(t \wedge 1)\rfloor$, $j \in \mathbb{N}$, completed with sets of $\mathbf{P}$-measure zero, and introduce filtrations $\mathbf{F}^n = (\mathcal{F}^n_t, t \in \mathbb{R}_+)$.

LEMMA 3.1. *Let $c_n \to c > 0$ as $n \to \infty$. Let $b_n \to \infty$ and $b_n/\sqrt{n} \to 0$ as $n \to \infty$. The following convergences hold as $n \to \infty$:*

$$\sup_{t\in[0,1]} |\bar{\varepsilon}_t^n| \overset{\mathbf{P}^{1/n}}{\to} 0, \qquad \sup_{t\in[0,1]} \frac{\sqrt{n}}{b_n}|\bar{\varepsilon}_t^n| \overset{\mathbf{P}^{1/b_n^2}}{\to} 0, \qquad \sup_{t\in[0,1]} \sqrt{n}|\bar{\varepsilon}_t^n| \overset{\mathbf{P}}{\to} 0$$

and

$$\sup_{t\in\mathbb{R}_+} \frac{|\varepsilon^n_{\lfloor n^{2/3}t\rfloor\wedge n}|}{n^{1/3}} \overset{\mathbf{P}}{\to} 0, \qquad \sup_{t\in\mathbb{R}_+} \frac{|\varepsilon^n_{\lfloor (nb_n)^{2/3}t\rfloor\wedge n}|}{n^{1/3}b_n^{4/3}} \overset{\mathbf{P}^{1/b_n^2}}{\to} 0.$$

PROOF. We prove the convergences on the first line. By (2.7) and (2.11),

$$\sup_{t\in[0,1]} |\bar{\varepsilon}_t^n| \le \frac{1}{n} + \frac{1}{n}\sum_{k=1}^{n} \xi^n_{k,n-k+1}. \tag{3.1}$$

The right-most convergence follows since $\mathbf{E}\xi^n_{k,n-k+1} = c_n/n$. Next, by (3.1) and the exponential Markov inequality for $\delta > 0$ and $\lambda > 0$,

$$\mathbf{P}\Bigl(\sup_{t\in[0,1]} |\bar{\varepsilon}_t^n| > \delta\Bigr)^{1/n} \le e^{\lambda/n}\mathbf{E}e^{\lambda\xi^n_{1,1}}e^{-\lambda\delta} \to e^{-\lambda\delta} \qquad \text{as } n \to \infty.$$

The left-most convergence in the statement of the theorem follows since λ is arbitrary. Finally,

$$\mathbf{P}\Bigl(\sup_{t\in[0,1]} \frac{\sqrt{n}}{b_n}|\bar{\varepsilon}_t^n| > \delta\Bigr)^{1/b_n^2} \le e^{1/b_n^2}\bigl(\mathbf{E}e^{\xi^n_{1,1}}\bigr)^{n/b_n^2}e^{-\delta\sqrt{n}/b_n} \to 0 \qquad \text{as } n \to \infty,$$

proving the convergence in the middle.

The convergences on the second line are proved similarly. □

In the next three lemmas we assume that $c > 0$.

LEMMA 3.2. 1. *The function $K_\rho(u)$, $u \in [0, 1]$, $\rho \in \mathbb{R}_+$, equals 0 when either $u = 0$ or $\rho = 0$, is strictly decreasing, strictly concave and strictly subadditive in each of the variables u and ρ when the other variable is positive. The function $L_c(u)$, $u \in [0, 1]$, equals 0 at $u = 0$ and is strictly increasing in u.*

2. *If $u \in [(1 - 1/c)^+, 1]$, then the function $K_c(x) + L_c(u + x)$ as a function of x is strictly increasing for $x \in [0, 1 - u]$. If $c > 1$ and $u \in [0, 1 - 1/c)$, then $K_c(x) + L_c(u + x)$ is strictly increasing for $x \in [0, \tilde{u}]$, is strictly decreasing for $x \in [\tilde{u}, u^*]$, and is strictly increasing for $x \in [u^*, 1 - u]$, where $\tilde{u} \in [0, 1 - 1/c - u]$ is the solution of the equation*

$$\frac{x}{1 - e^{-cx}} + x = 1 - u$$

and $u^* \in (1 - 1/c - u, \beta - u]$ *is the solution of the equation*

$$\frac{x}{1 - e^{-cx}} = 1 - u.$$

The values of the function at $x = u^*$ *and* $x = 0$ *coincide*: $K_c(u^*) + L_c(u + u^*) = L_c(u)$.

PROOF. Part 1 follows from the definitions. Part 2 follows by the equality

$$\frac{\partial}{\partial x}\big(K_c(x) + L_c(u + x)\big) = \big(c(1 - u - x) - \log\big(c(1 - u - x)\big)\big) - \left(\frac{cx}{1 - e^{-cx}} - \log\frac{cx}{1 - e^{-cx}}\right)$$

and the fact that the function $x - \log x$ is decreasing for $x \in (0, 1)$ and is increasing for $x > 1$. □

Let $0 \le s < t \le 1$ and $\Lambda_{s,t}$ denote the set of absolutely continuous real-valued functions $\mathbf{x} = (\mathbf{x}_p,\ p \in [s, t])$ with $\dot{\mathbf{x}}_p \ge -1$ a.e. and $1 - p - \mathbf{x}_p \ge 0$ on $[s, t]$. We denote for $\mathbf{x} \in \Lambda_{s,t}$,

$$I^S_{s,t}(\mathbf{x}) = \int_s^t \pi\left(\frac{\dot{\mathbf{x}}_p + 1}{c(1 - p - \mathbf{x}_p)}\right) c(1 - p - \mathbf{x}_p)\, dp.$$

Let also for $0 < \breve{s} < \breve{t}$, absolutely continuous real-valued $\mathbf{x} = (\mathbf{x}_p,\ p \in [\breve{s}, \breve{t}])$, and $\breve{\theta} \in \mathbb{R}$,

$$\breve{I}^S_{\breve{s},\breve{t}}(\mathbf{x}) = \tfrac{1}{2}\int_{\breve{s}}^{\breve{t}} (\dot{\mathbf{x}}_p + p - \breve{\theta})^2\, dp.$$

LEMMA 3.3. 1. *Given* $w \in (0, (t-s)^2/2)$, *the infimum of* $I^S_{s,t}(\mathbf{x})$ *over* $\mathbf{x} \in \Lambda_{s,t}$, *such that* $\mathbf{x}_s = \mathbf{x}_t = 0$ *and* $\int_s^t \mathbf{x}_p\, dp = w$, *is attained at*

$$\tilde{\mathbf{x}}_p(s, t) = s - p + \frac{t - s}{1 - e^{-\tilde{\rho}(t-s)}}\big(1 - e^{-\tilde{\rho}(p-s)}\big), \qquad p \in [s, t],$$

where $\tilde{\rho} \in \mathbb{R}_+$ *satisfies the equality* $\partial K_\rho(t - s)/\partial\rho = -w$, *that is,*

$$\frac{\tilde{\rho}(t - s)}{1 - e^{-\tilde{\rho}(t-s)}} = 1 + \frac{w\tilde{\rho}}{t - s} + \frac{1}{2}\tilde{\rho}(t - s).$$

The value of the infimum equals $K_{\tilde{\rho}}(t - s) + (\tilde{\rho} - c)w + L_c(t) - L_c(s) = \sup_{\rho\in\mathbb{R}_+}(K_\rho(t - s) + (\rho - c)w) + L_c(t) - L_c(s)$.

If $w = 0$, *then the infimum is attained at* $\tilde{\mathbf{x}}_p(s, t) = 0,\ p \in [s, t]$, *and is equal to* $L_c(t) - L_c(s)$.

2. *Given* $\breve{w} \in \mathbb{R}_+$, *the infimum of* $\breve{I}^S_{\breve{s},\breve{t}}(\mathbf{x})$ *over absolutely continuous real-valued functions* $\mathbf{x} = (\mathbf{x}_p,\ p \in [\breve{s}, \breve{t}])$, *such that* $\mathbf{x}_s = \mathbf{x}_t = 0$ *and* $\int_{\breve{s}}^{\breve{t}} \mathbf{x}_p\, dp = \breve{w}$, *is attained at*

$$\breve{\mathbf{x}}_p(\breve{s}, \breve{t}) = 6\breve{w}\, \frac{(p - \breve{s})(\breve{t} - p)}{(\breve{t} - \breve{s})^3}, \qquad p \in [\breve{s}, \breve{t}],$$

and equals

$$\frac{6\breve{w}^2}{(\breve{t} - \breve{s})^3} - \breve{w} + \frac{(\breve{t} - \breve{\theta})^3 - (\breve{s} - \breve{\theta})^3}{6}.$$

PROOF. Let C denote the closed convex subset of the Banach space of real-valued Lebesgue measurable functions $h = (h_p, p \in [s, t])$ with norm $\|h\| = \int_s^t |h_p|\, dp$, specified by the conditions $h_p \ge 0$ a.e., $\int_s^t h_p\, dp = t - s$, and $\int_s^t \int_s^p h_q\, dq\, dp = w + (t - s)^2/2$. We define a $[0, \infty]$-valued functional F on C by

$$F(h) = \int_s^t \pi\left(\frac{h_p}{c(1 - s - \int_s^p h_q\, dq)}\right) c\left(1 - s - \int_s^p h_q\, dq\right) dp.$$

On noting that for $h \in C$,

$$\begin{aligned} F(h) &= \int_s^t \left(h_p \log \frac{h_p}{c} + c\left(1 - s - \int_s^p h_q\, dq\right)\right) dp \\ &\quad + (1 - t)\log(1 - t) - (1 - s)\log(1 - s) \\ &= \int_s^t h_p \log h_p\, dp + (t - s)\big(c(1 - s) - \log c\big) - c\left(w + \frac{(t - s)^2}{2}\right) \\ &\quad + (1 - t)\log(1 - t) - (1 - s)\log(1 - s), \end{aligned}$$

we see that F is strictly convex on C. Therefore, the infimum of F on C is attained at a stationary point if the latter exists. The method of Lagrange multipliers shows that $\tilde{h}_p = (\tilde{\rho}(t - s)/(1 - e^{-\tilde{\rho}(t-s)}))e^{-\tilde{\rho}(p-s)}$ is such a point. The assertion of part 1 of the lemma follows.

For part 2 we apply the classical method of solving the isoperimetric problem, see, for example, Alekseev, Tikhomirov and Fomin (1987). □

LEMMA 3.4. 1. *Let* $a \in [0, 1]$ *and* $\tau \in [0, 1]$. *Then the infimum of*

$$\int_0^1 \pi\left(\frac{1 - \dot{\phi}_t}{c(1 - t)}\right) c(1 - t)\, dt$$

over absolutely continuous nondecreasing functions $\phi = (\phi_t,\ t \in [0, 1])$, *such that* $\phi_0 = 0$, $\phi_1 = a$, $\dot{\phi}_t \le 1$ *a.e., and the Lebesgue measure of the set where* $\dot{\phi}_t = 0$ *is at least* τ, *equals*

$$L_c\big((1 - 2a) \vee \tau\big) + \frac{c}{2}\big(1 - (1 - 2a) \vee \tau\big)^2 \pi\left(\frac{2(1 - a - (1 - 2a) \vee \tau)}{c(1 - (1 - 2a) \vee \tau)^2}\right).$$

2. *Let* $\breve{\tau} \in \mathbb{R}_+$ *and* $\breve{\theta} \in \mathbb{R}$. *Then the infimum of* $\int_0^\infty (-\dot{\phi}_t - \breve{\theta} + t)^2 \, dt/2$ *over absolutely continuous nondecreasing functions* $\phi = (\phi_t, \, t \in \mathbb{R}_+)$, *such that* $\phi_0 = 0$ *and the Lebesgue measure of the set where* $\dot{\phi}_t = 0$ *is at least* $\breve{\tau}$, *equals* $((\breve{\tau} - \breve{\theta})^3 \vee 0 + \breve{\theta}^3)/6$.

PROOF. We prove part 1. The optimizing integral can be written for a suitable function g as

$$\int_0^1 \bigl(c(1-t) - \log\bigl(c(1-t)\bigr)\bigr)\, dt + \int_0^1 g(\dot{\phi}_t)\, dt + \int_0^1 \dot{\phi}_t \log(1-t)\, dt.$$

Let $\dot{\phi}^*$ denote the increasing rearrangement of $\dot{\phi}$ defined by $\dot{\phi}_t^* = \sup\{\lambda \in \mathbb{R}_+ : \mu_\phi(\lambda) \le t\}$, where $\mu_\phi(\lambda)$ is the Lebesgue measure of those $t \in [0,1]$ for which $\dot{\phi}_t \le \lambda$. Since the function $\log(1-t)$ is decreasing, by a Hardy–Littlewood inequality, see Bennett and Sharpley (1988) or DeVore and Lorentz (1993), $\int_0^1 \dot{\phi}_t \log(1-t)\, dt \ge \int_0^1 \dot{\phi}_t^* \log(1-t)\, dt$. Also $\int_0^1 g(\dot{\phi}_t)\, dt = \int_0^1 g(\dot{\phi}_t^*)\, dt$. Therefore, the function $\dot{\phi}$ can be assumed nondecreasing, so $\dot{\phi}_t = 0$ for $t \in [0,\tau]$ and by the definition of L_c,

$$\int_0^1 \pi\left(\frac{1-\dot{\phi}_t}{c(1-t)}\right) c(1-t)\, dt = L_c(\tau) + I(\phi,\tau), \tag{3.2}$$

where

$$I(\phi,\tau) = \int_\tau^1 \pi\left(\frac{1-\dot{\phi}_t}{c(1-t)}\right) c(1-t)\, dt. \tag{3.3}$$

We now minimize $I(\phi,\tau)$ on the set $\Xi(\tau)$ of absolutely continuous functions ϕ with $\phi_\tau = 0$, $\phi_1 = a$, $\dot{\phi}_t \in [0,1]$ a.e., and $\dot{\phi}_t$ nondecreasing. Convexity considerations provide us with the lower bound

$$I(\phi,\tau) \ge \frac{c(1-\tau)^2}{2} \pi\left(\frac{2(1-\tau-a)}{c(1-\tau)^2}\right), \tag{3.4}$$

which is attained at

$$\dot{\tilde{\phi}}_t = 1 - \frac{2(1-\tau-a)}{(1-\tau)^2}(1-t), \qquad t \in [\tau,1].$$

If $\tau \ge 1-2a$, this function belongs to $\Xi(\tau)$ and delivers the infimum to $I(\phi,\tau)$ on $\Xi(\tau)$, implying the required.

However, if $\tau < 1-2a$ (hence, $2a < 1$), then $\dot{\tilde{\phi}}_t$ is negative for $t \in (\tau, 2 - (1-\tau)^2/(1-\tau-a) - \tau)$. We prove that for those τ, the infimum of $I(\phi,\tau)$ over $\phi \in \Xi(\tau)$ is attained at $\hat{\phi}$ defined by $\dot{\hat{\phi}}_t = 0$ when $t \in [\tau, 1-2a]$ and $\dot{\hat{\phi}}_t = 1 - (1-t)/(2a)$ when $t \in [1-2a, 1]$. Let us consider $\dot{\phi} = (\dot{\phi}_t, \, t \in [\tau,1])$ for $\phi \in \Xi(\tau)$ as an element of the Banach space of Lebesgue measurable functions $h = (h_t, \, t \in [\tau,1])$ with norm $\operatorname{ess\,sup}_{t\in[\tau,1]} |h_t|$. Let functional F on the subset of functions h with $0 \le h_t \le 1$ a.e., be defined by $F(h) = \int_\tau^1 \pi((1-h_t)/(c(1-t)))c(1-t)\,dt$. It is convex and has a Gâteau derivative at $\dot{\hat{\phi}}$ given by $\langle F'(\dot{\hat{\phi}}), h\rangle = -\int_\tau^1 \log((1-\dot{\hat{\phi}}_t)/(c(1-t)))h_t\,dt$. Therefore, for $\phi \in \Xi(\tau)$,

$$\langle F'(\dot{\hat{\phi}}), \dot{\phi} - \dot{\hat{\phi}}\rangle = \int_\tau^{1-2a} \log\bigl(c(1-t)\bigr)\dot{\phi}_t\,dt + \log(2ac)\int_{1-2a}^1 \bigl(\dot{\phi}_t - \dot{\hat{\phi}}_t\bigr)\,dt$$

$$\ge \log(2ac)\int_\tau^{1-2a} \dot{\phi}_t\,dt + \log(2ac)\int_{1-2a}^1 \bigl(\dot{\phi}_t - \dot{\hat{\phi}}_t\bigr)\,dt = 0,$$

implying [see, e.g., Ekeland and Temam (1976)] that $I(\hat{\phi},\tau) \le I(\phi,\tau)$ for $\phi \in \Xi(\tau)$, as claimed. The definition of $\hat{\phi}$ and (3.3) yield $I(\hat{\phi},\tau) = L_c(1-2a) - L_c(\tau) + 2a^2c\pi(1/(2ac))$, which in view of (3.2) implies the assertion of the lemma for the case $\tau < 1-2a$.

The proof of part 2 is similar, the infimum being attained at $\breve{\phi}$ with $\dot{\breve{\phi}}_t = 0$ for $t \in [0, \tau \vee \breve{\theta}]$ and $\dot{\breve{\phi}}_t = t - \breve{\theta}$ for $t > \tau \vee \breve{\theta}$. $\square$

LEMMA 3.5. *Subsets K of $\mathbb{R}_+^{\mathbb{N}}$ of sequences $\mathbf{u} = (u_1, u_2, \ldots)$, such that $\sup_{\mathbf{u}\in K}\sum_{i=1}^\infty u_i < \infty$ and $\lim_{i\to\infty}\sup_{\mathbf{u}\in K} u_i = 0$, are compact subsets of $\mathbb{S}$.*

PROOF. It suffices to check sequential compactness. Let $\mathbf{u}^n$, $n \in \mathbb{N}$, be a sequence of elements of K. The sequence $\mathbf{u}^n$, $n \in \mathbb{N}$, being compact for the product topology, let $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \ldots)$ denote an accumulation point. Passing if necessary to a subsequence, we may assume that $u_i^n \to \tilde{u}_i$ as $n \to \infty$ for $i \in \mathbb{N}$. We have that $\tilde{\mathbf{u}} \in K$. Let $B = \sup_{\mathbf{u}\in K}\sum_{i=1}^\infty u_i$. Given $\varepsilon > 0$, let $\delta > 0$ be such that $\chi(x) \le \varepsilon x/(2B)$ for $x \in [0,\delta]$ [we use that $\chi(x)/x \to 0$ as $x \to 0$], let k be such that $u_i \le \delta\varepsilon$ for $i \ge k$ and $\mathbf{u} \in K$, and let n_0 be such that $|u_i^n - \tilde{u}_i| \le \delta\varepsilon$ for $i = 1, 2, \ldots, k$ and $n \ge n_0$. We then have that for $n \ge n_0$,

$$\sum_{i=1}^\infty \chi\left(\frac{|u_i^n - \tilde{u}_i|}{\varepsilon}\right) \le \frac{1}{2B}\sum_{i=1}^\infty |u_i^n - \tilde{u}_i| \le 1,$$

proving by ε being arbitrary that $d_\chi(\mathbf{u}^n, \tilde{\mathbf{u}}) \to 0$ as $n \to \infty$. $\square$

4. Large deviation asymptotics for the basic processes. The main results of this section are LDPs for the stochastic processes $\overline{S}^n$ and $\overline{E}^n$. We also give without proofs LDPs for the $\overline{\Phi}^n$ and $\overline{Q}^n$, which are not used further. All these processes are well-defined random elements of $\mathbb{D}_C([0,1],\mathbb{R})$. For the notions and facts of idempotent probability theory used extensively in the below argument, the reader is referred to the Appendix [or Puhalskii (2001)].

THEOREM 4.1. *Let $c_n \to c > 0$ as $n \to \infty$. Then the processes $\overline{S}^n$ obey the LDP for scale n in $\mathbb{D}_C([0,1],\mathbb{R})$ with action functional I^S given by*

$$I^S(\mathbf{x}) = \int_0^1 \pi\left(\frac{\dot{\mathbf{x}}_t + 1}{c(1-t-\mathcal{R}(\mathbf{x})_t)}\right)c\bigl(1-t-\mathcal{R}(\mathbf{x})_t\bigr)\,dt$$

for absolutely continuous $\mathbf{x} = (\mathbf{x}_t,\ t \in [0,1])$, *with* $\mathbf{x}_0 = 0$, $\dot{\mathbf{x}}_t \geq -1$ *a.e., and* $\mathcal{R}(\mathbf{x})_t \leq 1 - t$ *for* $t \in [0,1]$, *and* $I^S(\mathbf{x}) = \infty$ *for other* $\mathbf{x}$.

PROOF. Let $A^n = (A_t^n,\ t \in [0,1])$ be defined by

$$A_t^n = \frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} \sum_{j=1}^{n - Q_{k-1}^n - (k-1)} \xi_{kj}^n. \tag{4.1}$$

We note that by (2.6) and the definition of $\overline{S}_t^n$,

$$\overline{S}_t^n = A_t^n - \frac{\lfloor nt \rfloor}{n}, \qquad t \in [0,1], \tag{4.2}$$

so an LDP for the $\overline{S}^n$ would follow from an LDP for the A^n. Let $\mathbf{e} = (t,\ t \in \mathbb{R}_+)$. We prove that the A^n as elements of $\mathbb{D}_C([0,1], \mathbb{R})$ obey the LDP for scale n with action functional

$$I^A(\mathbf{x}) = \int_0^1 \pi\left(\frac{\dot{\mathbf{x}}_t}{c(1 - t - \mathcal{R}(\mathbf{x} - \mathbf{e})_t)}\right) c\big(1 - t - \mathcal{R}(\mathbf{x} - \mathbf{e})_t\big)\, dt$$

if $\mathbf{x}$ is absolutely continuous, $\mathbf{x}_0 = 0$, $\dot{\mathbf{x}}_t \geq 0$ a.e., and $\mathcal{R}(\mathbf{x} - \mathbf{e})_t \leq 1 - t$ for $t \in [0,1]$, and $I^A(\mathbf{x}) = \infty$ otherwise.

Let us extend the time-domain of the processes A^n to $\mathbb{R}_+$ by letting $A_t^n = A_1^n$ for $t \geq 1$. We show that the extended A^n satisfy the hypotheses of Theorem 5.1.5 in Puhalskii (2001). By (4.1) A^n is a totally discontinuous $\mathbf{F}^n$-adapted semimartingale with predictable measure of jumps $(\nu^n([0,t], \Gamma),\ t \in \mathbb{R}_+,\ \Gamma \in \mathcal{B}(\mathbb{R}))$ given by

$$\nu^n([0,t], \Gamma) = \sum_{k=0}^{\lfloor n(t \wedge 1) \rfloor - 1} F^n\left(1 - \overline{Q}_{k/n}^n - \frac{k}{n}, \Gamma \setminus \{0\}\right), \qquad \Gamma \in \mathcal{B}(\mathbb{R}),$$

where

$$F^n(s, \Gamma') = \mathbf{P}\left(\frac{1}{n} \sum_{j=1}^{\lfloor ns \rfloor} \xi_{1j}^n \in \Gamma'\right), \qquad s \in \mathbb{R}_+,\ \Gamma' \in \mathcal{B}(\mathbb{R}). \tag{4.3}$$

Since the jumps of A^n are bounded from above by 1, A^n satisfies the Cramér condition, so its stochastic (or Doléans–Dade) exponential is well defined and has the form

$$\begin{aligned} \mathcal{E}_t^n(\lambda) &= \prod_{k=1}^{\lfloor n(t \wedge 1) \rfloor} \left(1 + \int_{\mathbb{R}} (e^{\lambda x} - 1) \nu^n\left(\left\{\frac{k}{n}\right\}, dx\right)\right) \\ &= \prod_{k=0}^{\lfloor n(t \wedge 1) \rfloor - 1} \int_{\mathbb{R}} e^{\lambda x} F^n\left(1 - \overline{Q}_{k/n}^n - \frac{k}{n}, dx\right), \end{aligned} \tag{4.4}$$

where $\lambda \in \mathbb{R}$. By (2.9) and (4.2),

$$\overline{Q}^n = \mathcal{R}(A^n - \mathbf{e}^n + \bar{\varepsilon}^n), \tag{4.5}$$

where $\mathbf{e}^n = (\lfloor nt \rfloor / n,\ t \in \mathbb{R}_+)$. Hence, recalling that the ξ_i^n are Bernoulli and equal 1 with probability c_n/n, we have by (4.3) and (4.4),

$$\begin{aligned}\frac{1}{n} \log \mathcal{E}_t^n(n\lambda) &= n \log\left(1 + (e^\lambda - 1)\frac{c_n}{n}\right) \\ &\quad \times \int_0^{\lfloor n(t\wedge 1)\rfloor/n} \left(1 - \mathcal{R}(A^n - \mathbf{e}^n + \bar{\varepsilon}^n)_s - \frac{\lfloor ns \rfloor}{n}\right) ds.\end{aligned} \tag{4.6}$$

Let us note that by the fact that $Q_k^n + k \le n$ and (2.9),

$$1 - \mathcal{R}(A^n - \mathbf{e}^n - \bar{\varepsilon}^n)_s - \frac{\lfloor ns \rfloor}{n} \ge 0 \qquad \text{for } s \in [0,1]. \tag{4.7}$$

Thus, denoting for $\mathbf{x} \in \mathbb{D}(\mathbb{R}_+, \mathbb{R})$,

$$G_t(\lambda, \mathbf{x}) = c(e^\lambda - 1) \int_0^{t\wedge 1} \big(1 - \mathcal{R}(\mathbf{x} - \mathbf{e})_s - s\big)\, ds, \tag{4.8}$$

we conclude by (4.6) and (4.7), the convergence $c_n \to c$, Lipshitz continuity of the reflection mapping on $\mathbb{D}_C([0,1], \mathbb{R}_+)$ and Lemma 3.1 that for arbitrary $T > 0$,

$$\sup_{t \in [0,T]} \left| \frac{1}{n} \log \mathcal{E}_t^n(n\lambda) - G_t(\lambda, A^n) \right| \overset{\mathbf{P}^{1/n}}{\to} 0 \qquad \text{as } n \to \infty.$$

Since $G_t(\lambda, \mathbf{x})$ satisfies the uniform continuity and majoration conditions of Theorem 5.1.5 of Puhalskii (2001), by the theorem the sequence of laws of the A^n on $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ is $\mathbb{C}$-exponentially tight (of order n), and its every large deviation accumulation point solves the maxingale problem $(0, G)$ with $G = (G_t(\lambda, \mathbf{x}),\ t \in \mathbb{R}_+, \lambda \in \mathbb{R}, \mathbf{x} \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}))$. Let deviability $\mathbf{\Pi}^A$ on $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ be a solution of $(0, G)$. We note that $\mathbf{\Pi}^A(\mathbb{D}(\mathbb{R}_+, \mathbb{R}) \setminus \mathbb{C}(\mathbb{R}_+, \mathbb{R})) = 0$ by the $\mathbb{C}$-exponential tightness of the laws of the A^n. Let deviability $\widehat{\mathbf{\Pi}}^A$ be the restriction of $\mathbf{\Pi}^A$ on $\mathbb{C}(\mathbb{R}_+, \mathbb{R})$. The claimed LDP will follow if for $\mathbf{x} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R})$,

$$\widehat{\mathbf{\Pi}}^A(\mathbf{x}) = \begin{cases} \exp\big(-I^A(p_1\mathbf{x})\big), & \text{if } \mathbf{x}_t = \mathbf{x}_{t\wedge 1},\ t \in \mathbb{R}_+, \\ \infty, & \text{otherwise}. \end{cases} \tag{4.9}$$

The idea of the proof of (4.9) is to translate the problem into a problem on uniqueness of idempotent processes. Let $\Upsilon = \mathbb{C}(\mathbb{R}_+, \mathbb{R}) \times \mathbb{C}(\mathbb{R}_+, \mathbb{R})$ and component idempotent processes $A = (A_t(\mathbf{x}, \mathbf{x}'),\ t \in \mathbb{R}_+, (\mathbf{x}, \mathbf{x}') \in \Upsilon)$ and $N = (N_t(\mathbf{x}, \mathbf{x}'),\ t \in \mathbb{R}_+, (\mathbf{x}, \mathbf{x}') \in \Upsilon)$ be defined by the respective equalities $A_t(\mathbf{x}, \mathbf{x}') = \mathbf{x}_t$ and $N_t(\mathbf{x}, \mathbf{x}') = \mathbf{x}'_t$. We will prove that there exists deviability $\mathbf{\Pi}$ on Υ such that A and N satisfy

$$A_t = N_{B_t(A)}, \qquad t \in \mathbb{R}_+\ \mathbf{\Pi}\text{-a.e.}, \tag{4.10}$$

where

$$B_t(\mathbf{x}) = c\int_0^t \big(1 - \mathcal{R}(\mathbf{x}-\mathbf{e})_s - s\big)^+ \, ds, \tag{4.11}$$

A has idempotent distribition $\widehat{\mathbf{\Pi}}^A$ and N is idempotent Poisson, that is, $\sup_{\mathbf{x}'\in\mathbb{C}(\mathbb{R}_+,\mathbb{R})} \mathbf{\Pi}(\mathbf{x},\mathbf{x}') = \widehat{\mathbf{\Pi}}^A(\mathbf{x})$ and $\sup_{\mathbf{x}\in\mathbb{C}(\mathbb{R}_+,\mathbb{R})} \mathbf{\Pi}(\mathbf{x},\mathbf{x}') = \mathbf{\Pi}^N(\mathbf{x}')$, where $\mathbf{\Pi}^N$ is the Poisson deviability. After that we will draw on Ethier and Kurtz [(1986), Theorem 1.1, Chapter 6] to conclude that (4.10) has a unique strong solution. That will imply that (4.10) has a unique weak solution in the sense that the idempotent distribution of A is specified uniquely and is given by (4.9). The reasoning used to establish (4.10) is also along the lines of the approaches developed in Ethier and Kurtz (1986).

By (4.7), Lemma 3.1 and $\mathbf{\Pi}^A$ being an LD accumulation point of the laws of the A^n, we have that

$$1 - \mathcal{R}(\mathbf{x}-\mathbf{e})_s - s \ge 0, \qquad s \in [0,1]\ \mathbf{\Pi}^A\text{-a.e.}, \tag{4.12}$$

so

$$G_t(\lambda;\mathbf{x}) = \widetilde{G}_t(\lambda;\mathbf{x}), \qquad t \in \mathbb{R}_+\ \widehat{\mathbf{\Pi}}^A\text{-a.e.}, \tag{4.13}$$

where for $\lambda \in \mathbb{R}$,

$$\widetilde{G}_t(\lambda;\mathbf{x}) = (e^\lambda - 1)B_t(\mathbf{x}), \qquad t \in \mathbb{R}_+,\ \mathbf{x} \in \mathbb{C}(\mathbb{R}_+,\mathbb{R}).$$

Given $\varepsilon > 0$, we define for $\mathbf{x},\mathbf{x}' \in \mathbb{C}(\mathbb{R}_+,\mathbb{R})$,

$$G_t^\varepsilon\big(\lambda;(\mathbf{x},\mathbf{x}')\big) = \widetilde{G}_t(\lambda;\mathbf{x}) + (e^\lambda - 1)\varepsilon t \tag{4.14}$$

and introduce an idempotent process $\widehat{A} = (\widehat{A}_t(\mathbf{x},\mathbf{x}'),\, t \in \mathbb{R}_+,\, (\mathbf{x},\mathbf{x}') \in \Upsilon)$ by

$$\widehat{A}_t(\mathbf{x},\mathbf{x}') = \mathbf{x}_t + \mathbf{x}'_t. \tag{4.15}$$

As the deviability $\mathbf{\Pi}^A$ is a solution of the maxingale problem $(0,G)$, $\mathbf{\Pi}^A$ is concentrated on $\mathbb{C}(\mathbb{R}_+,\mathbb{R})$, $\mathbf{\Pi}^A$ and $\widehat{\mathbf{\Pi}}^A$ coincide on $\mathbb{C}(\mathbb{R}_+,\mathbb{R})$, and Lemma A.2 and (4.13) hold, it follows that the idempotent process $(\exp(\lambda\mathbf{x}_t - \widetilde{G}_t(\lambda;\mathbf{x})),$ $t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}(\mathbb{R}_+,\mathbb{R}))$ is a $\mathbf{C}$-uniformly maximable exponential maxingale on $(\mathbb{C}(\mathbb{R}_+,\mathbb{R}), \widehat{\mathbf{\Pi}}^A)$, where $\mathbf{C} = (\mathcal{C}_t,\, t \in \mathbb{R}_+)$ is the canonical τ-flow. Next, the fact that $(\exp(\lambda\mathbf{x}_t - (e^\lambda - 1)t),\, t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}(\mathbb{R}_+,\mathbb{R}))$ is a $\mathbf{C}$-exponential maxingale on $(\mathbb{C}(\mathbb{R}_+,\mathbb{R}), \mathbf{\Pi}^N)$ implies that $(\exp(\lambda\mathbf{x}_t - (e^\lambda - 1)\varepsilon t),\, t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}(\mathbb{R}_+,\mathbb{R}))$ is a $\mathbf{C}$-exponential maxingale on $(\mathbb{C}(\mathbb{R}_+,\mathbb{R}), \mathbf{\Pi}^{N,\varepsilon})$, where $\mathbf{\Pi}^{N,\varepsilon}((\mathbf{x}_t,\, t \in \mathbb{R}_+)) = \mathbf{\Pi}^N((\mathbf{x}_{t/\varepsilon},\, t \in \mathbb{R}_+))$. By Lemma A.3, (4.14) and (4.15) under product deviability $\widehat{\mathbf{\Pi}}^A \times \mathbf{\Pi}^{N,\varepsilon}$, the idempotent process $(\exp(\lambda\widehat{A}_t(\mathbf{x},\mathbf{x}') - G_t^\varepsilon(\lambda;(\mathbf{x},\mathbf{x}'))),\ t \in \mathbb{R}_+,$ $(\mathbf{x},\mathbf{x}') \in \Upsilon)$ is an exponential maxingale relative to the τ-flow $\mathbf{A} = (\mathcal{A}_t,\, t \in \mathbb{R}_+)$, where $\mathcal{A}_t = \mathcal{C}_t \otimes \mathcal{C}_t$. Let

$$\sigma_t^\varepsilon(\mathbf{x},\mathbf{x}') = \inf\{s \in \mathbb{R}_+ : B_s(\mathbf{x}) + \varepsilon s \ge t\}. \tag{4.16}$$

The idempotent variables σ_t^ε, $t \in \mathbb{R}_+$, are bounded idempotent $\mathbf{A}$-stopping times and $G^\varepsilon_{\sigma_t^\varepsilon(\mathbf{x},\mathbf{x}')}(\lambda; (\mathbf{x},\mathbf{x}')) = (e^\lambda - 1)t$, so by Lemma A.1 the idempotent process $(\exp(\lambda N_t^\varepsilon(\mathbf{x},\mathbf{x}') - (e^\lambda - 1)t),\ t \in \mathbb{R}_+, (\mathbf{x},\mathbf{x}') \in \Upsilon)$, where $N_t^\varepsilon(\mathbf{x},\mathbf{x}') = \widehat{A}_{\sigma_t^\varepsilon(\mathbf{x},\mathbf{x}')}(\mathbf{x},\mathbf{x}')$, is an exponential maxingale on $(\Upsilon, \widehat{\mathbf{\Pi}}^A \times \mathbf{\Pi}^{N,\varepsilon})$ relative to the τ-flow $\mathbf{A}^\varepsilon = (\mathcal{A}_{\sigma_t^\varepsilon},\ t \in \mathbb{R}_+)$. Hence, $N^\varepsilon = (N_t^\varepsilon(\mathbf{x},\mathbf{x}'),\ t \in \mathbb{R}_+, (\mathbf{x},\mathbf{x}') \in \Upsilon)$ is an $\mathbf{A}^\varepsilon$-Poisson idempotent process, so it is a Poisson idempotent process on $(\Upsilon, \widehat{\mathbf{\Pi}}^A \times \mathbf{\Pi}^{N,\varepsilon})$. In view of (4.15), (4.16) and the definition of A_t, we can write that on Υ

$$A_t + \mathbf{x}'_t = N^\varepsilon_{B_t(A)+\varepsilon t}, \qquad t \in \mathbb{R}_+. \tag{4.17}$$

We now show that (4.10) is obtained as a limit of (4.17). The pair (A, N^ε) specifies a mapping of Υ into itself. Let $\mathbf{\Pi}^\varepsilon$ denote the image of $\widehat{\mathbf{\Pi}}^A \times \mathbf{\Pi}^{N,\varepsilon}$ under this mapping, that is, $\mathbf{\Pi}^\varepsilon(\mathbf{x},\mathbf{x}') = \sup_{(\mathbf{y},\mathbf{y}')\in\Upsilon : A(\mathbf{y},\mathbf{y}')=\mathbf{x},\, N^\varepsilon(\mathbf{y},\mathbf{y}')=\mathbf{x}'} \widehat{\mathbf{\Pi}}^A(\mathbf{y})\mathbf{\Pi}^{N,\varepsilon}(\mathbf{y}')$; briefly, $\mathbf{\Pi}^\varepsilon$ is the joint idempotent distribution of (A, N^ε) on $(\Upsilon, \widehat{\mathbf{\Pi}}^A \times \mathbf{\Pi}^{N,\varepsilon})$. Since the idempotent distributions of A and N^ε are deviabilities and do not depend on ε, the net $\mathbf{\Pi}^\varepsilon$, $\varepsilon \to 0$, of deviabilities on Υ is tight. It is thus relatively compact for weak convergence of idempotent probabilities. Let $\mathbf{\Pi}$ denote an accumulation point of the $\mathbf{\Pi}^\varepsilon$. By the continuous mapping theorem the marginal idempotent distributions of $\mathbf{\Pi}$ are equal to $\widehat{\mathbf{\Pi}}^A$ and $\mathbf{\Pi}^N$: $\sup_{\mathbf{x}'\in\mathbb{C}(\mathbb{R}_+,\mathbb{R})} \mathbf{\Pi}(\mathbf{x},\mathbf{x}') = \widehat{\mathbf{\Pi}}^A(\mathbf{x})$ and $\sup_{\mathbf{x}\in\mathbb{C}(\mathbb{R}_+,\mathbb{R})} \mathbf{\Pi}(\mathbf{x},\mathbf{x}') = \mathbf{\Pi}^N(\mathbf{x}')$. Next, by the definition of $\mathbf{\Pi}^\varepsilon$, (4.17) and (4.11) for $T > 0$ and $\eta > 0$,

$$\begin{aligned}
&\mathbf{\Pi}^\varepsilon\Bigl((\mathbf{x},\mathbf{x}') : \sup_{t\in[0,T]} \bigl|\mathbf{x}_t - \mathbf{x}'_{B_t(\mathbf{x})}\bigr| \ge \eta\Bigr)\\
&\quad= (\widehat{\mathbf{\Pi}}^A \times \mathbf{\Pi}^{N,\varepsilon})\Bigl(\sup_{t\in[0,T]} \bigl|A_t - N^\varepsilon_{B_t(A)}\bigr| \ge \eta\Bigr)\\
&\quad\le (\widehat{\mathbf{\Pi}}^A \times \mathbf{\Pi}^{N,\varepsilon})\Bigl(\sup_{t\in[0,T]} |\mathbf{x}'_t| \ge \frac{\eta}{2}\Bigr)\\
&\qquad\vee (\widehat{\mathbf{\Pi}}^A \times \mathbf{\Pi}^{N,\varepsilon})\Bigl(\sup_{s,t\in[0,(c+\varepsilon)T] : |s-t|\le\varepsilon T} |N^\varepsilon_s - N^\varepsilon_t| \ge \frac{\eta}{2}\Bigr)\\
&\quad= \mathbf{\Pi}^N\Bigl(\sup_{t\in[0,\varepsilon T]} |\mathbf{x}_t| \ge \frac{\eta}{2}\Bigr) \vee \sup_{s,t\in[0,(c+\varepsilon)T] : |s-t|\le\varepsilon T} \mathbf{\Pi}^N\Bigl(|\mathbf{x}_s - \mathbf{x}_t| \ge \frac{\eta}{2}\Bigr)\\
&\quad= \mathbf{\Pi}^N\Bigl(\mathbf{x}_{\varepsilon T} \ge \frac{\eta}{2}\Bigr),
\end{aligned} \tag{4.18}$$

where the latter two equalities use the definition of $\mathbf{\Pi}^{N,\varepsilon}$, the facts that N^ε is idempotent Poisson under $\widehat{\mathbf{\Pi}}^A \times \mathbf{\Pi}^{N,\varepsilon}$ and that idempotent Poisson processes have stationary increments. Given $L > 0$, we have by an exponential Markov inequality and the fact that $(\exp(L\mathbf{x}_t - (e^L - 1)t),\ t \in \mathbb{R}_+)$ is an exponential maxingale under $\mathbf{\Pi}^N$,

$$\mathbf{\Pi}^N\biggl(\mathbf{x}_{\varepsilon T} \geq \frac{\eta}{2}\biggr) \leq \mathbf{S}_{\mathbf{\Pi}^N}\bigl(\exp(L\mathbf{x}_{\varepsilon T})\bigr)\exp\biggl(-\frac{L\eta}{2}\biggr) = \exp\biggl((e^L - 1)\varepsilon T - \frac{L\eta}{2}\biggr),$$

where $\mathbf{S}_{\mathbf{\Pi}^N}$ denotes idempotent expectation with respect to $\mathbf{\Pi}^N$. Letting $\varepsilon \to 0$ and $L \to \infty$, we conclude that $\lim_{\varepsilon\to 0} \mathbf{\Pi}^N(\mathbf{x}_{\varepsilon T} \geq \eta/2) = 0$, so by (4.18) $\lim_{\varepsilon\to 0} \mathbf{\Pi}^\varepsilon((\mathbf{x}, \mathbf{x}') : \sup_{t\in[0,T]} |\mathbf{x}_t - \mathbf{x}'_{B_t(\mathbf{x})}| \geq \eta) = 0$. Since the $\mathbf{\Pi}^\varepsilon$ weakly converge along a subnet to $\mathbf{\Pi}$ and $\sup_{t\in[0,T]} |\mathbf{x}_t - \mathbf{x}'_{B_t(\mathbf{x})}|$ is a continuous function of $(\mathbf{x}, \mathbf{x}') \in \Upsilon$ so that the set $\{(\mathbf{x}, \mathbf{x}') \in \Upsilon : \sup_{t\in[0,T]} |\mathbf{x}_t - \mathbf{x}'_{B_t(\mathbf{x})}| > \eta\}$ is open, we conclude that $\mathbf{\Pi}((\mathbf{x}, \mathbf{x}') : \sup_{t\in[0,T]} |\mathbf{x}_t - \mathbf{x}'_{B_t(\mathbf{x})}| > \eta) = 0$. Consequently,

$$\begin{aligned}\mathbf{\Pi}\biggl((\mathbf{x}, \mathbf{x}') : \sup_{t\in[0,T]} \bigl|\mathbf{x}_t - \mathbf{x}'_{B_t(\mathbf{x})}\bigr| > 0\biggr)\\ = \sup_{\eta>0} \mathbf{\Pi}\biggl((\mathbf{x}, \mathbf{x}') : \sup_{t\in[0,T]} \bigl|\mathbf{x}_t - \mathbf{x}'_{B_t(\mathbf{x})}\bigr| > \eta\biggr) = 0,\end{aligned}$$

which is equivalent to (4.10) by A and N being the first and second component processes on Υ, respectively.

Equation (4.10) is of the form considered in Ethier and Kurtz [(1986), Theorem 1.1, Chapter 6]. The hypotheses of the theorem are seen to be met, which implies that (4.10) has a unique (strong) solution for A given by $A_t = N_{\sigma_t(N)}$, where $\sigma_t(\mathbf{x}') = \inf\{s \in [0,1] : \int_0^s (c(1 - \mathscr{R}(\mathbf{x}' - \mathbf{e})_p - p)^+)^{-1}\, dp \geq t\}$, $\mathbf{x}' \in \mathbb{C}(\mathbb{R}_+, \mathbb{R})$. Therefore, $\mathbf{\Pi}(\mathbf{x}, \mathbf{x}') = 0$ if $(\mathbf{x}_t,\ t \in \mathbb{R}_+) \neq (\mathbf{x}'_{\sigma_t(\mathbf{x}')},\ t \in \mathbb{R}_+)$, so the fact that $\sup_{\mathbf{x}\in\mathbb{C}(\mathbb{R}_+,\mathbb{R})} \mathbf{\Pi}(\mathbf{x}, \mathbf{x}') = \mathbf{\Pi}^N(\mathbf{x}')$ yields $\mathbf{\Pi}(\mathbf{x}, \mathbf{x}') = \mathbf{\Pi}^N(\mathbf{x}')$ if $(\mathbf{x}_t,\ t \in \mathbb{R}_+) = (\mathbf{x}'_{\sigma_t(\mathbf{x}')},\ t \in \mathbb{R}_+)$. Consequently, for $\mathbf{x} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R})$,

$$\begin{aligned}\widehat{\mathbf{\Pi}}^A(\mathbf{x}) &= \sup_{\mathbf{x}'\in\mathbb{C}(\mathbb{R}_+,\mathbb{R})} \mathbf{\Pi}(\mathbf{x}, \mathbf{x}')\\ &= \sup_{\mathbf{x}'\in\mathbb{C}(\mathbb{R}_+,\mathbb{R}) : \mathbf{x}_t = \mathbf{x}'_{\sigma_t(\mathbf{x}')}} \mathbf{\Pi}^N(\mathbf{x}') = \sup_{\mathbf{x}'\in\mathbb{C}(\mathbb{R}_+,\mathbb{R}) : \mathbf{x}_t = \mathbf{x}'_{B_t(\mathbf{x})}} \mathbf{\Pi}^N(\mathbf{x}').\end{aligned}\tag{4.19}$$

We have thus proved that $\widehat{\mathbf{\Pi}}^A$ is uniquely specified by the right-most side of (4.19). In particular, if $\mathbf{x}_t \neq \mathbf{x}_{t\wedge 1}$ for some $t \in \mathbb{R}_+$, the set over which the latter supremum is evaluated is empty, so $\widehat{\mathbf{\Pi}}^A(\mathbf{x}) = 0$. Let $\mathbf{x}_t = \mathbf{x}_{t\wedge 1}$, $t \in \mathbb{R}_+$. Recalling that $\mathbf{\Pi}^N(\mathbf{x}') = \exp(-I^N(\mathbf{x}'))$, where $I^N(\mathbf{x}') = \int_0^\infty \pi(\dot{\mathbf{x}}'_t)\, dt$ if $\mathbf{x}'$ is absolutely continuous, $\mathbf{x}'_0 = 0$, and $\dot{\mathbf{x}}'_t \geq 0$ a.e., and $I^N(\mathbf{x}') = \infty$ otherwise, we derive by a change of variables and (4.11) that the right-most side of (4.19) equals $\exp(-I^A(p_1\mathbf{x}))$ provided $1 - \mathscr{R}(\mathbf{x} - \mathbf{e})_s - s \geq 0$, $s \in [0, 1]$. If $\mathbf{x}$ does not meet the latter condition, then $\widehat{\mathbf{\Pi}}^A(\mathbf{x}) = 0$ according to (4.12). Equality (4.9) has been proved, so the LDP for the (extended) processes A^n has been proved. By the contraction principle the (nonextended) A^n obey the LDP in $\mathbb{D}_C([0,1],\mathbb{R})$ with I^A. (Note that the A^n are random elements of $\mathbb{D}_C([0,1],\mathbb{R})$.) The LDP for the $\overline{S}^n$ follows by (4.2) and the contraction principle. □

COROLLARY 4.1. *Let $c_n \to c > 0$ as $n \to \infty$. Then the processes $(\overline{S}^n, \overline{E}^n)$ obey the LDP for scale n in $\mathbb{D}_C([0,1],\mathbb{R}^2)$ with action functional $I^{S,E}$ given by*

$$I^{S,E}(\mathbf{x},\mathbf{y}) = I^S(\mathbf{x}) + I^E_{\mathbf{x}}(\mathbf{y}),$$

where $I^E_{\mathbf{x}}(\mathbf{y}) = \int_0^1 \pi(\dot{\mathbf{y}}_t/(c\mathcal{R}(\mathbf{x})_t))c\mathcal{R}(\mathbf{x})_t\,dt$ if $\mathbf{y} = (\mathbf{y}_t,\ t \in [0,1])$ is nondecreasing and absolutely continuous with $\mathbf{y}_0 = 0$ and $I^E_{\mathbf{x}}(\mathbf{y}) = \infty$ otherwise.

PROOF. Given a sequence $\mathbf{x}^n$, $n \in \mathbb{N}$, of elements of $\mathbb{D}_C([0,1],\mathbb{R})$, let

$$\overline{E}'^n_t = \frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=1}^{\lfloor n\mathcal{R}(\mathbf{x}^n)_{(i-1)/n}\rfloor - 1} \zeta^n_{ij}, \qquad t \in [0,1].$$

A standard argument [e.g., Theorem 2.3 in Puhalskii (1994)] shows that if $\mathbf{x}^n \to \mathbf{x}$ as $n \to \infty$, then the sequence $\overline{E}'^n$, $n \in \mathbb{N}$, obeys the LDP in $\mathbb{D}_C([0,1],\mathbb{R})$ for scale n with action functional $I^E_{\mathbf{x}}(\mathbf{y})$, $\mathbf{y} \in \mathbb{D}_C([0,1],\mathbb{R})$. The claim now follows by an argument as in Puhalskii [(1995), Theorem 2.2]; see also Chaganty (1997), (2.9), (2.12) and Lemma 3.1. □

REMARK 4.1. An application of the contraction principle yields LDPs for the $\overline{Q}^n$ and $\overline{\Phi}^n$:

1. The processes $\overline{Q}^n$ obey the LDP for scale n in $\mathbb{D}_C([0,1],\mathbb{R})$ with action functional I^Q given by

$$I^Q(\mathbf{x}) = \int_0^1 \pi\left(\frac{\dot{\mathbf{x}}_t + 1}{c(1-t-\mathbf{x}_t)}\right)c(1-t-\mathbf{x}_t)\mathbf{1}(\mathbf{x}_t > 0)\,dt + \int_0^{(1-1/c)^+} \pi\left(\frac{1}{c(1-t)}\right)c(1-t)\mathbf{1}(\mathbf{x}_t = 0)\,dt$$

for absolutely continuous $\mathbf{x} = (\mathbf{x}_t,\ t \in [0,1])$, with $\mathbf{x}_0 = 0$, $\dot{\mathbf{x}}_t \ge -1$ a.e. and $\mathbf{x}_t \in [0, 1-t]$ $t \in [0,1]$, and $I^Q(\mathbf{x}) = \infty$ for other $\mathbf{x}$.

2. The processes $\overline{\Phi}^n$ obey the LDP for scale n in $\mathbb{D}_C([0,1],\mathbb{R})$ with action functional I^Φ given by

$$I^\Phi(\phi) = \int_0^1 \pi\left(\frac{1-\dot{\phi}_t}{c(1-t)}\right)c(1-t)\,dt + \sum K_c(l_i)$$

if $\phi = (\phi_t,\ t \in [0,1])$ is absolutely continuous and nondecreasing, $\phi_0 = 0$ and $\dot{\phi}_t \le 1$ a.e., where the l_i are the lengths of the maximal intervals where ϕ is constant and summation is performed over all such intervals, and $I^\Phi(\phi) = \infty$ otherwise.

5. Large deviations for connected components. In this section we prove Theorem 2.1 and Corollaries 2.1–2.4. We need the following lemma. Let $a \in [0,1]$, $m \in \mathbb{N}$, $u_1, \dots, u_m$ be such that $u_i \in (0,1]$ and $\sum_{i=1}^m u_i \le 1$, $r_1, \dots, r_m$ belong to $\mathbb{R}_+$, and $\delta > 0$. We denote by $B_\delta^n(a; \{u_i, r_i\}_{i=1}^m)$ the event that there exist m connected components of $\mathcal{G}(n, c_n/n)$ of sizes in the intervals $(n(u_i - \delta), n(u_i + \delta))$ for $i = 1, 2, \dots, m$, the numbers of the excess edges of these components belong to the respective intervals $(n(r_i - \delta), n(r_i + \delta))$, the other connected components are of sizes less than $n\delta$, and the total number of components of the random graph belongs to the interval $(n(a-\delta), n(a+\delta))$. Let also $\widetilde{B}_\delta^n(a)$ denote the event that all the connected components are of sizes less than $n\delta$ and the total number of components belongs to the interval $(n(a-\delta), n(a+\delta))$.

LEMMA 5.1. *Let $c_n \to c > 0$ as $n \to \infty$. If $\sum_{i=1}^m u_i \le 1 - a$, then*

$$
\begin{aligned}
&\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \mathbf{P}\bigl(B_\delta^n(a; \{u_i, r_i\}_{i=1}^m)\bigr) \\
&\quad = \lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{n} \log \mathbf{P}\bigl(B_\delta^n(a; \{u_i, r_i\}_{i=1}^m)\bigr) \\
&\quad = -\Biggl[\sum_{i=1}^m \sup_{\rho\in\mathbb{R}_+} \biggl(K_\rho(u_i) + r_i \log\frac{\rho}{c}\biggr) + L_c\Biggl((1-2a) \vee \sum_{i=1}^m u_i\Biggr) \\
&\qquad\quad + \frac{c}{2}\Biggl(1 - (1-2a) \vee \sum_{i=1}^m u_i\Biggr)^2 \pi\biggl(\frac{2(1 - a - (1-2a)\vee \sum_{i=1}^m u_i)}{c(1 - (1-2a)\vee\sum_{i=1}^m u_i)^2}\biggr)\Biggr].
\end{aligned}
$$

If $\sum_{i=1}^m u_i > 1 - a$, then

$$
\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \mathbf{P}\bigl(B_\delta^n(a; \{u_i, r_i\}_{i=1}^m)\bigr) = -\infty.
$$

Also

$$
\begin{aligned}
&\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \mathbf{P}\bigl(\widetilde{B}_\delta^n(a)\bigr) \\
&\quad = \lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{n} \log \mathbf{P}\bigl(\widetilde{B}_\delta^n(a)\bigr) \\
&\quad = -\biggl[L_c\bigl((1-2a)^+\bigr) + \frac{c}{2}\bigl(1 - (1-2a)^+\bigr)^2 \pi\biggl(\frac{2(1 - a - (1-2a)^+)}{c(1-(1-2a)^+)^2}\biggr)\biggr].
\end{aligned}
$$

PROOF. We carry out the proof for the sets $B_\delta^n(a; \{u_i, r_i\}_{i=1}^m)$. A similar (and actually simpler) reasoning applies to the $\widetilde{B}_\delta^n(a)$. We denote throughout $B_\delta^n(a; \{u_i, r_i\}_{i=1}^m)$ as B_δ^n. Upper bounds are addressed first. Let $\delta \in (0, \min_{i=1,2,\dots,m} u_i)$, $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(m))$ denote a permutation of the set $\{1, 2, \dots, m\}$ and $B'_{\delta,\sigma}$ denote the set of functions $\mathbf{x} \in \mathbb{D}_C([0,1], \mathbb{R})$, with $\mathbf{x}_0 = 0$

such that $|\mathcal{T}(\mathbf{x})_1 - a| \le \delta$ and there exist points $0 = t'_0 \le t'_1 \le t'_2 \le \cdots \le t'_{2m} \le 1 = t'_{2m+1}$ with $|t'_{2i} - t'_{2i-1} - u_{\sigma(i)}| \le \delta$ for $i = 1, 2, \ldots, m$ for which $\mathcal{R}(\mathbf{x})_{t'_{2i-1}} = \mathcal{R}(\mathbf{x})_{t'_{2i}} = 0$, $\mathcal{T}(\mathbf{x})_{t'_{2i}-} - \mathcal{T}(\mathbf{x})_{t'_{2i-1}} = 0$, and $\mathcal{R}(\mathbf{x})$ is not strictly positive on any subinterval of $[t'_{2i}, t'_{2i+1}]$ of length δ for $i = 0, 1, \ldots, m$. Let $B_{\delta,\sigma}$ denote the set of functions $(\mathbf{x}, \mathbf{y}) \in \mathbb{D}_C([0,1], \mathbb{R}^2)$ such that $\mathbf{x} \in B'_{\delta,\sigma}$, $\mathbf{y}$ is nondecreasing with $\mathbf{y}_0 = 0$, and $|\mathbf{y}_{t'_{2i}} - \mathbf{y}_{t'_{2i-1}} - r_{\sigma(i)}| \le \delta$ for $i = 1, 2, \ldots, m$, where the t'_i are associated with $\mathbf{x}$, and let B_δ be the union of the $B_{\delta,\sigma}$ over all permutations σ. By the construction of Q^n and E^n, if there exists a connected component of size l of the random graph with k excess edges, then there exist integers k_1 and k_2 ranging in $\{0, 1, \ldots, n\}$ such that $k_2 - k_1 = l$, $Q^n_{k_1} = Q^n_{k_2} = 0$, $Q^n_i \ge 1$ for $i = k_1 + 1, \ldots, k_2 - 1$, and $E^n_{k_2} - E^n_{k_1} = k$. Also, Φ^n does not increase on $[k_1, k_2 - 1]$ and Φ^n_n equals the number of the connected components of $\mathcal{G}(n, c_n/n)$. Therefore, recalling (2.9) and (2.10), we have that $B^n_\delta \subset \{(\overline{S}^n + \bar{\varepsilon}^n, \overline{E}^n) \subset B_\delta\}$. Noting that B_δ and its closure in $\mathbb{D}_C([0,1], \mathbb{R}^2)$ have the same intersection with $\mathbb{C}([0,1], \mathbb{R}^2)$, we have by Corollary 4.1 and Lemma 3.1 that

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \log \mathbf{P}(B^n_\delta) &\le \limsup_{n\to\infty} \frac{1}{n} \log \mathbf{P}\big((\overline{S}^n + \bar{\varepsilon}^n, \overline{E}^n) \subset B_\delta\big) \\ &\le -\inf_{(\mathbf{x},\mathbf{y}) \in B_\delta \cap \mathbb{C}([0,1],\mathbb{R}^2)} \big(I^S(\mathbf{x}) + I^E_{\mathbf{x}}(\mathbf{y})\big).\end{aligned} \tag{5.1}$$

Let B'_σ denote the set of functions $\mathbf{x} \in \mathbb{C}([0,1], \mathbb{R})$ with $\mathbf{x}_0 = 0$ such that $\mathcal{T}(\mathbf{x})_1 = a$ and there exist points $0 = t_0 \le t_1 \le t_2 \le \cdots \le t_{2m} \le t_{2m+1} = 1$ with $t_{2i} - t_{2i-1} = u_{\sigma(i)}$ for $i = 1, 2, \ldots, m$ for which $\mathcal{R}(\mathbf{x})_{t_{2i-1}} = \mathcal{R}(\mathbf{x})_{t_{2i}} = 0$, $\mathcal{T}(\mathbf{x})_{t_{2i}} = \mathcal{T}(\mathbf{x})_{t_{2i-1}}$, and $\mathcal{R}(\mathbf{x})$ equals zero on the intervals $[t_{2i}, t_{2i+1}]$ for $i = 0, 1, \ldots, m$. Let $\widehat{B}_\sigma$ denote the set of functions $(\mathbf{x}, \mathbf{y}) \in \mathbb{C}([0,1], \mathbb{R}^2)$ such that $\mathbf{x} \in B'_\sigma$, $\mathbf{y}$ is nondecreasing with $\mathbf{y}_0 = 0$ and $\mathbf{y}_{t_{2i}} - \mathbf{y}_{t_{2i-1}} = r_{\sigma(i)}$ for $i = 1, 2, \ldots, m$ and the t_i associated with $\mathbf{x}$. Since $\bigcap_{\delta > 0} B_\delta \cap \mathbb{C}([0,1], \mathbb{R}^2) = \bigcup_\sigma \widehat{B}_\sigma$, we have by (5.1),

$$\limsup_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \mathbf{P}(B^n_\delta) \le -\inf_\sigma \inf_{(\mathbf{x},\mathbf{y}) \in \widehat{B}_\sigma} \big(I^S(\mathbf{x}) + I^E_{\mathbf{x}}(\mathbf{y})\big).$$

As the function π is convex and $\pi(1) = 0$, it follows by the form of $I^E_{\mathbf{x}}(\mathbf{y})$ in Corollary 4.1 that the infimum of $I^E_{\mathbf{x}}(\mathbf{y})$ over $\mathbf{y}$ such that $(\mathbf{x}, \mathbf{y}) \in \widehat{B}_\sigma$, where $\mathbf{x} \in B'_\sigma$ is fixed as well as the points t_i, is attained at $\hat{\mathbf{y}}$ defined by $\dot{\hat{\mathbf{y}}}_t = r_{\sigma(i)} \mathcal{R}(\mathbf{x})_t / \int_{t_{2i-1}}^{t_{2i}} \mathcal{R}(\mathbf{x})_s\,ds$ for $t \in [t_{2i-1}, t_{2i}]$, where $i = 1, \ldots, m$, and $\dot{\hat{\mathbf{y}}}_t = c\mathcal{R}(\mathbf{x})_t$ elsewhere, and is equal to $\sum_{i=1}^m \pi(r_{\sigma(i)}/(c\int_{t_{2i-1}}^{t_{2i}} \mathcal{R}(\mathbf{x})_s\,ds)) c \int_{t_{2i-1}}^{t_{2i}} \mathcal{R}(\mathbf{x})_s\,ds$. We can thus write

$$\begin{aligned}&\limsup_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \mathbf{P}(B^n_\delta) \\ &\quad\le -\inf_\sigma \inf_{\mathbf{x} \in B'_\sigma} \left(I^S(\mathbf{x}) + \sum_{i=1}^m \pi\left(\frac{r_{\sigma(i)}}{c\int_{t_{2i-1}}^{t_{2i}} \mathcal{R}(\mathbf{x})_s\,ds}\right) c \int_{t_{2i-1}}^{t_{2i}} \mathcal{R}(\mathbf{x})_s\,ds\right).\end{aligned} \tag{5.2}$$

We now evaluate the infimum over B'_σ. For $\mathbf{x} \in B'_\sigma$ with $I^S(\mathbf{x}) < \infty$, let $\phi = (\phi_t,\, t \in [0,1]) = \mathcal{T}(\mathbf{x})$. The condition $\dot{\mathbf{x}}_t \geq -1$ a.e. implies that $\dot{\phi}_t \leq 1$ a.e. The function ϕ does not increase on the intervals $[t_{2i-1}, t_{2i}]$, $i = 1, 2, \dots, m$, so $a = \phi_1 = \sum_{i=0}^m \int_{t_{2i}}^{t_{2i+1}} \dot{\phi}_t\, dt \leq 1 - \sum_{i=1}^m u_i$, implying that $I^S(\mathbf{x}) = \infty$ for $\mathbf{x} \in B'_\sigma$ if $\sum_{i=1}^m u_i > 1 - a$. This proves the second limit in the statement of the lemma. In the rest of the argument we assume that $\sum_{i=1}^m u_i \leq 1 - a$. We have, on using that $\dot{\mathbf{x}}_t \geq -1$ a.e.,

$$
\begin{aligned}
&\inf_{\mathbf{x} \in B'_\sigma} \left(I^S(\mathbf{x}) + \sum_{i=1}^m \pi\left(\frac{r_{\sigma(i)}}{c \int_{t_{2i-1}}^{t_{2i}} \mathcal{R}(\mathbf{x})_s\, ds} \right) c \int_{t_{2i-1}}^{t_{2i}} \mathcal{R}(\mathbf{x})_s\, ds \right) \\
&\qquad = \inf_{\substack{w_i \in [0, u_i^2/2),\\ i = 1,2,\dots,m}} \left(\inf_{\mathbf{x} \in B'_\sigma(w_1,\dots,w_m)} I^S(\mathbf{x}) + \sum_{i=1}^m \pi\left(\frac{r_i}{c w_i}\right) c w_i \right),
\end{aligned} \tag{5.3}
$$

where $B'_\sigma(w_1, \dots, w_m) = \{\mathbf{x} \in B'_\sigma : \int_{t_{2i-1}}^{t_{2i}} \mathcal{R}(\mathbf{x})_s\, ds = w_{\sigma(i)},\ i = 1, \dots, m\}$. We next prove that

$$
\begin{aligned}
&\inf_{\mathbf{x} \in B'_\sigma(w_1,\dots,w_m)} I^S(\mathbf{x}) \\
&\quad = \sum_{i=1}^m \sup_{\rho \in \mathbb{R}_+} \big(K_\rho(u_i) + (\rho - c) w_i\big) + L_c\left((1-2a) \vee \sum_{i=1}^m u_i \right) \\
&\qquad + \frac{c}{2}\left(1 - (1-2a) \vee \sum_{i=1}^m u_i \right)^2 \pi\left(\frac{2(1 - a - (1-2a) \vee \sum_{i=1}^m u_i)}{c(1 - (1-2a) \vee \sum_{i=1}^m u_i)^2} \right).
\end{aligned} \tag{5.4}
$$

Since for $\mathbf{x} \in B'_\sigma$, we have that $\mathcal{R}(\mathbf{x})_{t_{2i-1}} = 0$ and $\mathcal{T}(\mathbf{x})_{t_{2i}} = \mathcal{T}(\mathbf{x})_{t_{2i-1}}$ for $i = 1, 2, \dots, m$, it follows that $\mathcal{R}(\mathbf{x})_t = \mathbf{x}_t - \mathbf{x}_{t_{2i-1}}$ for $t \in [t_{2i-1}, t_{2i}]$. Hence, in view of the form of I^S in Theorem 4.1 and Lemma 3.3, if we change $\mathbf{x} \in B'_\sigma(w_1, \dots, w_m)$ with $I^S(\mathbf{x}) < \infty$ on intervals $[t_{2i-1}, t_{2i}]$ to $(\mathbf{x}_{t_{2i-1}} + \tilde{\mathbf{x}}_p(t_{2i-1}, t_{2i}),\ p \in [t_{2i-1}, t_{2i}])$, where $\tilde{\mathbf{x}}_p(t_{2i-1}, t_{2i})$ is defined in the statement of Lemma 3.3, this will not increase the value of $I^S(\mathbf{x})$. The altered function $\mathbf{x}$ will still belong to $B'_\sigma(w_1, \dots, w_m)$ (note that ϕ is not affected by this modification of $\mathbf{x}$). Since $\mathbf{x}_t + \phi_t = 0$ on $\bigcup_{i=0}^m [t_{2i}, t_{2i+1}]$, the function ϕ and the intervals $[t_{2i-1}, t_{2i}]$ uniquely determine the modified function $\mathbf{x}$. We may thus optimize over ϕ and the $[t_{2i}, t_{2i+1}]$, and assume, in view of Lemmas 3.2 and 3.3, Theorem 4.1 and the fact that $\dot{\phi}_t = 0$ a.e. on $\bigcup_{i=1}^m [t_{2i-1}, t_{2i}]$, that $\mathbf{x}$ is such that

$$
\begin{aligned}
I^S(\mathbf{x}) &= \sum_{i=1}^m \left(\sup_{\rho \in \mathbb{R}_+} \big(K_\rho(t_{2i} - t_{2i-1}) + (\rho - c) w_{\sigma(i)}\big) + L_c(t_{2i}) - L_c(t_{2i-1}) \right) \\
&\quad + \int_0^1 \mathbf{1}\left(t \in \bigcup_{i=0}^m [t_{2i}, t_{2i+1}] \right) \pi\left(\frac{1 - \dot{\phi}_t}{c(1-t)} \right) c(1-t)\, dt \\
&= \sum_{i=1}^m \sup_{\rho \in \mathbb{R}_+} \big(K_\rho(u_i) + (\rho - c) w_i\big) + \int_0^1 \pi\left(\frac{1 - \dot{\phi}_t}{c(1-t)} \right) c(1-t)\, dt,
\end{aligned}
$$

where for the latter equality we used the definition of L_c in (2.14). An application of Lemma 3.4 yields (5.4).

Now, a minimax argument [cf., e.g., Aubin and Ekeland (1984)] shows that

$$\begin{aligned}(5.5)\quad &\inf_{w_i\in[0,u_i^2/2)}\left(\sup_{\rho\in\mathbb{R}_+}\bigl(K_\rho(u_i)+(\rho-c)w_i\bigr)+\pi\left(\frac{r_i}{cw_i}\right)cw_i\right)\\ &\qquad=\sup_{\rho\in\mathbb{R}_+}\left(K_\rho(u_i)+r_i\log\frac{\rho}{c}\right).\end{aligned}$$

Thus, by (5.2)–(5.5), if $\sum_{i=1}^m u_i\le 1-a$, then

$$\begin{aligned}&\limsup_{\delta\to0}\limsup_{n\to\infty}\frac{1}{n}\log\mathbf{P}(B_\delta^n)\\ &\quad\le-\left[\sum_{i=1}^m\sup_{\rho\in\mathbb{R}_+}\left(K_\rho(u_i)+r_i\log\frac{\rho}{c}\right)+L_c\left((1-2a)\vee\sum_{i=1}^m u_i\right)\right.\\ &\qquad\left.+\frac{c}{2}\left(1-(1-2a)\vee\sum_{i=1}^m u_i\right)^2\pi\left(\frac{2(1-a-(1-2a)\vee\sum_{i=1}^m u_i)}{c(1-(1-2a)\vee\sum_{i=1}^m u_i)^2}\right)\right].\end{aligned}$$

We now establish the lower bound: if $\sum_{i=1}^m u_i\le 1-a$, then

$$\begin{aligned}&\liminf_{\delta\to0}\liminf_{n\to\infty}\frac{1}{n}\log\mathbf{P}(B_\delta^n)\\ (5.6)\quad &\quad\ge-\left[\sum_{i=1}^m\sup_{\rho\in\mathbb{R}_+}\left(K_\rho(u_i)+r_i\log\frac{\rho}{c}\right)+L_c\left((1-2a)\vee\sum_{i=1}^m u_i\right)\right.\\ &\qquad\left.+\frac{c}{2}\left(1-(1-2a)\vee\sum_{i=1}^m u_i\right)^2\pi\left(\frac{2(1-a-(1-2a)\vee\sum_{i=1}^m u_i)}{c(1-(1-2a)\vee\sum_{i=1}^m u_i)^2}\right)\right].\end{aligned}$$

Let $(\tilde{w}_i,\tilde{\rho}_i)$ denote the saddle point of the function on the left-hand side of (5.5) so that

$$(5.7)\quad K_{\tilde{\rho}_i}(u_i)+(\tilde{\rho}_i-c)\tilde{w}_i+\pi\left(\frac{r_i}{c\tilde{w}_i}\right)c\tilde{w}_i=\sup_{\rho\in\mathbb{R}_+}\left(K_\rho(u_i)+r_i\log\frac{\rho}{c}\right).$$

Calculations show that $\tilde{\rho}_i$ and $\tilde{w}_i$ are specified by the equalities

$$(5.8)\quad \frac{\tilde{\rho}_i u_i}{1-e^{-\tilde{\rho}_i u_i}}=1+\frac{r_i}{u_i}+\frac{\tilde{\rho}_i u_i}{2},\qquad \tilde{w}_i=\frac{r_i}{\tilde{\rho}_i},$$

with $\tilde{\rho}_i=\tilde{w}_i=0$ if $r_i=0$. Let $s_0=0$, $s_i=\sum_{j=1}^i u_j$, $j=1,\dots,m$. Motivated by the form of the optimal trajectory in Lemma 3.3, the definition of $\hat{\mathbf{y}}$ above and the definitions of $\tilde{\phi}$ and $\hat{\phi}$ in the proof of Lemma 3.4, we define, for $\eta \in (0, \min_{i=1,2,\dots,m} u_i)$, an absolutely continuous function $\breve{\mathbf{x}}^\eta$ by $\breve{\mathbf{x}}_0^\eta = 0$,

$$\dot{\breve{\mathbf{x}}}_t^\eta = -1 + \frac{u_i - \eta}{1 - e^{-(\tilde{\rho}_i \vee \eta)(u_i - \eta)}} (\tilde{\rho}_i \vee \eta) e^{-(\tilde{\rho}_i \vee \eta)(t - s_{i-1})}$$

for $t \in (s_{i-1}, s_i)$ and $i = 1, \dots, m$, $\dot{\breve{\mathbf{x}}}_t^\eta = -\eta$ for $t \in (s_m, s_m \vee (1 - 2a))$, and

$$\dot{\breve{\mathbf{x}}}_t^\eta = -1 + 2\frac{1 - s_m \vee (1 - 2a) - a}{(1 - s_m \vee (1 - 2a))^2}(1 - t)$$

for $t \in (s_m \vee (1 - 2a), 1)$, and we define an absolutely continuous function $\breve{\mathbf{y}}^\eta$ by $\dot{\breve{\mathbf{y}}}_t^\eta = r_i \mathcal{R}(\breve{\mathbf{x}}^\eta)_t / (\int_{s_{i-1}}^{s_i - \eta} \mathcal{R}(\breve{\mathbf{x}}^\eta)_s \, ds)$ for $t \in (s_{i-1}, s_i - \eta)$, $i = 1, \dots,$ and $\dot{\breve{\mathbf{y}}}_t^\eta = c\mathcal{R}(\mathbf{x})_t$ elsewhere.

Let us fix arbitrary $\delta \in (0, \min_{i=1,2,\dots,m} u_i)$. For $\varepsilon > 0$, let $\breve{B}_{\varepsilon,\eta}$ denote the ε-neighborhood of $(\breve{\mathbf{x}}^\eta, \breve{\mathbf{y}}^\eta)$ in $\mathbb{D}_C([0, 1], \mathbb{R}^2)$. It follows from the definitions of $\breve{\mathbf{x}}^\eta$, $\breve{\mathbf{y}}^\eta$ and the operator $\mathcal{R}$ that if ε and η are small enough, then for arbitrary $(\mathbf{x}, \mathbf{y}) \in \breve{B}_{\varepsilon,\eta}$ with $\mathbf{x}_0 = \mathbf{y}_0 = 0$ and $\mathbf{y}$ nondecreasing, there exist disjoint segments $(\tilde{s}_{i-1}, \tilde{s}_i)$, $i = 1, 2, \dots, m$ with $|\tilde{s}_i - \tilde{s}_{i-1} - u_i| < \delta$ such that the function $\mathcal{R}(\mathbf{x})$ is positive on these segments and equals zero at the endpoints, the other intervals where $\mathcal{R}(\mathbf{x})$ is positive are of lengths less than δ, and $|\mathbf{y}_{\tilde{s}_i} - \mathbf{y}_{\tilde{s}_{i-1}} - r_i| < \delta$. Furthermore, it may be assumed that $\mathcal{T}(\mathbf{x})_1 \in (a - \delta, a + \delta)$. We, therefore, have by (2.9) and (2.10) that $\{(\overline{S}^n + \bar{\varepsilon}^n, \overline{E}^n) \subset \breve{B}_{\varepsilon,\eta}\} \subset B_\delta^n$ for all small enough ε and η. As the set $\breve{B}_{\varepsilon,\eta}$ is open in $\mathbb{D}_C([0, 1], \mathbb{R}^2)$, in view of Lemma 3.1 and Corollary 4.1,

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{n} \log \mathbf{P}(B_\delta^n) &\geq \liminf_{n\to\infty} \frac{1}{n} \log \mathbf{P}\bigl((\overline{S}^n + \bar{\varepsilon}^n, \overline{E}^n) \subset \breve{B}_{\varepsilon,\eta}\bigr) \\
&\geq - \inf_{(\mathbf{x},\mathbf{y})\in\breve{B}_{\varepsilon,\eta}} \bigl(I^S(\mathbf{x}) + I_{\mathbf{x}}^E(\mathbf{y})\bigr) \qquad (5.9)\\
&\geq -\bigl(I^S(\breve{\mathbf{x}}^\eta) + I_{\breve{\mathbf{x}}^\eta}^E(\breve{\mathbf{y}}^\eta)\bigr).
\end{aligned}$$

By the definitions of $\breve{\mathbf{x}}^\eta$ and $\breve{\mathbf{y}}^\eta$, (5.7), (5.8), the form of I^S in Theorem 4.1, the form of I^E in Corollary 4.1, part 1 of Lemma 3.3 and part 1 of Lemma 3.4, we have that $I^S(\breve{\mathbf{x}}^\eta) + I_{\breve{\mathbf{x}}^\eta}^E(\breve{\mathbf{y}}^\eta)$ converges as $\eta \to 0$ to the sum on the right-hand side of (5.6), which together with (5.9) concludes the proof of (5.6). □

PROOF OF THEOREM 2.1. We check that the sequence $(\alpha^n/n, \overline{U}^n, \overline{R}^n)$, $n \in \mathbb{N}$, is exponentially tight (of order n) in $[0, 1] \times \mathbb{S}_1 \times \mathbb{S}$. By Lemma 3.5, the subsets of $[0, 1] \times \mathbb{S}_1 \times \mathbb{S}$ of elements $(a, \mathbf{u}, \mathbf{r})$, where $\mathbf{r} = (r_1, r_2, \dots)$, with the property that $\sum_{i=1}^\infty r_i \leq B$ for some $B > 0$ and $r_i \to 0$ as $i \to \infty$ uniformly, are compact. Therefore, it suffices to check that

$$\lim_{B\to\infty} \limsup_{n\to\infty} \mathbf{P}\left(\sum_{i=1}^\infty \overline{R}_i^n > B\right)^{1/n} = 0, \qquad (5.10)$$

$$\lim_{i\to\infty} \limsup_{n\to\infty} \mathbf{P}\left(\sup_{j=i,i+1,\dots} \overline{R}_j^n > \eta\right)^{1/n} = 0, \qquad \eta > 0. \qquad (5.11)$$

The first limit follows by exponential tightness of the $\overline{E}_1^n$ valid in view of Corollary 4.1 and the fact that $\sum_{i=1}^\infty \overline{R}_i^n = \overline{E}_1^n$. For the second limit, we note that $\overline{R}_i^n$ equals the increment of $\overline{E}_t^n$ over a time interval of length $\overline{U}_i^n$, so for $\delta > 0$,

$$\bigcup_{i=1}^{\infty}\{\overline{R}_i^n > \eta,\ \overline{U}_i^n \le \delta\} \subset \Big\{ \sup_{s,t\in[0,1]:\,|s-t|\le\delta} |\overline{E}_t^n - \overline{E}_s^n| > \eta \Big\}. \tag{5.12}$$

Since $u_i \le 1/i$ for an element $\mathbf{u} = (u_1, u_2, \ldots)$ of $\mathbb{S}_1$, we have that

$$\begin{aligned}\limsup_{i\to\infty}\limsup_{n\to\infty} \mathbf{P}\Big(\sup_{j=i,i+1,\ldots} \overline{R}_j^n > \eta\Big)^{1/n}\\ \le \limsup_{n\to\infty} \mathbf{P}\Big(\sup_{s,t\in[0,1]:\,|s-t|\le\delta} |\overline{E}_t^n - \overline{E}_s^n| > \eta\Big)^{1/n}.\end{aligned}$$

Therefore, (5.11) follows on using that by $\mathbb{C}$-exponential tightness of the $\overline{E}^n$,

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \mathbf{P}\Big(\sup_{s,t\in[0,1]:\,|s-t|\le\delta} |\overline{E}_t^n - \overline{E}_s^n| > \eta\Big)^{1/n} = 0. \tag{5.13}$$

It thus remains to check that

$$\begin{aligned}\lim_{\varepsilon\to 0}\limsup_{n\to\infty} \frac{1}{n}\log \mathbf{P}\Big(d\Big(\Big(\frac{\alpha^n}{n}, \overline{U}^n, \overline{R}^n\Big), (a,\mathbf{u},\mathbf{r})\Big) \le \varepsilon\Big)\\ = \lim_{\varepsilon\to 0}\liminf_{n\to\infty} \frac{1}{n}\log \mathbf{P}\Big(d\Big(\Big(\frac{\alpha^n}{n}, \overline{U}^n, \overline{R}^n\Big), (a,\mathbf{u},\mathbf{r})\Big) \le \varepsilon\Big)\\ = -I_c^{\alpha,U,R}(a,\mathbf{u},\mathbf{r}),\end{aligned}$$

where d is a product metric on $[0,1]\times\mathbb{S}_1\times\mathbb{S}$ and $(a,\mathbf{u},\mathbf{r}) \in [0,1]\times\mathbb{S}_1\times\mathbb{S}$. Let $\mathbf{u} = (u_1, u_2, \ldots)$ and $\mathbf{r} = (r_1, r_2, \ldots)$. If all the $u_i > 0$, then given $\delta > 0$, for all small enough $\varepsilon > 0$ and all large enough m,

$$\Big\{d\Big(\Big(\frac{\alpha^n}{n}, \overline{U}^n, \overline{R}^n\Big), (a,\mathbf{u},\mathbf{r})\Big) \le \varepsilon\Big\} \subset B_\delta^n(a; \{u_i, r_i\}_{i=1}^m). \tag{5.14}$$

If $u_1 > 0$ and $u_i = 0$ for all large i, then (5.14) holds for m that is the greatest index i with $u_i > 0$. If $u_1 = 0$, then we have the inclusion

$$\Big\{d\Big(\Big(\frac{\alpha^n}{n}, \overline{U}^n, \overline{R}^n\Big), (a,\mathbf{u},\mathbf{r})\Big) \le \varepsilon\Big\} \subset \widetilde{B}_\delta^n(a).$$

Therefore, Lemma 5.1 and the form of $I_c^{\alpha,U,R}(a,\mathbf{u},\mathbf{r})$ imply that, provided $r_i = 0$ when $u_i = 0$,

$$\begin{aligned}\limsup_{\varepsilon\to 0}\limsup_{n\to\infty} \frac{1}{n}\log \mathbf{P}\Big(d\Big(\Big(\frac{\alpha^n}{n}, \overline{U}^n, \overline{R}^n\Big), (a,\mathbf{u},\mathbf{r})\Big) \le \varepsilon\Big)\\ \le -I_c^{\alpha,U,R}(a,\mathbf{u},\mathbf{r}).\end{aligned} \tag{5.15}$$

If for some i we have that $u_i = 0$ and $r_i > 0$, then by (5.12) and (5.13) the left-hand side of (5.15) equals $-\infty$, so the required inequality holds as well.

For the lower bound,

$$\liminf_{\varepsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log \mathbf{P}\biggl(d\biggl(\biggl(\frac{\alpha^n}{n},\overline{U}^n,\overline{R}^n\biggr),(a,\mathbf{u},\mathbf{r})\biggr)\le\varepsilon\biggr)$$

$$\ge -I_c^{\alpha,U,R}(a,\mathbf{u},\mathbf{r}), \tag{5.16}$$

we may assume that $r_i = 0$ when $u_i = 0$. Let us be given $\varepsilon > 0$ and $B > 0$. If all the u_i are positive, then for all small enough $\delta > 0$, $\eta > 0$ and large enough m, we have the inclusion

$$B_\delta^n(a;\{u_i,r_i\}_{i=1}^m)\subset\biggl\{d\biggl(\biggl(\frac{\alpha^n}{n},\overline{U}^n,\overline{R}^n\biggr),(a,\mathbf{u},\mathbf{r})\biggr)\le\varepsilon\biggr\}$$

$$\cup\biggl\{\sum_{i=1}^\infty \overline{R}_i^n > B\biggr\}\cup\biggl\{\sup_{i=m+1,\dots}\overline{R}_i^n>\eta\biggr\}.$$

To see the latter we use the inequality $\sum_{i=m+1}^\infty \chi(u_i'/\varepsilon)\le \sup_{i=m+1,\dots}(\chi(u_i'/\varepsilon)/u_i')\sum_{i=m+1}^\infty u_i'$ for $(u_1',u_2',\dots)\in\mathbb{S}$ and the convergence $\chi(x)/x\to 0$ as $x\to 0$. Lemma 5.1, (5.10) and (5.11) imply (5.16). If $u_1 > 0$ and not all the u_i are positive, then by a similar argument

$$B_\delta^n(a;\{u_i,r_i\}_{i=1}^m)\subset\biggl\{d\biggl(\biggl(\frac{\alpha^n}{n},\overline{U}^n,\overline{R}^n\biggr),(a,\mathbf{u},\mathbf{r})\biggr)\le\varepsilon\biggr\}$$

$$\cup\biggl\{\sum_{i=1}^\infty \overline{R}_i^n > B\biggr\}\cup\bigcup_{i=m+1}^\infty\{\overline{R}_i^n>\eta,\ \overline{U}_i^n\le\delta\},$$

where m is the greatest index i with $u_i > 0$. If $u_1 = 0$, then

$$\widetilde{B}_\delta^n(a)\subset\biggl\{d\biggl(\biggl(\frac{\alpha^n}{n},\overline{U}^n,\overline{R}^n\biggr),(a,\mathbf{u},\mathbf{r})\biggr)\le\varepsilon\biggr\}$$

$$\cup\biggl\{\sum_{i=1}^\infty \overline{R}_i^n > B\biggr\}\cup\bigcup_{i=1}^\infty\{\overline{R}_i^n>\eta,\ \overline{U}_i^n\le\delta\}.$$

In either case, (5.16) follows by Lemma 5.1, (5.10), (5.12) and (5.13). □

Corollaries 2.1 and 2.2 follow by an application of the contraction principle. In some more detail, the infima of $I_c^{\alpha,U,R}(a,\mathbf{u},\mathbf{r})$ and $I_c^{\alpha,U}(a,\mathbf{u})$ over $a\in[0,1]$ are attained at $a^* = 1/(2c)$ if $\sum_{i=1}^\infty u_i < 1-1/c$ and at $a^* = 1-\sum_{i=1}^\infty u_i - c(1-\sum_{i=1}^\infty u_i)^2/2$ if $\sum_{i=1}^\infty u_i \ge 1-1/c$; the infimum of $I_c^{\alpha,U,R}(a,\mathbf{u},\mathbf{r})$ over $\mathbf{r}\in\mathbb{S}$ is found by a minimax argument [it is actually attained at $\mathbf{r}^* = (r_1^*, r_2^*,\dots)$ with $r_i^* = cu_i^2/(1-\exp(-cu_i)) - cu_i^2/2 - u_i$), cf. Aubin and Ekeland (1984)]. The expression for $I_c^\alpha(a)$ is obtained on noting that subadditivity of $K_c(u)$ in u implies that $\sum_{i=1}^{\infty} K_c(u_i) \ge K_c(\sum_{i=1}^{\infty} u_i)$, so one should minimize $I_c^{\alpha,U}(a,\mathbf{u})$ with respect to $\sum_{i=1}^{\infty} u_i$, and that $K_c(u)$ is monotonically decreasing in u, so the infimum can be taken over $\sum_{i=1}^{\infty} u_i \ge 1-2a$. We provide more detail as to the proofs of Corollaries 2.3 and 2.4.

PROOF OF COROLLARY 2.3. Let $A_\delta(u_1,\dots,u_m)$, for $\delta \in (0, \min_{i=1,\dots,m} u_i/2)$, denote the subset of $\mathbb{S}_1$ of vectors $\tilde{\mathbf{u}} = (\tilde u_1, \tilde u_2, \dots)$ such that there exist distinct $j_i \in \{1,2,\dots,\lfloor 2/u_i \rfloor\}$ with $|\tilde u_{j_i} - u_i| < \delta$ for $i = 1,2,\dots,m$. Let a set $A(u_1,\dots,u_m)$ be defined as the set of $\tilde{\mathbf{u}} = (\tilde u_1, \tilde u_2, \dots) \in \mathbb{S}_1$ such that $\tilde u_{j_i} = u_i$, $i = 1,2,\dots,m$, for some $j_1,\dots,j_m$. Since $A(u_1,\dots,u_m)$ equals the intersection of the closures of the $A_\delta(u_1,\dots,u_m)$ over $\delta > 0$, the sets $A_\delta(u_1,\dots,u_m)$ are open in $\mathbb{S}_1$, and $A_\delta^n(u_1,\dots,u_m) = \{\overline{U}^n \in A_\delta(u_1,\dots,u_m)\}$, we have, by Corollary 2.2 and the definition of the LDP,

$$\begin{aligned}
&\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \mathbf{P}\big(A_\delta^n(u_1,\dots,u_m)\big)\\
&\quad= \lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{n} \log \mathbf{P}\big(A_\delta^n(u_1,\dots,u_m)\big)\\
&\quad= -\inf_{\mathbf{u}\in A(u_1,\dots,u_m)} I_c^U(\mathbf{u}).
\end{aligned}$$

We evaluate the latter infimum. Since $I_c^U(\mathbf{u})$ is invariant with respect to permutations of the entries of $\mathbf{u}$, we may replace $\mathbf{u}$ with its permutation that has $u_1,\dots,u_m$ as the first m entries. By subadditivity of $K_c(u)$ in u, we have that $\sum_{i=m+1}^{\infty} K_c(u_i) \ge K_c(\sum_{i=m+1}^{\infty} u_i)$, so it is optimal to assume that $u_{m+2} = u_{m+3} = \cdots = 0$. We thus need to find optimal u_{m+1}. If $\sum_{i=1}^{m} u_i \ge 1 - 1/c$, then $I_c^U(\mathbf{u}) = \sum_{i=1}^{m+1} K_c(u_i) + L_c(\sum_{i=1}^{m+1} u_i)$. By Lemma 3.2 $K_c(u_{m+1}) + L_c(\sum_{i=1}^{m+1} u_i) > L_c(\sum_{i=1}^{m} u_i)$ for any $u_{m+1} > 0$, so it is optimal to take $u_{m+1} = 0$, accordingly, $\inf_{\mathbf{u}\in A(u_1,\dots,u_m)} I_c^U(\mathbf{u}) = \sum_{i=1}^{m} K_c(u_i) + L_c(\sum_{i=1}^{m} u_i)$. If $\sum_{i=1}^{m} u_i < 1 - 1/c$, then Lemma 3.2 implies that for $u^* > 0$, such that $u^*/(1-\exp(-cu^*)) = 1 - \sum_{i=1}^{m} u_i$, we have $K_c(u^*) + L_c(\sum_{i=1}^{m} u_i + u^*) = L_c(\sum_{i=1}^{m} u_i)$. Also $\sum_{i=1}^{m} u_i + u^* > 1 - 1/c$, so the choice of u^* as u_{m+1} yields the value of the action functional $\sum_{i=1}^{m} K_c(u_i) + K_c(u^*) + L_c(\sum_{i=1}^{m} u_i + u^*) = \sum_{i=1}^{m} K_c(u_i) + L_c(\sum_{i=1}^{m} u_i)$. If $u_{m+1} \ne u^*$ and is such that $\sum_{i=1}^{m+1} u_i \ge 1 - 1/c$, then $I_c^U(\mathbf{u}) = \sum_{i=1}^{m+1} K_c(u_i) + L_c(\sum_{i=1}^{m+1} u_i)$, which is greater than $\sum_{i=1}^{m} K_c(u_i) + L_c(\sum_{i=1}^{m} u_i)$ by Lemma 3.2. Finally, if u_{m+1} is such that $\sum_{i=1}^{m+1} u_i < 1 - 1/c$, then with the use of Lemma 3.2, $I_c^U(\mathbf{u}) = \sum_{i=1}^{m+1} K_c(u_i) + L_c(1-1/c) > \sum_{i=1}^{m+1} K_c(u_i) + L_c(\sum_{i=1}^{m+1} u_i) \ge \sum_{i=1}^{m} K_c(u_i) + L_c(\sum_{i=1}^{m} u_i)$. Therefore, u^* is the optimal value of u_{m+1}. Thus, $\inf_{\mathbf{u}\in A(u_1,\dots,u_m)} I_c^U(\mathbf{u}) = \sum_{i=1}^{m} K_c(u_i) + L_c(\sum_{i=1}^{m} u_i)$ and it is attained at a unique point $\mathbf{u}^*$ given by $\mathbf{u}^* = (u_1,u_2,\dots,u_m,0,0,\dots)$ if $\sum_{i=1}^{m} u_i \ge 1 - 1/c$ and $\mathbf{u}^* = (u_1,u_2,\dots,u_m,u^*,0,0,\dots)$ if $\sum_{i=1}^{m} u_i < 1 - 1/c$. We also have by the form of $I_c^{U,R}$ in Corollary 2.1 that the infimum of $I_c^{U,R}(\mathbf{u}^*,\mathbf{r})$ over

$\mathbf{r}$ equals $I_c^U(\mathbf{u}^*)$ and is attained at the unique point $\mathbf{r}^* = (r_1^*, \dots, r_m^*, 0, 0, \dots)$ if $\sum_{i=1}^m u_i \geq 1 - 1/c$ and $\mathbf{r}^* = (r_1^*, \dots, r_m^*, r^*, 0, 0, \dots)$ if $\sum_{i=1}^m u_i < 1 - 1/c$. Therefore, letting $\tilde{d}$ denote a metric on $\mathbb{S}_1 \times \mathbb{S}$,

$$\begin{aligned}
&\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \mathbf{P}\big(A_\delta^n(u_1, \dots, u_m)\big) \\
&\quad = \lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{n} \log \mathbf{P}\big(A_\delta^n(u_1, \dots, u_m)\big) \\
&\quad = \lim_{\eta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \mathbf{P}\big(\tilde{d}\big((\overline{U}^n, \overline{R}^n), (\mathbf{u}^*, \mathbf{r}^*)\big) < \eta\big) \\
&\quad = \lim_{\eta\to 0} \liminf_{n\to\infty} \frac{1}{n} \log \mathbf{P}\big(\tilde{d}\big((\overline{U}^n, \overline{R}^n), (\mathbf{u}^*, \mathbf{r}^*)\big) < \eta\big) \\
&\quad = -\left(\sum_{i=1}^m K_c(u_i) + L_c\left(\sum_{i=1}^m u_i\right)\right).
\end{aligned}$$

In addition, $\liminf_{\delta\to 0} \lim_{n\to\infty} \mathbf{P}(\{\tilde{d}((\overline{U}^n, \overline{R}^n), (\mathbf{u}^*, \mathbf{r}^*)) < \eta\} | A_\delta^n(u_1, \dots, u_m)) = 1$ for $\eta > 0$ as in Freidlin and Wentzell [(1998), Theorem 3.4 of Chapter 3]. The proof is completed by noting that $\{\tilde{d}((\overline{U}^n, \overline{R}^n), (\mathbf{u}^*, \mathbf{r}^*)) < \eta\} \subset \widetilde{A}_{\delta,\varepsilon}^n(u_1, \dots, u_m) \subset A_\delta^n(u_1, \dots, u_m)$ for all small enough $\eta > 0$. □

PROOF OF COROLLARY 2.4. By Theorem 2.1 and the contraction principle,

$$I_c^{\alpha,\beta,\gamma}(a, u, r) = \inf_{(\mathbf{u},\mathbf{r})\in O(u,r)} I_c^{\alpha,U,R}(a, \mathbf{u}, \mathbf{r}), \tag{5.17}$$

where $O(u, r) = \{(\mathbf{u}, \mathbf{r}) \in \mathbb{S}_1 \times \mathbb{S} : u_1 = u,\ r_1 = r\}$. The assertion of the corollary for $u = 0$ follows. Let us assume now that $u > 0$. The infimum of $\sup_{\rho\in\mathbb{R}_+}(K_\rho(x) + r\log(\rho/c))$ over $r \in \mathbb{R}_+$ equals $K_c(x)$, therefore, it suffices to minimize over $u_2, u_3, \dots$ the function

$$\begin{aligned}
&\sum_{i=1}^\infty K_c(u_i) + L_c\left((1-2a) \vee \sum_{i=1}^\infty u_i\right) \\
&\qquad + \frac{c}{2}\left(1 - (1-2a) \vee \sum_{i=1}^\infty u_i\right)^2 \pi\left(\frac{2(1 - a - (1-2a) \vee \sum_{i=1}^\infty u_i)}{c(1 - (1-2a) \vee \sum_{i=1}^\infty u_i)^2}\right).
\end{aligned}$$

By the fact that $K_c(x) < 0$ for $x > 0$ and is decreasing in x (Lemma 3.2), we can assume that in an optimal configuration $\sum_{i=1}^\infty u_i \geq 1 - 2a$. Next, since $K_c(x)$ is concave in x, $K_c(0) = 0$ and $u_i \leq u$, we have that

$$\sum_{i=1}^\infty K_c(u_i) \geq \left\lfloor \frac{\sum_{i=1}^\infty u_i}{u} \right\rfloor K_c(u) + K_c\left(\sum_{i=1}^\infty u_i - u\left\lfloor \frac{\sum_{i=1}^\infty u_i}{u} \right\rfloor\right). \tag{5.18}$$

Hence, by Theorem 2.1,

$$I_c^{\alpha,\beta,\gamma}(a,u,r) = \sup_{\rho\in\mathbb{R}_+}\left(K_\rho(u) + r\log\frac{\rho}{c}\right) - K_c(u)$$
$$(5.19)\qquad + \inf_{\tau\in[(1-2a)\vee u,1-a]}\left(\left\lfloor\frac{\tau}{u}\right\rfloor K_c(u) + K_c\left(\tau - u\left\lfloor\frac{\tau}{u}\right\rfloor\right)\right.$$
$$\left. + L_c(\tau) + \frac{c}{2}(1-\tau)^2\pi\left(\frac{2(1-a-\tau)}{c(1-\tau)^2}\right)\right),$$

as required. Part 1 has been proved.

We prove part 2. By the contraction principle the sequence $(\beta^n/n, \gamma^n/n), n\in\mathbb{N}$, obeys the LDP for scale n with action functional $I_c^{\beta,\gamma}(u,r) = \inf_{a\in[0,1]} I_c^{\alpha,\beta,\gamma}(a, u,r)$, which yields the assertion of part 2 for $(u,r) = (0,0)$. Let $u>0$. The infimum of the right-most term on the right-hand side of (5.19) over $a\in[(1-\tau)/2, 1-\tau]$ is attained at $(1-\tau)/2$ if $\tau < 1-1/c$ and at $1-\tau-c(1-\tau)^2/2$ if $\tau\ge 1-1/c$ with respective values $c(1-\tau)^2/2\pi(1/(c(1-\tau)))$ and 0. If $\tau < 1-1/c$, then by Lemma 3.2 there exists $\tau^*\in(0,1-\tau)$ such that $\tau+\tau^* > 1-1/c$ and $L_c(\tau) = K_c(\tau^*) + L_c(\tau+\tau^*)$. Therefore, in analogy with (5.18),

$$\left\lfloor\frac{\tau}{u}\right\rfloor K_c(u) + K_c\left(\tau - u\left\lfloor\frac{\tau}{u}\right\rfloor\right)$$
$$+ L_c(\tau) \ge \left\lfloor\frac{\tau+\tau^*}{u}\right\rfloor K_c(u) + K_c\left(\tau+\tau^* - u\left\lfloor\frac{\tau+\tau^*}{u}\right\rfloor\right) + L_c(\tau+\tau^*),$$

which implies that we may disregard the domain $\tau < 1-1/c$. Hence, (5.19) yields

$$I_c^{\beta,\gamma}(u,r) = \sup_{\rho\in\mathbb{R}_+}\left(K_\rho(u) + r\log\frac{\rho}{c}\right) - K_c(u)$$
$$(5.20)\qquad + \inf_{\tau\in[(1-1/c)\vee u,1]}\left(\left\lfloor\frac{\tau}{u}\right\rfloor K_c(u) + K_c\left(\tau - u\left\lfloor\frac{\tau}{u}\right\rfloor\right) + L_c(\tau)\right).$$

If $u\ge 1-1/c$, then for $\tau\ge u$ by Lemma 3.2 $\lfloor\tau/u\rfloor K_c(u) + K_c(\tau - \lfloor\tau/u\rfloor u) + L_c(\tau) \ge K_c(u) + K_c(\tau-u) + L_c(\tau) \ge K_c(u) + L_c(u)$, so

$$(5.21)\qquad I_c^{\beta,\gamma}(u) = \sup_{\rho\in\mathbb{R}_+}\left(K_\rho(u) + r\log\frac{\rho}{c}\right) + L_c(u).$$

Let us now assume that $u < 1-1/c$, so $c>1$. If $\tau\ge\lfloor(1-1/c)/u\rfloor u + u$, then by the fact that $\lfloor(1-1/c)/u\rfloor u + u > 1-1/c$ and Lemma 3.2,

$$\left\lfloor\frac{\tau}{u}\right\rfloor K_c(u) + K_c\left(\tau - \left\lfloor\frac{\tau}{u}\right\rfloor u\right) + L_c(\tau)$$
$$\ge \left(\left\lfloor\frac{1}{u}\left(1-\frac{1}{c}\right)\right\rfloor + 1\right)K_c(u) + K_c\left(\tau - \left\lfloor\frac{1}{u}\left(1-\frac{1}{c}\right)\right\rfloor u - u\right) + L_c(\tau)$$
$$\ge \left(\left\lfloor\frac{1}{u}\left(1-\frac{1}{c}\right)\right\rfloor + 1\right)K_c(u) + L_c\left(\left\lfloor\frac{1}{u}\left(1-\frac{1}{c}\right)\right\rfloor u + u\right),$$

so by (5.20),

$$I_c^\beta(u) = \sup_{\rho\in\mathbb{R}_+}\left(K_\rho(u) + r\log\frac{\rho}{c}\right) - K_c(u)$$
$$(5.22)\qquad + \inf_{\tau\in[1-1/c,\lfloor(1-1/c)/u\rfloor u+u]}\left(\left\lfloor\frac{\tau}{u}\right\rfloor K_c(u) + K_c\left(\tau - u\left\lfloor\frac{\tau}{u}\right\rfloor\right) + L_c(\tau)\right).$$

By subadditivity of $K_c(x)$ in x, for $\tau \geq 1-1/c$,

$$(5.23)\qquad \begin{aligned}&\left\lfloor\frac{\tau}{u}\right\rfloor K_c(u) + K_c\left(\tau - \left\lfloor\frac{\tau}{u}\right\rfloor u\right)\\ &\quad\geq \left\lfloor\frac{1}{u}\left(1-\frac{1}{c}\right)\right\rfloor K_c(u) + K_c\left(\tau - \left\lfloor\frac{1}{u}\left(1-\frac{1}{c}\right)\right\rfloor u\right).\end{aligned}$$

By Lemma 3.2 and the definition of $\hat{u}$ for $\tau \in [1-1/c, \lfloor(1-1/c)/u\rfloor u + u]$,

$$K_c\left(\tau - \left\lfloor\frac{1}{u}\left(1-\frac{1}{c}\right)\right\rfloor u\right) + L_c(\tau) \geq K_c(\hat{u}\wedge u) + L_c\left(\left\lfloor\frac{1}{u}\left(1-\frac{1}{c}\right)\right\rfloor u + \hat{u}\wedge u\right),$$

which implies by (5.23) that the minimum in (5.22) is attained at $\hat{\tau} = \lfloor(1-1/c)/u\rfloor u + \hat{u}\wedge u$, completing the proof of part 2.

Part 3 follows by minimizing $I_c^{\beta,\gamma}(u,r)$ over $r\in\mathbb{R}_+$. $\square$

6. Normal and moderate deviations for the largest component. In this section we prove Theorems 2.2 and 2.3. We start by establishing a law-of-large-numbers result. Let

$$(6.1)\qquad \overline{M}_t^n = \frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=1}^{n-Q_{i-1}^n-(i-1)}\left(\xi_{ij}^n - \frac{c_n}{n}\right), \qquad t\in[0,1],$$

$$(6.2)\qquad \overline{L}_t^n = \frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=1}^{Q_{i-1}^n-1}\left(\zeta_{ij}^n - \frac{c_n}{n}\right), \qquad t\in[0,1],$$

so that by (2.5), (2.6), (2.11) and (2.12),

$$(6.3)\qquad \overline{Q}_t^n = \int_0^{\lfloor nt\rfloor/n}\left(c_n\left(1-\overline{Q}_s^n - \frac{\lfloor ns\rfloor}{n}\right) - 1\right)ds + \bar{\varepsilon}_t^n + \overline{M}_t^n + \overline{\Phi}_t^n,$$

$$(6.4)\qquad \overline{E}_t^n = c_n\int_0^{\lfloor nt\rfloor/n}\overline{Q}_s^n\,ds + \overline{L}_t^n - \frac{c_n}{n}\int_0^{\lfloor nt\rfloor/n}\mathbf{1}(\overline{Q}_s^n > 0)\,ds.$$

The processes $\overline{M}^n = (\overline{M}_t^n,\, t\in[0,1])$ and $\overline{L}^n = (\overline{L}_t^n,\, t\in[0,1])$ are orthogonal square integrable martingales relative to the filtration $(\mathcal{F}_t^n,\, t\in[0,1])$ with respective predictable quadratic characteristics

$$(6.5)\qquad \langle\overline{M}^n\rangle_t = \frac{c_n}{n}\left(1-\frac{c_n}{n}\right)\int_0^{\lfloor nt\rfloor/n}\left(1-\overline{Q}_s^n - \frac{\lfloor ns\rfloor}{n}\right)ds,$$

$$\langle \overline{L}^n \rangle_t = \frac{c_n}{n}\left(1 - \frac{c_n}{n}\right) \int_0^{\lfloor nt \rfloor/n} \left(\overline{Q}_s^n - \frac{1}{n}\right)^+ ds. \tag{6.6}$$

Let functions $\bar{q} = (\bar{q}_t,\ t \in [0,1])$, $\bar{\phi} = (\bar{\phi}_t,\ t \in [0,1])$ and $\bar{e} = (\bar{e}_t,\ t \in [0,1])$ be defined by

$$\bar{q}_t = \begin{cases} 1 - t - e^{-ct}, & \text{if } t \in [0,\beta], \\ 0, & \text{otherwise}, \end{cases} \tag{6.7}$$

$$\bar{\phi}_t = \begin{cases} \dfrac{c}{2}(t^2 - \beta^2) - (c-1)(t-\beta), & \text{if } t \in [\beta, 1], \\ 0, & \text{otherwise}, \end{cases} \tag{6.8}$$

and

$$\bar{e}_t = e^{-c(t\wedge\beta)} - 1 + c(t\wedge\beta) - \frac{c(t\wedge\beta)^2}{2}. \tag{6.9}$$

Equivalently, the pair $(\bar{q}, \bar{\phi})$ can be defined as the solution to the Skorohod problem

$$\bar{q}_t = \int_0^t \big(c(1 - \bar{q}_s - s) - 1\big)\, ds + \bar{\phi}_t \quad \text{and} \quad \bar{\phi}_t = \int_0^t \mathbf{1}(\bar{q}_s = 0)\, d\bar{\phi}_s. \tag{6.10}$$

We note that

$$\begin{aligned} \bar{q}_t &= \int_0^t \big(c(1 - \bar{q}_s - s) - 1\big)\, ds \qquad \text{for } t \in [0,\beta] \quad \text{and} \\ \bar{e}_t &= c\int_0^t \bar{q}_s\, ds \qquad \text{for } t \in [0,1]. \end{aligned} \tag{6.11}$$

LEMMA 6.1. *Let $c_n \to c > 0$ as $n \to \infty$. Then the processes $\overline{Q}^n$, $\overline{\Phi}^n$ and $\overline{E}^n$ converge in probability uniformly on $[0,1]$ to the functions $\bar{q}$, $\bar{\phi}$ and $\bar{e}$, respectively.*

PROOF. By (6.5), (6.6) and Doob's inequality, the $\overline{M}^n$ and $\overline{L}^n$ converge to 0 in probability uniformly over $[0,1]$ as $n \to \infty$. Also, the $\bar{\varepsilon}^n$ converge in probability to 0 uniformly on $[0,1]$ by Lemma 3.1. Now, a standard tightness argument applied to (6.3) and (6.4) shows that the sequence $(\overline{Q}^n, \overline{\Phi}^n, \overline{E}^n)$, $n \in \mathbb{N}$, is $\mathbb{C}$-tight in $\mathbb{D}_C([0,1], \mathbb{R}^3)$, where a limit point $(\tilde{q}, \tilde{\phi}, \tilde{e})$ is such that $\tilde{q}_t = \int_0^t (c(1 - \tilde{q}_s - s) - 1)\, ds + \tilde{\phi}_t$, $\tilde{\phi}$ is nondecreasing with $\tilde{\phi}_t = \int_0^t \mathbf{1}(\tilde{q}_s = 0)\, d\tilde{\phi}_s$ and $\tilde{e}_t = c\int_0^t \tilde{q}_s\, ds$. Hence, $(\tilde{q}, \tilde{\phi}, \tilde{e}) = (\bar{q}, \bar{\phi}, \bar{e})$, concluding the proof. □

REMARK 6.1. The convergences $\overline{Q}^n \xrightarrow{\mathbf{P}} \bar{q}$ and $\overline{\Phi}^n \xrightarrow{\mathbf{P}} \bar{\phi}$ also follow from Remark 4.1 since the action functionals I^Q and I^Φ are equal to 0 at $\bar{q}$ and $\bar{\phi}$, respectively.

We now prove a diffusion limit theorem, which will lead to the proof of Theorem 2.2. Let us define processes $M^n = (M_t^n,\ t \in [0,1])$, $L^n = (L_t^n,\ t \in$

$[0,1])$, $X^n = (X_t^n,\ t \in [0,1])$, $Y^n = (Y_t^n,\ t \in [0,1])$ and $Z^n = (Z_t^n,\ t \in [0,1])$ by the respective equalities $M_t^n = \sqrt{n}\overline{M}_t^n$, $L_t^n = \sqrt{n}\overline{L}_t^n$, $X_t^n = \sqrt{n}(\overline{Q}_t^n - \bar{q}_t)$, $Y_t^n = \sqrt{n}(\overline{\Phi}_t^n - \bar{\phi}_t)$ and $Z_t^n = \sqrt{n}(\overline{E}_t^n - \bar{e}_t)$. By (6.3), (6.4), (6.7), (6.10) and (6.11), these processes satisfy the equations

$$X_t^n = -c_n \int_0^t X_s^n\,ds + \sqrt{n}(c_n - c)\int_0^t \sigma_s^2\,ds + M_t^n + \tilde{\varepsilon}_t^n + Y_t^n, \tag{6.12}$$

$$Z_t^n = c_n \int_0^t X_s^n\,ds + \sqrt{n}(c_n - c)\int_0^t \bar{q}_s\,ds + L_t^n + \tilde{\delta}_t^n, \tag{6.13}$$

where

$$\sigma_t^2 = \begin{cases} e^{-ct}, & \text{if } t \in [0,\beta], \\ 1 - t, & \text{if } t \in [\beta, 1], \end{cases} \tag{6.14}$$

$$\begin{aligned} \tilde{\varepsilon}_t^n &= \sqrt{n}\bar{\varepsilon}_t^n + \sqrt{n}\int_t^{\lfloor nt\rfloor/n} \left(c_n\left(1 - \overline{Q}_s^n - \frac{\lfloor ns\rfloor}{n}\right) - 1\right) ds \\ &\quad + \sqrt{n}\int_0^t \left(s - \frac{\lfloor ns\rfloor}{n}\right) ds, \end{aligned} \tag{6.15}$$

$$\tilde{\delta}_t^n = \sqrt{n}\,c_n \int_t^{\lfloor nt\rfloor/n} \overline{Q}_s^n\,ds - \frac{c_n}{\sqrt{n}}\int_0^{\lfloor nt\rfloor/n} \mathbf{1}(\overline{Q}_s^n > 0)\,ds. \tag{6.16}$$

We note that Lemma 3.1 implies that if $c_n \to c$ as $n \to \infty$, then for arbitrary $\eta > 0$,

$$\sup_{t\in[0,1]} |\tilde{\varepsilon}_t^n| \xrightarrow{\mathbf{P}} 0, \tag{6.17}$$

also

$$\sup_{t\in[0,1]} |\tilde{\delta}_t^n| \le \frac{c_n}{\sqrt{n}}. \tag{6.18}$$

Let $W^{(1)} = (W_t^{(1)},\ t \in [0,1])$ and $W^{(2)} = (W_t^{(2)},\ t \in [0,1])$ be independent Wiener processes, and processes $H = (H_t,\ t \in [0,1])$ and $Z = (Z_t,\ t \in [0,1])$ be specified by the equations

$$H_t = -c\int_0^{t\wedge\beta} H_s\,ds + \theta\int_0^t \sigma_s^2\,ds + \sqrt{c}\int_0^t \sigma_s\,dW_s^{(1)}, \tag{6.19}$$

$$Z_t = c\int_0^{t\wedge\beta} H_s\,ds + \theta\int_0^t \bar{q}_s\,ds + \sqrt{c}\int_0^t \sqrt{\bar{q}_s}\,dW_s^{(2)}. \tag{6.20}$$

We also define processes $M = (M_t,\ t \in [0,1])$ and $L = (L_t,\ t \in [0,1])$ by $M_t = \sqrt{c}\int_0^t \sigma_s\,dW_s^{(1)}$ and $L_t = \sqrt{c}\int_0^t \sqrt{\bar{q}_s}\,dW_s^{(2)}$.

LEMMA 6.2. *Let $\sqrt{n}(c_n - c) \to \theta \in \mathbb{R}$ as $n \to \infty$, where $c > 0$. Then*

$$\lim_{B\to\infty} \limsup_{n\to\infty} \mathbf{P}\left(\sup_{t\in[0,1]} |X_t^n| > B\right) = 0.$$

Also the following holds:

1. *If* $\beta > 0$, *then for* $\delta \in (0, \beta \wedge (1-\beta))$, *the processes* M^n, L^n, $(X_t^n,\ t \in [0, \beta-\delta])$, $(Y_t^n,\ t \in [\beta+\delta, 1])$ *and* $(Z_t^n,\ t \in [0,1])$ *jointly converge in distribution in* $\mathbb{D}_C([0,1], \mathbb{R}^2) \times \mathbb{D}_C([0, \beta-\delta], \mathbb{R}) \times \mathbb{D}_C([\beta+\delta, 1], \mathbb{R}) \times \mathbb{D}_C([0,1], \mathbb{R})$ *to the respective processes* M, L, $(H_t,\ t \in [0, \beta-\delta])$, $(-H_t,\ t \in [\beta+\delta, 1])$ *and* Z. *In addition*, $\lim_{n\to\infty} \mathbf{P}(\sup_{t\in[0,\beta-\delta]} |Y_t^n| > \delta) = 0$.

2. *If* $\beta = 0$, *then the processes* Y^n *converge in distribution in* $\mathbb{D}_C([0,1], \mathbb{R})$ *to the process* $-H$.

PROOF. We start by proving that the processes (M^n, L^n) converge in distribution in $\mathbb{D}_C([0,1], \mathbb{R}^2)$ to the process (M, L). The processes M^n and L^n are orthogonal square integrable martingales relative to the filtration $(\mathcal{F}_t^n,\ t \in [0,1])$, whose respective predictable quadratic characteristics $n\langle \overline{M}^n \rangle_t$ and $n\langle \overline{L}^n \rangle_t$ converge in probability as $n \to \infty$ to $c\int_0^t \sigma_s^2\, ds$ and $c\int_0^t \bar{q}_s\, ds$, respectively, in view of (6.5), (6.6), (6.7), (6.14) and Lemma 6.1. The predictable measure of jumps of (M^n, L^n) is given by

$$\begin{aligned}
&\tilde{\nu}^n([0,t], \Gamma \times \Gamma') \\
&\quad = \sum_{k=0}^{\lfloor nt \rfloor - 1} \widetilde{F}^n\biggl(1 - \frac{Q_k^n}{n} - \frac{k}{n}, \Gamma \setminus \{0\}\biggr) \widetilde{F}^n\biggl(\biggl(\frac{Q_k^n}{n} - \frac{1}{n}\biggr)^+, \Gamma' \setminus \{0\}\biggr), \\
&\hspace{24em} \Gamma, \Gamma' \in \mathcal{B}(\mathbb{R}),
\end{aligned}$$

where

$$\widetilde{F}^n(s,\ \Gamma'') = \mathbf{P}\biggl(\frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor ns \rfloor} \biggl(\xi_{1j}^n - \frac{c_n}{n}\biggr) \in \Gamma''\biggr), \qquad s \in [0,1],\ \Gamma'' \in \mathcal{B}(\mathbb{R}).$$

Therefore, for $\varepsilon > 0$ and n large enough,

$$\begin{aligned}
&\int_0^1 \int_{\mathbb{R}^2} |x|^2 \mathbf{1}(|x| > \varepsilon) \tilde{\nu}^n(ds, dx) \\
&\quad \le \frac{1}{\varepsilon^2} \int_0^1 \int_{\mathbb{R}^2} |x|^4 \tilde{\nu}^n(ds, dx) \\
&\quad \le \frac{2}{\varepsilon^2} \sum_{k=1}^n \int_{\mathbb{R}} |x|^4 \widetilde{F}^n\biggl(1 - \frac{Q_{k-1}^n}{n} - \frac{k-1}{n},\ dx\biggr) \\
&\qquad + \frac{2}{\varepsilon^2} \sum_{k=1}^n \int_{\mathbb{R}} |x|^4 \widetilde{F}^n\biggl(\biggl(\frac{Q_{k-1}^n}{n} - \frac{1}{n}\biggr)^+,\ dx\biggr) \le \frac{4(2c_n + 3c_n^2)}{n^2 \varepsilon^2},
\end{aligned}$$

which converges to 0 as $n \to \infty$. Therefore, extending the (M^n, L^n) to processes with trajectories in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^2)$ by setting $(M_t^n, L_t^n) = (M_1^n, L_1^n)$, $t \ge 1$, we see by Jacod and Shiryaev [(1987), Theorem VIII.3.22] that these processes converge in distribution to the extension of (M, L) defined as $(M_t, L_t) = (M_1, L_1)$, $t \geq 1$. Since the projection p_1 from $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^2)$ to $\mathbb{D}_C([0,1], \mathbb{R}^2)$ is continuous at continuous functions from $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^2)$, we conclude that the (nonextended) processes (M^n, L^n) converge in distribution in $\mathbb{D}_C([0,1], \mathbb{R}^2)$ to the process (M, L).

By (6.3), (6.10) and Lipshitz continuity of reflection for $r \in [0, 1]$,

$$|\overline{Q}^n_r - \bar{q}_r| \leq 2 \sup_{t \in [0,r]} \biggl| \int_0^{\lfloor nt \rfloor/n} \biggl(c_n \biggl(1 - \overline{Q}^n_s - \frac{\lfloor ns \rfloor}{n} \biggr) - 1 \biggr) ds + \bar{\varepsilon}^n_t + \overline{M}^n_t - \int_0^t \bigl(c(1 - \bar{q}_s - s) - 1 \bigr) ds \biggr|,$$

so the definitions of X^n_t and M^n_t, (6.7), (6.14) and (6.15) yield

$$|X^n_t| \leq 2c_n \int_0^t |X^n_s| \, ds + 2 \sup_{s \in [0,t]} |M^n_s| + 2\sqrt{n}|c_n - c| \int_0^t \sigma_s^2 \, ds + 2 \sup_{s \in [0,1]} |\tilde{\varepsilon}^n_s|, \qquad t \in [0,1]. \tag{6.21}$$

In view of $\mathbb{C}$-tightness of the M^n, the convergence $\sqrt{n}(c_n - c) \to \theta$, (6.17) and Gronwall's inequality, (6.21) yields the asymptotic boundedness in probability of the $\sup_{t \in [0,1]} |X^n_t|$ asserted in the first display of the statement of the lemma. This implies by (6.13), the convergence $\sqrt{n}(c_n - c) \to \theta$, (6.18) and $\mathbb{C}$-tightness of the L^n that the sequence Z^n, $n \in \mathbb{N}$, is $\mathbb{C}$-tight in $\mathbb{D}([0,1], \mathbb{R})$.

We next show that for arbitrary $\delta \in (0, 1 - \beta)$,

$$\lim_{n \to \infty} \mathbf{P}\Bigl(\sup_{t \in [\beta + \delta, 1]} |X^n_t| > \delta \Bigr) = 0. \tag{6.22}$$

On recalling the definition of Y^n_t, we write (6.12) in the following form:

$$X^n_t = -c_n \int_0^t X^n_s \, ds + \sqrt{n}(c_n - c) \int_0^t \sigma_s^2 \, ds + M^n_t + \tilde{\varepsilon}^n_t - \sqrt{n}\,\bar{\phi}_t + \sqrt{n}\,\overline{\Phi}^n_t. \tag{6.23}$$

Since $X^n_t = \sqrt{n}\,\overline{Q}^n_t$ for $t \in [\beta, 1]$, $\bar{\phi}_\beta = 0$, and $\overline{\Phi}^n_t$ increases only when $\overline{Q}^n_t = 0$, (6.23) implies that $(X^n_t, t \in [\beta, 1])$ is the reflection of the process $(X^n_\beta - c_n \int_\beta^t X^n_s \, ds + \sqrt{n}(c_n - c) \int_\beta^t \sigma_s^2 \, ds + (M^n_t - M^n_\beta) - \sqrt{n}\bar{\phi}_t + (\tilde{\varepsilon}^n_t - \tilde{\varepsilon}^n_\beta), t \in [\beta, 1])$, so by X^n_s being nonnegative on $[\beta, 1]$, it is not greater than the reflection of $(X^n_\beta + \sqrt{n}(c_n - c) \int_\beta^t \sigma_s^2 \, ds + (M^n_t - M^n_\beta) - \sqrt{n}\bar{\phi}_t + (\tilde{\varepsilon}^n_t - \tilde{\varepsilon}^n_\beta), t \in [\beta, 1])$. Therefore,

$$X^n_t \leq \sup_{s \in [\beta, t]} \biggl(\sqrt{n}(c_n - c) \int_s^t \sigma_p^2 \, dp + (M^n_t - M^n_s) + \sqrt{n}(\bar{\phi}_s - \bar{\phi}_t) + (\tilde{\varepsilon}^n_t - \tilde{\varepsilon}^n_s) \biggr) \vee \biggl(X^n_\beta + \sqrt{n}(c_n - c) \int_\beta^t \sigma_s^2 \, ds + (M^n_t - M^n_\beta) - \sqrt{n}\bar{\phi}_t + (\tilde{\varepsilon}^n_t - \tilde{\varepsilon}^n_\beta) \biggr), \quad t \in [\beta, 1]. \tag{6.24}$$

Hence, for $t \ge \beta + \delta$ and $\eta \in (0,\delta)$,

$$X_t^n \le \Big(\big|\sqrt{n}(c_n - c)\big| \int_\beta^1 \sigma_s^2\,ds + 2 \sup_{s\in[\beta,1]} |M_s^n| \tag{6.25}$$
$$+ 2 \sup_{s\in[\beta,1]} |\tilde{\varepsilon}_s^n| + X_\beta^n + \sqrt{n}(\bar{\phi}_{t-\eta} - \bar{\phi}_t) \Big)$$
$$\vee \sup_{s\in[t-\eta,t]} \Big(\big|\sqrt{n}(c_n - c)\big| \int_s^t \sigma_p^2\,dp + |M_t^n - M_s^n| + |\tilde{\varepsilon}_t^n - \tilde{\varepsilon}_s^n| \Big).$$

Limit (6.22) follows by (6.25), (6.17), $\mathbb{C}$-tightness of the M^n, asymptotic boundedness in probability of the $\sup_{t\in[0,1]}|X_t^n|$, the convergence $\sqrt{n}(c_n - c) \to \theta$ and convergence of $\sup_{t\in[\beta+\delta,1]} \sqrt{n}(\bar{\phi}_{t-\eta} - \bar{\phi}_t)$ to $-\infty$ as $n \to \infty$. Now, (6.22) implies by (6.12), (6.17), the convergence $\sqrt{n}(c_n - c) \to \theta$, asymptotic boundedness in probability of the $\sup_{t\in[0,1]}|X_t^n|$ and $\mathbb{C}$-tightness of the M^n that the processes Y^n restricted to $[\beta+\delta,1]$ are $\mathbb{C}$-tight in $\mathbb{D}([\beta+\delta,1],\mathbb{R})$.

Let us now assume that $\beta > 0$. By (6.23), the definition of X_t^n and the definition of the reflection mapping for $t \in [0,1]$,

$$\sqrt{n}\,\overline{\Phi}_t^n = -\inf_{s\in[0,t]} \Big(-c_n \int_0^s X_p^n\,dp + \sqrt{n}(c_n - c) \int_0^s \sigma_p^2\,dp \tag{6.26}$$
$$+ M_s^n + \tilde{\varepsilon}_s^n + \sqrt{n}\bar{q}_s - \sqrt{n}\bar{\phi}_s \Big) \wedge 0.$$

Convergence in distribution of the M^n to a continuous-path process implies that for $\delta > 0$, $\lim_{\eta\to 0} \limsup_{n\to\infty} \mathbf{P}(\sup_{t\in[0,\eta]} |M_t^n| > \delta) = 0$. Therefore, given $\delta \in (0,\beta)$, we derive from (6.26), taking into consideration the convergences $\sqrt{n}(c_n - c) \to \theta$ and $\sqrt{n}\inf_{t\in[\eta,\beta-\delta]} \bar{q}_t \to \infty$ as $n \to \infty$, where $\eta \in (0,\beta-\delta)$, the fact that $\bar{\phi}_t = 0$ for $t \in [0,\beta]$, (6.17) and asymptotic boundedness in probability of the $\sup_{t\in[0,1]}|X_t^n|$ and $\sup_{t\in[0,1]}|M_t^n|$ that

$$\lim_{n\to\infty} \mathbf{P}\Big(\sup_{t\in[0,\beta-\delta]} |Y_t^n| > \delta \Big) = 0. \tag{6.27}$$

Putting together (6.12), (6.17), (6.27), the convergence $\sqrt{n}(c_n - c) \to \theta$, asymptotic boundedness in probability of the $\sup_{t\in[0,1]}|X_t^n|$ and $\mathbb{C}$-tightness of the M^n, we conclude that the X^n restricted to $[0,\beta-\delta]$ are $\mathbb{C}$-tight in $\mathbb{D}_C([0,\beta-\delta],\mathbb{R})$.

We have thus established that for $\beta > 0$ and $\delta \in (0, \beta\wedge(1-\beta))$, the processes M^n, L^n, X^n restricted to $[0,\beta-\delta]$, Y^n restricted to $[\beta+\delta,1]$ and Z^n are $\mathbb{C}$-tight in the associated function spaces, so they are jointly tight as random elements with values in the product space. Convergence in distribution in $\mathbb{D}_C([0,1],\mathbb{R}^2) \times \mathbb{D}_C([0,\beta-\delta],\mathbb{R}) \times \mathbb{D}_C([\beta+\delta,1],\mathbb{R}) \times \mathbb{D}_C([0,1],\mathbb{R})$ of the $(M^n, L^n, (X_t^n,\, t \in [0,\beta-\delta]), (Y_t^n,\, t\in[\beta+\delta,1]), Z^n)$ to $(M, L, (H_t,\, t\in[0,\beta-\delta]), (-H_t,\, t \in [\beta+\delta,1]), Z)$ now follows by (6.12), (6.13), (6.17)–(6.20), (6.22), (6.27), the convergence $\sqrt{n}(c_n - c) \to \theta$, convergence in distribution of the (M^n, L^n) to (M, L) and uniqueness of the solution (H, Z) to (6.19) and (6.20).

Let us now assume that $\beta = 0$. Inequality (6.21), in view of asymptotic boundedness in probability of the $\sup_{t\in[0,1]} |X^n_t|$, $\mathbb{C}$-tightness of the M^n, limits (6.17), (6.22) and $\sqrt{n}(c_n - c) \to \theta$, yields the limit

$$\lim_{\eta\to 0} \limsup_{n\to\infty} \mathbf{P}\Big(\sup_{t\in[0,\eta]} |X^n_t| > \delta\Big) = 0$$

for $\delta > 0$, so by (6.22) $\lim_{n\to\infty} \mathbf{P}(\sup_{t\in[0,1]} |X^n_t| > \delta) = 0$. Therefore, by (6.12), the convergence $\sqrt{n}(c_n - c) \to \theta$, and convergence in distribution of the M^n to M, the Y^n converge in distribution in $\mathbb{D}_C([0,1], \mathbb{R})$ to $-H$. $\square$

REMARK 6.2. A slight modification of the proof allows one to strengthen the assertion of the lemma for $\beta > 0$ to the joint convergence in distribution in $\mathbb{D}_C([0,1], \mathbb{R}^2) \times \mathbb{D}_C([0, \beta - \delta], \mathbb{R})^2 \times \mathbb{D}_C([\beta + \delta, 1], \mathbb{R})^2 \times \mathbb{R}^2 \times \mathbb{D}_C([0,1], \mathbb{R})$ of the M^n, L^n, $(X^n_t,\ t \in [0, \beta - \delta])$, $(Y^n_t,\ t \in [0, \beta - \delta])$, $(X^n_t,\ t \in [\beta + \delta, 1])$, $(Y^n_t,\ t \in [\beta + \delta, 1])$, X^n_β, Y^n_β and Z^n to the respective random elements M, L, $(X_t,\ t \in [0, \beta - \delta])$, $(Y_t,\ t \in [0, \beta - \delta])$, $(X_t,\ t \in [\beta + \delta, 1])$, $(Y_t,\ t \in [\beta + \delta, 1])$, X_β, Y_β and Z, where

$$X_t = \begin{cases} H_t, & \text{for } t \in [0, \beta), \\ H_\beta \vee 0, & \text{for } t = \beta, \\ 0, & \text{for } t \in (\beta, 1], \end{cases} \quad \text{and} \quad Y_t = \begin{cases} 0, & \text{for } t \in [0, \beta), \\ (-H_\beta) \vee 0, & \text{for } t = \beta, \\ -H_t, & \text{for } t \in (\beta, 1]. \end{cases}$$

We thus have convergence in distribution with unmatched jumps in the limit process mentioned in the Introduction.

PROOF OF THEOREM 2.2. Let $c > 1$, so $\beta > 0$. We prove that as $n \to \infty$,

$$\begin{aligned} &\Big(\sqrt{n}\Big(\frac{\alpha^n}{n} - \alpha\Big), \sqrt{n}\Big(\frac{\beta^n}{n} - \beta\Big), \sqrt{n}\Big(\frac{\gamma^n}{n} - \gamma\Big)\Big) \\ &\quad \xrightarrow{d} \Big(-H_1, \frac{H_\beta}{1 - c(1-\beta)}, Z_\beta\Big), \end{aligned} \tag{6.28}$$

which implies the assertion of part 2 of the theorem.

Let τ^n be the last time t before $\beta/2$ when $\overline{Q}^n_t = 0$ and $\tilde{\beta}^n$ be the first time t not before $\beta/2$ when $\overline{Q}^n_t = 0$. By Lemma 6.1 and (6.7), $\overline{Q}^n_t > 0$ for $t \in [\delta, \beta - \delta]$ with probability tending to 1 as $n \to \infty$ for arbitrary $\delta \in (0, \beta/2)$, so $\mathbf{P}(\tau^n \le \delta) \to 1$ and $\mathbf{P}(\tilde{\beta}^n \ge \beta - \delta) \to 1$. Also, noting that $\overline{\Phi}^n_t = \overline{\Phi}^n_{\tau^n}$ for $t \in (\tau^n, \tilde{\beta}^n)$, Lemma 6.1 and (6.8),

$$\limsup_{n\to\infty} \mathbf{P}(\tilde{\beta}^n > \beta + \delta) \le \limsup_{n\to\infty} \mathbf{P}(\overline{\Phi}^n_{\tau^n} = \overline{\Phi}^n_{\beta+\delta}) \le \mathbf{1}(0 = \bar{\phi}_{\beta+\delta}) = 0,$$

so as $n \to \infty$,

$$\tilde{\beta}^n \xrightarrow{\mathbf{P}} \beta. \tag{6.29}$$

Similarly, the event that there exists an excursion of $\overline{Q}^n$ of duration greater than η, where $\eta \in (0, 1-\beta)$, which ends at some time after $\beta+\eta$, is contained in the event $\{\inf_{t\in[\beta,1-\eta]}(\overline{\Phi}^n_{t+\eta} - \overline{\Phi}^n_t) = 0\}$. Lemma 6.1 and the fact that $\bar{\phi}_t$ is strictly increasing on $[\beta, 1]$, in view of (6.8), imply that the probability of the latter event tends to 0 as $n \to \infty$. As the sizes of the connected components of $\mathcal{G}(n, c_n/n)$ are equal to n multiplied by the excursion lengths of $\overline{Q}^n$, we see that, with probability tending to 1 as $n \to \infty$, the largest component "starts" at $n\tau^n$ and "ends" at $n\tilde{\beta}^n$, so

$$\mathbf{P}\Big(\frac{\beta^n}{n} = \tilde{\beta}^n - \tau^n\Big) \to 1, \tag{6.30}$$

$$\mathbf{P}\Big(\frac{\gamma^n}{n} = \overline{E}^n_{\tilde{\beta}^n} - \overline{E}^n_{\tau^n}\Big) \to 1. \tag{6.31}$$

By (6.12) and the facts that $X^n_{\tau^n} = -\sqrt{n}\bar{q}_{\tau^n}$ and $X^n_{\tilde{\beta}^n} = -\sqrt{n}\bar{q}_{\tilde{\beta}^n}$,

$$\begin{aligned} -\sqrt{n}\bar{q}_{\tau^n} &= -c_n \int_0^{\tau^n} X^n_s\,ds + \sqrt{n}(c_n - c)\int_0^{\tau^n} \sigma_s^2\,ds \\ &\quad + M^n_{\tau^n} + \tilde{\varepsilon}^n_{\tau^n} + Y^n_{\tau^n}, \end{aligned} \tag{6.32}$$

$$\begin{aligned} -\sqrt{n}\bar{q}_{\tilde{\beta}^n} &= -c_n \int_0^{\tilde{\beta}^n} X^n_s\,ds + \sqrt{n}(c_n - c)\int_0^{\tilde{\beta}^n} \sigma_s^2\,ds \\ &\quad + M^n_{\tilde{\beta}^n} + \tilde{\varepsilon}^n_{\tilde{\beta}^n} + \sqrt{n}\,\overline{\Phi}^n_{\tilde{\beta}^n} - \sqrt{n}\,\bar{\phi}_{\tilde{\beta}^n}. \end{aligned} \tag{6.33}$$

Since $\tau^n \xrightarrow{\mathbf{P}} 0$, the right-hand side of (6.32) converges in probability to zero by (6.17) and Lemma 6.2, so $\sqrt{n}\bar{q}_{\tau^n} \xrightarrow{\mathbf{P}} 0$ and, consequently, by (6.11) and the fact that $c > 1$,

$$\sqrt{n}\tau^n \xrightarrow{\mathbf{P}} 0. \tag{6.34}$$

Since $\overline{\Phi}^n_{\tilde{\beta}^n} = \overline{\Phi}^n_{\tau^n} + 1/n$ [see (2.8)], $\sqrt{n}\overline{\Phi}^n_{\tau^n} \xrightarrow{\mathbf{P}} 0$ and $\bar{q}_{\tilde{\beta}^n} - \bar{\phi}_{\tilde{\beta}^n} = \int_0^{\tilde{\beta}^n} (c(1-\bar{q}_s - s) - 1)\,ds = \int_\beta^{\tilde{\beta}^n} (c(1-\bar{q}_s - s) - 1)\,ds$ [see (6.10)], we derive from (6.33), on using (6.29), (6.17) and Lemma 6.2, that

$$\begin{aligned} &\sqrt{n}\int_\beta^{\tilde{\beta}^n} \big(c(1-\bar{q}_s - s) - 1\big)\,ds \\ &\qquad - c_n \int_0^{\tilde{\beta}^n} X^n_s\,ds + \sqrt{n}(c_n - c)\int_0^{\tilde{\beta}^n} \sigma_s^2\,ds + M^n_{\tilde{\beta}^n} \xrightarrow{\mathbf{P}} 0. \end{aligned} \tag{6.35}$$

Since $\alpha = \bar{\phi}_1$ [see (6.8)] and $\alpha^n = \Phi^n_n$, we also have that

$$\sqrt{n}\Big(\frac{\alpha^n}{n} - \alpha\Big) = Y^n_1. \tag{6.36}$$

Convergence (6.28) follows by (6.29)–(6.31) and (6.34)–(6.36), the observation that $\gamma = \bar{e}_\beta$ [see (6.9)], asymptotic boundedness in probability of the $\sup_{t\in[0,1]} |X_t^n|$, the convergence $\sqrt{n}(c_n - c) \to \theta$, the joint convergence in distribution $(M^n, Y_1^n, (X_s^n, s \in [0, \beta-\delta]), Z^n) \xrightarrow{d} (M, -H_1, (H_s, s \in [0, \beta-\delta]), Z)$ in $\mathbb{D}_C([0,1],\mathbb{R}) \times \mathbb{R} \times \mathbb{D}_C([0,\beta-\delta],\mathbb{R}) \times \mathbb{D}_C([0,1],\mathbb{R})$ valid by Lemma 6.2 and the continuous mapping theorem.

If $c \le 1$, the Y_1^n converge in distribution to $-H_1$ by part 2 of Lemma 6.2, which completes the proof of part 1. □

We now prove Theorem 2.3. As mentioned above, the proof is along the lines of the proof of Theorem 2.2, so we begin with an idempotent analogue of Lemma 6.2. We recall that b_n, $n \in \mathbb{N}$, is a real-valued sequence such that $b_n \to \infty$ and $b_n/\sqrt{n} \to 0$ as $n \to \infty$, and introduce processes $\widehat{M}^n = (\widehat{M}_t^n, t \in [0,1])$, $\widehat{L}^n = (\widehat{L}_t^n, t \in [0,1])$, $\widehat{X}^n = (\widehat{X}_t^n, t \in [0,1])$, $\widehat{Y}^n = (\widehat{Y}_t^n, t \in [0,1])$ and $\widehat{Z}^n = (\widehat{Z}_t^n, t \in [0,1])$ by the respective equalities $\widehat{M}_t^n = M_t^n/b_n$, $\widehat{L}_t^n = L_t^n/b_n$, $\widehat{X}_t^n = X_t^n/b_n$, $\widehat{Y}_t^n = Y_t^n/b_n$ and $\widehat{Z}_t^n = Z_t^n/b_n$. Dividing (6.12) and (6.13) through by b_n yields for $t \in [0,1]$,

$$\widehat{X}_t^n = -c_n \int_0^t \widehat{X}_s^n \, ds + \frac{\sqrt{n}}{b_n}(c_n - c)\int_0^t \sigma_s^2 \, ds + \widehat{M}_t^n + \hat{\varepsilon}_t^n + \widehat{Y}_t^n, \tag{6.37}$$

$$\widehat{Z}_t^n = c_n \int_0^t \widehat{X}_s^n \, ds + \frac{\sqrt{n}}{b_n}(c_n - c)\int_0^t \bar{q}_s \, ds + \widehat{L}_t^n + \hat{\delta}_t^n, \tag{6.38}$$

where

$$\hat{\varepsilon}_t^n = \frac{\tilde{\varepsilon}_t^n}{b_n}, \qquad \hat{\delta}_t^n = \frac{\tilde{\delta}_t^n}{b_n}. \tag{6.39}$$

We note that by (6.15), (6.16), (6.39) and Lemma 3.1,

$$\sup_{t\in[0,1]} |\hat{\varepsilon}_t^n| \overset{\mathbf{P}^{1/b_n^2}}{\to} 0, \tag{6.40}$$

provided $c_n \to c$ as $n \to \infty$, and

$$\sup_{t\in[0,1]} |\hat{\delta}_t^n| \le \frac{c_n}{b_n\sqrt{n}}. \tag{6.41}$$

Let $\widehat{W}^{(1)} = (\widehat{W}_t^{(1)}, t \in [0,1])$ and $\widehat{W}^{(2)} = (\widehat{W}_t^{(2)}, t \in [0,1])$ be independent idempotent Wiener processes on an idempotent probability space $(\Upsilon, \mathbf{\Pi})$ adapted to a complete τ-flow $\mathbf{A}$, idempotent processes $\widehat{M} = (\widehat{M}_t, t \in [0,1])$ and $\widehat{L} = (\widehat{L}_t, t \in [0,1])$ be defined by $\widehat{M}_t = \sqrt{c}\int_0^t \sigma_s \dot{\widehat{W}}_s^{(1)} \, ds$ and $\widehat{L}_t = \sqrt{c}\int_0^t \sqrt{\bar{q}_s}\, \dot{\widehat{W}}_s^{(2)} \, ds$, respectively, an idempotent process $\widehat{H} = (\widehat{H}_t, t \in [0,1])$ be the Luzin strong solution of the equation

$$\widehat{H}_t = -c \int_0^{t\wedge\beta} \widehat{H}_s \, ds + \hat{\theta}\int_0^t \sigma_s^2 \, ds + \sqrt{c}\int_0^t \sigma_s \dot{\widehat{W}}_s^{(1)} \, ds, \tag{6.42}$$

and an idempotent process $\widehat{Z}=(\widehat{Z}_t,\ t\in[0,1])$ be given by

$$\widehat{Z}_t = c\int_0^{t\wedge\beta}\widehat{H}_s\,ds+\hat{\theta}\int_0^t \bar{q}_s\,ds+\sqrt{c}\int_0^t\sqrt{\bar{q}_s}\,\dot{\widehat{W}}_s^{(2)}\,ds. \tag{6.43}$$

LEMMA 6.3. *Let $(\sqrt{n}/b_n)(c_n-c)\to\hat{\theta}\in\mathbb{R}$ as $n\to\infty$, where $c>0$, $b_n\to\infty$ and $b_n/\sqrt{n}\to 0$. Then for arbitrary $\eta>0$,*

$$\lim_{n\to\infty}\mathbf{P}\Big(\sup_{t\in[0,1]}|\overline{Q}_t^n-\bar{q}_t|>\eta\Big)^{1/b_n^2}=0,$$

$$\lim_{n\to\infty}\mathbf{P}\Big(\sup_{t\in[0,1]}|\overline{\Phi}_t^n-\bar{\phi}_t|>\eta\Big)^{1/b_n^2}=0$$

and

$$\lim_{B\to\infty}\limsup_{n\to\infty}\mathbf{P}\Big(\sup_{t\in[0,1]}|\widehat{X}_t^n|>B\Big)^{1/b_n^2}=0.$$

Also the following hold:

1. *If $\beta>0$, then, for $\delta\in(0,\beta\wedge(1-\beta))$, the stochastic processes $\widehat{M}^n$, $\widehat{L}^n$, $(\widehat{X}_t^n,\ t\in[0,\beta-\delta])$, $(\widehat{Y}_t^n,\ t\in[\beta+\delta,1])$ and $(\widehat{Z}_t^n,\ t\in[0,1])$ jointly LD converge in distribution at rate b_n^2 in $\mathbb{D}_C([0,1],\mathbb{R}^2)\times\mathbb{D}_C([0,\beta-\delta],\mathbb{R})\times\mathbb{D}_C([\beta+\delta,1],\mathbb{R})\times\mathbb{D}_C([0,1],\mathbb{R})$ to the respective idempotent processes $\widehat{M}$, $\widehat{L}$, $(\widehat{H}_t,\ t\in[0,\beta-\delta])$, $(-\widehat{H}_t,\ t\in[\beta+\delta,1])$ and $\widehat{Z}$. In addition, $\lim_{n\to\infty}\mathbf{P}(\sup_{t\in[0,\beta-\delta]}|\widehat{Y}_t^n|>\delta)^{1/b_n^2}=0$.*
2. *If $\beta=0$, then the stochastic processes $\widehat{Y}^n$ LD converge in distribution at rate b_n^2 in $\mathbb{D}_C([0,1],\mathbb{R})$ to the idempotent process $-\widehat{H}$.*

PROOF. We have by (6.1) and (6.2),

$$\widehat{M}_t^n=\frac{1}{b_n\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=1}^{n-Q_{i-1}^n-(i-1)}\Big(\xi_{ij}^n-\frac{c_n}{n}\Big),\qquad t\in[0,1],$$

$$\widehat{L}_t^n=\frac{1}{b_n\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=1}^{Q_{i-1}^n-1}\Big(\zeta_{ij}^n-\frac{c_n}{n}\Big),\qquad t\in[0,1].$$

Therefore, the $\mathbf{F}^n$-predictable measure of jumps of $(\widehat{M}^n,\widehat{L}^n)$ has the form

$$\begin{aligned}\hat{\nu}^n([0,t],\Gamma\times\Gamma')&=\sum_{k=0}^{\lfloor nt\rfloor-1}\widehat{F}^n\Big(1-\overline{Q}_{k/n}^n-\frac{k}{n},\Gamma\setminus\{0\}\Big)\\&\quad\times\widehat{F}^n\Big(\Big(\overline{Q}_{k/n}^n-\frac{1}{n}\Big)^+,\ \Gamma'\setminus\{0\}\Big),\qquad \Gamma,\Gamma'\in\mathcal{B}(\mathbb{R}),\end{aligned}\tag{6.44}$$

where

$$\widehat{F}^n(s,\Gamma'') = \mathbf{P}\biggl(\frac{1}{b_n\sqrt{n}}\sum_{j=1}^{\lfloor ns\rfloor}\Bigl(\xi_{1j}^n - \frac{c_n}{n}\Bigr) \in \Gamma''\biggr), \qquad s\in[0,1],\ \Gamma''\in\mathcal{B}(\mathbb{R}). \tag{6.45}$$

Accordingly, the stochastic exponential $(\widehat{\mathcal{E}}_t^n(\lambda),\ t\in[0,1])$, where $\lambda\in\mathbb{R}$, associated with $\widehat{M}^n$ is given by

$$\begin{aligned}\log\widehat{\mathcal{E}}_t^n(\lambda) &= \sum_{k=1}^{\lfloor nt\rfloor}\log\biggl(1+\int_{\mathbb{R}}(e^{\lambda x}-1)\hat{\nu}^n\biggl(\Bigl\{\frac{k}{n}\Bigr\}, dx\times\mathbb{R}\biggr)\biggr)\\ &= n\log\biggl(\mathbf{E}\exp\biggl(\frac{\lambda}{b_n\sqrt{n}}\Bigl(\xi_{11}^n-\frac{c_n}{n}\Bigr)\biggr)\biggr)\sum_{k=0}^{\lfloor nt\rfloor-1}\Bigl(1-\overline{Q}_{k/n}^n-\frac{k}{n}\Bigr).\end{aligned}$$

Since, for $B>0$, by Doob's inequality,

$$\begin{aligned}\mathbf{P}\Bigl(\sup_{t\in[0,1]}|\widehat{M}_t^n|>B\Bigr)^{1/b_n^2} &\le e^{-B}\Bigl(\bigl(\mathbf{E}e^{b_n^2\widehat{M}_1^n}\bigr)^{1/b_n^2}+\bigl(\mathbf{E}e^{-b_n^2\widehat{M}_1^n}\bigr)^{1/b_n^2}\Bigr)\\ &\le e^{-B}\bigl(\bigl(\mathbf{E}\widehat{\mathcal{E}}_1^n(2b_n^2)\bigr)^{1/(2b_n^2)}+\bigl(\mathbf{E}\widehat{\mathcal{E}}_1^n(-2b_n^2)\bigr)^{1/(2b_n^2)}\bigr)\end{aligned}$$

and $(n/b_n)^2\log\mathbf{E}\exp(\pm(2b_n/\sqrt{n})(\xi_{11}^n-c_n/n))\to 2c$ as $n\to\infty$, we conclude that

$$\lim_{B\to\infty}\limsup_{n\to\infty}\mathbf{P}\Bigl(\sup_{t\in[0,1]}|\widehat{M}_t^n|>B\Bigr)^{1/b_n^2}=0. \tag{6.46}$$

Dividing (6.21) through by b_n and recalling (6.39) yields

$$\begin{aligned}|\widehat{X}_t^n| &\le 2c_n\int_0^t|\widehat{X}_s^n|\,ds+2\frac{\sqrt{n}}{b_n}|c_n-c|\int_0^t\sigma_s^2\,ds\\ &\quad+2\sup_{s\in[0,t]}|\widehat{M}_s^n|+2\sup_{s\in[0,1]}|\hat{\varepsilon}_s^n|, \qquad t\in[0,1].\end{aligned} \tag{6.47}$$

Applying Gronwall's inequality to (6.47), we have by (6.40), (6.46) and the convergence $(\sqrt{n}/b_n)(c_n-c)\to\hat{\theta}$ that

$$\lim_{B\to\infty}\limsup_{n\to\infty}\mathbf{P}\Bigl(\sup_{t\in[0,1]}|\widehat{X}_t^n|>B\Bigr)^{1/b_n^2}=0, \tag{6.48}$$

proving the third display in the statement of the lemma. As a consequence of (6.48), the definition of $\widehat{X}_t^n$ and the convergence $\sqrt{n}/b_n\to\infty$,

$$\lim_{n\to\infty}\mathbf{P}\Bigl(\sup_{t\in[0,1]}|\overline{Q}_t^n-\bar{q}_t|>\eta\Bigr)^{1/b_n^2}=0, \tag{6.49}$$

and then by (6.3), (6.10), (6.40) and (6.46),

$$\lim_{n\to\infty} \mathbf{P}\Big(\sup_{t\in[0,1]} |\overline{\Phi}_t^n - \bar{\phi}_t| > \eta\Big)^{1/b_n^2} = 0$$

for arbitrary $\eta > 0$, proving the other claimed super-exponential convergences in probability.

We now prove that the $(\widehat{M}^n, \widehat{L}^n)$ LD converge in distribution at rate b_n^2 to $(\widehat{M}, \widehat{L})$ in $\mathbb{D}_C([0,1], \mathbb{R}^2)$. This is accomplished by checking the conditions of Corollary 4.3.13 in Puhalskii (2001). Extending $\widehat{M}^n$ and $\widehat{L}^n$ to processes defined on $\mathbb{R}_+$ by letting $\widehat{M}_t^n = \widehat{M}_1^n$ and $\widehat{L}_t^n = \widehat{L}_1^n$ for $t \ge 1$, we have by (6.5) and (6.6) that $\widehat{M}^n$ and $\widehat{L}^n$ are orthogonal $\mathbf{F}^n$-square integrable martingales with respective $\mathbf{F}^n$-predictable quadratic characteristics

$$\langle \widehat{M}^n\rangle_t = \frac{c_n}{b_n^2}\Big(1 - \frac{c_n}{n}\Big)\int_0^{\lfloor n(t\wedge 1)\rfloor/n} \Big(1 - \overline{Q}_s^n - \frac{\lfloor ns\rfloor}{n}\Big)\,ds,$$

$$\langle \widehat{L}^n\rangle_t = \frac{c_n}{b_n^2}\Big(1 - \frac{c_n}{n}\Big)\int_0^{\lfloor nt\rfloor/n} \Big(\overline{Q}_s^n - \frac{1}{n}\Big)^+\,ds,$$

so by (6.7), (6.14) and (6.49), for $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}\Big(\Big|b_n^2\langle \widehat{M}^n\rangle_t - c\int_0^{t\wedge 1}\sigma_s^2\,ds\Big| > \varepsilon\Big)^{1/b_n^2} = 0,$$

$$\lim_{n\to\infty} \mathbf{P}\Big(\Big|b_n^2\langle \widehat{L}^n\rangle_t - c\int_0^{t\wedge 1}\bar{q}_s\,ds\Big| > \varepsilon\Big)^{1/b_n^2} = 0,$$

checking condition (C_0') of the corollary. The processes $(\widehat{M}^n, \widehat{L}^n)$ satisfy the Cramér condition by (6.44) and (6.45). We check condition (L_e):

(6.50)
$$\lim_{n\to\infty} \mathbf{P}\Big(\frac{1}{b_n^2}\int_0^1\int_{\mathbb{R}^2} e^{\lambda b_n^2|x|}\mathbf{1}(b_n^2|x| > \varepsilon)\hat{\nu}^n(ds,dx) > \eta\Big)^{1/b_n^2} = 0,$$
$$\lambda > 0,\ \varepsilon > 0,\ \eta > 0.$$

We have for n large enough by (6.44) and (6.45),

$$\frac{1}{b_n^2}\int_0^1\int_{\mathbb{R}^2} e^{\lambda b_n^2|x|}\mathbf{1}(b_n^2|x| > \varepsilon)\hat{\nu}^n(ds,dx)$$
$$\le \frac{e^{-\varepsilon\sqrt{n}/b_n}}{b_n^2}\int_0^1\int_{\mathbb{R}^2} e^{(\lambda+\varepsilon)b_n\sqrt{n}|x|}\hat{\nu}^n(ds,dx)$$

$$\le \frac{e^{-\varepsilon\sqrt{n}/b_n}}{2b_n^2}\sum_{k=0}^{n-1}\biggl(\int_{\mathbb{R}} e^{2(\lambda+\varepsilon)b_n\sqrt{n}|x|}\widehat{F}^n\Bigl(1-\overline{Q}^n_{k/n}-\frac{k}{n},\,dx\Bigr)$$

$$+\int_{\mathbb{R}} e^{2(\lambda+\varepsilon)b_n\sqrt{n}|x|}\widehat{F}^n\Bigl(\Bigl(\overline{Q}^n_{k/n}-\frac{1}{n}\Bigr)^+,\,dx\Bigr)\biggr)$$

$$\le e^{-\varepsilon\sqrt{n}/b_n}\frac{n}{b_n^2}e^{c_n(\exp(2(\lambda+\varepsilon))-1+2(\lambda+\varepsilon))}.$$

Since the latter expression converges to 0 as $n\to\infty$, convergence (6.50) holds. Conditions (0) and $(\sup B')$ of the corollary trivially hold. Thus, the extended $(\widehat{M}^n,\widehat{L}^n)$ LD converge in distribution in $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^2)$ at rate b_n^2 to $(\widehat{M},\widehat{L})$. Since the projection p_1 from $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^2)$ to $\mathbb{D}_C([0,1],\mathbb{R}^2)$ is continuous at continuous functions from $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^2)$, we conclude by the contraction principle that the processes $(\widehat{M}^n,\widehat{L}^n)$ LD converge in distribution at rate b_n^2 in $\mathbb{D}_C([0,1],\mathbb{R}^2)$ to the idempotent process $(\widehat{M},\widehat{L})$. As a byproduct of $\mathbb{C}$-exponential tightness of the $\widehat{L}^n$, we deduce by (6.48), (6.38), the convergence $(\sqrt{n}/b_n)(c_n-c)\to\hat{\theta}$ and (6.41) that the sequence $\widehat{Z}^n$, $n\in\mathbb{N}$, is $\mathbb{C}$-exponentially tight in $\mathbb{D}([0,1],\mathbb{R})$.

We next show that for arbitrary $\delta>0$,

$$\lim_{n\to\infty}\mathbf{P}\Bigl(\sup_{t\in[\beta+\delta,1]}|\widehat{X}^n_t|>\delta\Bigr)^{1/b_n^2}=0. \tag{6.51}$$

Dividing (6.25) through by b_n yields for $t\ge\beta+\delta$ and $\eta\in(0,\delta)$,

$$\widehat{X}^n_t\le\biggl(\frac{\sqrt{n}}{b_n}|c_n-c|\int_\beta^1\sigma_s^2\,ds+2\sup_{s\in[\beta,1]}|\widehat{M}^n_s|$$

$$+2\sup_{s\in[\beta,1]}|\hat{\varepsilon}^n_s|+\widehat{X}^n_\beta+\frac{\sqrt{n}}{b_n}(\bar{\phi}_{t-\eta}-\bar{\phi}_t)\biggr)$$

$$\vee\sup_{s\in[t-\eta,t]}\biggl(\frac{\sqrt{n}}{b_n}|c_n-c|\int_s^t\sigma_p^2\,dp+|\widehat{M}^n_t-\widehat{M}^n_s|+|\hat{\varepsilon}^n_t-\hat{\varepsilon}^n_s|\biggr).$$

Convergence (6.51) follows if we recall that the $\widehat{M}^n$ are $\mathbb{C}$-exponentially tight of order b_n^2, $(\sqrt{n}/b_n)(c_n-c)\to\hat{\theta}$, (6.40) and (6.48) hold, and use that $\sup_{t\in[\beta+\delta,1]}(\bar{\phi}_{t-\eta}-\bar{\phi}_t)<0$. Consequently, by (6.37), (6.40), (6.48), (6.51), $\mathbb{C}$-exponential tightness of the $\widehat{M}^n$ and the convergence $(\sqrt{n}/b_n)(c_n-c)\to\hat{\theta}$, the processes $\widehat{Y}^n$ restricted to $[\beta+\delta,1]$ are $\mathbb{C}$-exponentially tight of order b_n^2.

Next, let us assume that $\beta>0$. Representation (6.26) implies that for $t\in[0,1]$,

$$\frac{\sqrt{n}}{b_n}\overline{\Phi}^n_t=-\inf_{s\in[0,t]}\biggl(-c_n\int_0^s\widehat{X}^n_p\,dp+\frac{\sqrt{n}}{b_n}(c_n-c)\int_0^s\sigma_p^2\,dp$$

$$+\widehat{M}^n_s+\hat{\varepsilon}^n_s+\frac{\sqrt{n}}{b_n}\bar{q}_s-\frac{\sqrt{n}}{b_n}\bar{\phi}_s\biggr)\wedge 0. \tag{6.52}$$

In view of LD convergence in distribution at rate b_n^2 of the $\widehat{M}^n$ to a continuous-path idempotent process $\lim_{\eta\to 0}\limsup_{n\to\infty}\mathbf{P}(\sup_{t\in[0,\eta]}|\widehat{M}_t^n|>\delta)^{1/b_n^2}=0$ for $\delta>0$. Therefore, given $\delta\in(0,\beta)$, we derive from (6.52), taking into consideration the convergences $(\sqrt{n}/b_n)(c_n-c)\to\hat{\theta}$ and $(\sqrt{n}/b_n)\inf_{t\in[\eta,\beta-\delta]}\bar{q}_t\to\infty$ as $n\to\infty$, where $\eta\in(0,\beta-\delta)$, the fact that $\bar{\phi}_t=0$ for $t\in[0,\beta]$, (6.40), (6.46), (6.48) and $\mathbb{C}$-exponential tightness of the $\widehat{M}^n$ that for $\delta\in(0,\beta)$,

$$\lim_{n\to\infty}\mathbf{P}\biggl(\sup_{t\in[0,\beta-\delta]}|Y_t^n|>\delta\biggr)^{1/b_n^2}=0. \tag{6.53}$$

Putting together (6.37), (6.40), (6.48), (6.53), the convergence $(\sqrt{n}/b_n)(c_n-c)\to\hat{\theta}$ and LD convergence in distribution at rate b_n^2 of the $\widehat{M}^n$ to $\widehat{M}$, we conclude that the sequence of laws of the $\widehat{X}^n$ restricted to $[0,\beta-\delta]$ is $\mathbb{C}$-exponentially tight of order b_n^2 in $\mathbb{D}([0,\beta-\delta],\mathbb{R})$.

We have thus established that for $\beta>0$ and $\delta\in(0,\beta\wedge(1-\beta))$, the processes $\widehat{M}^n$, $\widehat{L}^n$, $\widehat{X}^n$ restricted to $[0,\beta-\delta]$, $\widehat{Y}^n$ restricted to $[\beta+\delta,1]$ and $\widehat{Z}^n$ are $\mathbb{C}$-exponentially tight of order b_n^2 in the associated function spaces, so they are jointly exponentially tight of order b_n^2 as random elements with values in the product space. Now, LD convergence in distribution at rate b_n^2 in $\mathbb{D}_C([0,1],\mathbb{R}^2)\times\mathbb{D}_C([0,\beta-\delta],\mathbb{R})\times\mathbb{D}_C([\beta+\delta,1],\mathbb{R})\times\mathbb{D}_C([0,1],\mathbb{R})$ of the $(\widehat{M}^n,\widehat{L}^n,(\widehat{X}_t^n,\ t\in[0,\beta-\delta]),(\widehat{Y}_t^n,\ t\in[\beta+\delta,1]),\widehat{Z}^n)$ to $(\widehat{M},\widehat{L},(\widehat{H}_t,\ t\in[0,\beta-\delta]),(-\widehat{H}_t,\ t\in[\beta+\delta,1]),\widehat{Z})$ follows by (6.37), (6.38), (6.40)–(6.43), (6.51), (6.53), the convergence $(\sqrt{n}/b_n)(c_n-c)\to\hat{\theta}$, LD convergence in distribution of the $(\widehat{M}^n,\widehat{L}^n)$ to $(\widehat{M},\widehat{L})$ and strong uniqueness of the solution $(\widehat{H},\widehat{L})$ of (6.42) and (6.43).

Let us now assume that $\beta=0$. In view of limits (6.40), (6.48), the convergence $(\sqrt{n}/b_n)(c_n-c)\to\hat{\theta}$ and LD convergence in distribution at rate b_n^2 of the $\widehat{M}^n$ to $\widehat{M}$, we have by (6.47) the convergence $\lim_{\eta\to 0}\limsup_{n\to\infty}\mathbf{P}(\sup_{t\in[0,\eta]}|\widehat{X}_t^n|>\delta)^{1/b_n^2}=0$ for $\delta>0$, so by (6.51), $\lim_{n\to\infty}\mathbf{P}(\sup_{t\in[0,1]}|\widehat{X}_t^n|>\delta)^{1/b_n^2}=0$. Therefore, by (6.37), the convergence $(\sqrt{n}/b_n)(c_n-c)\to\hat{\theta}$, and LD convergence in distribution at rate b_n^2 of the $\widehat{M}^n$ to $\widehat{M}$ the $\widehat{Y}^n$ LD converge in distribution at rate b_n^2 in $\mathbb{D}_C([0,1],\mathbb{R})$ to $-\widehat{H}$. $\square$

REMARK 6.3. A slight modification of the proof shows that for $\beta>0$ and $\delta\in(0,\beta\wedge(1-\beta))$, the random elements $\widehat{M}^n$, $\widehat{L}^n$, $(\widehat{X}_t^n,\ t\in[0,\beta-\delta])$, $(\widehat{Y}_t^n,\ t\in[0,\beta-\delta])$, $(\widehat{X}_t^n,\ t\in[\beta+\delta,1])$, $(\widehat{Y}_t^n,\ t\in[\beta+\delta,1])$, $\widehat{X}_\beta^n$, $\widehat{Y}_\beta^n$ and $\widehat{Z}$ jointly LD converge in distribution at rate b_n^2 in $\mathbb{D}_C([0,1],\mathbb{R}^2)\times\mathbb{D}_C([0,\beta-\delta],\mathbb{R})^2\times\mathbb{D}_C([\beta+\delta,1],\mathbb{R})^2\times\mathbb{R}^2\times\mathbb{D}_C([0,1],\mathbb{R})$ to the respective idempotent elements $\widehat{M}$, $\widehat{L}$, $(\widehat{X}_t,t\in[0,\beta-\delta])$, $(\widehat{Y}_t,\ t\in[0,\beta-\delta])$, $(\widehat{X}_t,\ t\in[\beta+\delta,1])$, $(\widehat{Y}_t,\ t\in[\beta+\delta,1])$, $\widehat{X}_\beta$, $\widehat{Y}_\beta$ and $\widehat{Z}$, where idempotent processes $\widehat{X}=(\widehat{X}_t,\ t\in[0,1])$ and $\widehat{Y}=(\widehat{Y}_t,\ t\in[0,1])$ are defined by

$$\widehat{X}_t=\begin{cases}\widehat{H}_t, & \text{for } t\in[0,\beta),\\ \widehat{H}_\beta\vee 0, & \text{for } t=\beta,\\ 0, & \text{for } t\in(\beta,1],\end{cases}\qquad\text{and}\qquad \widehat{Y}_t=\begin{cases}0, & \text{for } t\in[0,\beta),\\ (-\widehat{H}_\beta)\vee 0, & \text{for } t=\beta,\\ -\widehat{H}_t, & \text{for } t\in(\beta,1].\end{cases}$$

PROOF. Proof of Theorem 2.3 The proof replicates the proof of Theorem 2.2. We begin by proving that in analogy with (6.28) if $c > 1$, then as $n \to \infty$,

$$\left(\frac{\sqrt{n}}{b_n}\left(\frac{\alpha^n}{n} - \alpha\right), \frac{\sqrt{n}}{b_n}\left(\frac{\beta^n}{n} - \beta\right), \frac{\sqrt{n}}{b_n}\left(\frac{\gamma^n}{n} - \gamma\right)\right) \xrightarrow[b_n^2]{ld} \left(-\widehat{H}_1, \frac{\widehat{H}_\beta}{1 - c(1-\beta)}, \widehat{Z}_\beta\right). \tag{6.54}$$

As in the proof of Theorem 2.2, we let τ^n be the last time t before $\beta/2$ when $\overline{Q}_t^n = 0$ and $\tilde{\beta}^n$ be the first time t not before $\beta/2$ when $\overline{Q}_t^n = 0$. The argument of the proof of Theorem 2.2 with the super-exponential limits in probability of Lemma 6.3 used in place of Lemma 6.1 implies that under the hypotheses as $n \to \infty$,

$$\tau^n \overset{\mathbf{P}^{1/b_n^2}}{\to} 0, \qquad \tilde{\beta}^n \overset{\mathbf{P}^{1/b_n^2}}{\to} \beta, \qquad \mathbf{P}\left(\frac{\beta^n}{n} \neq \tilde{\beta}^n - \tau^n\right)^{1/b_n^2} \to 0, \qquad \mathbf{P}\left(\frac{\gamma^n}{n} \neq \overline{E}^n_{\tilde{\beta}^n} - \overline{E}^n_{\tau^n}\right)^{1/b_n^2} \to 0. \tag{6.55}$$

By (6.32) and (6.33) with the use of (6.39),

$$-\frac{\sqrt{n}}{b_n}\bar{q}_{\tau^n} = -c_n \int_0^{\tau^n} \widehat{X}_s^n \, ds + \frac{\sqrt{n}}{b_n}(c_n - c)\int_0^{\tau^n} \sigma_s^2 \, ds + \widehat{M}^n_{\tau^n} + \hat{\varepsilon}^n_{\tau^n} + \widehat{Y}^n_{\tau^n}, \tag{6.56}$$

$$-\frac{\sqrt{n}}{b_n}\bar{q}_{\tilde{\beta}^n} = -c_n \int_0^{\tilde{\beta}^n} \widehat{X}_s^n \, ds + \frac{\sqrt{n}}{b_n}(c_n - c)\int_0^{\tilde{\beta}^n} \sigma_s^2 \, ds + \widehat{M}^n_{\tilde{\beta}^n} + \hat{\varepsilon}^n_{\tilde{\beta}^n} + \frac{\sqrt{n}}{b_n}\overline{\Phi}^n_{\tilde{\beta}^n} - \frac{\sqrt{n}}{b_n}\bar{\phi}_{\tilde{\beta}^n}. \tag{6.57}$$

The left-most convergence in (6.55) implies by Lemma 6.3, (6.40) and the convergence $(\sqrt{n}/b_n)(c_n - c) \to \hat{\theta}$ that the right-hand side of (6.56) converges super-exponentially in probability at rate b_n^2 to 0, which yields the convergence

$$\frac{\sqrt{n}}{b_n}\tau^n \overset{\mathbf{P}^{1/b_n^2}}{\to} 0. \tag{6.58}$$

Next, (6.55), (6.57) and Lemma 6.3 imply by an argument along the lines of the one used for deriving (6.35) that

$$\frac{\sqrt{n}}{b_n}\int_\beta^{\tilde{\beta}^n} \big(c(1 - \bar{q}_s - s) - 1\big)\, ds - c_n \int_0^{\tilde{\beta}^n} \widehat{X}_s^n \, ds + \frac{\sqrt{n}}{b_n}(c_n - c)\int_0^{\tilde{\beta}^n} \sigma_s^2 \, ds + \widehat{M}^n_{\tilde{\beta}^n} \overset{\mathbf{P}^{1/b_n^2}}{\to} 0. \tag{6.59}$$

Also by the definition of $\widehat{Y}^n$ and (6.36),

$$\frac{\sqrt{n}}{b_n}\left(\frac{\alpha^n}{n}-\alpha\right)=\widehat{Y}_1^n. \tag{6.60}$$

Convergence (6.54) follows by (6.59), (6.60), the convergence $(\sqrt{n}/b_n)(c_n - c) \to \hat{\theta}$, the joint LD convergence in distribution $(\widehat{M}^n, \widehat{Y}_1^n, (\widehat{X}_s^n, s \in [0, \beta-\delta]), \widehat{Z}^n) \xrightarrow[b_n^2]{ld} (\widehat{M}, -\widehat{H}_1, (\widehat{H}_s, s \in [0, \beta-\delta]), \widehat{Z})$ in $\mathbb{D}_C([0,1],\mathbb{R}) \times \mathbb{R} \times \mathbb{D}_C([0,\beta-\delta],\mathbb{R}) \times \mathbb{D}_C([0,1],\mathbb{R})$, the third super-exponential convergence in probability in the statement of Lemma 6.3, the last three convergences in (6.55), (6.58) and the contraction principle.

If $c \le 1$, then the Y_1^n LD converge in distribution to $-\widehat{H}_1$ by part 2 of Lemma 6.3.

We complete the proof by showing that the right-hand side of (6.54) is idempotent Gaussian with parameters (μ, Σ), that is,

$$\mathbf{S}\exp\left(-\lambda_1\widehat{H}_1+\lambda_2\frac{\widehat{H}_\beta}{1-c(1-\beta)}+\lambda_3\widehat{Z}_\beta\right)=\exp\left(\lambda^T\mu+\frac{1}{2}\lambda^T\Sigma\lambda\right), \tag{6.61}$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)^T \in \mathbb{R}^3$ and $\mathbf{S}$ denotes idempotent expectation with respect to $\mathbf{\Pi}$. By (6.42), (6.43), (6.7) and (6.14),

$$\widehat{H}_\beta=\hat{\theta}\beta e^{-\beta c}+\sqrt{c}e^{-\beta c}\int_0^\beta e^{cs/2}\dot{\widehat{W}}_s^{(1)}\,ds,$$
$$\widehat{Z}_\beta=\frac{\hat{\theta}\beta^2}{2}+\sqrt{c}\int_0^\beta (1-e^{c(s-\beta)})e^{-cs/2}\dot{\widehat{W}}_s^{(1)}\,ds+\sqrt{c}\int_0^\beta \sqrt{\bar{q}_s}\dot{\widehat{W}}_s^{(2)}\,ds.$$

On noting that by (6.42) and (6.14), $\widehat{H}_1 = \widehat{H}_\beta + \hat{\theta}\int_\beta^1 (1-s)\,ds + \sqrt{c}\int_\beta^1 \sqrt{1-s} \times \dot{\widehat{W}}_s^{(1)}\,ds$, $\widehat{W}^{(1)}$ and $\widehat{W}^{(2)}$ are independent, we can write using Lemma A.4,

$$\begin{aligned}
&\mathbf{S}\exp\left(-\lambda_1\widehat{H}_1+\lambda_2\frac{\widehat{H}_\beta}{1-c(1-\beta)}+\lambda_3\widehat{Z}_\beta\right)\\
&\quad=\exp\left(-\frac{\lambda_1\hat{\theta}(1-\beta)^2}{2}+\left(\frac{\lambda_2}{1-c(1-\beta)}-\lambda_1\right)\hat{\theta}\beta e^{-\beta c}+\lambda_3\frac{\hat{\theta}\beta^2}{2}\right)\\
&\qquad\times\mathbf{S}\exp\left(\left(\frac{\lambda_2}{1-c(1-\beta)}-\lambda_1-\lambda_3\right)\sqrt{c}e^{-\beta c}\int_0^\beta e^{cs/2}\dot{\widehat{W}}_s^{(1)}\,ds\right.\\
&\qquad\qquad\left.+\lambda_3\sqrt{c}\int_0^\beta e^{-cs/2}\dot{\widehat{W}}_s^{(1)}\,ds\right)\\
&\qquad\times\mathbf{S}\exp\left(-\lambda_1\sqrt{c}\int_\beta^1\sqrt{1-s}\dot{\widehat{W}}_s^{(1)}\,ds\right)\mathbf{S}\exp\left(\lambda_3\sqrt{c}\int_0^\beta\sqrt{\bar{q}_s}\dot{\widehat{W}}_s^{(2)}\,ds\right).
\end{aligned} \tag{6.62}$$

Lemma A.4 also yields

$$\mathbf{S}\exp\left(\sqrt{c}\int_0^\beta\left(\left(\frac{\lambda_2}{1-c(1-\beta)}-\lambda_1-\lambda_3\right)e^{cs/2-\beta c}+\lambda_3 e^{-cs/2}\right)\dot{\widehat{W}}_s^{(1)}\,ds\right) \tag{6.63}$$
$$=\exp\left(\frac{c}{2}\int_0^\beta\left(\left(\frac{\lambda_2}{1-c(1-\beta)}-\lambda_1-\lambda_3\right)e^{cs/2-\beta c}+\lambda_3 e^{-cs/2}\right)^2 ds\right),$$

$$\mathbf{S}\exp\left(-\lambda_1\sqrt{c}\int_\beta^1\sqrt{1-s}\,\dot{\widehat{W}}_s^{(1)}\,ds\right)=\exp\left(\frac{c\lambda_1^2}{2}\int_\beta^1(1-s)\,ds\right), \tag{6.64}$$

$$\mathbf{S}\exp\left(\lambda_3\sqrt{c}\int_0^\beta\sqrt{\bar{q}_s}\,\dot{\widehat{W}}_s^{(2)}\,ds\right)=\exp\left(\frac{c\lambda_3^2}{2}\int_0^\beta\bar{q}_s\,ds\right). \tag{6.65}$$

Equality (6.61) follows on substituting (6.63), (6.64) and (6.65) into (6.62) and recalling (6.7). □

REMARK 6.4. Equality (6.61) admits also a direct proof by solving the variational problem on the left.

7. The critical random graph. In this section we prove Theorem 2.4, so the notation of the theorem is adopted. We denote $\widetilde{S}_t^n = S^n_{\lfloor n^{2/3}t\rfloor\wedge n}/n^{1/3}$, $\widetilde{E}_t^n = E^n_{\lfloor n^{2/3}t\rfloor\wedge n}$, $\widetilde{Q}_t^n = Q^n_{\lfloor n^{2/3}t\rfloor\wedge n}/n^{1/3}$, $\breve{S}_t^n = S^n_{\lfloor (nb_n)^{2/3}t\rfloor\wedge n}/(n^{1/3}b_n^{4/3})$, $\breve{E}_t^n = E^n_{\lfloor (nb_n)^{2/3}t\rfloor\wedge n}/b_n^2$ and $\breve{Q}_t^n = Q^n_{\lfloor (nb_n)^{2/3}t\rfloor\wedge n}/(n^{1/3}b_n^{4/3})$ for $t\in\mathbb{R}_+$, and introduce processes $\widetilde{S}^n = (\widetilde{S}_t^n,\ t\in\mathbb{R}_+)$, $\widetilde{E}^n = (\widetilde{E}_t^n,\ t\in\mathbb{R}_+)$, $\widetilde{Q}^n = (\widetilde{Q}_t^n,\ t\in\mathbb{R}_+)$, $\breve{S}^n = (\breve{S}_t^n,\ t\in\mathbb{R}_+)$, $\breve{E}^n = (\breve{E}_t^n,\ t\in\mathbb{R}_+)$ and $\breve{Q}^n = (\breve{Q}_t^n,\ t\in\mathbb{R}_+)$. Let stochastic processes $\widetilde{S} = (\widetilde{S}_t,\ t\in\mathbb{R}_+)$ and $\widetilde{E} = (\widetilde{E}_t,\ t\in\mathbb{R}_+)$ be defined by the respective equalities $\widetilde{S}_t = W_t + \tilde{\theta}t - t^2/2$ and $\widetilde{E}_t = N_{\int_0^t \widetilde{X}_s\,ds}$. Let idempotent processes $\breve{S} = (\breve{S}_t,\ t\in\mathbb{R}_+)$ and $\breve{E} = (\breve{E}_t,\ t\in\mathbb{R}_+)$ be defined by the respective equalities $\breve{S}_t = \breve{W}_t + \breve{\theta}t - t^2/2$ and $\breve{E}_t = \breve{N}_{\int_0^t \mathcal{R}(\breve{S})_p\,dp}$, where $\breve{W} = (\breve{W}_t,\ t\in\mathbb{R}_+)$ and $\breve{N} = (\breve{N}_t,\ t\in\mathbb{R}_+)$ are independent Wiener and Poisson idempotent processes, respectively. The first assertion of part 1 of the next lemma is in the theme of Aldous [(1997), equation (31)].

LEMMA 7.1. 1. *If $n^{1/3}(c_n-1)\to\tilde{\theta}\in\mathbb{R}$ as $n\to\infty$, then the $(\widetilde{S}^n,\widetilde{E}^n)$ converge in distribution in $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^2)$ as $n\to\infty$ to $(\widetilde{S},\widetilde{E})$. If $\sqrt{n}(c_n-1)\to\theta\in\mathbb{R}$ as $n\to\infty$, then the $(\sqrt{n}(\alpha^n/n-1/2),\widetilde{S}^n,\widetilde{E}^n)$ converge in distribution in $\mathbb{R}\times\mathbb{D}(\mathbb{R}_+,\mathbb{R}^2)$ to $(\tilde{\alpha},\widetilde{S},\widetilde{E})$, where $(\widetilde{S},\widetilde{E})$ correspond to $\tilde{\theta}=0$ and are independent of $\tilde{\alpha}$.*

2. *If $(n^{1/3}/b_n^{2/3})(c_n-1)\to\breve{\theta}\in\mathbb{R}$ as $n\to\infty$, then the $(\breve{S}^n,\breve{E}^n)$ LD converge in distribution in $\mathbb{D}_C(\mathbb{R}_+,\mathbb{R}^2)$ at rate b_n^2 to $(\breve{S},\breve{E})$. If $(\sqrt{n}/b_n)(c_n-1)\to\hat{\theta}\in\mathbb{R}$ as*

$n \to \infty$, then the $((\sqrt{n}/b_n)(\alpha^n/n - 1/2), \breve{S}^n, \breve{E}^n)$ LD converge in distribution at rate b_n^2 in $\mathbb{R} \times \mathbb{D}_C(\mathbb{R}_+, \mathbb{R}^2)$ to $(\breve{\alpha}, \breve{S}, \breve{E})$, where $(\breve{S}, \breve{E})$ correspond to $\breve{\theta} = 0$, $\breve{\alpha}$ is idempotent Gaussian with parameters $(-\hat{\theta}/2, 1/2)$ and is independent of $(\breve{S}, \breve{E})$.

PROOF. We begin with the proof of part 1. By (2.6),

$$\begin{aligned} \widetilde{S}_t^n &= \widetilde{M}_t^n + n^{1/3}(c_n - 1)\frac{\lfloor n^{2/3}t \rfloor \wedge n}{n^{2/3}} \\ &\quad - c_n \int_0^{\lfloor n^{2/3}t \rfloor \wedge n/n^{2/3}} \frac{\lfloor n^{2/3}s \rfloor}{n^{2/3}}\, ds - \frac{c_n}{n^{1/3}} \int_0^{\lfloor n^{2/3}t \rfloor \wedge n/n^{2/3}} \widetilde{Q}_s^n\, ds, \end{aligned} \tag{7.1}$$

where

$$\widetilde{M}_t^n = \frac{1}{n^{1/3}} \sum_{i=1}^{\lfloor n^{2/3}t \rfloor \wedge n} \sum_{j=1}^{n - Q_{i-1}^n - (i-1)} \left(\xi_{ij}^n - \frac{c_n}{n} \right). \tag{7.2}$$

Let $\widetilde{\mathcal{F}}_t^n$, $t \in \mathbb{R}_+$, denote the σ-algebras generated by the ξ_{ij}^n, ζ_{ij}^n, $i = 1, 2, \dots, \lfloor n^{2/3}t \rfloor \wedge n$, $j \in \mathbb{N}$, completed with sets of $\mathbf{P}$-measure zero. Then $\widetilde{M}^n = (\widetilde{M}_t^n, t \in \mathbb{R}_+)$ is a square-integrable martingale relative to the filtration $\widetilde{\mathbf{F}}^n = (\widetilde{\mathcal{F}}_t^n, t \in \mathbb{R}_+)$ with predictable quadratic characteristic

$$\langle \widetilde{M}^n \rangle_t = \frac{1}{n^{2/3}} \frac{c_n}{n} \left(1 - \frac{c_n}{n} \right) \sum_{i=1}^{\lfloor n^{2/3}t \rfloor \wedge n} \big(n - Q_{i-1}^n - (i-1) \big). \tag{7.3}$$

By Lemma 6.1, $\langle \widetilde{M}^n \rangle_t \xrightarrow{\mathbf{P}} t$ as $n \to \infty$. The predictable measure of jumps of $\widetilde{M}^n$ is given by

$$\breve{\nu}^n([0,t], \Gamma) = \sum_{k=0}^{\lfloor n^{2/3}t \rfloor \wedge n-1} \breve{F}^n\left(1 - \frac{Q_k^n}{n} - \frac{k}{n}, \Gamma \setminus \{0\} \right), \qquad \Gamma \in \mathcal{B}(\mathbb{R}),$$

where

$$\breve{F}^n(s, \Gamma') = \mathbf{P}\left(\frac{1}{n^{1/3}} \sum_{j=1}^{\lfloor ns \rfloor} \left(\xi_{1j}^n - \frac{c_n}{n} \right) \in \Gamma' \right), \qquad s \in \mathbb{R}_+, \Gamma' \in \mathcal{B}(\mathbb{R}).$$

Therefore, for $\varepsilon > 0$ and n large enough,

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}} |x|^2 \mathbf{1}(|x| > \varepsilon) \breve{\nu}^n(ds, dx) \\ &\qquad \le \frac{1}{\varepsilon^2} \sum_{k=1}^{\lfloor n^{2/3}t \rfloor \wedge n} \int_{\mathbb{R}} |x|^4 \breve{F}^n\left(1 - \frac{Q_{k-1}^n}{n} - \frac{k-1}{n},\, dx \right) \le \frac{(2c_n + 3c_n^2)t}{n^{2/3}\varepsilon^2}, \end{aligned}$$

which converges to 0 as $n \to \infty$. Consequently, by Liptser and Shiryaev [(1989), Theorem 7.1.4] the processes $\widetilde{M}^n$ converge in distribution in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ to the process W as $n \to \infty$. Hence, the processes $\widetilde{S}'^n = (\widetilde{S}_t'^n,\ t \in \mathbb{R}_+)$, where

$$\widetilde{S}_t'^n = \widetilde{M}_t^n + n^{1/3}(c_n - 1)\frac{\lfloor n^{2/3}t\rfloor \wedge n}{n^{2/3}} - c_n \int_0^{\lfloor n^{2/3}t\rfloor \wedge n/n^{2/3}} \frac{\lfloor n^{2/3}s\rfloor}{n^{2/3}}\, ds$$

converge in distribution to the process $\widetilde{S}$.

Let $\tilde{\tilde{\varepsilon}}^n = (\tilde{\tilde{\varepsilon}}_t^n,\ t \in \mathbb{R}_+)$ be defined by $\tilde{\tilde{\varepsilon}}_t^n = \varepsilon^n_{\lfloor n^{2/3}t\rfloor \wedge n}/n^{1/3}$. According to (2.9) and (2.11),

$$\widetilde{Q}^n = \mathcal{R}(\widetilde{S}^n + \tilde{\tilde{\varepsilon}}^n). \tag{7.4}$$

Besides, by Lemma 3.1,

$$\sup_{s \in \mathbb{R}_+} |\tilde{\tilde{\varepsilon}}_s^n| \xrightarrow{\mathbf{P}} 0 \qquad \text{as } n \to \infty. \tag{7.5}$$

Since the difference $\widetilde{S}_t'^n - \widetilde{S}_t^n$ is nonnegative and nondecreasing in t, it follows by (7.4) that the values of the process $\widetilde{Q}^n$ are not greater than the corresponding values of the reflection of $\widetilde{S}'^n + \tilde{\tilde{\varepsilon}}^n$. On using that the $\sup_{r \in [0,t]} |\widetilde{S}_r'^n|$ are asymptotically bounded in probability and that (7.5) holds, we conclude that the $\sup_{s \in [0,t]} \widetilde{Q}_s^n$ are asymptotically bounded in probability, so the right-most term of (7.1) tends in probability to 0 uniformly over bounded intervals as $n \to \infty$, implying that the $\widetilde{S}^n$ converge in distribution to $\widetilde{S}$.

Next, according to (2.12),

$$\widetilde{E}_t^n = \sum_{i=1}^{\lfloor n^{2/3}t\rfloor \wedge n} \sum_{j=1}^{Q_{i-1}^n - 1} \zeta_{ij}^n, \qquad t \in \mathbb{R}_+. \tag{7.6}$$

Given a sequence $\mathbf{x}^n$, $n \in \mathbb{N}$, of elements of $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$, let

$$\widetilde{E}_t'^n = \sum_{i=1}^{\lfloor n^{2/3}t\rfloor \wedge n} \sum_{j=1}^{\lfloor n^{1/3}\mathcal{R}(\mathbf{x}^n)_{(i-1)/n^{2/3}}\rfloor - 1} \zeta_{ij}^n, \qquad t \in \mathbb{R}_+.$$

The $\widetilde{E}'^n = (\widetilde{E}_t'^n,\ t \in \mathbb{R}_+)$ are jump processes with $\widetilde{\mathbf{F}}^n$-predictable measures of jumps $\tilde{\nu}'^n([0,t], \Gamma) = \sum_{i=0}^{\lfloor n^{2/3}t\rfloor \wedge n - 1} \widetilde{F}'^n(\mathcal{R}(\mathbf{x}^n)_{i/n^{2/3}}, \Gamma \setminus \{0\})$, $\Gamma \in \mathcal{B}(\mathbb{R})$, where $\widetilde{F}'^n(y, \Gamma') = \mathbf{P}(\sum_{j=1}^{\lfloor n^{1/3}y\rfloor - 1} \zeta_{1j}^n \in \Gamma')$, $\Gamma' \in \mathcal{B}(\mathbb{R})$. Theorem VII.3.7 in Jacod and Shiryaev (1987) implies that if $\mathbf{x}^n \to \mathbf{x}$ as $n \to \infty$ in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$, then the sequence $\widetilde{E}'^n$, $n \in \mathbb{N}$, converges in distribution in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ to a compound Poisson process with compensator $\int_0^t \mathcal{R}(\mathbf{x})_s\, ds$. On noting that, in view of independence of $\widetilde{S}^n$ and the ζ_{ij}^n, (7.4) and (7.6), the $\widetilde{E}'^n$ are distributed according to the regular conditional distributions of $\widetilde{E}^n$ given that $\widetilde{S}^n + \tilde{\tilde{\varepsilon}}^n = \mathbf{x}^n$, we conclude by (7.4), (7.5) and (7.6) that the $(\widetilde{S}^n, \widetilde{E}^n)$ jointly converge in distribution in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^2)$ to $(\widetilde{S}, \widetilde{E})$ as $n \to \infty$. The first assertion of part 1 has been proved.

For the second assertion, let in analogy with (6.1) for $\eta > 0$,

$$\widetilde{M}_t^{n,\eta} = \frac{\mathbf{1}(t \geq \eta)}{n} \sum_{i=\lfloor n\eta \rfloor+1}^{\lfloor nt \rfloor} \sum_{j=1}^{n-\lfloor n\bar{q}_{(i-1)/n}\rfloor-(i-1)} \left(\xi_{ij}^n - \frac{c_n}{n}\right), \qquad t \in [0,1], \tag{7.7}$$

and $\widetilde{Q}^{n,\eta} = (\widetilde{Q}_t^{n,\eta}, t \in [0,1])$ be defined in analogy with (6.3) by the condition that it is the reflection of the process $\int_0^t (c_n(1 - \widetilde{Q}_s^{n,\eta} - s) - 1)\, ds + \widetilde{M}_t^{n,\eta}$, that is, $\widetilde{Q}_t^{n,\eta} \geq 0$ and

$$\widetilde{Q}_t^{n,\eta} = \int_0^t \big(c_n(1 - \widetilde{Q}_s^{n,\eta} - s) - 1\big)\, ds + \widetilde{M}_t^{n,\eta} + \widetilde{\Phi}_t^{n,\eta}, \tag{7.8}$$

where $\widetilde{\Phi}^{n,\eta} = (\widetilde{\Phi}_t^{n,\eta}, t \in [0,1])$ is nondecreasing with $\widetilde{\Phi}_t^{n,\eta} = \int_0^t \mathbf{1}(\widetilde{Q}_s^{n,\eta} = 0)\, d\widetilde{\Phi}_s^{n,\eta}$. (For existence of $\widetilde{Q}^{n,\eta}$, one can first prove that a solution exists between the jumps of $\widetilde{M}^{n,\eta}$ by using the method of successive approximations and making use of Lipshitz continuity of the reflection mapping and Gronwall's inequality, and then account for the jumps by introducing, if necessary, jumps in $\widetilde{\Phi}^{n,\eta}$. Strong uniqueness for $\widetilde{Q}^{n,\eta}$ follows by Lipshitz continuity of the reflection mapping and Gronwall's inequality too.) By (6.1), (7.7) and the convergence of the $\overline{Q}^n$ to $\bar{q}$ (Lemma 6.1) for $\tilde{\eta} > 0$,

$$\lim_{\eta \to 0} \limsup_{n\to\infty} \mathbf{P}\Big(\sup_{t\in[0,1]} \sqrt{n}|\widetilde{M}_t^{n,\eta} - \overline{M}_t^n| > \tilde{\eta}\Big) = 0,$$

which implies by (6.3), (7.8), Lemma 3.1, Lipshitz continuity of the reflection mapping and Gronwall's inequality that

$$\lim_{\eta \to 0} \limsup_{n\to\infty} \mathbf{P}\Big(\sup_{t\in[0,1]} \sqrt{n}|\widetilde{Q}_t^{n,\eta} - \overline{Q}_t^n| > \tilde{\eta}\Big) = 0,$$

and, consequently,

$$\lim_{\eta \to 0} \limsup_{n\to\infty} \mathbf{P}\big(\sqrt{n}|\widetilde{\Phi}_1^{n,\eta} - \overline{\Phi}_1^n| > \tilde{\eta}\big) = 0. \tag{7.9}$$

Since $\widetilde{\Phi}^{n,\eta}$ is independent of the ξ_{ij}^n, $i = 1, 2, \dots, \lfloor n\eta \rfloor$, $j \in \mathbb{N}$, and ζ_{ij}^n, $i \in \mathbb{N}$, $j \in \mathbb{N}$, and the $(\widetilde{S}_t^n, \widetilde{E}_t^n)$ are measurable functions of ξ_{ij}^n, $i = 1, 2, \dots, \lfloor n^{2/3} t \rfloor \wedge n$, $j \in \mathbb{N}$, and ζ_{ij}^n, $i = 1, 2, \dots, \lfloor n^{2/3} t \rfloor \wedge n$, $j \in \mathbb{N}$, it follows that $\widetilde{\Phi}_1^{n,\eta}$ and finite-dimensional distributions of the $(\widetilde{S}^n, \widetilde{E}^n)$ are independent for all large n, which yields by (7.9) the asymptotic independence of $\sqrt{n}(\overline{\Phi}_1^n - \bar{\phi}_1)$ and finite-dimensional distributions of the $(\widetilde{S}^n, \widetilde{E}^n)$. The proof of part 1 is over.

The proof of part 2 is similar. In analogy with (7.1) and (7.2),

$$
\begin{aligned}
\breve{S}_t^n = \breve{M}_t^n &+ \frac{n^{1/3}}{b_n^{2/3}}(c_n - 1)\frac{\lfloor (nb_n)^{2/3} t\rfloor \wedge n}{(nb_n)^{2/3}} \\
&- c_n \int_0^{\lfloor (nb_n)^{2/3} t\rfloor \wedge n/(nb_n)^{2/3}} \frac{\lfloor (nb_n)^{2/3} s\rfloor}{(nb_n)^{2/3}}\, ds \\
&- c_n \frac{b_n^{2/3}}{n^{1/3}} \int_0^{\lfloor (nb_n)^{2/3} t\rfloor \wedge n/(nb_n)^{2/3}} \breve{Q}_s^n\, ds,
\end{aligned}
\tag{7.10}
$$

where

$$
\breve{M}_t^n = \frac{1}{n^{1/3} b_n^{4/3}} \sum_{i=1}^{\lfloor (nb_n)^{2/3} t\rfloor \wedge n} \sum_{j=1}^{n - Q_{i-1}^n - (i-1)} \left(\xi_{ij}^n - \frac{c_n}{n}\right).
\tag{7.11}
$$

Let $\breve{\mathcal{F}}_t^n$, $t \in \mathbb{R}_+$, denote the σ-algebras generated by the ξ_{ij}^n, ζ_{ij}^n, $i = 1, 2, \dots, \lfloor (nb_n)^{2/3} t\rfloor \wedge n$, $j \in \mathbb{N}$, completed with sets of $\mathbf{P}$-measure zero. Then $\breve{M}^n = (\breve{M}_t^n,\ t \in \mathbb{R}_+)$ is a square-integrable martingale relative to the filtration $\breve{\mathbf{F}}^n = (\breve{\mathcal{F}}_t^n,\ t \in \mathbb{R}_+)$ with predictable quadratic characteristic

$$
\langle \breve{M}^n \rangle_t = \frac{1}{n^{2/3} b_n^{8/3}} \frac{c_n}{n}\left(1 - \frac{c_n}{n}\right) \sum_{i=1}^{\lfloor (nb_n)^{2/3} t\rfloor \wedge n} \big(n - Q_{i-1}^n - (i-1)\big)
\tag{7.12}
$$

and predictable measure of jumps

$$
\breve{\nu}^n([0,t], \Gamma) = \sum_{k=0}^{\lfloor (nb_n)^{2/3} t\rfloor \wedge n-1} \breve{F}^n\left(1 - \frac{Q_k^n}{n} - \frac{k}{n}, \Gamma \setminus \{0\}\right), \qquad \Gamma \in \mathcal{B}(\mathbb{R}),
\tag{7.13}
$$

where

$$
\breve{F}^n(s, \Gamma') = \mathbf{P}\left(\frac{1}{n^{1/3} b_n^{4/3}} \sum_{j=1}^{\lfloor ns\rfloor} \left(\xi_{1j}^n - \frac{c_n}{n}\right) \in \Gamma'\right), \qquad s \in \mathbb{R}_+,\ \Gamma' \in \mathcal{B}(\mathbb{R}).
\tag{7.14}
$$

By (7.12) and the first super-exponential convergence in probability in Lemma 6.3, $b_n^2 \langle \breve{M}^n \rangle_t \overset{\mathbf{P}^{1/b_n^2}}{\to} t$ as $n \to \infty$. Next, in analogy with (6.50), it is established that

$$
\lim_{n\to\infty} \mathbf{P}\left(\frac{1}{b_n^2}\int_0^t \int_{\mathbb{R}} e^{\lambda b_n^2 |x|} \mathbf{1}(b_n^2 |x| > \varepsilon)\breve{\nu}^n(ds, dx) > \eta\right)^{1/b_n^2} = 0,
$$

$$
\lambda > 0,\ \varepsilon > 0,\ \eta > 0,\ t > 0.
$$

By Corollary 4.3.13 in Puhalskii (2001), we thus have that the $\breve{M}^n$ LD converge in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ at rate b_n^2 to the idempotent process $\breve{W}$ as $n \to \infty$. Since in analogy with (7.4) $\breve{Q}^n = \mathcal{R}(\breve{S}^n + \breve{\varepsilon}^n)$, where

$$\breve{\varepsilon}^n_t = \frac{\varepsilon^n_{\lfloor (nb_n)^{2/3} t\rfloor \wedge n}}{n^{1/3} b_n^{4/3}}, \tag{7.15}$$

and $\sup_{t\in\mathbb{R}_+} |\tilde{\breve{\varepsilon}}^n_t| \overset{\mathbf{P}^{1/b_n^2}}{\to} 0$ as $n \to \infty$ by Lemma 3.1, we conclude by an argument replicating the one used in the first part of the proof that the $\breve{S}^n$ LD converge to $\breve{S}$. Finally, a "conditional" argument modeled on those used in the proofs of part 1 and Corollary 4.1 shows that $(\breve{S}^n, \breve{E}^n) \underset{b_n^2}{\overset{ld}{\to}} (\breve{S}, \breve{E})$ in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^2)$. Convergence in $\mathbb{D}_C(\mathbb{R}_+, \mathbb{R}^2)$ follows by continuity of $(\widetilde{S}, \widetilde{E})$ and $(\breve{S}^n, \breve{E}^n)$ being a random element of $\mathbb{D}_C(\mathbb{R}_+, \mathbb{R}^2)$. The proof of the second assertion of part 2 is similar to the proof of the second assertion of part 1. $\square$

PROOF OF THEOREM 2.4. We begin with part 1, so we assume that $n^{1/3}(c_n - 1) \to \tilde{\theta}$. The below reasoning repeatedly invokes the property that for almost every trajectory of $\widetilde{S}$, the process $\mathcal{T}(\widetilde{S})$ is increasing in arbitrarily small neighborhoods to the left of the initial point and to the right of the terminal point of an excursion of $\mathcal{R}(\widetilde{S})$; equivalently, the value of $\widetilde{S}$ at the initial point is strictly less than at any point to its left and the infimum of the values of $\widetilde{S}$ in an arbitrary neighborhood to the right of the terminal point is strictly less than the value of $\widetilde{S}$ at the terminal point. [The stated property can be proved by using the decomposition of the Wiener process into excursions, see, e.g., Ikeda and Watanabe (1989).]

We denote $\widetilde{U}^n_i = U^n_i / n^{2/3}$ and $\widetilde{R}^n_i = R^n_i / n^{2/3}$. Given intervals $[\underline{u}_i, \bar{u}_i]$ and $[\bar{r}_i, \underline{r}_i]$, where $0 < \underline{u}_i < \bar{u}_i$ and $0 \le \underline{r}_i < \bar{r}_i$ for $i = 1, \dots, m$, let $\overline{B}^n$ denote the event that there exist m connected components of $\mathcal{G}(n, c_n/n)$ of sizes in the intervals $[n^{2/3}\underline{u}_i, n^{2/3}\bar{u}_i]$ for $i = 1, 2, \dots, m$ and the numbers of the excess edges of these components belong to the respective intervals $[n^{2/3}\underline{r}_i, n^{2/3}\bar{r}_i]$. Let $\overline{B}_T$, for $T > 0$, denote the set of functions $(\mathbf{x}, \mathbf{y}) \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}^2)$ with $\mathbf{x}_0 = 0$, $\mathbf{y}_0 = 0$ and $\mathbf{y}$ nondecreasing such that there exist nonoverlapping intervals $[s_i, t_i]$ with $t_i - s_i \in [\underline{u}_i, \bar{u}_i]$ and $t_i \le T$ for which $\mathcal{R}(\mathbf{x})_{s_i} = \mathcal{R}(\mathbf{x})_{t_i} = 0$, $\mathcal{T}(\mathbf{x})_{t_i-} = \mathcal{T}(\mathbf{x})_{s_i}$ and $\mathbf{y}_{t_i} - \mathbf{y}_{s_i} \in [\underline{r}_i, \bar{r}_i]$ for $i = 1, 2, \dots, m$. Since the connected components of $\mathcal{G}(n, c_n/n)$ correspond to excursions of $\widetilde{Q}^n$ and may occur either before time T or after it, we have $\overline{B}^n \subset \{(\widetilde{S}^n + \tilde{\tilde{\varepsilon}}^n, \widetilde{E}^n) \in \overline{B}_T\} \cup \{\sup_{t \ge T}(\widetilde{S}^n_t + \tilde{\tilde{\varepsilon}}^n_t - \widetilde{S}^n_{t-\eta} - \tilde{\tilde{\varepsilon}}^n_{t-\eta}) > 0\}$ for $\eta \in (0, T \wedge \min_{i=1,2,\dots,m} \underline{u}_i)$. Since the set $\overline{B}_T$ and its closure [in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^2)$] have the same intersection with $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^2)$, Lemma 7.1 implies that

$$\begin{aligned}&\limsup_{n\to\infty} \mathbf{P}(\overline{B}^n)\\ &\quad \le \mathbf{P}\big((\widetilde{S}, \widetilde{E}) \in \overline{B}_T\big) + \limsup_{n\to\infty} \mathbf{P}\Big(\sup_{t\ge T}(\widetilde{S}^n_t + \tilde{\tilde{\varepsilon}}^n_t - \widetilde{S}^n_{t-\eta} - \tilde{\tilde{\varepsilon}}^n_{t-\eta}) > 0\Big).\end{aligned} \tag{7.16}$$

We show that

$$\lim_{T\to\infty}\limsup_{n\to\infty}\mathbf{P}\Big(\sup_{t\geq T}(\widetilde{S}^n_t+\tilde{\tilde{\varepsilon}}^n_t-\widetilde{S}^n_{t-\eta}-\tilde{\tilde{\varepsilon}}^n_{t-\eta})>0\Big)=0. \tag{7.17}$$

By (7.1), (7.3), (2.7) and Doob's inequality for all n and T large enough,

$$\begin{aligned}
&\mathbf{P}\Big(\sup_{t\geq T}(\widetilde{S}^n_t+\tilde{\tilde{\varepsilon}}^n_t-\widetilde{S}^n_{t-\eta}-\tilde{\tilde{\varepsilon}}^n_{t-\eta})>0\Big)\\
&\quad\leq\sum_{k=0}^{\infty}\mathbf{P}\Big(\sup_{t\in[T+k\eta,T+(k+1)\eta]}(\widetilde{M}^n_t-\widetilde{M}^n_{t-\eta}+\tilde{\tilde{\varepsilon}}^n_t-\tilde{\tilde{\varepsilon}}^n_{t-\eta})\\
&\qquad\qquad>c_n\eta\big(T+(k-1)\eta\big)-2\eta n^{1/3}|c_n-1|\Big)\\
&\quad\leq\sum_{k=0}^{\infty}\mathbf{P}\Big(2\sup_{s\in[0,\eta]}(|\widetilde{M}^n_{T+(k-1)\eta+s}-\widetilde{M}^n_{T+(k-1)\eta}|\\
&\qquad\qquad+|\tilde{\tilde{\varepsilon}}^n_{T+(k-1)\eta+s}-\tilde{\tilde{\varepsilon}}^n_{T+(k-1)\eta}|)\\
&\qquad\qquad+\sup_{s\in[0,\eta]}(|\widetilde{M}^n_{T+k\eta+s}-\widetilde{M}^n_{T+k\eta}|+|\tilde{\tilde{\varepsilon}}^n_{T+k\eta+s}-\tilde{\tilde{\varepsilon}}^n_{T+k\eta}|)\\
&\qquad\qquad>c_n\eta\big(T+(k-1)\eta\big)-2\eta n^{1/3}|c_n-1|\Big)\\
&\quad\leq\sum_{k=-1}^{\infty}\mathbf{P}\Big(\sup_{s\in[0,\eta]}(|\widetilde{M}^n_{T+k\eta+s}-\widetilde{M}^n_{T+k\eta}|+|\tilde{\tilde{\varepsilon}}^n_{T+k\eta+s}-\tilde{\tilde{\varepsilon}}^n_{T+k\eta}|)\\
&\qquad\qquad>\frac{c_n\eta}{3}(T+k\eta)-\frac{2\eta}{3}n^{1/3}|c_n-1|\Big)\\
&\qquad+\sum_{k=-1}^{\infty}\mathbf{P}\Big(\sup_{s\in[0,\eta]}(|\widetilde{M}^n_{T+(k+1)\eta+s}-\widetilde{M}^n_{T+(k+1)\eta}|\\
&\qquad\qquad+|\tilde{\tilde{\varepsilon}}^n_{T+(k+1)\eta+s}-\tilde{\tilde{\varepsilon}}^n_{T+(k+1)\eta}|)\\
&\qquad\qquad>\frac{c_n\eta}{3}(T+k\eta)-\frac{2\eta}{3}n^{1/3}|c_n-1|\Big)\\
&\quad\leq 2\sum_{k=-1}^{\infty}\Big(4\mathbf{E}\big(\langle\widetilde{M}^n\rangle_{T+(k+1)\eta}-\langle\widetilde{M}^n\rangle_{T+k\eta}\big)\\
&\qquad\qquad+\mathbf{E}\sup_{s\in[0,\eta]}|\tilde{\tilde{\varepsilon}}^n_{T+k\eta+s}-\tilde{\tilde{\varepsilon}}^n_{T+k\eta}|^2\Big)\\
&\qquad\qquad\times\big(\big((c_n\eta)/3(T+k\eta)-(2\eta)/3n^{1/3}|c_n-1|\big)^2\big)^{-1}
\end{aligned} \tag{7.18}$$

$$+2\sum_{k=-1}^{\infty}\Big(4\mathbf{E}\big(\langle\widetilde{M}^n\rangle_{T+(k+2)\eta}-\langle\widetilde{M}^n\rangle_{T+(k+1)\eta}\big)$$

$$+\mathbf{E}\sup_{s\in[0,\eta]}\big|\tilde{\tilde{\varepsilon}}^n_{T+(k+1)\eta+s}-\tilde{\tilde{\varepsilon}}^n_{T+(k+1)\eta}\big|^2\Big)$$

$$\times\big(((c_n\eta)/3(T+k\eta)-(2\eta)/3n^{1/3}|c_n-1|)^2\big)^{-1}$$

$$\le 4\sum_{k=-1}^{\infty}\frac{8c_n\eta+((1+c_n)^2+c_n)n^{-2/3}}{(c_n\eta/3(T+k\eta)-2\eta/3n^{1/3}|c_n-1|)^2}$$

$$\le 4\sum_{k=-1}^{\infty}\frac{12^2(16\eta+1)}{(T+k\eta)^2\eta^2}.$$

The latter sum converges to 0 as $T\to\infty$, so (7.17) follows.

Denoting $\overline{B}=\bigcup_{T>0}\overline{B}_T$, we deduce from (7.16) and (7.17) that

$$\limsup_{n\to\infty}\mathbf{P}(\overline{B}^n)\le\mathbf{P}\big((\widetilde{S},\widetilde{E})\in\overline{B}\big).$$

By the cited property, for almost all $\omega\in\Omega$, any interval $[s,t]$ such that

$$\mathcal{R}(\widetilde{S})_s(\omega)=\mathcal{R}(\widetilde{S})_t(\omega)=0\quad\text{and}\quad\mathcal{T}(\widetilde{S})_s(\omega)=\mathcal{T}(\widetilde{S})_t(\omega)$$

is an excursion of $\mathcal{R}(\widetilde{S})(\omega)$. Therefore, $\mathbf{P}((\widetilde{S},\widetilde{E})\in\overline{B})=P_{\{[\underline{u}_i,\bar{u}_i],[\bar{r}_i,\underline{r}_i]\}_{i=1}^m}$, where $P_{\{[\underline{u}_i,\bar{u}_i],[\bar{r}_i,\underline{r}_i]\}_{i=1}^m}$ denotes the probability that there exist m excursions of $\mathcal{R}(\widetilde{S})=\widetilde{X}$ with lengths in the respective intervals $[\underline{u}_i,\bar{u}_i]$ and the increments of $\widetilde{E}$ over these excursions belong to the respective intervals $[\underline{r}_i,\bar{r}_i]$. Hence,

$$\limsup_{n\to\infty}\mathbf{P}(\overline{B}^n)\le P_{\{[\underline{u}_i,\bar{u}_i],[\bar{r}_i,\underline{r}_i]\}_{i=1}^m}. \tag{7.19}$$

Next, let $\overset{o}{B}{}^n$ denote the event that there exist m connected components of $\mathcal{G}(n,c_n/n)$ of sizes in the segments $(n^{2/3}\underline{u}_i,n^{2/3}\bar{u}_i)$ for $i=1,2,\dots,m$ and the numbers of the excess edges of these components belong to the respective segments $(n^{2/3}\underline{r}_i,n^{2/3}\bar{r}_i)$. Let $\overset{o}{B}$ denote the set of functions $(\mathbf{x},\mathbf{y})\in\mathbb{D}(\mathbb{R}_+,\mathbb{R}^2)$ for which there exist disjoint intervals $[s_i,t_i]$ with $t_i-s_i\in(\underline{u}_i,\bar{u}_i)$ such that $\mathbf{x}_{s_i}=\mathbf{x}_{t_i}<\inf_{p\in[0,(s_i-\eta)^+]}\mathbf{x}_p$ and $\mathbf{x}_{t_i}>\inf_{p\in[t_i,t_i+\eta]}\mathbf{x}_p$ for arbitrary $\eta>0$, $\mathbf{x}_p>\mathbf{x}_{s_i}$ for $p\in(s_i,t_i)$, and $\mathbf{y}_{t_i}-\mathbf{y}_{s_i}\in(\underline{r}_i,\bar{r}_i)$ for $i=1,2,\dots,m$. Since continuous functions from $\overset{o}{B}$ are interior points of $\overset{o}{B}$ and $\{(\widetilde{S}^n+\tilde{\tilde{\varepsilon}}^n,\widetilde{E}^n)\in\overset{o}{B}\}\subset\overset{o}{B}{}^n$, by Lemma 7.1, $\liminf_{n\to\infty}\mathbf{P}(\overset{o}{B}{}^n)\ge\mathbf{P}((\widetilde{S},\widetilde{E})\in\overset{o}{B})$. If a sample event $\omega\in\Omega$ is such that $\widetilde{X}(\omega)$ has m excursions of lengths in the respective segments $(\underline{u}_i,\bar{u}_i)$ and the increments of $\widetilde{E}(\omega)$ over these excursions belong to the respective segments $(\underline{r}_i,\bar{r}_i)$, then by the cited property, $(\widetilde{X}(\omega),\widetilde{E}(\omega))\in\overset{o}{B}$ with probability 1. Therefore, denoting the probability of the set of these ω as $P_{\{(\underline{u}_i,\bar{u}_i),(\bar{r}_i,\underline{r}_i)\}_{i=1}^m}$, we deduce that

$$\liminf_{n\to\infty} \mathbf{P}\big(\overset{o}{B^n}\big) \geq P_{\{(\underline{u}_i,\bar{u}_i),(\bar{r}_i,\underline{r}_i)\}_{i=1}^m}. \tag{7.20}$$

The assertion in part 1 of the theorem convergence of $(\widetilde{U}^n, \widetilde{R}^n)$ follows by (7.19), (7.20) and the observation that the right-hand sides of these inequalities coincide. The assertion of the theorem for the case $\sqrt{n}(c_n - 1) \to \theta$ follows by a similar argument with the use of part 1 of Theorem 2.2 and the second assertion of part 1 of Lemma 7.1.

The proof of part 2 is obtained by combining the approaches of the proofs of part 1 and Theorem 2.1. We first note that the action functional $\breve{I}^{S,E}(\mathbf{x},\mathbf{y})$ associated with $(\breve{S},\breve{E})$ is of the form $\breve{I}^{S,E}(\mathbf{x},\mathbf{y}) = \int_0^\infty (\dot{\mathbf{x}}_t - \breve{\theta} + t)^2\,dt/2 + \int_0^\infty \pi(\dot{\mathbf{y}}_t/\mathcal{R}(\mathbf{x})_t)\mathcal{R}(\mathbf{x})_t\,dt$ if $\mathbf{x}$ and $\mathbf{y}$ are absolutely continuous with $\mathbf{x}_0 = \mathbf{y}_0 = 0$ and $\mathbf{y}$ nondecreasing, and $\breve{I}^{S,E}(\mathbf{x},\mathbf{y}) = \infty$ otherwise. Then the proof is carried out along the lines of the proof of Theorem 2.1, where the proof of an analogue of Lemma 5.1 uses parts 2 of Lemmas 3.3 and 3.4, instead of respective parts 1 of these lemmas. In addition, the proof of an analogue of (5.1), as in the argument just given, uses the convergence

$$\lim_{T\to\infty}\limsup_{n\to\infty} \mathbf{P}\Big(\sup_{t\geq T}(\breve{S}_t^n + \breve{\varepsilon}_t^n - \breve{S}_{t-\eta}^n - \breve{\varepsilon}_{t-\eta}^n) > 0\Big)^{1/b_n^2} = 0, \qquad \eta > 0. \tag{7.21}$$

We omit most of the details and only show the latter. Arguing as in (7.18),

$$\begin{aligned}
&\mathbf{P}\Big(\sup_{t\geq T}(\breve{S}_t^n + \breve{\varepsilon}_t^n - \breve{S}_{t-\eta}^n - \breve{\varepsilon}_{t-\eta}^n) > 0\Big)^{1/b_n^2}\\
&\quad\leq \sum_{k=-1}^{\infty} \mathbf{P}\Big(\sup_{s\in[0,\eta]}(|\breve{M}_{T+k\eta+s}^n - \breve{M}_{T+k\eta}^n| + |\breve{\varepsilon}_{T+k\eta+s}^n - \breve{\varepsilon}_{T+k\eta}^n|)\\
&\qquad\qquad > \frac{c_n\eta}{3}(T+k\eta) - \frac{2\eta}{3}\frac{n^{1/3}}{b_n^{2/3}}|c_n - 1|\Big)^{1/b_n^2}\\
&\qquad + \sum_{k=-1}^{\infty} \mathbf{P}\Big(\sup_{s\in[0,\eta]}\big(|\breve{M}_{T+(k+1)\eta+s}^n - \breve{M}_{T+(k+1)\eta}^n|\\
&\qquad\qquad + |\breve{\varepsilon}_{T+(k+1)\eta+s}^n - \breve{\varepsilon}_{T+(k+1)\eta}^n|\big)\\
&\qquad\qquad > \frac{c_n\eta}{3}(T+k\eta) - \frac{2\eta}{3}\frac{n^{1/3}}{b_n^{2/3}}|c_n - 1|\Big)^{1/b_n^2}\\
&\quad\leq \Bigg(\sum_{i=1}^{2}\sup_{t\in\mathbb{R}_+}\big(\mathbf{E}\exp\big((-1)^i b_n^2(\breve{M}_{t+\eta}^n - \breve{M}_t^n)\big)\big)^{1/b_n^2}\\
&\qquad + \Big(\sup_{t\in\mathbb{R}_+}\mathbf{E}\exp\Big(b_n^2\sup_{s\in[0,\eta]}|\varepsilon_{t+s}^n - \varepsilon_t^n|\Big)\Big)^{1/b_n^2}\Bigg)\\
&\qquad + \sum_{k=-1}^{\infty}\exp\Big(-\Big(\frac{c_n\eta}{6}(T+k\eta) - \frac{\eta}{3}\frac{n^{1/3}}{b_n^{2/3}}|c_n - 1|\Big)\Big).
\end{aligned} \tag{7.22}$$

Let $\breve{\mathcal{E}}_t^n(\lambda)$, $t \in \mathbb{R}_+$, $\lambda \in \mathbb{R}$, denote the stochastic exponential of $\breve{M}^n$ so that by (7.13) and (7.14),

$$\log \breve{\mathcal{E}}_t^n(\lambda) = n \log \mathbf{E} \exp\left(\frac{\lambda}{n^{1/3} b_n^{4/3}}\left(\xi_{11}^n - \frac{c_n}{n}\right)\right) \sum_{k=0}^{\lfloor (nb_n)^{2/3} t\rfloor \wedge n-1} \left(1 - \frac{Q_k^n}{n} - \frac{k}{n}\right).$$

Hence, for $t \in \mathbb{R}_+$ and n large enough,

$$\begin{aligned} \frac{1}{b_n^2} \log \mathbf{E} \exp\left(\pm b_n^2 (\breve{M}_{t+\eta}^n - \breve{M}_t^n)\right) &\le \frac{1}{2b_n^2} \log \mathbf{E} \frac{\breve{\mathcal{E}}_{t+\eta}^n(\pm 2b_n^2)}{\breve{\mathcal{E}}_t^n(\pm 2b_n^2)} \\ &\le \frac{n^{5/3}\eta}{b_n^{4/3}}\left(\log \mathbf{E} \exp\left(\pm \frac{2b_n^{2/3}}{n^{1/3}}\left(\xi_{11}^n - \frac{c_n}{n}\right)\right)\right)^+, \end{aligned}$$

so since $\log \mathbf{E} \exp(\pm 2b_n^{2/3}(\xi_{11}^n - c_n/n)/n^{1/3})$ is asymptotically equivalent to $2c_n b_n^{4/3}/n^{5/3}$ as $n \to \infty$, we conclude that

$$\limsup_{n\to\infty} \sup_{t\in\mathbb{R}_+} \left(\mathbf{E} \exp\left(\pm b_n^2 (\breve{M}_{t+\eta}^n - \breve{M}_t^n)\right)\right)^{1/b_n^2} \le e^{2\eta}. \tag{7.23}$$

Also by (2.7) and the definition of $\breve{\varepsilon}_t^n$ in (7.15),

$$\limsup_{n\to\infty} \sup_{t\in\mathbb{R}_+} \left(\mathbf{E} \exp\left(b_n^2 \sup_{s\in[0,\eta]} |\breve{\varepsilon}_{t+s}^n - \breve{\varepsilon}_t^n|\right)\right)^{1/b_n^2} \le 1. \tag{7.24}$$

Limit (7.21) follows by (7.22)–(7.24) and the convergence $(n^{1/3}/b_n^{2/3})(c_n - 1) \to \breve{\theta}$. □

Corollary 2.5 follows by the contraction principle, in particular, part 2 is proved in analogy with part 2 of Corollary 2.4. [Note that in the expression for $\breve{I}_{\breve{\theta}}^{\beta}$ the role of $K_c(u)$ and $L_c(u)$ are played by the functions $-u^3/24$ and $((u - \breve{\theta})^3 + \breve{\theta}^3)/6$, respectively, and an analogue of Lemma 3.2 holds with $2(\breve{\theta} - u)$ as u^*.]

APPENDIX

Summary of idempotent probability. This appendix relates some facts of idempotent probability theory. More detailed exposition is given in Puhalskii (2001).

Let Υ be a set. A function $\mathbf{\Pi}$ from the power set of Υ to $[0, 1]$ is called an idempotent probability if $\mathbf{\Pi}(\Gamma) = \sup_{\upsilon\in\Gamma} \mathbf{\Pi}(\{\upsilon\})$, $\Gamma \subset \Upsilon$ and $\mathbf{\Pi}(\Upsilon) = 1$. If, in addition, Υ is a metric space and the sets $\{\upsilon \in \Upsilon : \mathbf{\Pi}(\upsilon) \ge a\}$ are compact for all $a \in (0, 1]$, then $\mathbf{\Pi}$ is called a deviability. Obviously, $\mathbf{\Pi}$ is a deviability if and only if $I(\upsilon) = -\log \mathbf{\Pi}(\{\upsilon\})$ is an action functional. Below, we denote $\mathbf{\Pi}(\upsilon) = \mathbf{\Pi}(\{\upsilon\})$ and assume, unless mentioned otherwise, that $\mathbf{\Pi}$ is an idempotent probability on Υ. A property $\mathcal{P}(\upsilon)$, $\upsilon \in \Upsilon$, pertaining to the elements of Υ is said to hold $\mathbf{\Pi}$-a.e. if $\mathbf{\Pi}(\mathcal{P}(\upsilon)$ does not hold$) = 0$. A τ-algebra $\mathcal{A}$ on Υ is defined as a subset of the power set of Υ for which there exists a partitioning of Υ into disjoint sets such that every element of $\mathcal{A}$ is a union of the elements of the partitioning. We call the elements of the partitioning the atoms of $\mathcal{A}$ and denote as $[\upsilon]$ the atom containing υ. The power set of Υ is called the discrete τ-algebra. A τ-algebra $\mathcal{A}$ is called complete (or $\mathbf{\Pi}$-complete, or complete with respect to $\mathbf{\Pi}$ if idempotent probability needs to be specified) if each one-point set $\{\upsilon\}$ with $\mathbf{\Pi}(\upsilon) = 0$ is an atom of $\mathcal{A}$; the completion (or the $\mathbf{\Pi}$-completion, or the completion with respect to $\mathbf{\Pi}$ if idempotent probability needs to be specified) of a τ-algebra $\mathcal{A}$ is defined as the τ-algebra obtained by taking as the atoms the points of idempotent probability 0 and set-differences of the atoms of $\mathcal{A}$ and sets of idempotent probability 0; the completion of a τ-algebra is a complete τ-algebra. If Υ' is another set equipped with idempotent probability $\mathbf{\Pi}'$ and τ-algebra $\mathcal{A}'$, then the product idempotent probability $\mathbf{\Pi} \times \mathbf{\Pi}'$ on $\Upsilon \times \Upsilon'$ is defined by $(\mathbf{\Pi} \times \mathbf{\Pi}')(\upsilon, \upsilon') = \mathbf{\Pi}(\upsilon)\mathbf{\Pi}'(\upsilon')$ for $(\upsilon, \upsilon') \in \Upsilon \times \Upsilon'$, the product τ-algebra $\mathcal{A} \otimes \mathcal{A}'$ is defined as having the atoms $[\upsilon] \times [\upsilon']$, where $\upsilon \in \Upsilon$ and $\upsilon' \in \Upsilon'$.

A function f from a set Υ equipped with idempotent probability $\mathbf{\Pi}$ to a set Υ' is called an idempotent variable. If Υ and Υ' are equipped with τ-algebras $\mathcal{A}$ and $\mathcal{A}'$, respectively, the idempotent variable f is said to be $(\mathcal{A}/\mathcal{A}')$-measurable, or simply measurable if the τ-algebras are understood, if $f^{-1}([\upsilon']) \in \mathcal{A}$ for any $\upsilon' \in \Upsilon'$. We say that f is $\mathcal{A}$-measurable if it is measurable for the discrete τ-algebra on Υ'. The τ-algebra of Υ generated by f is defined by the atoms $\{\upsilon \in \Upsilon : f(\upsilon) = \upsilon'\}$, $\upsilon' \in \Upsilon'$. The idempotent variable f is thus $\mathcal{A}$-measurable if $\{\upsilon \in \Upsilon : f(\upsilon) = \upsilon'\} \in \mathcal{A}$ for all $\upsilon' \in \Upsilon'$. As in probability theory, we routinely omit the argument υ in the notation for an idempotent variable. The idempotent distribution of an idempotent variable f is defined as the set function $\mathbf{\Pi} \circ f^{-1}(\Gamma) = \mathbf{\Pi}(f \in \Gamma)$, $\Gamma \subset \Upsilon'$; it is also called the image of $\mathbf{\Pi}$ under f. If Υ is a metric space, $\mathbf{\Pi}$ is a deviability on Υ, and f is a continuous mapping from Υ to a metric space Υ', then $\mathbf{\Pi} \circ f^{-1}$ is a deviability on Υ'. In particular, if $\Upsilon' \subset \Upsilon$ with induced metric and $\mathbf{\Pi}(\Upsilon \setminus \Upsilon') = 0$, then the restriction $\mathbf{\Pi}|_{\Upsilon'}$ of $\mathbf{\Pi}$ to Υ' defined by $\mathbf{\Pi}|_{\Upsilon'}(\upsilon) = \mathbf{\Pi}(\upsilon)$ for $\upsilon \in \Upsilon'$ is a deviability on Υ'. In general, f is said to be Luzin if $\mathbf{\Pi} \circ f^{-1}$ is a deviability on Υ'.

Subsets A and A' of Υ are said to be independent if $\mathbf{\Pi}(A \cap A') = \mathbf{\Pi}(A)\mathbf{\Pi}(A')$; τ-algebras $\mathcal{A}$ and $\mathcal{A}'$ are said to be independent if sets A and A' are independent for any $A \in \mathcal{A}$ and $A' \in \mathcal{A}'$; Υ'-valued idempotent variables f and f' are said to be independent if $\mathbf{\Pi}(f = \upsilon', f' = \upsilon'') = \mathbf{\Pi}(f = \upsilon')\mathbf{\Pi}(f' = \upsilon'')$ for all $\upsilon', \upsilon'' \in \Upsilon'$. An idempotent variable f and a τ-algebra $\mathcal{A}$ are said to be independent (or f to be independent of $\mathcal{A}$) if the τ-algebra generated by f and $\mathcal{A}$ are independent. If f is $\mathbb{R}_+$-valued, the idempotent expectation of f is defined by $\mathbf{S}f = \sup_{\upsilon \in \Upsilon} f(\upsilon)\mathbf{\Pi}(\upsilon)$ and it is also denoted as $\mathbf{S}_{\mathbf{\Pi}} f$ if the reference idempotent probability needs to be indicated. The following analogue of the Markov inequality holds: $\mathbf{\Pi}(f \geq a) \leq \mathbf{S}f/a$, where $a > 0$. If $\mathbb{R}_+$-valued idempotent variables f and f' are independent, then $\mathbf{S}(ff') = \mathbf{S}f\,\mathbf{S}f'$. An $\mathbb{R}_+$-valued idempotent variable f is said to be maximable if $\lim_{b\to\infty}\mathbf{S}(f\mathbf{1}(f>b)) = 0$. A collection f_α of $\mathbb{R}_+$-valued idempotent variables is called uniformly maximable if $\lim_{b\to\infty}\sup_\alpha \mathbf{S}(f_\alpha\mathbf{1}(f_\alpha > b)) = 0$. The conditional idempotent expectation of an $\mathbb{R}_+$-valued idempotent variable f, given a τ-algebra $\mathcal{A}$, is defined as

$$\mathbf{S}(f|\mathcal{A})(\upsilon) = \begin{cases} \sup_{\upsilon'\in[\upsilon]} f(\upsilon')\dfrac{\mathbf{\Pi}(\upsilon')}{\mathbf{\Pi}([\upsilon])}, & \text{if } \mathbf{\Pi}([\upsilon]) > 0, \\ f'(\upsilon), & \text{if } \mathbf{\Pi}([\upsilon]) = 0, \end{cases}$$

where $f'(\upsilon)$ is an $\mathbb{R}_+$-valued function constant on the atoms of $\mathcal{A}$. Conditional idempotent expectation is thus specified $\mathbf{\Pi}$-a.e. It has many of the properties of conditional expectation, in particular, $\mathbf{S}(f|\mathcal{A})$ is $\mathcal{A}$-measurable, if f is $\mathcal{A}$-measurable, then $\mathbf{S}(f|\mathcal{A}) = f$ $\mathbf{\Pi}$-a.e., and if f and $\mathcal{A}$ are independent, then $\mathbf{S}(f|\mathcal{A}) = \mathbf{S}f$ $\mathbf{\Pi}$-a.e. [Puhalskii (2001), Lemma 1.6.21]. If for an $\mathbb{R}^d$-valued idempotent variable f, the conditional idempotent expectation $\mathbf{S}(\exp(\lambda^T f)|\mathcal{A})$ is $\mathbf{\Pi}$-a.e. constant on Υ for all $\lambda \in \mathbb{R}^d$ and is an essentially smooth function of λ, then f and $\mathcal{A}$ are independent [Puhalskii (2001), Corollary 1.11.9].

An $\mathbb{R}^d$-valued idempotent variable f on $(\Upsilon, \mathbf{\Pi})$ is said to be Gaussian with parameters (m, Σ), where $m \in \mathbb{R}^d$ and Σ is a positive semi-definite $d \times d$ matrix, if $\mathbf{S}\exp(\lambda^T f) = \exp(\lambda^T m + \lambda^T\Sigma\lambda/2)$ for all $\lambda \in \mathbb{R}^d$. Equivalently, $\mathbf{\Pi}(f = z) = \exp(-(z-m)^T\Sigma^{\oplus}(z-m)/2)$ if $z - m$ belongs to the range of Σ and $\mathbf{\Pi}(f = z) = 0$ otherwise, where $\Sigma^{\oplus}$ denotes the pseudo-inverse of Σ [Puhalskii (2001), Lemma 1.11.12].

A flow of τ-algebras, or a τ-flow, on Υ is defined as a collection $\mathbf{A} = (\mathcal{A}_t, t \in \mathbb{R}_+)$ of τ-algebras on Υ such that $\mathcal{A}_s \subset \mathcal{A}_t$ for $s \le t$; the latter condition is equivalent to the atoms of $\mathcal{A}_s$ being unions of the atoms of $\mathcal{A}_t$. A τ-flow is called complete if it consists of complete τ-algebras, the completion of a τ-flow is obtained by completing its τ-algebras; the completion of a τ-flow is a complete τ-flow. An idempotent variable $\sigma : \Upsilon \to \mathbb{R}_+$ is called an idempotent $\mathbf{A}$-stopping time, or a stopping time relative to $\mathbf{A}$, if $\{\upsilon : \sigma(\upsilon) = t\} \in \mathcal{A}_t$ for $t \in \mathbb{R}_+$. Given a τ-flow $\mathbf{A}$ and an idempotent $\mathbf{A}$-stopping time σ, we define $\mathcal{A}_\sigma$ as the τ-algebra with atoms $[\upsilon]_{\mathcal{A}_{\sigma(\upsilon)}}$. If $\Upsilon = \mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$, the canonical τ-flow is the τ-flow $\mathbf{C} = (\mathcal{C}_t, t \in \mathbb{R}_+)$ with the $\mathcal{C}_t$ having the atoms $p_t^{-1}\mathbf{x}$, $\mathbf{x} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$, where $p_t : \mathbb{C}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$ is defined by $(p_t\mathbf{x})_s = \mathbf{x}_{s\wedge t}$, $s \in \mathbb{R}_+$.

A collection $(X_t, t \in \mathbb{R}_+)$ of $\mathbb{R}^d$-valued idempotent variables on Υ is called an idempotent process. The functions $(X_t(\upsilon), t \in \mathbb{R}_+)$ for various $\upsilon \in \Upsilon$ are called trajectories (or paths) of X. An idempotent process $(X_t, t \in \mathbb{R}_+)$ is said to be $\mathbf{A}$-adapted if the X_t are $\mathcal{A}_t$-measurable for $t \in \mathbb{R}_+$. If $(X_t, t \in \mathbb{R}_+)$ is $\mathbf{A}$-adapted and real-valued with unbounded above continuous paths, then $\sigma = \inf\{t \in \mathbb{R}_+ : X_t \ge a\}$, where $a \in \mathbb{R}$, is an idempotent $\mathbf{A}$-stopping time [Puhalskii (2001), Lemma 2.2.18]. If $\Upsilon = \mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$, the canonical idempotent process is defined by $X_t(\mathbf{x}) = \mathbf{x}_t$. An $\mathbf{A}$-adapted $\mathbb{R}_+$-valued idempotent process $M =$

$(M_t,\, t \in \mathbb{R}_+)$ is said to be an **A**-exponential maxingale, or an exponential maxingale relative to **A**, if the M_t are maximable and $\mathbf{S}(M_t|\mathcal{A}_s) = M_s$ $\mathbf{\Pi}$-a.e., for $s \le t$. If, in addition, the collection $M_t,\, t \in \mathbb{R}_+$, is uniformly maximable, then M is said to be a uniformly maximable exponential maxingale. An **A**-adapted $\mathbb{R}_+$-valued idempotent process $M = (M_t,\, t \in \mathbb{R}_+)$ is called an **A**-local exponential maxingale, or a local exponential maxingale relative to **A**, if there exists a sequence τ_n of idempotent **A**-stopping times such that $\tau_n \uparrow \infty$ as $n \to \infty$ and the stopped idempotent processes $(M_{t\wedge\tau_n},\, t \in \mathbb{R}_+)$ are uniformly maximable **A**-exponential maxingales.

LEMMA A.1. *Let $M = (M_t,\, t \in \mathbb{R}_+)$ be an exponential maxingale relative to a τ-flow $\mathbf{A} = (\mathcal{A}_t,\, t \in \mathbb{R}_+)$ and $\sigma_t,\, t \in \mathbb{R}_+$, be a collection of bounded idempotent* **A***-stopping times such that $\sigma_s \le \sigma_t$ for $s \le t$. Then the idempotent process $(M_{\sigma_t},\, t \in \mathbb{R}_+)$ is an exponential maxingale relative to the τ-flow $(\mathcal{A}_{\sigma_t},\, t \in \mathbb{R}_+)$.*

PROOF. By Corollary 2.3.10 in Puhalskii (2001), $\mathbf{S}(M_{\sigma_t}|\mathcal{A}_{\sigma_s}) = M_{\sigma_s}$ $\mathbf{\Pi}$-a.e. for $s \le t$. Each M_{σ_t} is maximable since by the boundedness of σ_t, there exists $T \ge \sigma_t$, so $M_{\sigma_t} = \mathbf{S}(M_T|\mathcal{A}_{\sigma_t})$, which is maximable by maximability of M_T, inclusion $\mathcal{A}_{\sigma_t} \subset \mathcal{A}_T$ and Lemma 1.6.21 in Puhalskii (2001). □

Given an $\mathbb{R}$-valued function $G = (G_t(\lambda; \mathbf{x}),\, t \in \mathbb{R}_+,\, \mathbf{x} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}),\, \lambda \in \mathbb{R})$, where $G_t(\lambda; \mathbf{x})$ is $\mathcal{C}_t$-measurable in $\mathbf{x}$, we say that a deviability $\mathbf{\Pi}$ on $\mathbb{C}(\mathbb{R}_+, \mathbb{R})$ solves the maxingale problem (x, G), where $x \in \mathbb{R}$, if $X_0 = x$ $\mathbf{\Pi}$-a.e. and $(\exp(\lambda X_t - G_t(\lambda; X)),\, t \in \mathbb{R}_+)$ is a **C**-local exponential maxingale under $\mathbf{\Pi}$, where $X = (X_t,\, t \in \mathbb{R}_+)$ is the canonical idempotent process on $\mathbb{C}(\mathbb{R}_+, \mathbb{R})$. We have the following lemma.

LEMMA A.2. *Let $\mathbf{\Pi}$ solve the maxingale problem (x, G). If the function $(G_t(\lambda; \mathbf{x}),\, t \in \mathbb{R}_+,\, \mathbf{x} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}))$ is bounded in $(t, \mathbf{x})$ for all $\lambda \in \mathbb{R}$, then the process $(\exp(\lambda X_t - G_t(\lambda; X)),\, t \in \mathbb{R}_+)$ is a* **C***-uniformly maximable exponential maxingale under $\mathbf{\Pi}$.*

PROOF. Let $M_t(\lambda) = \exp(\lambda X_t - G_t(\lambda; X))$. By Lemma 2.3.13(3) in Puhalskii (2001) it is enough to prove that the collection $(M_t(\lambda),\, t \in \mathbb{R}_+)$ is uniformly maximable. The definition of a local exponential maxingale and Lemma 1.6.22 in Puhalskii (2001) imply that $\mathbf{S}_{\mathbf{\Pi}} M_t(2\lambda) \le 1$. Therefore, denoting by b an upper bound for $(\exp(-2G_t(\lambda; \mathbf{x})),\, t \in \mathbb{R}_+,\, \mathbf{x} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}))$ and $(\exp(G_t(2\lambda; \mathbf{x})),\, t \in \mathbb{R}_+,\, \mathbf{x} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}))$, we have

$$\mathbf{S}_{\mathbf{\Pi}} M_t(\lambda)^2 = \mathbf{S}_{\mathbf{\Pi}}\big(M_t(2\lambda) \exp\big(G_t(2\lambda; X)\big) \exp\big(-2G_t(\lambda; X)\big)\big) \le b^2.$$

The uniform maximability now follows by Corollary 1.4.15 in Puhalskii (2001). □

LEMMA A.3. *Let* $(M_t,\, t \in \mathbb{R}_+)$ *and* $(M'_t,\, t \in \mathbb{R}_+)$ *be exponential maxingales on* $(\Upsilon, \mathbf{\Pi})$ *and* $(\Upsilon', \mathbf{\Pi}')$, *respectively, relative to the respective τ-flows* $(\mathcal{A}_t,\, t \in \mathbb{R}_+)$ *and* $(\mathcal{A}'_t,\, t \in \mathbb{R}_+)$. *Then* $(M_t M'_t,\, t \in \mathbb{R}_+)$ *is an exponential maxingale on* $(\Upsilon \times \Upsilon', \mathbf{\Pi} \times \mathbf{\Pi}')$ *relative to the τ-flow* $(\mathcal{A}_t \otimes \mathcal{A}'_t,\, t \in \mathbb{R}_+)$.

PROOF. By Puhalskii [(2001), Lemma 1.6.28], $\mathbf{S}_{\mathbf{\Pi}\times\mathbf{\Pi}'}(M_t M'_t|\mathcal{A}_s \otimes \mathcal{A}'_s) = \mathbf{S}_{\mathbf{\Pi}}(M_t|\mathcal{A}_s)\mathbf{S}_{\mathbf{\Pi}'}(M'_t|\mathcal{A}'_s)$ $\mathbf{\Pi} \times \mathbf{\Pi}'$-a.e. for $s \le t$. Maximability of $(M_t M'_t,\, t \in \mathbb{R}_+)$ under $\mathbf{\Pi} \times \mathbf{\Pi}'$ is obvious. □

Poisson idempotent probability (or Poisson deviability) is a deviability on $\mathbb{C}(\mathbb{R}_+, \mathbb{R})$ defined by

$$\mathbf{\Pi}^N(\mathbf{x}) = \begin{cases} \exp\Big(-\displaystyle\int_0^\infty \pi(\dot{\mathbf{x}}_t)\,dt\Big), & \text{if } \mathbf{x} \text{ is absolutely continuous} \\ & \text{and nondecreasing, and } \mathbf{x}_0 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

A Poisson idempotent process on $(\Upsilon, \mathbf{\Pi})$ is defined as an idempotent process with idempotent distribution $\mathbf{\Pi}^N$. Thus, a Poisson idempotent process has absolutely continuous nondecreasing trajectories $\mathbf{\Pi}$-a.e. The definition implies that the canonical idempotent process on $\mathbb{C}(\mathbb{R}_+, \mathbb{R})$ is Poisson under $\mathbf{\Pi}^N$. If N is a Poisson idempotent process on $(\Upsilon, \mathbf{\Pi})$, then the idempotent process $M^N(\lambda) = (M^N_t(\lambda),\, t \in \mathbb{R}_+)$ defined by $M^N_t(\lambda) = \exp(\lambda N_t - (e^\lambda - 1)t)$ is an exponential maxingale relative to the τ-flow $(\mathcal{A}^N_t,\, t \in \mathbb{R}_+)$, where the $\mathcal{A}^N_t$ are the τ-algebras generated by the N_s, $s \le t$ [Puhalskii (2001), Theorem 2.4.16]. We say that a continuous-path idempotent process N is Poisson relative to a τ-flow $\mathbf{A}$ if $N_0 = 0$ and the idempotent process $M^N(\lambda)$ is an $\mathbf{A}$-exponential maxingale for all $\lambda \in \mathbb{R}$. If N is idempotent Poisson relative to $\mathbf{A}$, then it is idempotent Poisson [Puhalskii (2001), Corollary 2.4.19].

Wiener idempotent probability (or Wiener deviability) is a deviability on $\mathbb{C}(\mathbb{R}_+, \mathbb{R})$ defined by

$$\mathbf{\Pi}^W(\mathbf{x}) = \begin{cases} \exp\Big(-\frac{1}{2}\displaystyle\int_0^\infty \dot{\mathbf{x}}_t^2\,dt\Big), & \text{if } \mathbf{x} \text{ is absolutely continuous and } \mathbf{x}_0 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

A Wiener idempotent process on $(\Upsilon, \mathbf{\Pi})$ is defined as an idempotent process with idempotent distribution $\mathbf{\Pi}^W$. Thus, a Wiener idempotent process has $\mathbf{\Pi}$-a.e. absolutely continuous paths. The definition implies that the canonical idempotent process on $\mathbb{C}(\mathbb{R}_+, \mathbb{R})$ is Wiener under $\mathbf{\Pi}^W$.

Let $W = (W_t,\, t \in \mathbb{R}_+)$ be a Wiener idempotent process on $(\Upsilon, \mathbf{\Pi})$. Then the idempotent process $(\exp(\lambda W_t - \lambda^2 t/2),\, t \in \mathbb{R}_+)$ is an exponential maxingale relative to the flow $\mathbf{A}^W = (\mathcal{A}^W_t,\, t \in \mathbb{R}_+)$, where the $\mathcal{A}^W_t$ are the τ-algebras generated by W_s, $s \le t$ [Puhalskii (2001), Theorem 2.4.2]. We say that a continuous-path idempotent process W is Wiener relative to a τ-flow $\mathbf{A}$ if $W_0 = 0$

and the idempotent process $(\exp(\lambda W_t - \lambda^2 t/2),\ t \in \mathbb{R}_+)$ is an **A**-exponential maxingale for all $\lambda \in \mathbb{R}$. If W is idempotent Wiener relative to **A**, then it is idempotent Wiener [Puhalskii (2001), Corollary 2.4.6]. In particular, $W_t - W_s$, for $t \geq s$, is independent of $\mathcal{A}_s$ by the fact that $\mathbf{S}(\exp(\lambda(W_t - W_s))|\mathcal{A}_s) = \exp(\lambda^2(t-s)/2)$, which is a smooth function of λ.

Given a bounded $\mathbb{R}$-valued idempotent process σ_t, $t \in \mathbb{R}_+$, we define the idempotent Ito integral $(\sigma \diamond W)_t$ by

$$(\sigma \diamond W)_t(\upsilon) = \begin{cases} \displaystyle\int_0^t \sigma_s(\upsilon)\dot{W}_s(\upsilon)\,ds, & \text{if } \mathbf{\Pi}(\upsilon) > 0, \\ Y(\upsilon), & \text{otherwise,} \end{cases}$$

where $Y(\upsilon)$ is an $\mathbb{R}$-valued idempotent variable and $\dot{W}_s(\upsilon)$ denotes the Radon–Nikodym derivative in s of the Wiener idempotent trajectory. The integral is thus specified uniquely $\mathbf{\Pi}$-a.e. The idempotent process $((\sigma \diamond W)_t,\ t \in \mathbb{R}_+)$ has $\mathbf{\Pi}$-a.e. continuous paths. If $(W_t,\ t \in \mathbb{R}_+)$ and $(\sigma_t,\ t \in \mathbb{R}_+)$ are adapted to a complete τ-flow **A**, then $((\sigma \diamond W)_t,\ t \in \mathbb{R}_+)$ is **A**-adapted. For clarity, we further use $\int_0^t \sigma_s \dot{W}_s\,ds$ for $(\sigma \diamond W)_t$. In the next lemma, $\int_s^t \sigma_p \dot{W}_p\,dp = \int_0^t \sigma_p \mathbf{1}(r \in [s,t])\dot{W}_p\,dp$.

LEMMA A.4. *Let σ_s, $s \in \mathbb{R}_+$ be an $\mathbb{R}$-valued bounded Lebesgue-measurable function and $W = (W_t,\ t \in \mathbb{R}_+)$ be a Wiener idempotent process on $(\Upsilon, \mathbf{\Pi})$ relative to a complete τ-flow* **A**. *Then the idempotent process $M = (M_t,\ t \in \mathbb{R}_+)$, where $M_t = \exp(\lambda \int_0^t \sigma_s \dot{W}_s\,ds - \lambda^2 \int_0^t \sigma_s^2\,ds/2)$, is an* **A**-*exponential maxingale. In particular, $\int_s^t \sigma_p \dot{W}_p\,dp$ is independent of $\mathcal{A}_s$ for $s \leq t$.*

PROOF. The idempotent process M is **A**-adapted by M_t being constant on the atoms of $\mathcal{A}_t$ for $t \in \mathbb{R}_+$, see Puhalskii [(2001), Lemma 2.2.17]. If the function σ_s, $s \in \mathbb{R}_+$, is piecewise constant, the maxingale property follows by the properties of conditional idempotent expectations in a standard manner. A limit argument shows that this property carries over to continuous σ_s, $s \in \mathbb{R}_+$. The case of a Lebesgue measurable σ_s, $s \in \mathbb{R}_+$, follows via Luzin's theorem. Maximability of the M_t follows by Lemma A.2. Finally, $\int_s^t \sigma_p \dot{W}_p\,dp$ is independent of $\mathcal{A}_s$, for $s \leq t$, since by the maxingale property $\mathbf{S}(\exp(\lambda \int_s^t \sigma_p \dot{W}_p\,dp)|\mathcal{A}_s) = \exp((\lambda^2/2)\int_s^t \sigma_p^2\,dp)$, where the latter is a smooth function of λ. □

Let $\sigma_t(x)$, $x \in \mathbb{R}$, $t \in \mathbb{R}_+$, and $b_t(x)$, $x \in \mathbb{R}$, $t \in \mathbb{R}_+$, be real-valued functions, which are continuous in x and Lebesgue-measurable in t. Let W be a Wiener idempotent process on an idempotent probability space $(\Upsilon, \mathbf{\Pi})$ relative to a complete τ-flow **A** and let $\overline{\mathcal{C}}_t^W$, for $t \in \mathbb{R}_+$, denote the completion of $\mathcal{C}_t$ with respect to the Wiener deviability on $\mathbb{C}(\mathbb{R}_+, \mathbb{R})$. We say that, given $x \in \mathbb{R}$, an idempotent process X on $(\Upsilon, \mathbf{\Pi})$ is a strong solution to the Itô idempotent equation

$$X_t = x + \int_0^t b_s(X_s)\,ds + \int_0^t \sigma_s(X_s)\dot{W}_s\,ds, \qquad t \in \mathbb{R}_+, \tag{A.1}$$

where integrals are understood as Lebesgue integrals, if equality (A.1) holds $\mathbf{\Pi}$-a.e. and there exists a function $J : \mathbb{C}(\mathbb{R}_+, \mathbb{R}) \to \mathbb{C}(\mathbb{R}_+, \mathbb{R})$, which is $(\overline{\mathcal{C}}_t^W/\mathcal{C}_t)$-measurable for every $t \in \mathbb{R}_+$, such that $X = J(W)$ $\mathbf{\Pi}$-a.e. As a consequence, X is $\mathbf{A}$-adapted. A strong solution is called Luzin if the function J is continuous in restriction to the sets $\{\mathbf{x} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}) : \mathbf{\Pi}^W(\mathbf{x}) \geq a\}$ for $a \in (0, 1]$. We say that there exists a unique strong solution (resp. Luzin strong solution) if any strong solution (resp. Luzin strong solution) can be written as $X = J(W)$ $\mathbf{\Pi}$-a.e. for the same function J. Let us assume that $\sigma_t(x)$ and $b_t(x)$ are locally Lipshitz-continuous in x, that is, for every $a > 0$, there exists an $\mathbb{R}_+$-valued Lebesgue-measurable in t function k_t^a, $t \in \mathbb{R}_+$, with $\int_0^t k_s^a\, ds < \infty$ for $t \in \mathbb{R}_+$ such that $|b_t(x) - b_t(y)| \leq k_t^a|x - y|$ and $|\sigma_t(x) - \sigma_t(y)|^2 \leq k_t^a|x - y|^2$ if $|x| \leq a$ and $|y| \leq a$, and satisfy the linear-growth condition that there exists an $\mathbb{R}_+$-valued Lebesgue-measurable function l_t, $t \in \mathbb{R}_+$, with $\int_0^t l_s\, ds < \infty$ for $t \in \mathbb{R}_+$ such that $|b_t(x)| \leq l_t(1 + |x|)$ and $\sigma_t(x)^2 \leq l_t(1 + |x|^2)$ for $x \in \mathbb{R}$. Then (A.1) has a unique strong solution, which is also a Luzin strong solution [Puhalskii (2001), Theorems 2.6.21, 2.6.22 and 2.6.26].

Let Υ be a metric space. A net $\mathbf{\Pi}^\psi$, $\psi \in \Psi$, where Ψ is a directed set, of idempotent probabilities on Υ is said to converge weakly to idempotent probability $\mathbf{\Pi}$ on Υ if $\lim_{\psi \in \Psi} \mathbf{S}_{\mathbf{\Pi}^\psi} f = \mathbf{S}_{\mathbf{\Pi}} f$ for every nonnegative bounded and continuous function f on Υ; equivalently, see Puhalskii [(2001), Theorem 1.9.2], $\limsup_{\psi \in \Psi} \mathbf{\Pi}^\psi(F) \leq \mathbf{\Pi}(F)$ for all closed sets $F \subset \Upsilon$ and $\liminf_{\psi \in \Psi} \mathbf{\Pi}^\psi(G) \geq \mathbf{\Pi}(G)$ for all open sets $G \subset \Upsilon$. A net of idempotent variables with values in the same metric space is said to converge in idempotent distribution if their idempotent distributions weakly converge. One has a continuous mapping theorem for convergence in idempotent distribution: if a net X^ψ, $\psi \in \Psi$, of idempotent variables with values in Υ converges in idempotent distribution to an idempotent variable X with values in Υ and f is a continuous function from Υ to a metric space Υ', then the net $f(X^\psi)$, $\psi \in \Psi$, converges in idempotent distribution to $f(X)$. A net $\mathbf{\Pi}^\psi$, $\psi \in \Psi$, of deviabilities on Υ is said to be tight if $\inf_{K \in \mathcal{K}} \limsup_{\psi \in \Psi} \mathbf{\Pi}^\psi(\Upsilon \setminus K) = 0$, where $\mathcal{K}$ denotes the collection of compact subsets of Υ. A tight net of deviabilities contains a subnet that converges weakly to a deviability, see Puhalskii [(2001), Theorem 1.9.27] (if $\mathbf{\Pi}^\psi$ is a sequence, then it contains a weakly convergent subsequence).

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