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SOME PRODUCT BESSEL DENSITY DISTRIBUTIONS

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Abstract. Three new Bessel function distributions are introduced by taking products of a Bessel function pdf of the first kind and a Bessel function pdf of the second kind. Various particular cases and expressions for moments are derived for each distribution. The work is motivated by recent developments in the electrical and electronic engineering literature.

1. INTRODUCTION

Univariate Bessel function distributions are due to McKay (1932). There are two kinds of univariate Bessel function distributions. Bessel function distribution of the first kind has the pdf given by

$$f(x) = \frac{\left|1-c^2\right|^{m+1/2} x^m}{\sqrt{\pi} 2^m b^{m+1} \Gamma(m+1/2)} \exp\left(-\frac{cx}{b}\right) I_m\left(\frac{x}{b}\right) \tag{1}$$

for $x > 0$, $b > 0$, $c > 1$ and $m > 1$, where

$$I_m(x) = \frac{x^m}{\sqrt{\pi} 2^m \Gamma(m+1/2)} \int_{-1}^{1} \left(1-t^2\right)^{m-1/2} \exp(\pm xt)\, dt$$

is the modified Bessel function of the first kind. Bessel function distribution of the second kind has the pdf given by

$$f(x) = \frac{\left|1-c^2\right|^{m+1/2} \mid x \mid^m}{\sqrt{\pi} 2^m b^{m+1} \Gamma(m+1/2)} \exp\left(-\frac{cx}{b}\right) K_m\left(\left|\frac{x}{b}\right|\right) \tag{2}$$

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for $-\infty < x < \infty$, $b > 0$, $\mid c \mid < 1$ and $m > 1$, where

$$K_m(x) = \frac{\sqrt{\pi} x^m}{2^m \Gamma(m+1/2)} \int_1^\infty \left(t^2 - 1\right)^{m-1/2} \exp(-xt)\, dt$$

is the modified Bessel function of the second kind. The properties of these distributions including estimation issues and extensions have been the subject of many papers. Just to mention a few, see Pearson et al. (1929, 1932), Bhattacharyya (1942), Sastry (1948), Laha (1954), McNolty (1967), Gupta (1976, 1978), Ong and Lee (1979) and McLeish (1982). See also the books Johnson et al. (1994) and Kotz et al. (2001).

Recently, there has been much activity in the electrical and electronic engineering literature that requires the study of further generalizations of (1) and (2). Here, we discuss just two of the published work to motivate the need for further generalizations.

The first example is based on the most recent work by Eltoft (2006) with respect to modeling of amplitude statistics of ultrasonic images. Consider the complex envelope of a signal $S = X + jY$ with X and Y components that are uncorrelated and distributed with the common pdf

$$f(x) = \frac{\alpha \exp\{p(x)\}}{\pi q(x)} K_1(\delta \alpha q(x)), \tag{3}$$

where

$$p(x) = \delta \sqrt{\alpha^2 - \beta^2} + \beta x$$

and

$$p(x) = \delta^{-1} \sqrt{x^2 + \delta^2}.$$

The pdf in (3) is the normal inverse Gaussian pdf introduced by Barndorff–Nielsen (1997). By the known properties of this distribution, one can express X and Y as compound random variables of the form

$$X = \beta_x Z + \sqrt{Z} N_x$$

and

$$Y = \beta_y Z + \sqrt{Z} N_y,$$

where (N_x, N_y) has the bivariate normal distribution $N(\mathbf{0}, \mathbf{I}_2)$ and Z has the standard inverse gaussian distribution. Now, let $R = \sqrt{X^2 + Y^2}$ denote the amplitude of the envelope. It is easy to see that the conditional distribution of $R \mid Z$ has the well–known Rice distribution with the pdf

$$f_{R|Z}(r \mid z) = \frac{r}{z} \exp\left(-\frac{1}{2z}\left(r^2 + \beta^2 z^2\right)\right) I_0(\beta r),$$

where $\beta = \sqrt{\beta_x^2 + \beta_y^2}$. The pdf of the amplitude, R, can be obtained by calculating

$$f_R(r) = \int f_{R|Z}(r \mid z) f_Z(z) dz.$$

The exact form of $f_R(\cdot)$ is complicated. However, as $r \to \infty$, one can see that

$$f_R(r) \sim \sqrt{\frac{2}{\pi}} \alpha^{3/2} \exp(\delta\gamma) \, r^{-1/2} K_{3/2}(\alpha r) \, I_0(\beta r), \tag{4}$$

where $\gamma = \sqrt{\alpha^2 - \beta^2}$. It is evident that the pdf (4) takes the form of a product of (1) and (2).

The second example concerns a published work by Corona et al. (2004) with respect to modeling of the electromagnetic field in reverberating chambers. The variable of interest is the scattered field received at a linearly polarized receiving antenna. Under suitable assumptions, it can be shown that the pdf of the square magnitude of the total linear field, denoted h, is given by

$$f(h) = \frac{1}{2} I_0\left(\frac{\nu\sqrt{h}}{\alpha}\right) \int_0^\infty \Psi(s) J_0\left(s\sqrt{h}\right) s ds, \tag{5}$$

where $J_0(\cdot)$ denotes the modified Bessel function of the first kind of zeroth–order and $\Psi(\cdot)$ is a function that depends on the mean number of elementary field scattering contributions. If the mean number of elementary field scattering contributions tends to infinity, which is the case most expected in a reverberating chamber, then (5) can be reduced to

$$\begin{aligned} f(h) = {} & \frac{2\alpha^{(\alpha+1)/2} h^{(\alpha-1)/2}}{\Gamma(\alpha)\eta^{\alpha+1}} \left(1 + \frac{\nu^2}{4\alpha}\right)^{(1-\alpha)/2} \\ & I_0\left(\frac{\nu\sqrt{h}}{\eta}\right) K_{\alpha-1}\left(\frac{2\sqrt{\alpha h}}{\eta}\sqrt{1 + \frac{\nu^2}{4\alpha}}\right). \end{aligned} \tag{6}$$

It is evident again that the pdf (6) takes the form of a product of (1) and (2). See Bisceglie et al. (1999) for an earlier work of the same kind that gives rise to a distribution of the form (6).

In the light of the above examples, it is of interest to study the distributions that result by taking the product of the densities (1) and (2). In this paper, we introduce three new Bessel function distributions with their pdf taken to be: the product of two densities of the form (1); product of two densities of the form (2); and, product of a density of the form (1) and another density of the form (2). For each new distribution, we derive various expressions for its particular forms and its moments.

2. Product Bessel Distribution of the First Kind

Here, we introduce a new Bessel function distribution with its pdf taken to be the product of two densities of the form (1), i.e.

$$f(x) = C x^{m+n} \exp(-px) I_m\left(\frac{x}{b}\right) I_n\left(\frac{x}{\beta}\right) \tag{7}$$

for $x > 0$, $b > 0$, $\beta > 0$, $p > 1/b + 1/\beta$, $m > 1$ and $n > 1$, where C denotes the normalizing constant. Application of equation (2.15.20.2) in Prudnikov et al. (1986, volume 2) shows that one can determine C as

$$\begin{aligned} \frac{1}{C} = & \frac{\Gamma(2m+2n+1)(2b)^{-m}(2\beta)^{-n}}{p^{2m+2n+1}\Gamma(m+1)\Gamma(n+1)} \\ & F_4\left(m+n+\frac{1}{2}, m+n+1; m+1, n+1; \frac{1}{p^2b^2}, \frac{1}{p^2\beta^2}\right), \end{aligned} \tag{8}$$

where F_4 denotes the Appell function of the fourth kind defined by

$$F_4(a, b; c, d; x, y) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(a)_{k+l}(b)_{k+l} x^k y^l}{(c)_k (d)_l k! l!}.$$

Using special properties of the Appell function, one can obtain simpler expressions for (8): if $b = \beta$ then (8) can be reduced to

$$\begin{aligned} \frac{1}{C} = & \frac{\Gamma(2m+2n+1)(2b)^{-(m+n)}}{p^{2m+2n+1}\Gamma(m+1)\Gamma(n+1)} \\ & {}_3F_2\left(\frac{m+n+1}{2}, \frac{m+n}{2}+1, m+n+\frac{1}{2}; m+1, n+1; \frac{4}{b^2p^2}\right) \end{aligned}$$

and if both $b = \beta$ and $m = n$ then

$$\frac{1}{C} = \frac{2^{4m}\Gamma(m+1/2)\Gamma(2m+1/2)}{\pi b^{2m} p^{4m+1}\Gamma(m+1)} \, {}_2F_1\left(m+\frac{1}{2}, 2m+\frac{1}{2}; m+1; \frac{4}{b^2p^2}\right),$$

where ${}_3F_2$ and ${}_2F_1$ are the hypergeometric functions defined by

$${}_3F_2(a, b, c; d, e; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k (c)_k}{(d)_k (e)_k} \frac{x^k}{k!}$$

and

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},$$

respectively, where $(f)_k = f(f+1)\cdots(f+k-1)$ denotes the ascending factorial.

When m and n take half-integer values one can reduce (7) to elementary forms. Note that

$$I_{3/2}(x) = \sqrt{\frac{2}{\pi}}\frac{x\cosh(x)-\sinh(x)}{x^{3/2}},$$

$$I_{5/2}(x) = \sqrt{\frac{2}{\pi}}\frac{\left(x^2+3\right)\sinh(x)-3x\cosh(x)}{x^{5/2}},$$

$$I_{7/2}(x) = \sqrt{\frac{2}{\pi}}\frac{x\left(x^2+15\right)\cosh(x)-3\left(2x^2+5\right)\sinh(x)}{x^{7/2}}$$

and

$$I_{9/2}(x) = \sqrt{\frac{2}{\pi}}\frac{\left(x^4+45x^2+105\right)\sinh(x)-5x\left(2x^2+21\right)\cosh(x)}{x^{9/2}}.$$

More generally, if $\nu - 1/2 \geq 1$ is an integer then

$$\begin{aligned} I_\nu(x) &= \sqrt{2}\sqrt{x\pi}\exp\left\{\frac{\pi i}{2}\left(\frac{1}{2}-\nu\right)\right\}\left[\sinh\left(\frac{\pi x}{2}\left(\frac{1}{2}-\nu\right)-x\right)\right. \\ &\quad \times \sum_{k=0}^{[(2|\nu|-1)/4]} \frac{(\mid\nu\mid+2k-1/2)!}{(2k)!\,(\mid\nu\mid-2k-1/2)!(2x)^{2k}} \\ &\quad \left. +\cosh\left(\frac{\pi x}{2}\left(\frac{1}{2}-\nu\right)-x\right)\sum_{k=0}^{[(2|\nu|-3)/4]}\frac{(\mid\nu\mid+2k+1/2)!(2x)^{-2k-1}}{(2k+1)!\,(\mid\nu\mid-2k-3/2)!}\right]. \end{aligned}$$

Thus, several particular forms of (7) can be obtained for half-integer values of m and n. For example, if $m = 3/2$ and $n = 3/2$ then (7) reduces to

$$\begin{aligned} f(x) &= \frac{2Cb^{3/2}\beta^{3/2}\exp(-px)}{\pi}\left[\left(\frac{x}{b}\right)\cosh\left(\frac{x}{b}\right)-\sinh\left(\frac{x}{b}\right)\right] \\ &\quad \left[\left(\frac{x}{\beta}\right)\cosh\left(\frac{x}{\beta}\right)-\sinh\left(\frac{x}{\beta}\right)\right]. \end{aligned}$$

If $m = 3/2$ and $n = 5/2$ then (7) reduces to

$$\begin{aligned} f(x) &= \frac{2Cb^{3/2}\beta^{5/2}\exp(-px)}{\pi} \\ &\quad \left[\left(\frac{x}{b}\right)\cosh\left(\frac{x}{b}\right)-\sinh\left(\frac{x}{b}\right)\right] \\ &\quad \times\left[\left\{\left(\frac{x}{\beta}\right)^2+3\right\}\sinh\left(\frac{x}{\beta}\right)-3\frac{x}{\beta}\cosh\left(\frac{x}{\beta}\right)\right]. \end{aligned}$$

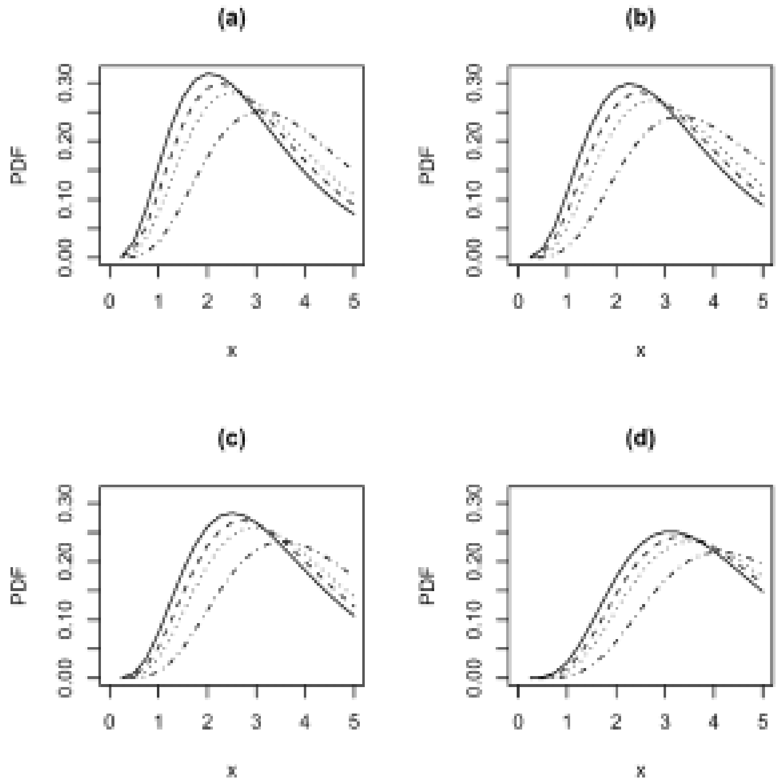

Fig. 1. Plots of the pdf (7) for $b = 1$, $\beta = 1/2$ and (a) $m = 1.1$; (b) $m = 1.3$; (c) $m = 1.5$; and, (d) $m = 2$. The four curves in each plot from the left to the right correspond to $n = 1.1, 1.3, 1.5, 2$.

Illustrates possible shapes of the pdf (7) for selected values of m and n. The four curves in each plot correspond to selected values of n. Note that the shapes are unimodal and that the densities appear to shrink with increasing values of both m and n.

If X is a random variable with pdf (7) then its kth moment can be expressed as

$$E\left(X^k\right) = C\int_0^\infty x^{m+n+k}\exp\left(-px\right) I_m\left(\frac{x}{b}\right) I_n\left(\frac{x}{\beta}\right) dx.$$

By application of equation (2.15.20.2) in Prudnikov et al. (1986, volume 2), the above can be calculated as

$$E\left(X^k\right) = \frac{C\Gamma(2m+2n+k+1)(2b)^{-m}(2\beta)^{-n}}{p^{2m+2n+k+1}\Gamma(m+1)\Gamma(n+1)} \times F_4\left(m+n+\frac{k+1}{2}, m+n+1+\frac{k}{2}; m+1, n+1; \frac{1}{p^2b^2}, \frac{1}{p^2\beta^2}\right). \quad (9)$$

If $b = \beta$ then, using special properties of the Appell function, one can reduce (9) to the simpler form:

$$
\begin{aligned}
E\left(X^k\right) &= \frac{C\Gamma(2m+2n+k+1)(2b)^{-(m+n)}}{p^{2m+2n+k+1}\Gamma(m+1)\Gamma(n+1)} \\
&\quad {}_4F_3\left(\frac{m+n+1}{2}, \frac{m+n}{2}+1, m+n+\frac{k+1}{2}, \right. \\
&\quad \left. m+n+1+\frac{k}{2}; m+n+1, m+1, n+1; \frac{4}{b^2p^2}\right),
\end{aligned}
\tag{10}
$$

where ${}_4F_3$ is the hypergeometric function defined by

$$
{}_4F_3\left(a, b, c, d; e, f, g; x\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k (c)_k (d)_k}{(e)_k (f)_k (g)_k} \frac{x^k}{k!}.
$$

Using special properties of this hypergeometric function, one can reduce (10) to simpler forms when m and n take integer or half-integer values. If either both m and n are half-integers or m is an integer and n is a half-integer or m is a half-integer and n is an integer then (10) can be reduced to an elementary form. On the other hand, if both m and n are integers then one can express (10) in terms of the complete elliptical integral of the first kind and the complete elliptical integral of the second kind defined by

$$
\mathrm{EllipticK}(a) = \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-a^2x^2}}
$$

and

$$
\mathrm{EllipticE}(a) = \int_0^1 \frac{\sqrt{1-a^2x^2}}{\sqrt{1-x^2}} dx,
$$

respectively. For instance, if $m = 3/2$ and $n = 3/2$ then the first four moments are

$$
\begin{aligned}
E\Big(X\Big) &= 32C\Big\{35 - 14x + 3x^2\Big\}\Big/\Big\{b^3p^8\pi\Big(-1+x\Big)^4\Big\}, \\
E\Big(X^2\Big) &= 128C\Big\{-70 + 35x - 16x^2 + 3x^3\Big\}\Big/\Big\{b^3p^9\pi\Big(-1+x\Big)^5\Big\}, \\
E\Big(X^3\Big) &= 384C\Big\{210 - 105x + 81x^2 + 5x^4 - 31x^3\Big\}\Big/\Big\{b^3p^{10}\pi\Big(-1+x\Big)^6\Big\}, \\
E\Big(X^4\Big) &= 768C\Big\{-1050 + 420x - 567x^2 - 107x^4 + 329x^3 + 15x^5\Big\} \\
&\qquad \Big/\Big\{b^3p^{11}\pi\Big(-1+x\Big)^7\Big\},
\end{aligned}
$$

where the normalizing constant $C = b^3 p^7 \pi (x-1)^3 / \{32(x-5)\}$ and $x = 4/(b^2p^2)$. If $m = 3/2$ and $n = 2$ then the first four moments are

$$\begin{aligned}
E\left(X\right) &= 15\sqrt{2}C\Big\{4\cos(y)\sqrt{x}\sqrt{1-x} + 79\cos(y)\,x^{3/2}\sqrt{1-x} \\
&\quad -34\cos(y)\,x^{5/2}\sqrt{1-x} + 7\cos(y)\,x^{7/2}\sqrt{1-x} - 34\sin(y)\,x^3 \\
&\quad +72\sin(y)\,x + 75\sin(y)\,x^2 + 7\sin(y)\,x^4 \\
&\quad -8\sin(y)\Big\}\Big/\Big\{x^{3/2}b^{7/2}p^9\sqrt{\pi}\,(1-x)^{9/2}\Big\},
\end{aligned}$$

$$\begin{aligned}
E\left(X^2\right) &= -15/\sqrt{2}C\Big\{63\cos(y)\,x^{9/2}\sqrt{1-x} - 352\cos(y)\,x^{7/2}\sqrt{1-x} \\
&\quad +831\cos(y)\,x^{5/2}\sqrt{1-x} \\
&\quad -1502\cos(y)\,x^{3/2}\sqrt{1-x} - 48\cos(y)\sqrt{x}\sqrt{1-x} \\
&\quad +63\sin(y)\,x^5 - 359\sin(y)\,x^4 \\
&\quad +853\sin(y)\,x^3 - 1569\sin(y)\,x^2 - 1100\sin(y)\,x + 96\sin(y)\Big\} \\
&\quad \Big/\Big\{x^{3/2}b^{7/2}p^{10}\sqrt{\pi}\,(1-x)^{11/2}\Big\},
\end{aligned}$$

$$\begin{aligned}
E\left(X^3\right) &= (315\sqrt{2}C/4)\Big\{1492\cos(y)\,x^{3/2}\sqrt{1-x} - 875\cos(y)\,x^{5/2}\sqrt{1-x} \\
&\quad +587\cos(y)\,x^{7/2}\sqrt{1-x} - 213\cos(y)\,x^{9/2}\sqrt{1-x} \\
&\quad +33\cos(y)\,x^{11/2}\sqrt{1-x} \\
&\quad +32\cos(y)\sqrt{x}\sqrt{1-x} - 945\sin(y)\,x^3 + 619\sin(y)\,x^4 - 219\sin(y)\,x^5 \\
&\quad +33\sin(y)\,x^6 + 1784\sin(y)\,x^2 + 904\sin(y)\,x - 64\sin(y)\Big\} \\
&\quad \Big/\Big\{x^{3/2}b^{7/2}p^{11}\sqrt{\pi}\,(1-x)^{13/2}\Big\},
\end{aligned}$$

$$\begin{aligned}
E\left(X^4\right) &= -(315\sqrt{2}C/8)\Big\{-33920\cos(y)\,x^{3/2}\sqrt{1-x} \\
&\quad 17982\cos(y)\,x^{5/2}\sqrt{1-x} \\
&\quad -18415\cos(y)\,x^{7/2}\sqrt{1-x} + 429\cos(y)\,x^{13/2}\sqrt{1-x} \\
&\quad -512\cos(y)\sqrt{x}\sqrt{1-x} \\
&\quad -3165\cos(y)\,x^{11/2}\sqrt{1-x} + 10145\cos(y)\,x^{9/2}\sqrt{1-x} \\
&\quad -17408\sin(y)\,x \\
&\quad -46780\sin(y)\,x^2 + 20525\sin(y)\,x^3 - 3264\sin(y)\,x^6 \\
&\quad +1024\sin(y) \\
&\quad +10798\sin(y)\,x^5 - 20236\sin(y)\,x^4 + 429\sin(y)\,x^7\Big\} \\
&\quad \Big/\Big\{x^{3/2}b^{7/2}p^{12}\sqrt{\pi}\,(1-x)^{15/2}\Big\},
\end{aligned}$$

where $x = 4/(b^2p^2)$, $y = (1/2)\arcsin(\sqrt{x})$, and the normalizing constant C satisfies

$$\begin{aligned}\frac{1}{C} &= -6\sqrt{2}\Big\{-2\cos(y)\sqrt{x}\sqrt{1-x} - 23\cos(y)\,x^{3/2}\sqrt{1-x}\\ &\quad +5\cos(y)\,x^{5/2}\sqrt{1-x} - 27\sin(y)\,x\\ &\quad -22\sin(y)\,x^2 + 5\sin(y)\,x^3 + 4\sin(y)\Big\}\Big/\Big\{x^{3/2}b^{7/2}p^8\sqrt{\pi}\,(1-x)^{7/2}\Big\}.\end{aligned}$$

Finally, if $m = 2$ and $n = 2$ then the first four moments are

$$\begin{aligned}E\Big(X\Big) &= 96C\Big\{70\text{EllipticK}\Big(\sqrt{x}\Big)x - 42\,\text{EllipticK}\Big(\sqrt{x}\Big)x^2\\ &\quad -22\text{EllipticK}\Big(\sqrt{x}\Big)x^3 + 4\text{EllipticK}\Big(\sqrt{x}\Big)x^4 - 10\text{EllipticK}\Big(\sqrt{x}\Big)\\ &\quad -65\,\text{EllipticE}\Big(\sqrt{x}\Big)x - 108\text{EllipticE}\Big(\sqrt{x}\Big)x^2 + 43\,\text{EllipticE}\Big(\sqrt{x}\Big)x^3\\ &\quad -8\text{EllipticE}\Big(\sqrt{x}\Big)x^4 + 10\,\text{EllipticE}\Big(\sqrt{x}\Big)\Big\}\\ &\qquad \Big/\Big\{b^4p^{10}x^2\Big(-1+x\Big)^5\pi\Big\},\\ E\Big(X^2\Big) &= -480C\Big\{103\text{EllipticK}\Big(\sqrt{x}\Big)x - 25\text{EllipticK}\Big(\sqrt{x}\Big)x^2\\ &\quad -87\text{EllipticK}\Big(\sqrt{x}\Big)x^3 + 25\text{EllipticK}\Big(\sqrt{x}\Big)x^4 - 4\text{EllipticK}\Big(\sqrt{x}\Big)x^5\\ &\quad -12\text{EllipticK}\Big(\sqrt{x}\Big) - 97\text{EllipticE}\Big(\sqrt{x}\Big)x - 259\text{EllipticE}\Big(\sqrt{x}\Big)x^2\\ &\quad +129\,\text{EllipticE}\Big(\sqrt{x}\Big)x^3 - 49\text{EllipticE}\Big(\sqrt{x}\Big)x^4 + 8\text{EllipticE}\Big(\sqrt{x}\Big)x^5\\ &\quad +12\text{EllipticE}\Big(\sqrt{x}\Big)\Big\} \quad \Big/\Big\{b^4p^{11}x^2\Big(-1+x\Big)^6\pi\Big\},\\ E\Big(X^3\Big) &= -1440C\Big\{-287\text{EllipticK}\Big(\sqrt{x}\Big)x - 108\text{EllipticK}\Big(\sqrt{x}\Big)x^2\\ &\quad +494\text{EllipticK}\Big(\sqrt{x}\Big)x^3 - 176\text{EllipticK}\Big(\sqrt{x}\Big)x^4\\ &\quad +57\text{EllipticK}\Big(\sqrt{x}\Big)x^5 - 8\text{EllipticK}\Big(\sqrt{x}\Big)x^6 + 28\text{EllipticK}\Big(\sqrt{x}\Big)\\ &\quad +273\text{EllipticE}\Big(\sqrt{x}\Big)x + 1109\text{EllipticE}\Big(\sqrt{x}\Big)x^2 - 573\text{EllipticE}\Big(\sqrt{x}\Big)x^3\\ &\quad +339\text{EllipticE}\Big(\sqrt{x}\Big)x^4 - 112\text{EllipticE}\Big(\sqrt{x}\Big)x^5 + 16\text{EllipticE}\Big(\sqrt{x}\Big)x^6\\ &\quad -28\text{EllipticE}\Big(\sqrt{x}\Big)\Big\}\Big/\Big\{b^4p^{12}x^2\Big(-1+x\Big)^7\pi\Big\},\\ E\Big(X^4\Big) &= 1440C\Big\{-2688\text{EllipticK}\Big(\sqrt{x}\Big)x - 3459\text{EllipticK}\Big(\sqrt{x}\Big)x^2\\ &\quad +7780\text{EllipticK}\Big(\sqrt{x}\Big)x^3 - 3058\text{EllipticK}\Big(\sqrt{x}\Big)x^4\end{aligned}$$

$$+1596\text{EllipticK}\Big(\sqrt{x}\Big)x^5 - 451\text{EllipticK}\Big(\sqrt{x}\Big)x^6 + 56\text{EllipticK}\Big(\sqrt{x}\Big)x^7$$
$$+224\text{EllipticK}\Big(\sqrt{x}\Big) + 2576\text{EllipticE}\Big(\sqrt{x}\Big)x + 15128\text{EllipticE}\Big(\sqrt{x}\Big)x^2$$
$$-6992\text{EllipticE}\Big(\sqrt{x}\Big)x^3 + 6160\text{EllipticE}\Big(\sqrt{x}\Big)x^4 - 3088\text{EllipticE}\Big(\sqrt{x}\Big)x^5$$
$$+888\text{EllipticE}\Big(\sqrt{x}\Big)x^6 - 112\text{EllipticE}\Big(\sqrt{x}\Big)x^7 - 224\text{EllipticE}\Big(\sqrt{x}\Big)\Big\}$$
$$\Big/\Big\{b^4p^{13}x^2\Big(-1+x\Big)^8\pi\Big\},$$

where $x = 4/(b^2p^2)$ and the normalizing constant C satisfies

$$\frac{1}{C} = 96\Big\{-11\text{EllipticK}\Big(\sqrt{x}\Big)x + 8\text{EllipticK}\Big(\sqrt{x}\Big)x^2 + \text{EllipticK}\Big(\sqrt{x}\Big)x^3$$
$$+2\text{EllipticK}\Big(\sqrt{x}\Big) + 10\text{EllipticE}\Big(\sqrt{x}\Big)x + 10\text{EllipticE}\Big(\sqrt{x}\Big)x^2$$
$$-2\text{EllipticE}\Big(\sqrt{x}\Big)x^3 - 2\text{EllipticE}\Big(\sqrt{x}\Big)\Big\} \quad \Big/\Big\{b^4p^9x^2\Big(-1+x\Big)^4\pi\Big\}.$$

3. Product Bessel Distributions of the Second Kind

Here, we introduce a new Bessel function distribution with its pdf taken to be the product of two densities of the form (2), i.e.

$$f(x) = C \mid x \mid^{m+n} K_m\left(\left|\frac{x}{b}\right|\right)K_n\left(\left|\frac{x}{\beta}\right|\right) \tag{11}$$

for $-\infty < x < \infty$, $b > 0$, $\beta > 0$, $m > 1$ and $n > 1$, where C denotes the normalizing constant. Application of equation (2.16.33.1) in Prudnikov et al. (1986, volume 2) shows that one can determine C as

$$\begin{aligned}\frac{1}{C} &= \sqrt{\pi}2^{m+n-1}b^{-m}\beta^{2m+n+1}\Gamma\left(m+n+1/2\right)B\left(m+1/2,n+1/2\right) \\ &\quad \times {}_2F_1\left(m+n+\frac{1}{2}, n+\frac{1}{2}; m+n+1; 1-\frac{\beta^2}{b^2}\right).\end{aligned} \tag{12}$$

Using special properties of the ${}_2F_1$ hypergeometric function, one can obtain simpler expressions for (12). For instance, if $m = n$ then (12) can be reduced to

$$\begin{aligned}\frac{1}{C} &= 4^m\exp\left(im\pi\right)\left|\frac{1}{b^2}-\frac{1}{\beta^2}\right|^{-(m+1/2)}\Gamma\left(m+\frac{1}{2}\right) \\ &\quad \Gamma\left(2m+\frac{1}{2}\right)Q_{m-1/2}^{-m}\left(\frac{\beta^2+b^2}{\beta^2-b^2}\right),\end{aligned}$$

where $Q_\nu^\mu(\cdot)$ is the Legendre function defined by

$$\begin{aligned} Q_\nu^\mu(x) \;=\; & \frac{\sqrt{\pi}\exp\left(i\mu\pi\right)\Gamma(\mu+\nu+1)}{2^{\nu+1}\Gamma\left(\nu+3/2\right)} x^{-\mu-\nu-1}\left(x^2-1\right)^{\mu/2} \\ & {}_2F_1\left(\frac{\mu+\nu+1}{2}, \frac{\mu+\nu}{2}+1; \nu+\frac{3}{2}; \frac{1}{x^2}\right). \end{aligned}$$

When m and n take half-integer values one can reduce (11) to elementary forms. Note that

$$K_{3/2}(x) \;=\; \sqrt{\frac{\pi}{2}} \frac{\exp(-x)(x+1)}{x^{3/2}},$$

$$K_{5/2}(x) \;=\; \sqrt{\frac{\pi}{2}} \frac{\exp(-x)\left(x^2+3x+3\right)}{x^{5/2}},$$

$$K_{7/2}(x) \;=\; \sqrt{\frac{\pi}{2}} \frac{\exp(-x)\left(x^3+6x^2+15x+15\right)}{x^{7/2}}$$

and

$$K_{9/2}(x) \;=\; \sqrt{\frac{\pi}{2}} \frac{\exp(-x)\left(x^4+10x^3+45x^2+105x+105\right)}{x^{9/2}}.$$

More generally, if $\nu - 1/2 \geq 1$ is an integer then

$$I_\nu(x) \;=\; \sqrt{\pi}\exp(-x)\sqrt{2x} \sum_{j=0}^{[|\nu|-1/2]} \frac{(j+\mid \nu \mid -1/2)!(2x)^{-j}}{j!\left(\mid \nu \mid -j-1/2\right)!}.$$

Thus, several particular forms of (11) can be obtained for half-integer values of m and n. For example, if $m = 3/2$ and $n = 3/2$ then (11) reduces to

$$f(x) \;=\; \frac{C\pi(b\beta)^{3/2}}{2} \exp\left(-\frac{\mid x \mid}{b}\right)\left(\frac{\mid x \mid}{b}+1\right)\exp\left(-\frac{\mid x \mid}{\beta}\right)\left(\frac{\mid x \mid}{\beta}+1\right).$$

If $m = 3/2$ and $n = 5/2$ then (11) reduces to

$$f(x) \;=\; \frac{C\pi b^{3/2}\beta^{5/2}}{2} \exp\left(-\frac{\mid x \mid}{b}\right)\left(\frac{\mid x \mid}{b}+1\right)\exp\left(-\frac{\mid x \mid}{\beta}\right)\left(\frac{x^2}{\beta^2}+\frac{3\mid x \mid}{\beta}+3\right).$$

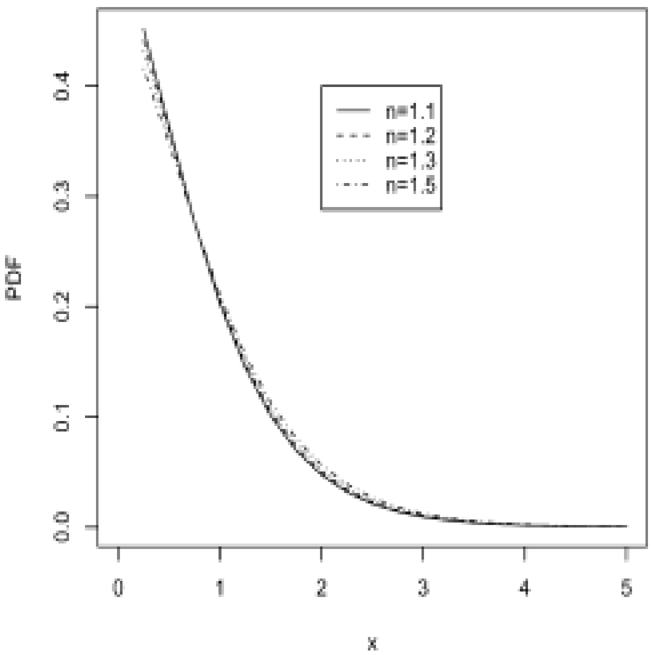

Fig. 2. Plots of the pdf (11) for $b = 1$, $\beta = 1$, $m = 1.1$ and $n = 1.1, 1.2, 1.3, 1.5$ (only the positive side is shown since the pdf is symmetric around zero).

Illustrates possible shapes of the pdf (11) for selected values of n. Only the positive side of the pdf is shown since the pdf is symmetric around zero. The effect of the parameter n is evident. When the plots were redrawn for other values of m (not shown in the figure) it was noticed the densities shrunk with increasing m.

If X is a random variable with pdf (11) then its kth moment can be expressed as

$$E\left(X^k\right) = C \int_{-\infty}^{\infty} x^k \mid x \mid^{m+n} K_m\Big(\left|\frac{x}{b}\right|\Big) K_n\Big(\left|\frac{x}{\beta}\right|\Big) dx. \tag{13}$$

Clearly, the above is zero if k is an odd integer. If k is an even integer then an application of equation (2.16.33.1) in Prudnikov et al. (1986, volume 2) shows that (12) can be calculated as

$$\begin{aligned} E\left(X^k\right) &= 2C \int_0^{\infty} x^{m+n+k} K_m\Big(\frac{x}{b}\Big) K_n\Big(\frac{x}{\beta}\Big) dx \\ &= C 2^{k+m+n-1} b^{-m} \beta^{k+2m+n+1} \Gamma\left(\frac{k+1}{2}\right) \\ &\quad \Gamma\left(m+n+\frac{k+1}{2}\right) B\left(m+\frac{k+1}{2}, n+\frac{k+1}{2}\right) \\ &\quad \times {}_2F_1\left(m+n+\frac{k+1}{2}, n+\frac{k+1}{2}; k+m+n+1; 1-\frac{\beta^2}{b^2}\right). \end{aligned} \tag{14}$$

Using special properties of the ${}_2F_1$ hypergeometric function, one can derive several simpler forms of (13) as discussed in the following. If $m = n$ then (13) reduces to:

$$E\left(X^k\right) = C2^{k+2m}\exp\left(im\pi\right)\left|\frac{1}{b^2} - \frac{1}{\beta^2}\right|^{-(k+2m+1)/2}\Gamma\left(m + \frac{k+1}{2}\right)$$
$$\times\Gamma\left(2m + \frac{k+1}{2}\right)Q^{-m}_{m+(k-1)/2}\left(\frac{\beta^2+b^2}{\beta^2-b^2}\right).$$

One can reduce (13) to elementary forms if either both m and n are half-integers or m is an integer and n is a half-integer. On the other hand, if either both m and n are integers or m is a half-integer and n is an integer then one can express (13) in terms of the complete elliptical integral of the first kind and the complete elliptical integral of the second kind. For instance, if $m = 3/2$ and $n = 3/2$ then the first four even order moments are

$$E\left(X^2\right) = -(81\sqrt{6\pi}C/512)\beta^{17/2}\Big\{88x^{7/2} + 16x^{9/2} - 1154x^{5/2} + 1995x^{3/2}$$
$$-945\sqrt{x} + 720\sqrt{1-x}yx^2 - 1680\sqrt{1-x}yx + 945\sqrt{1-x}y\Big\}$$
$$\Big/\Big\{x^{11/2}b^2(x-1)^2\Big\},$$
$$E\left(X^4\right) = (6561\sqrt{6\pi}C/8192)\beta^{21/2}\Big\{384x^{9/2} + 32x^{11/2} - 10524x^{7/2} + 34748x^{5/2}$$
$$-39655x^{3/2} + 15015\sqrt{x} + 5600\sqrt{1-x}yx^3 - 25200\sqrt{1-x}yx^2$$
$$+34650\sqrt{1-x}yx - 15015\sqrt{1-x}y\Big\}\Big/\Big\{x^{15/2}b^2(x-1)^2\Big\},$$
$$E\left(X^6\right) = -(295245\sqrt{6\pi}C/131072)\beta^{25/2}\Big\{5248x^{11/2} + 256x^{13/2} - 254192x^{9/2}$$
$$+1377816x^{7/2} - 2780778x^{5/2} + 2417415x^{3/2} - 765765\sqrt{x} + 120960$$
$$\sqrt{1-x}yx^4 - 887040\sqrt{1-x}yx^3 + 2162160\sqrt{1-x}yx^2 - 2162160$$
$$\sqrt{1-x}yx + 765765\sqrt{1-x}y\Big\}\Big/\Big\{x^{19/2}b^2(x-1)^2\Big\},$$
$$E\left(X^8\right) = (18600435\sqrt{6\pi}C/2097152)\beta^{29/2}\Big\{79360x^{13/2} + 2560x^{15/2}$$
$$-6081056x^{11/2} + 49228608x^{9/2} - 153295428x^{7/2} + 226462236x^{5/2}$$
$$-160044885x^{3/2} + 43648605\sqrt{x} + 2661120\sqrt{1-x}yx^5 - 28828800$$
$$\sqrt{1-x}yx^4 + 108108000\sqrt{1-x}yx^3 - 183783600\sqrt{1-x}yx^2$$
$$+145495350\sqrt{1-x}yx - 43648605\sqrt{1-x}y\Big\}\Big/\Big\{x^{23/2}b^2(x-1)^2\Big\},$$

where $x = 1 - \beta^2/b^2$, $y = \arcsin(\sqrt{x})$, and the normalizing constant C satisfies

$$\begin{aligned} \frac{1}{C} &= (9\sqrt{6}/32)\beta^{13/2}\Big\{4\,x^{5/2} + 4\,x^{7/2} - 23x^{3/2} + 15\,\sqrt{x} + 18\sqrt{1-x}yx \\ &\quad -15\sqrt{1-x}y\Big\} \quad \Big/ \Big\{x^{7/2}b^2\,(x-1)^2\Big\}. \end{aligned}$$

If $m = 2$ and $n = 2$ then the first four even order moments are

$$\begin{aligned} E\left(X^2\right) &= -15\beta^9\pi C\Big\{640\,\mathrm{EllipticK}\left(\sqrt{x}\right)x - 510\mathrm{EllipticK}\left(\sqrt{x}\right)x^2 \\ &\quad +125\mathrm{EllipticK}\left(\sqrt{x}\right)x^3 + \mathrm{EllipticK}\left(\sqrt{x}\right)x^4 - 256\mathrm{EllipticK}\left(\sqrt{x}\right) \\ &\quad -2\mathrm{EllipticE}\left(\sqrt{x}\right)x^4 - 14\mathrm{EllipticE}\left(\sqrt{x}\right)x^3 + 270\mathrm{EllipticE}\left(\sqrt{x}\right)x^2 \\ &\quad -512\mathrm{EllipticE}\left(\sqrt{x}\right)x + 256\mathrm{EllipticE}\left(\sqrt{x}\right)\Big\}\Big/\Big\{b^2x^6\left(x-1\right)^2\Big\}, \\ E\left(X^4\right) &= 315\beta^{11}\pi C\Big\{-6144\,\mathrm{EllipticK}\left(\sqrt{x}\right)x + 6592\mathrm{EllipticK}\left(\sqrt{x}\right)x^2 \\ &\quad -2944\mathrm{EllipticK}\left(\sqrt{x}\right)x^3 + 447\mathrm{EllipticK}\left(\sqrt{x}\right)x^4 + \mathrm{EllipticK}\left(\sqrt{x}\right)x^5 \\ &\quad +2048\mathrm{EllipticK}\left(\sqrt{x}\right) - 2\mathrm{EllipticE}\left(\sqrt{x}\right)x^5 - 28\mathrm{EllipticE}\left(\sqrt{x}\right)x^4 \\ &\quad +1120\mathrm{EllipticE}\left(\sqrt{x}\right)x^3 - 4160\mathrm{EllipticE}\left(\sqrt{x}\right)x^2 \\ &\quad +5120\mathrm{EllipticE}\left(\sqrt{x}\right)x - 2048\mathrm{EllipticE}\left(\sqrt{x}\right)\Big\} \quad \Big/\Big\{\left(x-1\right)^2x^8b^2\Big\}, \\ E\left(X^6\right) &= -2835\beta^{13}\pi C\Big\{229376\,\mathrm{EllipticK}\left(\sqrt{x}\right)x - 306432\mathrm{EllipticK}\left(\sqrt{x}\right)x^2 \\ &\quad +192640\mathrm{EllipticK}\left(\sqrt{x}\right)x^3 - 55798\mathrm{EllipticK}\left(\sqrt{x}\right)x^4 \\ &\quad +5745\mathrm{EllipticK}\left(\sqrt{x}\right)x^5 + 5\mathrm{EllipticK}\left(\sqrt{x}\right)x^6 - 65536\mathrm{EllipticK}\left(\sqrt{x}\right) \\ &\quad -10\mathrm{EllipticE}\left(\sqrt{x}\right)x^6 - 230\mathrm{EllipticE}\left(\sqrt{x}\right)x^5 + 15846\mathrm{EllipticE}\left(\sqrt{x}\right)x^4 \\ &\quad -96768\mathrm{EllipticE}\left(\sqrt{x}\right)x^3 + 212224\mathrm{EllipticE}\left(\sqrt{x}\right)x^2 \\ &\quad -196608\mathrm{EllipticE}\left(\sqrt{x}\right)x + 65536\mathrm{EllipticE}\left(\sqrt{x}\right)\Big\}\Big/\Big\{b^2x^{10}\left(x-1\right)^2\Big\}, \\ E\left(X^8\right) &= 155925\beta^{15}\pi C\Big\{-2097152\,\mathrm{EllipticK}\left(\sqrt{x}\right)x + 3342336\mathrm{EllipticK}\left(\sqrt{x}\right)x^2 \\ &\quad -2686976\mathrm{EllipticK}\left(\sqrt{x}\right)x^3 + 1131248\mathrm{EllipticK}\left(\sqrt{x}\right)x^4 \\ &\quad -230880\mathrm{EllipticK}\left(\sqrt{x}\right)x^5 + 17129\mathrm{EllipticK}\left(\sqrt{x}\right)x^6 \end{aligned}$$

$$+7\text{EllipticK}\left(\sqrt{x}\right)x^7 + 524288\text{EllipticK}\left(\sqrt{x}\right) - 14\text{EllipticE}\left(\sqrt{x}\right)x^7$$
$$-476\text{EllipticE}\left(\sqrt{x}\right)x^6 + 50664\text{EllipticE}\left(\sqrt{x}\right)x^5$$
$$-459760\text{EllipticE}\left(\sqrt{x}\right)x^4 + 1556480\text{EllipticE}\left(\sqrt{x}\right)x^3$$
$$-2457600\text{EllipticE}\left(\sqrt{x}\right)x^2 + 1835008\text{EllipticE}\left(\sqrt{x}\right)x$$
$$-524288\text{EllipticE}\left(\sqrt{x}\right)\Big\} \Big/ \Big\{b^2 x^{12}\left(x-1\right)^2\Big\},$$

where $x = 1-\beta^2/b^2$ and the normalizing constant C satisfies

$$\begin{aligned} \frac{1}{C} &= 3\beta^7\pi\Big\{-32\text{EllipticK}\left(\sqrt{x}\right)x + 15\text{EllipticK}\left(\sqrt{x}\right)x^2 + \text{EllipticK}\left(\sqrt{x}\right)x^3 \\ &\quad +16\text{EllipticK}\left(\sqrt{x}\right) - 4\text{EllipticE}\left(\sqrt{x}\right)x^2 - 2\text{EllipticE}\left(\sqrt{x}\right)x^3 \\ &\quad +24\text{EllipticE}\left(\sqrt{x}\right)x - 16\text{EllipticE}\left(\sqrt{x}\right)\Big\} \Big/ \Big\{b^2 x^4\left(x-1\right)^2\Big\}. \end{aligned}$$

4. Produce Bessel Distributions of the First and Second Kind

Here, we introduce a new Bessel function distribution with its pdf taken to be the product of two densities of the form (1) and (2), i.e.

$$f(x) = Cx^{m+n} I_m\left(\frac{x}{b}\right) K_n\left(\frac{x}{\beta}\right) \tag{15}$$

for $x > 0$, $0 < \beta < b$, $m > 1$ and $n > 1$, where C denotes the normalizing constant. Application of equation (2.16.28.1) in Prudnikov et al. (1986, volume 2) shows that one can determine C as

$$\begin{aligned} \frac{1}{C} &= \frac{2^{m+n-1}\beta^{2m+n+1}}{b^m\Gamma(m+1)}\Gamma\left(m+n+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right) \\ &\quad {}_2F_1\left(m+n+\tfrac{1}{2}, m+\tfrac{1}{2}; m+1; \tfrac{\beta^2}{b^2}\right). \end{aligned} \tag{16}$$

Using special properties of the ${}_2F_1$ hypergeometric function, one can obtain simpler expressions for (16). For instance, if $m = n$ then (16) can be reduced to

$$\begin{aligned} \frac{1}{C} &= \pi^{-1/2}2^{m+n-1}(b\beta)^{m+n+1/2}\left(b^2-\beta^2\right)^{-(m+n)/2}\Gamma\left(\frac{m+n+1}{2}\right) \\ &\quad \times\exp\left(\frac{(m+n)i\pi}{2}\right)Q_{n-1/2}^{(m+n)/2}\left(\frac{b^2+\beta^2}{2b\beta}\right), \end{aligned}$$

where $Q_\nu^\mu(\cdot)$ is the Legendre function defined by

$$Q_\nu^\mu(x) = \frac{\sqrt{\pi}\exp(i\mu\pi)\Gamma(\mu+\nu+1)}{2^{\nu+1}\Gamma(\nu+3/2)} x^{-\mu-\nu-1}\left(x^2-1\right)^{\mu/2}$$

$$ {}_2F_1\left(\frac{\mu+\nu+1}{2}, \frac{\mu+\nu}{2}+1; \nu+\frac{3}{2}; \frac{1}{x^2}\right).$$

Using the special properties of the modified Bessel functions given in Sections 2 and 3, several particular forms of (15) can be obtained for half-integer values of m and n. For example, if $m = 3/2$ and $n = 3/2$ then (15) reduces to

$$f(x) = C(b\beta)^{3/2}\left\{\frac{x}{b}\cosh\left(\frac{x}{b}\right) - \sinh\left(\frac{x}{b}\right)\right\}\exp\left(-\frac{x}{\beta}\right)\left(\frac{x}{\beta}+1\right).$$

If $m = 3/2$ and $n = 5/2$ then (15) reduces to

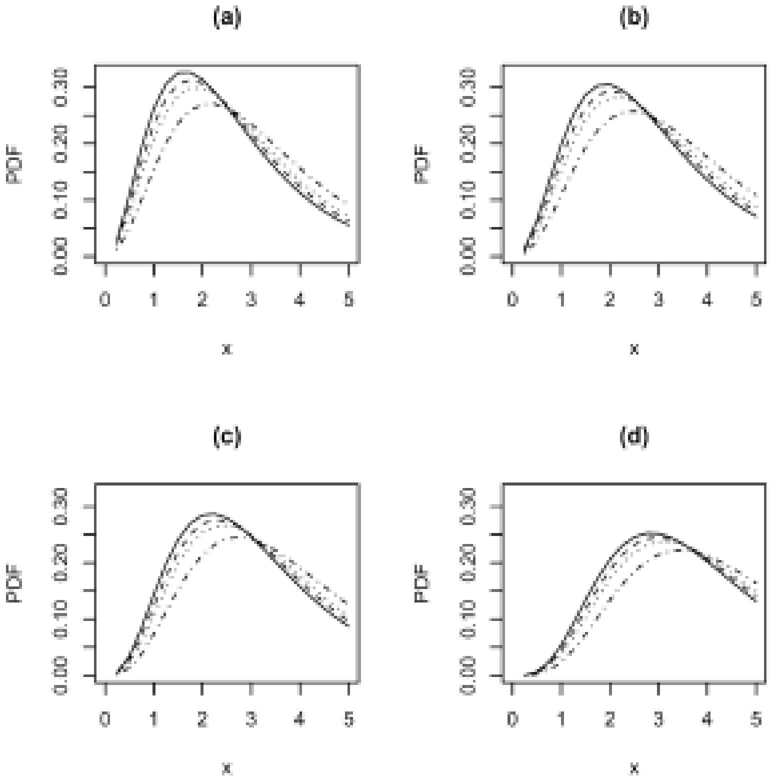

Fig. 3. Plots of the pdf (15) for $b = 1$, $\beta = 1/2$ and (a) $m = 1.1$; (b) $m = 1.3$; (c) $m = 1.5$; and, (d) $m = 2$. The four curves in each plot from the left to the right correspond to $n = 1.1, 1.3, 1.5, 2$.

$$f(x) = Cb^{3/2}\beta^{5/2}\left\{\frac{x}{b}\cosh\left(\frac{x}{b}\right) - \sinh\left(\frac{x}{b}\right)\right\}\exp\left(-\frac{x}{\beta}\right)\left(\frac{x^2}{\beta^2} + \frac{3x}{\beta} + 3\right).$$

Fig. 3 Illustrates possible shapes of the pdf (15) for selected values of m and n. The four curves in each plot correspond to selected values of n. Note that the shapes are unimodal and that the densities appear to shrink with increasing values of both m and n.

If X is a random variable with pdf (15) then its kth moment can be expressed as

$$E\left(X^k\right) = C\int_0^\infty x^{k+m+n} I_m\left(\frac{x}{b}\right) K_n\left(\frac{x}{\beta}\right) dx. \tag{17}$$

Application of equation (2.16.28.1) in Prudnikov et al. (1986, volume 2) shows that (17) can be calculated as

$$\begin{aligned} E\left(X^k\right) &= \frac{C2^{k+m+n-1}\beta^{k+2m+n+1}}{b^m\Gamma(m+1)}\Gamma\left(m+n+\frac{k+1}{2}\right)\Gamma\left(m+\frac{k+1}{2}\right) \\ &\quad \times {}_2F_1\left(m+n+\frac{k+1}{2}, m+\frac{k+1}{2}; m+1; \frac{\beta^2}{b^2}\right). \end{aligned} \tag{18}$$

Using special properties of the ${}_2F_1$ hypergeometric function, one can derive several simpler forms of (18) as discussed in the following. If $m = n$ then (18) reduces to:

$$\begin{aligned} E\left(X^k\right) &= C\pi^{-1/2}2^{k+m+n-1}(b\beta)^{k+m+n+1/2}\left(b^2-\beta^2\right)^{-(k+m+n)/2} \\ &\quad \times\Gamma\left(\frac{k+m+n+1}{2}\right)\exp\left(\frac{(k+m+n)i\pi}{2}\right)Q_{n-1/2}^{(k+m+n)/2}\left(\frac{b^2+\beta^2}{2b\beta}\right). \end{aligned}$$

If $k \geq 1$ is odd then (18) can be reduced to the elementary form:

$$\begin{aligned} E\left(X^k\right) &= \frac{C2^{k+m+n-1}b^{k+m+2n+1}\beta^{k+2m+n+1}}{\left(b^2-\beta^2\right)^{m+n+(k+1)/2}} \\ &\quad \Gamma(m+1)\Gamma\left(m+n+\frac{k+1}{2}\right)\Gamma\left(m+\frac{k+1}{2}\right) \\ &\quad \times {}_2F_1\left(m+n+\frac{k+1}{2}, \frac{1-k}{2}; m+1; \frac{\beta^2}{\beta^2-b^2}\right) \\ &= \frac{C2^{k+m+n-1}b^{k+m+2n+1}\beta^{k+2m+n+1}}{\left(b^2-\beta^2\right)^{m+n+(k+1)/2}} \\ &\quad \Gamma(m+1)\Gamma\left(m+n+\frac{k+1}{2}\right)\Gamma\left(m+\frac{k+1}{2}\right) \\ &\quad \times \sum_{j=0}^{(k-1)/2}\frac{(m+n+(k+1)/2)_j\,((1-k)/2)_j}{(m+1)_j}\left(\frac{\beta^2}{\beta^2-b^2}\right)^j. \end{aligned}$$

When k is even, one can reduce (18) to simpler forms when m and n take integer or half-integer values. If either both m and n are half-integers or m is an integer and n is a half-integer or m is a half-integer and n is an integer then (18) can be reduced to an elementary form. On the other hand, if both m and n are integers then one can express (18) in terms of the complete elliptical integral of the first kind and the complete elliptical integral of the second kind. For instance, if $m = 3/2$ and $n = 3/2$ then the first four even order moments are

$$\begin{aligned}
E\left(X^2\right) &= 8C\beta^{15/2}\left(-35-14x+x^2\right) \Big/ \left\{b^{3/2}\left(-1+x\right)^5\right\}, \\
E\left(X^4\right) &= 144C\beta^{19/2}\left(-105-189\,x-27\,x^2+x^3\right) \Big/ \left\{b^{3/2}\left(-1+x\right)^7\right\}, \\
E\left(X^6\right) &= 5760C\beta^{23/2}\left(-231-924x-594\,x^2-44\,x^3+x^4\right) \Big/ \left\{b^{3/2}\left(-1+x\right)^9\right\}, \\
E\left(X^8\right) &= 403200C\beta^{27/2}\left(-429-3003x-4290\,x^2-1430x^3-65x^4+x^5\right) \\
&\quad \Big/ \left\{b^{3/2}\left(-1+x\right)^{11}\right\},
\end{aligned}$$

where $x = \beta^2/b^2$ and the normalizing constant $C = 2\beta^{11/2}(-5+x)/\{b^{3/2}(-1+x)^3\}$. If $m = 2$ and $n = 2$ then the first four even order moments are x

$$\begin{aligned}
E\left(X^2\right) &= 15C\beta^9\Big\{-23\,\mathrm{EllipticK}\left(\sqrt{x}\right)x - 87\mathrm{EllipticK}\left(\sqrt{x}\right)x^2 \\
&\quad +107\mathrm{EllipticK}\left(\sqrt{x}\right)x^3 + \mathrm{EllipticK}\left(\sqrt{x}\right)x^4 + 2\,\mathrm{EllipticK}\left(\sqrt{x}\right) \\
&\quad +22\,\mathrm{EllipticE}\left(\sqrt{x}\right)x + 216\,\mathrm{EllipticE}\left(\sqrt{x}\right)x^2 + 22\,\mathrm{EllipticE}\left(\sqrt{x}\right)x^3 \\
&\quad -2\,\mathrm{EllipticE}\left(\sqrt{x}\right)x^4 - 2\,\mathrm{EllipticE}\left(\sqrt{x}\right)\Big\} \Big/ \left\{x^2b^2\left(-1+x\right)^6\right\}, \\
E\left(X^4\right) &= 315C\beta^{11}\Big\{-39\,\mathrm{EllipticK}\left(\sqrt{x}\right)x - 536\mathrm{EllipticK}\left(\sqrt{x}\right)x^2 \\
&\quad +158\mathrm{EllipticK}\left(\sqrt{x}\right)x^3 + 414\mathrm{EllipticK}\left(\sqrt{x}\right)x^4 + \mathrm{EllipticK}\left(\sqrt{x}\right)x^5 \\
&\quad +2\mathrm{EllipticK}\left(\sqrt{x}\right) + 38\mathrm{EllipticE}\left(\sqrt{x}\right)x + 988\mathrm{EllipticE}\left(\sqrt{x}\right)x^2 \\
&\quad +988\mathrm{EllipticE}\left(\sqrt{x}\right)x^3 + 38\mathrm{EllipticE}\left(\sqrt{x}\right)x^4 - 2\mathrm{EllipticE}\left(\sqrt{x}\right)x^5 \\
&\quad -2\mathrm{EllipticE}\left(\sqrt{x}\right)\Big\} \Big/ \left\{x^2b^2\left(-1+x\right)^8\right\}, \\
E\left(X^6\right) &= 2835C\beta^{13}\Big\{-295\,\mathrm{EllipticK}\left(\sqrt{x}\right)x - 8771\mathrm{EllipticK}\left(\sqrt{x}\right)x^2 \\
&\quad -8886\mathrm{EllipticK}\left(\sqrt{x}\right)x^3 + 12452\mathrm{EllipticK}\left(\sqrt{x}\right)x^4 \\
&\quad +5485\mathrm{EllipticK}\left(\sqrt{x}\right)x^5 + 5\mathrm{EllipticK}\left(\sqrt{x}\right)x^6 + 10\mathrm{EllipticK}\left(\sqrt{x}\right)
\end{aligned}$$

$$
\begin{aligned}
& +290\text{EllipticE}\left(\sqrt{x}\right)x + 14546\text{EllipticE}\left(\sqrt{x}\right)x^2 \\
& +35884\text{EllipticE}\left(\sqrt{x}\right)x^3 + 290\text{EllipticE}\left(\sqrt{x}\right)x^5 \\
& +14546\text{EllipticE}\left(\sqrt{x}\right)x^4 - 10\text{EllipticE}\left(\sqrt{x}\right)x^6 \\
& -10\text{EllipticE}\left(\sqrt{x}\right)\Big\} \Big/ \Big\{\Big(-1+x\Big)^{10} x^2 b^2\Big\},
\end{aligned}
$$

$$
\begin{aligned}
E\left(X^8\right) = {} & 155925 C\beta^{15}\Big\{14\text{EllipticK}\left(\sqrt{x}\right) - 581\text{EllipticK}\left(\sqrt{x}\right)x \\
& -30336\text{EllipticK}\left(\sqrt{x}\right)x^2 - 86111\text{EllipticK}\left(\sqrt{x}\right)x^3 \\
& +19958\text{EllipticK}\left(\sqrt{x}\right)x^4 + 80445\text{EllipticK}\left(\sqrt{x}\right)x^5 \\
& +16604\text{EllipticK}\left(\sqrt{x}\right)x^6 + 7\text{EllipticK}\left(\sqrt{x}\right)x^7 \\
& -14\text{EllipticE}\left(\sqrt{x}\right) + 574\text{EllipticE}\left(\sqrt{x}\right)x \\
& +47514\text{EllipticE}\left(\sqrt{x}\right)x^2 + 214070\text{EllipticE}\left(\sqrt{x}\right)x^3 \\
& +47514\text{EllipticE}\left(\sqrt{x}\right)x^5 + 214070\text{EllipticE}\left(\sqrt{x}\right)x^4 \\
& -14\text{EllipticE}\left(\sqrt{x}\right)x^7 + 574\text{EllipticE}\left(\sqrt{x}\right)x^6\Big\} \\
& \Big/\Big\{\Big(-1+x\Big)^{12} x^2 b^2\Big\},
\end{aligned}
$$

where $x = \beta^2/b^2$ and the normalizing constant C satisfies

$$
\begin{aligned}
\frac{1}{C} = {} & 3\beta^7\Big\{-11\text{EllipticK}\left(\sqrt{x}\right)x + 8\text{EllipticK}\left(\sqrt{x}\right)x^2 + \text{EllipticK}\left(\sqrt{x}\right)x^3 \\
& +2\text{EllipticK}\left(\sqrt{x}\right) + 10\text{EllipticE}\left(\sqrt{x}\right)x + 10\text{EllipticE}\left(\sqrt{x}\right)x^2 \\
& -2\text{EllipticE}\left(\sqrt{x}\right)x^3 - 2\text{EllipticE}\left(\sqrt{x}\right)\Big\}\Big/\Big\{x^2 b^2\Big(-1+x\Big)^4\Big\}.
\end{aligned}
$$

5. Conclusions

We have introduced three new Bessel function distributions by taking products of a Bessel function pdf of the first kind and a Bessel function pdf of the second kind. We have studied the moments and various particular cases of each proposed distribution. It is expected that these results will be useful especially for electrical and electronic engineers.

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