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# $K$-CYCLIC EVEN CYCLE SYSTEMS OF THE COMPLETE GRAPH

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**Abstract.** An $(m_1, \ldots, m_r)$-cycle is the union of edge-disjoint $m_i$-cycles for $1 \leq i \leq r$. An $(m_1, \ldots, m_r)$-cycle system of the complete graph $K_v$, $(\boldsymbol{V}, \boldsymbol{C})$, is said to be $k$-cyclic if $\boldsymbol{V} = Z_v$ and for $k \in Z_v$, $C + k \in \boldsymbol{C}$ whenever $C \in \boldsymbol{C}$.

Let $m_i$ ($1 \leq i \leq r$ ) be even integers ($> 2$) and let $\sum_{i=1}^{r} m_i = m = ks$ with $\gcd(k, s) = 1$ and $k$ odd. Suppose $v$ is the least positive integer such that $v(v-1) \equiv 0 \pmod{2m}$ and $\gcd(v, m) = k$. In this paper, it is proved that if there is a $k$-cyclic $(m_1, \ldots, m_r)$-cycle system of order $v$, then for any positive integer $p$, a $k$-cyclic $(m_1, \ldots, m_r)$ cycle system of order $2pm + v$ exists.

As the main consequence of this paper, the necessary and sufficient conditions for the existence of a $k$-cyclic $(m_1, \ldots, m_r)$-cycle system of order $v$ with $m_i$ even and $\sum_{i=1}^{r} m_i \leq 20$ are given.

## 1. INTRODUCTION

An $m$-cycle, written $(c_0, c_1, \ldots, c_{m-1})$, consists of $m$ distinct vertices $c_0$, $c_1$, …, $c_{m-1}$, and $m$ edges $\{c_i, c_{i+1}\}$, $0 \leq i \leq m-2$, and $\{c_0, c_{m-1}\}$. Let $m_1$, …, $m_r$ be integers greater than 2. An $(m_1, \ldots, m_r)$*-cycle* is the union of edge-disjoint $m_i$-cycles for $1 \leq i \leq r$. An $(m_1, \ldots, m_r)$*-cycle system* of a graph $G$ is a pair $(\boldsymbol{V}, \boldsymbol{C})$, where $\boldsymbol{V}$ is the vertex set of $G$ and $\boldsymbol{C}$ is a collection of $(m_1, \ldots, m_r)$-cycles whose edges partition the edges of $G$.

If $G = K_v$, the complete graph with $v$ vertices, then such an $(m_1, \ldots, m_r)$-cycle system is called an $(m_1, \ldots, m_r)$*-cycle system of order* $v$. In particular, If $m_1 = \cdots = m_r = m$, it is known as an $m$*-cycle system*.

Given an $m$-cycle $C_m = (c_0, c_1, \ldots, c_{m-1})$, by $C_m + j$ we mean $(c_0 + j, c_1 + j, \ldots, c_{m-1} + j)$, where $j \in Z_v$. Analogously, if $C = \{C_{m_1}, \ldots, C_{m_r}\}$ is an $(m_1, ..., m_r)$-cycle, we use $C + j$ instead of $\{C_{m_1} + j, \ldots, C_{m_r} + j\}$.

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An $(m_1, \ldots, m_r)$-cycle system of order $v$, $(\boldsymbol{V}, \boldsymbol{C})$, is said to be $k$-*cyclic* if $\boldsymbol{V} = Z_v$ and for $k \in Z_v$, $C + k \in \boldsymbol{C}$ whenever $C \in \boldsymbol{C}$. In particular, if $k = 1$, then it is simply called *cyclic*. A cyclic $(m_1, \ldots, m_r)$-cycle system, of course, is also a $k$-cyclic $(m_1, \ldots, m_r)$-cycle system for $k \in Z_v$.

The study of $m$-cycle systems of the complete graph has been one of the most interesting problems in graph decomposition. The existence question for $m$-cycle systems of the complete graph has been completely settled by Alspach and Gavlas [1] in the case of $m$ odd and by Šajna [10] in the even case.

The existence question for cyclic $m$-cycle systems of order $v$ has been completely solved for $m = 3$ [7], 5 and 7 [9]. For $m$ even and $v \equiv 1 \pmod{2m}$, cyclic $m$-cycle systems of order $v$ was proved for $m \equiv 0 \pmod 4$ [6] and for $m \equiv 2 \pmod 4$ [8]. Recently, it has been shown in [2, 4, 5] that for each pair of integers $(m, n)$, there exists a cyclic $m$-cycle system of order $2mn + 1$, and in particular, for each odd prime $p$, there exists a cyclic $p$-cycle system [2, 5]. For $v \equiv m \pmod{2m}$, cyclic $m$-cycle systems of order $v$ are presented for $m \notin M$ [3], where $M = \{p^\alpha \mid p \text{ is prime}, \alpha > 1\} \cup \{15\}$, and in [11] for $m \in M$. More recently, combining the known results, it has also been proved in [12] that for $3 \le m \le 32$, there exists a cyclic $m$-cycle system and there exists a cyclic $2q$-cycle system with $q$ a prime power. Moreover, Fu and Wu [5] proved the following result.

**Theorem 1.1.** [5] *]If $m_1, \ldots, m_r$ are integers with $\sum_{i=1}^r m_i = m$, then there exists a cyclic $(m_1, \ldots, m_r)$-cycle system of order $2m + 1$.*

The main result of this article is the following.

**Theorem 1.2.** *Let $m_i$ $(1 \le i \le r)$ be even integers $(> 2)$ and let $\sum_{i=1}^r m_i = ks \le 20$ with $\gcd(k, s) = 1$ and $k$ odd. Then for each admissible value $v$ such that $v(v-1) \equiv 0 \pmod{2m}$ and $\gcd(v, m) = k$, there exists a $k$-cyclic $(m_1, \ldots, m_r)$-cycle system of order $v$ with the exception that $(v; m_1, \ldots, m_r) = (9; 4, 6, 8)$ and $(9; 4, 4, 4, 6)$.*

## 2. The Necessary Conditions

All graphs considered here have vertices in $Z_v$. In what follows, assume $3 \le m_1 \le \cdots \le m_r$ and $\sum_{i=1}^r m_i = m$. The necessary conditions for the existence of an $(m_1, \ldots, m_r)$-cycle system of order $v$ is that $v \ge m_r$, $m$ divides $v(v-1)/2$, and the degree of each vertex is even. Obviously, $v$ must be odd.

Given any positive integers $m_1, \ldots, m_r$ with $\sum_{i=1}^r m_i = m$, it is not easy to find each admissible value $v$ such that an $(m_1, \ldots, m_r)$-cycle system of order $v$ exists. However, if we fix an odd factor of $m$, say $k$, so that $m = ks$ with $\gcd(k, s) = 1$ and suppose $\gcd(v, m) = k$, then it turns out to be a simpler work.

A method to find out each admissible value $v$ will be given. First, we need one basic fact from number theory.

**Fact.** The linear congruence $ax \equiv 1 \pmod{m}$ has a unique integral solution modulo $m$ if and only if $\gcd(a, m) = 1$.

Throughout this paper, let $\sum_{i=1}^{r} m_i = m = ks$ with $k$ odd and $\gcd(k, s) = 1$.

**Proposition 2.1.** *Let $m$ and $v$ be positive integers with $\gcd(m, v) = k$ and let $c$ be the least positive integral solution of the linear congruence $kx \equiv 1 \pmod{2s}$ satisfying $kc \geq m_r$. If $v$ is any admissible value of an $(m_1, \ldots, m_r)$-cycle system, then*

$$v = 2pm + kc$$

*for some integer $p \geq 0$.*

*Proof.* Since the value of $v$ is admissible, we have $2m|v(v-1)$, and since $m = ks$ and $\gcd(m, v) = k$, it implies that $2s|v-1$ or, equivalently, $v \equiv 1 \pmod{2s}$ or, equivalently, $kx \equiv 1 \pmod{2s}$ for some positive integer $x$. Note that $x$ is odd and $\gcd(x, 2s) = 1$. Now, by the fact stated above, the linear congruence $kx \equiv 1 \pmod{2s}$ has a unique least positive integral solution $c$, that is, $v = 2pm + kc$ for some integer $p \geq 0$ because $\gcd(m, v) = k$. ■

As usual, we use $Spec(m)$ to denote the set of all admissible values $v$. By Proposition 2.1, if $m$ has $n$ distinct odd factors then the number of residue classes (modulo $2m$) in $Spec(m)$ is $2^n$. Consider, for instance, the $m$-cycle system with $m = 180$. It is clear that all possible values of $k$ are $1$, $3^2$, $5$, or $3^2 \cdot 5$ and we have four residue classes modulo 360. An easy verification shows that $Spec(180) = \{v|v \equiv 1, 81, 145, \text{or } 225 \pmod{360}\}$.

As a consequence of Proposition 2.1, which will be used later, we have $Spec(m)$ for $m = 6$, $2^k$ ($k \geq 2$), 10, 12, 14, 18, and 20.

**Corollary 2.2.**

(1) $Spec(6) = \{v|v \equiv 1, 9 \pmod{12}\}$.

(2) $Spec(2^k) = \{v|v \equiv 1 \pmod{2^{k+1}}\}$ *for* $k \geq 2$.

(3) $Spec(10) = \{v|v \equiv 1, 5 \pmod{20}\}$.

(4) $Spec(12) = \{v|v \equiv 1, 9 \pmod{24}\}$.

(5) $Spec(14) = \{v|v \equiv 1, 21 \pmod{28}\}$.

(6) $Spec(18) = \{v|v \equiv 1, 9 \pmod{36}\}$.

(7) $Spec(20) = \{v|v \equiv 1, 25 \pmod{40}\}$.

For any cycle with vertices in $Z_v$, it is proved in [13] that the sum of absolute differences of edges in $C$ must be even.

**Lemma 2.3.** [13] *Let $C = (c_0, c_1, \ldots, c_{m-1})$ be an $m$-cycle with $c_i \in Z_v$ where $0 \leq i \leq m-1$ and $v$ is any positive integer. Then the sum of absolute differences of edges in $C$ is even.*

*Proof.* The proof follows immediately from the fact that

$$\sum_{i=1}^{m} |c_i - c_{i-1}| \equiv \sum_{i=1}^{m} (c_i - c_{i-1}) \equiv 0 \pmod 2 \qquad \blacksquare$$

## 3. Definitions and Preliminaries

Assume $\{a, b\}$ to be any edge in $K_v$, we shall use $\pm|a-b|$ to denote the *difference* of the edge $\{a, b\}$.

Given a subset $\Omega$ of $Z_v \setminus \{0\}$ with $\Omega = -\Omega$, let $G_v[\Omega]$ denote the subgraph of $K_v$ which contains the edges $\{a, a+b\}$ with $a \in Z_v$ and $b \in \Omega$. Where it is clear what $v$ is, the subscript will be omitted and just write $G[\Omega]$.

Let $C$ be an $(m_1, \ldots, m_r)$-cycle in a $k$-cyclic $(m_1, \ldots, m_r)$-cycle system of order $v$. The *$(m_1, \ldots, m_r)$-cycle orbit* $\grave{O}$ of $C$ is defined as the set of distinct $(m_1, \ldots, m_r)$-cycles $\{C + ik | i \in Z_v\}$. The *length* of an $(m_1, \ldots, m_r)$-cycle orbit is its cardinality, i.e., the minimum positive integer $p$ such that $C + pk = C$. An $(m_1, \ldots, m_r)$-cycle orbit of length $v$ is called *full*, otherwise *short*. A *base $(m_1, \ldots, m_r)$-cycle* of an $(m_1, \ldots, m_r)$-cycle orbit $\grave{O}$ is an $(m_1, \ldots, m_r)$-cycle $C \in \grave{O}$ that is chosen arbitrarily. A *base $(m_1, \ldots, m_r)$-cycle* corresponding to an $(m_1, \ldots, m_r)$-cycle orbit $\grave{O}$ is said to be *full* (resp. *short*) if $\grave{O}$ is full (resp. short). An $(m_1, \ldots, m_r)$-cycle orbit $\grave{O}$ is full if and only if the differences of edges of any base $(m_1, \ldots, m_r)$-cycle in $\grave{O}$ are distinct. Any $k$-cyclic $(m_1, \ldots, m_r)$-cycle system of order $v$ could be generated from full or short base $(m_1, \ldots, m_r)$-cycles.

Throughout this paper, we will restrict our attention to the case where $m_i$ $(1 \leq i \leq r)$ are all even $(> 3)$.

**Lemma 3.1.** *Let $G[\Omega]$ be a subgraph of $K_{2pm+kc}$ with $\Omega = \pm\{a_1, \ldots, a_t\}$ and $m$ even. If there exists a $k$-cyclic $(m_1, \ldots, m_r)$-cycle system of $G[\Omega]$, then $t$ is even and $m$ divides $kt$.*

*Proof.* Let $\{a, b\}$ be any edge of $G[\Omega]$. Note that the edges $\{a, b\}$ and $\{a+i, b+i\}$ in $G[\Omega]$ with $i \in Z_{2pm+kc}$ have the same difference. Since $G[\Omega]$ has a $k$-cyclic $(m_1, \ldots, m_r)$-cycle system, this means that the number of edges with the same difference occurring in the union of base $(m_1, \ldots, m_r)$-cycles, say $C$, is precisely $k$, and so the number of edges in $C$ is equal to $kt$, which is a multiple of $m$. It follows that $t$ must be even since $m$ is even. ■

A set of integers is said to be a *complete residue system modulo* $k$ if every integer is congruent modulo $k$ to exactly one integer of the set. For instance, the set $\{0, 1, 7, 8, 4\}$ is a complete residue system of modulo $5$.

Given a subgraph $H$ of $G[\Omega]$ with $ks$ edges, $\gcd(k, s) = 1$, and $k$ odd, the graph $H$ is called *modulo $k$-complete* on $G[\Omega]$ if the following conditions hold:

(1) The edge set of $H$ can be partitioned into $s$ subsets such that each subset contains $k$ edges;

(2) All $k$ edges in each subset have the same difference;

(3) If $\{a_1, b_1\}, \ldots, \{a_k, b_k\}$ with $a_i < b_i$ are distinct edges in a subset, then both of the sets $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$ are complete residue systems modulo $k$; and

(4) For each edge $\{a, b\}$ in $H$, the absolute difference of $a$ and $b$ is less than or equal to $\lfloor (v + 1)/2 \rfloor$.

**Example 1.** Consider a $(4, 8)$-cycle $C = \{(1, 2, 3, 5), (0, 1, 4, 6, 2, 5, 7, 3)\}$, which is a subgraph of $K_9$ with $3 \cdot 4$ edges. It is easy to check that the $(4, 8)$-cycle $C$ is modulo 3-complete on $K_9$, and by virtue of the fact that $C$ is modulo 3-complete on $K_9$, it follows that there exists a 3-cyclic $(4, 8)$-cycle system of order 9.

The following consequence plays a crucial role for the construction of a $k$-cyclic $(m_1, \ldots, m_r)$-cycle system and its proof follows immediately from the definition of modulo $k$-completeness on $G[\Omega]$.

**Proposition 3.2.** *Let $C$ be the union of $(m_1, \ldots, m_r)$-cycles with $ks$ edges, $\gcd(k, s) = 1$, and $k$ odd. If $C$ is modulo $k$-complete on $G_{kc}[\Omega]$, then there exists a k-cyclic $(m_1, \ldots, m_r)$-cycle system of $G_{kc}[\Omega]$.*

By $D$ we mean the difference set in an $(m_1, \ldots, m_r)$-cycle. The following consequences can be found in [14].

**Lemma 3.3.** [14] *For any positive integers $s$ and $t$, there exists a $4s$-cycle with $D = \pm\{t, t+1, \ldots, t+4s-1\}$ in $K_v$ where $v$ is odd with $v \geq 2(t+4s-1)+1$.*

Note that, by Lemma 2.3, there does not exist a $(4s + 2)$-cycle with $D = \pm\{t, t + 1, \ldots, t + 4s + 1\}$ for any positive integer $t$.

**Lemma 3.4.** [14] *Let $s$ and $t$ be any positive integers.*

(1) *There exists a* $(4s+2)$*-cycle with* $D = \pm\{t, t+1, \ldots, t+4s, t+4s+2\}$ *in* $K_v$ *where* $v$ *is odd with* $v \geq 2(t+4s+2)+1$.

(2) *There exists a* $(4s+2)$*-cycle with* $D = \pm\{t, t+2, \ldots, t+4s+2\}$ *in* $K_v$ *where* $v$ *is odd with* $v \geq 2(t+4s+2)+1$.

In order to construct cycles of even length with consecutive differences, one may utilize cycles of length congruent to $0$ modulo $4$ and/or an even number of cycles of length congruent to $2$ modulo $4$.

As immediate consequences of Lemmas 3.3 and 3.4, we have the following two preliminary results.

**Corollary 3.5.** *If* $m_i \equiv 0 \pmod 4$ *for* $1 \leq i \leq r$*, then for any positive integer* $t$*, there exists an* $(m_1, \ldots, m_r)$*-cycle with* $D = \pm\{t, t+1, \ldots, t+m-1\}$ *in* $K_v$ *where* $v$ *is odd with* $v \geq 2(t+m-1)+1$.

**Corollary 3.6.** *Let* $t$ *be any positive integer.*

(1) *If* $r$ *is even and* $m_i \equiv 2 \pmod 4$ *for* $1 \leq i \leq r$*, then there exists an* $(m_1, \ldots, m_r)$*-cycle with* $D = \pm\{t, t+1, ..., t+m-1\}$ *in* $K_v$ *where* $v$ *is odd with* $v \geq 2(t+m-1)+1$.

(2) *If* $r$ *is even and* $m_i \equiv 2 \pmod 4$ *for* $1 \leq i \leq r$*, then there exists an* $(m_1, \ldots, m_r)$*-cycle with* $D = \pm\{t, t+2, \ldots, t+m-1, t+m+1\}$ *in* $K_v$ *where* $v$ *is odd with* $v \geq 2(t+m+1)+1$.

(3) *If* $r$ *is odd and* $m_i \equiv 2 \pmod 4$ *for* $1 \leq i \leq r$*, then there exists an* $(m_1, \ldots, m_r)$*-cycle with* $D = \pm\{t, t+1, \ldots, t+m-2, t+m\}$ *in* $K_v$ *where* $v$ *is odd with* $v \geq 2(t+m)+1$.

(4) *If* $r$ *is odd and* $m_i \equiv 2 \pmod 4$ *for* $1 \leq i \leq r$*, then there exists an* $(m_1, \ldots, m_r)$*-cycle with* $D = \pm\{t, t+2, \ldots, t+m\}$ *in* $K_v$ *where* $v$ *is odd with* $v \geq 2(t+m)+1$.

By $N$ we mean the number of cycles in an $(m_1, \ldots, m_r)$-cycle with length congruent to $2$ modulo $4$.

**Proposition 3.7.** *If there exists a* $k$*-cyclic* $(m_1, \ldots, m_r)$*-cycle system of order* $kc$*, then for any positive integer* $p$*, there exists a* $k$*-cyclic* $(m_1, \ldots, m_r)$*-cycle system of order* $2pm+kc$.

*Proof.* Set $q = \lfloor kc/2 \rfloor$. Note that by Lemma 3.1, $q$ is congruent to $0$ or $2$ $\pmod 4$. Let $\Omega_1 = \pm\{1, 2, \ldots, q\}$, $\Omega_2 = \pm\{q+1, q+2, \ldots, pm+q\}$, $\Omega_3 = \pm\{1, 2, \ldots, q-1, q+1\}$, and $\Omega_4 = \pm\{q, q+2, \ldots, pm+q\}$. It is clear that $K_{2pm+kc}$ is isomorphic to the union of $G[\Omega_1]$ and $G[\Omega_2]$ or $G[\Omega_3]$ and $G[\Omega_4]$. Since there exists a $k-$cyclic $(m_1, \ldots, m_r)$-cycle system of order $kc$, it suffices to show that there exists a cyclic $(m_1, \dots, m_r)$-cycle system of $G[\Omega_2]$ or $G[\Omega_4]$ with full base $(m_1, \dots, m_r)$-cycles. Without loss of generality, we may assume $m_i \equiv 0 \pmod 4$ for $1 \le i \le w$ and $m_i \equiv 2 \pmod 4$ for $w+1 \le i \le r$, where $0 \le w \le r$. Let us set $N = r - w$. We divide the proof into two cases depending on whether $q \equiv 0$ or $2 \pmod 4$.

**Case 1.** $q \equiv 0 \pmod 4$.

If $N$ is even or if $N$ is odd and $p$ is even, then by Corollaries 3.5 and 3.6-(1), we obtain the graph $C$, the union of $p$ edge-disjoint $(m_1, \dots, m_r)$-cycles, with $D = \pm\{q+1, q+2, \dots, pm+q\}$; if $N$ is odd and $p$ is odd, by Corollaries 3.5 and 3.6-(3), we also have the graph $C$ with $D = \pm\{q+1, q+2, \dots, pm+q\}$.

**Case 2.** $q \equiv 2 \pmod 4$.

If $N$ is even or if $N$ is odd and $p$ is even, then by Corollaries 3.5 and 3.6-(2), the graph $C$ with $D = \pm\{q, q+2, q+3, \dots, pm+q\}$ is given; if $N$ is odd and $p$ is odd, by Corollaries 3.5 and 3.6-(4), we obtain the graph $C$ with $D = \pm\{q, q+2, q+3, \dots, pm+q\}$.

Then use the graph $C$ constructed in each case as the base $(m_1, \dots, m_r)$-cycles and the desired $(m_1, \dots, m_r)$-cycle system follows. ■

For clarity, we give some examples to demonstrate the construction of a $k$-cyclic $(m_1, \dots, m_r)$-cycle system stated above.

**Example 2.** Consider the $(4, 8)$-cycle $C$ in Example 1. Use $C$ as the base $(4, 8)$-cycle and a 3-cyclic $(4, 8)$-cycle system of $G_9[\Omega]$ with $\Omega = \pm\{1, 2, 3, 4\}$ then follows. Since the absolute difference of each edge $\{a, b\}$ in $C$ is less than or equal to 4, it means that the $(4, 8)$-cycle $C$ can also be used as the base $(4, 8)$-cycle of a 3-cyclic $(4, 8)$-cycle system of $G_{24p+9}[\Omega]$ and hence, a 3-cyclic $(4, 8)$-cycle system of $G_{24p+9}[\Omega]$ does exist for $p \ge 0$.

**Example 3.** For $1 \le i \le 5$, let $C_i$ be $(4, 10)$-cycles with $7 \cdot 2$ edges given as

$C_1 = \{(4, 5, 8, 10), (2, 8, 16, 9, 17, 6, 10, 7, 14, 3)\}$;

$C_2 = \{(8, 17, 10, 19), (1, 2, 4, 3, 6, 14, 5, 9, 18, 7)\}$;

$C_3 = \{(5, 10, 18, 11), (1, 6, 12, 20, 13, 9, 7, 3, 8, 4)\}$;

$C_4 = \{(5, 7, 8, 6), (0, 5, 16, 7, 11, 19, 12, 8, 15, 6)\}$; and

$C_5 = \{(4, 7, 6, 9), (2, 5, 3, 9, 20, 11, 6, 4, 15, 7)\}$.

Assume $C$ to be the union of $(4, 10)$-cycles $C_1, \dots, C_5$ and let $v = 21 = 7 \cdot 3$. An easy verification shows that $C$ is modulo 7-complete on $K_{21}$ and a 7-cyclic $(4, 10)$-cycle system of order $28t + 21$ exist for each $t \ge 0$.

## 4. Proof of the Main Result

We are now in a position to prove the main result in this paper.

*Proof of Theorem 1.2.* By virtue of Theorem 1.1, it follows that there exists a cyclic $(m_1, \ldots, m_r)$-cycle system of order $2pm+1$ for $p \geq 1$. So, we need only consider the remaining case. By Propositions 2.1 and 3.7, it is enough to show that there exists a $k$-cyclic $(m_1, \ldots, m_r)$-cycle system of order $kc$ if $kc < 2m+1$, or that there exists a $k$-cyclic $(m_1, \ldots, m_r)$-cycle system of $G_{kc}[\Omega]$ if $kc \geq 2m+1$, where $\Omega = \pm\{1, 2, \ldots, \lfloor kc/2 \rfloor - m\}$ or $\pm\{1, 2, \ldots, \lfloor kc/2 \rfloor - m - 1, \lfloor kc/2 \rfloor - m + 1\}$.

In order to accomplish this objective, by Proposition 3.2, it suffices to prove that $C$, the union of $(m_1, \ldots, m_r)$-cycles, is modulo $k$-complete on $G_{kc}[\Omega]$. For the convenience of notation, by $(G_{kc}[\Omega]; C_{m_1}, \ldots, C_{m_r}; k)$-CS we mean the union of $(m_1, \ldots, m_r)$-cycles which is modulo $k$-complete on $G_{kc}[\Omega]$. The proof is split into 6 cases depending on $m = 6$, $10$, $12$, $14$, $18$, or $20$. We recall that for each $m$, the $Spec(m)$ is given in Corollary 2.2.

**Case 1.** Suppose that $\sum_{i=1}^{r} m_i = 6$.

$(K_9; C_6; 3)$-CS $= \{(0,1,2,5,7,3), (1,5,3,2,6,4)\}$.

**Case 2.** Suppose that $\sum_{i=1}^{r} m_i = 10$.

$(G_{25}[\pm\{1,3\}]; C_{10}; 5)$-CS $= \{(0,3,6,9,8,7,4,5,2,1)\}$.

$(G_{25}[\pm\{1,3\}]; C_4, C_6; 5)$-CS $= \{(0,1,2,3), (3,4,7,10,9,6)\}$.

**Case 3.** Suppose that $\sum_{i=1}^{r} m_i = 12$.

$(G_{33}[\pm\{1,2,3,4\}]; C_{12}; 3)$-CS $= \{(1,5,8,9,12,14,16,15,11,10,7,3)\}$.

$(K_9; C_4, C_8; 3)$-CS $= \{(1,2,3,5), (0,1,4,6,2,5,7,3)\}$.

$(K_9; C_6, C_6; 3)$-CS $= \{(0,3,5,8,4,1), (1,2,6,5,7,3)\}$.

$(K_9; C_4, C_4, C_4; 3)$-CS $= \{(0,1,5,2), (1,2,6,3), (2,3,7,4)\}$.

**Case 4.** Suppose that $\sum_{i=1}^{r} m_i = 14$.

$$
\begin{aligned}
(K_{21}; C_{14}; 7)\text{-CS} = \{&(2,3,5,4,15,7,9,17,6,11,18,10,19,8),\\
&(1,2,4,3,8,10,6,14,5,11,20,9,18,7),\\
&(3,14,13,12,19,11,7,10,5,9,16,8,4,6),\\
&(0,5,2,7,3,9,13,16,11,14,15,8,12,6),
\end{aligned}
$$

$$(1,4,10,17,8,5,16,7,14,12,20,13,15,6)\}.$$

$$\begin{aligned}(K_{21};C_4,C_{10};7)\text{-CS} = \{&\{(4,5,8,10),(2,8,16,9,17,6,10,7,14,3)\}\cup\\ &\{(8,17,10,19),(1,2,4,3,6,14,5,9,18,7)\}\cup\\ &\{(5,10,18,11),(1,6,12,20,13,9,7,3,8,4)\}\cup\\ &\{(5,7,8,6),(0,5,16,7,11,19,12,8,15,6)\}\cup\\ &\{(4,7,6,9),(2,5,3,9,20,11,6,4,15,7)\}\}.\end{aligned}$$

$$\begin{aligned}(K_{21};C_6,C_8;7)\text{-CS} = \{&\{(6,10,7,18,9,17),(3,8,4,15,6,11,20,9)\}\cup\\ &\{(7,11,19,12,8,15),(0,6,1,4,10,19,8,5)\}\cup\\ &\{(6,12,20,13,9,7),(2,5,4,3,14,6,8,7)\}\cup\\ &\{(1,10,5,14,7,3),(2,3,5,9,4,7,16,8)\}\cup\\ &\{(3,10,18,11,5,6),(1,2,4,6,9,16,5,7)\}\}\end{aligned}$$

$$\begin{aligned}(K_{21};C_4,C_4,C_6;7)\text{-CS} = \{&\{(1,4,15,6),(6,9,20,11),(2,8,16,9,5,3)\}\cup\\ &\{(2,5,16,7),(0,5,14,6),(6,10,7,18,9,17)\}\cup\\ &\{(4,5,8,10),(5,7,8,6),(7,11,19,12,8,15)\}\cup\\ &\{(8,17,10,19),(3,9,4,8),(6,12,20,13,9,7)\}\cup\\ &\{(5,10,18,11),(3,4,7,14),(1,2,4,6,3,7)\}\}.\end{aligned}$$

**Case 5.** Suppose that $\sum_{i=1}^{r} m_i = 18$.
It follows from Lemma 3.1 that there does not exist 3-cyclic $(4,6,8)$- and $(4,4,4,6)$-cycle systems of order 9

$$\begin{aligned}&(G_{45}[\pm\{1,2,3,4\}];C_{18};9)\text{-CS} =\\ &\qquad\{(1,2,5,3,7,6,9,10,12,15,19,17,16,14,13,11,8,4),\\ &\qquad(7,11,9,12,13,15,17,21,20,24,23,26,27,25,22,18,14,10)\}\end{aligned}$$

$$\begin{aligned}&(G_{45}[\pm\{1,2,3,4\}];C_4,C_{14};9)\text{-CS} =\\ &\qquad\{\{(8,9,10,12),(1,2,3,7,10,14,12,13,17,15,11,9,6,4)\}\cup\\ &\qquad\{(4,5,8,7),(5,6,10,8,11,14,18,22,20,16,15,12,9,7)\}.\end{aligned}$$

$$\begin{aligned}&(G_{45}[\pm\{1,2,3,4\}];C_6,C_{12};9)\text{-CS} =\\ &\qquad\{\{(5,7,10,9,8,6),(1,5,9,12,14,17,13,10,8,7,6,2)\}\cup\\ &\qquad\{(3,7,11,9,6,4),(2,5,4,7,9,13,11,8,12,10,6,3)\}\}\end{aligned}$$

$$(G_{45}[\pm\{1,2,3,4\}];C_8,C_{10};9)\text{-CS} =$$

$\{\{(9,10,14,12,13,17,15,11),(4,5,8,12,10,6,9,13,11,7)\}\cup$

$\{(7,8,11,14,16,15,12,9),(1,2,3,7,10,8,9,5,6,4)\}\}.$

$(G_{45}[\pm\{1,2,3,4\}];C_4,C_4,C_{10};9)$-CS $=$

$\{\{(0,1,5,3),(3,4,8,6),(4,5,8,12,10,6,9,13,11,7)\}\cup$

$\{(7,9,11,8),(2,6,7,5),(1,2,3,7,10,8,9,5,6,4)\}\}.$

$(G_{45}[\pm\{1,2,3,4\}];C_4,C_6,C_8;9)$-CS $=$

$\{\{(0,1,5,3),(6,7,10,12,8,9),(1,2,5,7,8,6,3,4)\}\cup$

$\{(5,6,10,8),(4,5,9,13,11,7),(2,3,7,9,11,8,4,6)\}\}.$

$(G_{45}[\pm\{1,2,3,4\}];C_4,C_4,C_4,C_6;9)$-CS $=$

$\{\{(0,1,5,3),(1,2,6,4),(2,3,7,5),(6,7,10,12,8,9)\}\cup$

$\{(3,4,8,6),(5,6,10,8),(7,9,11,8),(4,5,9,13,11,7)\}\}.$

**Case 6.** Suppose that $\sum_{i=1}^{r}=20$.

$(K_{25};C_{20};5)$-CS $=$

$\{(1,6,15,9,8,4,14,19,22,11,17,20,21,18,13,5,16,7,12,2),$

$(0,4,2,13,11,9,16,8,15,10,20,18,12,24,17,5,7,3,14,6),$

$(2,3,12,5,4,16,6,18,11,15,24,21,17,9,20,8,14,23,13,10)\}.$

$(K_{25};C_4,C_{16};5)$-CS $=$

$\{\{(4,11,16,12),(2,12,7,16,17,11,22,19,14,4,8,9,15,6,18,13)\}\cup$

$\{(0,6,5,4),(2,4,16,5,7,3,14,8,20,18,12,24,15,11,13,10)\}\cup$

$\{(2,3,12,5),(0,8,18,9,1,11,3,10,5,15,4,6,13,16,19,12)\}\}.$

$(K_{25};C_6,C_{14};5)$-CS $=$

$\{\{(4,5,6,12,16,11),(0,8,18,9,1,11,5,15,4,6,13,16,19,12)\}\cup$

$\{(5,9,17,16,7,12),(2,5,10,3,14,4,8,20,18,12,24,15,11,13)\}\cup$

$\{(2,3,7,5,16,4),(2,12,3,11,22,19,14,8,9,15,6,18,13,10)\}\}.$

$(K_{25};C_8,C_{12};5)$-CS $=$

$\{\{(1,13,4,9,6,12,3,5),(0,12,2,3,1,6,7,9,8,4,16,5)\}\cup$

$\{(1,12,4,5,2,10,3,9),(0,8,2,9,15,3,6,4,14,5,12,10)\}\cup$

$\{(0,1,11,2,14,3,13,6),(2,6,15,4,11,3,7,12,9,5,8,13)\}\}.$

$(K_{25}; C_{10}, C_{10}; 5)$-CS $=$

$\{\{(0,5,16,4,9,7,6,1,3,12),(2,11,3,7,12,9,5,8,13,6)\}\cup$

$\{(0,10,12,5,4,6,3,9,2,8),(1,9,15,3,10,2,5,14,4,12)\}\cup$

$\{(0,1,5,3,14,2,13,4,15,6),(1,11,4,8,9,6,12,2,3,13)\}\}.$

$(K_{25}; C_4, C_4, C_{12}; 5)$-CS $=$

$\{\{(3,12,4,9),(1,5,2,3),(0,12,2,11,1,6,7,9,8,4,16,5)\}\cup$

$\{(2,10,3,14),(1,9,6,12),(0,8,2,9,15,3,6,4,14,5,12,10)\}\cup$

$\{(0,1,13,6),(3,5,4,13),(2,6,15,4,11,3,7,12,9,5,8,13)\}\}.$

$(K_{25}; C_4, C_6, C_{10}; 5)$-CS $=$

$\{\{(3,11,4,15),(2,12,7,6,4,13),(0,10,12,4,5,9,7,3,13,6)\}\cup$

$\{(1,9,2,5),(0,1,6,2,3,12),(0,8,2,11,1,3,9,4,16,5)\}\cup$

$\{(2,10,3,14),(5,12,6,15,9,8),(1,12,9,6,3,5,14,4,8,13)\}\}.$

$(K_{25}; C_4, C_8, C_8; 5)$-CS $=$

$\{\{(3,11,4,15),(0,8,2,11,1,9,3,12),(0,5,16,4,9,2,6,1)\}\cup$

$\{(1,5,2,3),(0,6,7,3,13,4,12,10),(4,8,9,15,6,12,5,14)\}\cup$

$\{(2,10,3,14),(1,12,9,6,3,5,8,13),(2,13,6,4,5,9,7,12)\}\}.$

$(K_{25}; C_6, C_6, C_8; 5)$-CS $=$

$\{\{(1,9,5,4,8,13),(2,9,7,3,14,5),(0,10,12,4,15,3,13,6)\}\cup$

$\{(2,12,7,6,4,13),(0,1,6,2,3,12),(5,12,6,8,14,9,21,10)\}\cup$

$\{(1,12,9,6,3,5),(2,10,3,11,4,14),(5,13,7,16,6,15,9,8)\}\}.$

$(K_{25}; C_4, C_4, C_4, C_8; 5)$-CS $=$

$\{\{(4,14,5,8),(3,11,4,15),(0,4,2,7),(0,8,2,11,1,9,3,12)\}\cup$

$\{(1,5,2,3),(1,4,3,8),(1,12,2,13),(0,5,16,4,9,2,6,1)\}\cup$

$\{(3,5,4,6),(2,10,3,14),(1,7,4,10),(0,6,7,3,13,4,12,10)\}\}.$

$(K_{25}; C_4, C_4, C_6, C_6; 5)$-CS $=$

$\{\{(4,14,5,8),(1,5,2,3),(1,9,3,15,4,11),(0,8,2,11,3,12)\}\cup$

$\{(1,4,3,8),(1,12,2,13),(0,5,16,4,6,1),(2,6,3,5,4,9)\}\cup$

$\{(2,10,3,14),(1,7,4,10),(0,6,7,3,13,4),(0,10,12,4,2,7)\}\}.$ ■

## 5. Concluding Remark

So far, for each even $m_i$ with $\sum_{i=1}^{r} m_i \leq 20$, a $k$-cyclic $(m_1, \ldots, m_r)$-cycle system is given, but we can not find out an ingenious method to construct a $k$-cyclic $(m_1, \ldots, m_r)$-cycle in general. It is natural, however, to pose the following problem.

**Conjecture.** Suppose $\sum_{i=1}^{r} m_i = ks$ with $m_i$ even, $\gcd(k, s) = 1$, and $k$ odd and let $c$ be the least positive integral solution of $kx \equiv 1 \pmod{2s}$ satisfying $kc \geq m_r$. Then there exists a $k$-cyclic $(m_1, \ldots, m_r)$-cycle system of order $kc$.

Moreover, we may ask whether the values of $m_i$'s in an $(m_1, \ldots, m_r)$-cycle could be odd. In fact, we believe that the existence problem for $k$-cyclic $(m_1, \ldots, m_r)$-cycle system is still correct even though some of $m_i$'s in an $(m_1, \ldots, m_r)$-cycle are odd. It turns out, however, a much more difficult problem. We conclude this paper with an example.

Let $C$ be a $(3, 4, 5)$-cycle with $3 \cdot 4$ edges and $v = 33 = 3 \cdot 11$. The base $(3, 4, 5)$-cycles of the 3-cyclic $(3, 4, 5)$-cycle system of order 33 are:

$(G_{33}[\pm\{1, 2, 3, 4\}]; C_3, C_4, C_5; 3)$-CS $= \{(0, 1, 2), (1, 5, 2, 3), (2, 4, 7, 3, 6)\}$, and

$(G_{33}[\pm\{5, 6, \ldots, 16\}]; C_3, C_4, C_5; 1)$-CS $= \{(0, 5, 11), (0, 12, 28, 13), (0, 7, 15, 24, 10)\}$.

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