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POSITIVE SOLUTIONS OF P-LAPLACIAN M-POINT BOUNDARY VALUE PROBLEMS ON TIME SCALES

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Abstract. Let $\mathbb{T}$ be a time scale such that $0, T \in \mathbb{T}$, $a_i \geq 0$ for $i = 1, \ldots, m-2$. Let ξ_i satisfy $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < \rho(T)$ and $\sum_{i=1}^{m-2} a_i < 1$. We consider the following p-Laplacian m-point boundary value problem on time scales

$$\left(\varphi_p\left(u^{\Delta}(t)\right)\right)^{\nabla} + a(t)f(t, u(t)) = 0, t \in (0, T),$$
$$u^{\Delta}(0) = 0,\ u(T) = \sum_{i=1}^{m-2} a_i u(\xi_i),$$

where $a \in C_{ld}[(0,T), [0,\infty)]$ and $f \in C\left((0,T) \times [0,\infty), [0,\infty)\right)$. Some new results are obtained for the existence of at least single, twin or triple positive solutions of the above problem by applying Krasnosel'skii's fixed point theorem, new fixed point theorem due to Avery and Henderson and Leggett-Williams fixed point theorem. In particular, our criteria extend and improve some known results.

1. INTRODUCTION

The study of dynamic equations on time scales goes back to its founder Stefan Hilger [12], and is a new area of still fairly theoretical exploration in mathematics. Motivating the subject is the notion that dynamic equations on time scales can build bridges between continuous and discrete mathematics. Further, the study of time scales has led to several important applications, e.g., in the study of insect population models, neural networks, heat transfer and epidemic models [1].

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We begin by presenting some basic definitions which can be found in [1, 6, 12, 14]. Another excellent source on dynamic equations on time scales is the book [7].

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\} \text{ and } \rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\}$$

(supplemented by $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$) are well defined. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t, \rho(t) < t, \sigma(t) = t$, $\sigma(t) > t$, respectively. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_\kappa = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^\kappa = \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^\kappa = \mathbb{T}$. The forward graininess is $\mu(t) := \sigma(t) - t$. Similarly, the backward graininess is $\upsilon(t) := t - \rho(t)$.

We make the blanket assumption that $0, T$ are points in $\mathbb{T}$. By an interval $(0, T)$ we always mean $(0, T) \cap \mathbb{T}$. Other type of intervals are defined similarly.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, the delta derivative [6] of f at t, denoted by $f^\Delta(t)$, is the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s] \right| \leq \epsilon \left| \sigma(t) - s \right|,$$

for all $s \in U$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, the nabla derivative [4] of f at t, denoted by $f^\nabla(t)$, is the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\rho(t)) - f(s) - f^\nabla(t)[\rho(t) - s] \right| \leq \epsilon \left| \rho(t) - s \right|,$$

for all $s \in U$.

In the case $\mathbb{T} = \mathbb{R}$, $f^\Delta(t) = f'(t) = f^\nabla(t)$; when $\mathbb{T} = \mathbb{Z}$, $f^\Delta(t) = f(t+1) - f(t)$ and $f^\nabla(t) = f(t) - f(t-1)$.

A function $f : \mathbb{T} \to \mathbb{R}$ is ld-continuous provided it is continuous at left dense points in $\mathbb{T}$ and its right sided limit exists (finite) at right dense points in $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous provided it is continuous at right dense points in $\mathbb{T}$ and its left sided limit exists (finite) at left dense points in $\mathbb{T}$. If $\mathbb{T} = \mathbb{R}$, then f is ld-continuous if and only if f is continuous. If $\mathbb{T} = \mathbb{Z}$, then any function is ld-continuous and rd-continuous. It is known [4] that if f is ld-continuous, then there is a function $F(t)$ such that $F^\nabla(t) = f(t)$. In this case, we define

$$\int_a^b f(\tau)\nabla\tau = F(b) - F(a).$$

Very recently, there is an increasing attention paid to question of positive solution for second order three point boundary value problems on time scales [3, 8, 19, 20, 22-24], other related results are referred to [10, 16, 25, 26]. In particular, we would like to mention some results of Anderson [3], Kaufmann [19], and Sun and Li [23], which motivate us to consider one-dimensional p-Laplacian m-point boundary value problem on time scales.

In [3], for the problem

$$u^{\Delta\nabla}(t) + f(t, u(t)) = 0, t \in (0, T), \tag{1.1}$$

$$u(0) = 0,\ \alpha u(\eta) = u(T), \tag{1.2}$$

where $f : (0, T) \times [0, \infty) \to [0, \infty)$ is ld-continuous, $\alpha > 0$, $\eta \in (0, \rho(T))$ and $\alpha\eta < T$, Anderson developed some existence criteria of three positive solutions by Leggett-Williams fixed point theorem [15]. He also used Krasnosel'skii's fixed point theorem [9, 13] to obtain the existence of at least one positive solution to the problem

$$u^{\Delta\nabla}(t) + a(t)f(u(t)) = 0, t \in (0, T), \tag{1.3}$$

with boundary condition (1.2) when f is superlinear or sublinear. In [19], Kaufmann gave the existence results of at least two positive solutions of (1.3) and (1.2). In [23], Sun and Li discussed the dynamic equation (1.3) with the boundary condition

$$u^{\Delta}(0) = 0,\ \alpha u(\eta) = u(T), \tag{1.4}$$

where $0 < \alpha < 1$ and got the existence of at least one or two positive solutions.

However, to the best of our knowledge, there are not any results on the existence of positive solution for p-Laplacian multi-point boundary value problems on time scales.

In this paper we discuss the existence of positive solutions for the one-dimensional p-Laplacian m-point boundary value problem on time scales

$$\left(\varphi_p\left(u^{\Delta}(t)\right)\right)^{\nabla} + a(t)f(t, u(t)) = 0,\ t \in (0, T), \tag{1.5}$$

$$u^{\Delta}(0) = 0,\ u(T) = \sum_{i=1}^{m-2} a_i u(\xi_i), \tag{1.6}$$

where $\varphi_p(u)$ is p-Laplacian operator, i.e., $\varphi_p(u) = |u|^{p-2} u$, $p > 1, (\varphi_p)^{-1} = \varphi_q$, $1/p + 1/q = 1$. Some new and more general results are obtained for the existence of at least one, two and three positive solutions for the above problem by using Krasnosel'skii's fixed theorem [9, 13], new fixed point theorems due to Avery and Henderson [5] and Leggett-Williams fixed point theorem [15]. The results are even new for the special cases of difference equations and differential equations, as well as in the general time scale setting. In particular, our results in Section 3 include and extend many results of Sun and Li [23] $(p = 2, m = 3, f(t, u) = f(u))$ for the general time scale $\mathbb{T}$; Liu [17] $(p = 2, m = 3, f(t, u) = f(u))$, and Webb [28] $(p = 2, m = 3, f(t, u) = f(u))$ in the case $\mathbb{T} = \mathbb{R}$.

It is also noted that when $\sum_{i=1}^{m-2} a_i = 0$, boundary value problem (1.5) and (1.6) reduce to one dimension p-Laplacian two point boundary value problem. Thus, our results in Section 3 include the main results of He [11] $(f(t, u) = f(u))$ in the case of $\mathbb{T} = \mathbb{Z}$ and Sun and Ge [27] $(f(t, u) = f(u))$ in the case of $\mathbb{T} = \mathbb{R}$.

The rest of the paper is organized as follows. In Section 2, we first give four lemmas which are needed throughout this paper and then state several fixed point results: Krasnosel'skii's fixed point theorem in a cone, new fixed point theorem due to Avery and Henderson and Leggett-Williams fixed point theorem. In Section 3 we use Krasnosel'skii's fixed point theorem to obtain the existence of at least one or two positive solutions of problem (1.5) and (1.6). Section 4 will further discuss the existence of twin positive solutions of problem (1.5) and (1.6). Two new results and some corollaries will be presented by new fixed point theorem due to Avery and Henderson. Section 5 is due to develop existence criteria for (at least) three positive and arbitrary odd positive solutions of problem (1.5) and (1.6). In particular, our results in this section are new when $\mathbb{T} = \mathbb{R}$ (the continual case) and $\mathbb{T} = \mathbb{Z}$ (the discrete case).

For convenience, we list the following hypotheses:

(A_1) $a_i \geq 0$ for $i = 1, \ldots, m-2$, $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < \rho(T)$ and $d = 1 - \sum_{i=1}^{m-2} a_i > 0$;

(A_2) $a : (0, T) \to [0, \infty)$ is ld-continuous such that $a(t_0) > 0$ for at least one $t_0 \in [0, T)$ and $f : (0, T) \times [0, \infty) \to [0, \infty)$ is continuous.

2. Some Lemmas

To prove the main results in this paper, we will employ several lemmas. These lemmas are based on the linear boundary value problem

$$(2.1) \qquad \left(\varphi_p\left(u^{\Delta}(t)\right)\right)^{\nabla} + h(t) = 0, t \in (0, T),$$

$$(2.2) \qquad u^{\Delta}(0) = 0, u(T) = \sum_{i=1}^{m-2} a_i u(\xi_i).$$

Lemma 2.1. *If $d \neq 0$, then for $h \in C_{ld}[0, T]$ the boundary value problem*

(2.1) *and* (2.2) *has the unique solution*

$$\begin{aligned} u(t) = &-\int_0^t \varphi_q\left(\int_0^s h(\tau)\nabla\tau\right)\Delta s + \frac{1}{d}\int_0^T \varphi_q\left(\int_0^s h(\tau)\nabla\tau\right)\Delta s \\ &-\frac{1}{d}\sum_{i=1}^{m-2} a_i\int_0^{\xi_i}\varphi_q\left(\int_0^s h(\tau)\nabla\tau\right)\Delta s. \end{aligned} \tag{2.3}$$

Proof. The proof is similar to that of Lemma 2.1 in [24]. For convenience, we list it here.

Let u be as in (2.3). Routine calculations verify that u satisfies the boundary conditions in (2.2). By Theorem 2.10 (i) in [4] or Theorem 8.50 (i) in [6, p 333],

$$\left(\int_a^t f(t,s)\Delta s\right)^{\Delta} = f(\sigma(t),t) + \int_a^t f^{\Delta}(t,s)\Delta s,$$

if f and f^{Δ} are continuous. Using this theorem to take the delta derivative of (2.3) we have

$$u^{\Delta}(t) = -\varphi_q\left(\int_0^t h(\tau)\nabla\tau\right).$$

Thus

$$\varphi_p(u^{\Delta}(t)) = -\int_0^t h(\tau)\nabla\tau.$$

Taking the nabla derivative of this expression and applying Theorem 2.10 (iv) in [4] or Theorem 8.50 (iv) in [6, p 333],

$$\left(\int_a^t f(t,s)\nabla s\right)^{\nabla} = f(\rho(t),t) + \int_a^t f^{\nabla}(t,s)\nabla s$$

yields $\left(\varphi_p\left(u^{\Delta}(t)\right)\right)^{\nabla} = -h(t)$, so that u given in (2.3) is a solution of (2.1) and (2.2).

It is easy to see that boundary value problem $\left(\varphi_p\left(x^{\Delta}(t)\right)\right)^{\nabla} = 0$, $x^{\Delta}(0) = 0, x(T) = \sum_{i=1}^{m-2} a_i x(\xi_i)$ has only the trivial solution if $d \neq 0$. Thus u in (2.3) is the unique solution of (2.1) and (2.2). The proof is complete.

Lemma 2.2. *Let* $d > 0$. *If* $h \in C_{ld}[0,T]$ *and* $h \geq 0$, *then the unique solution* u *of (2.1) and (2.2) satisfies*

$$u(t) \geq 0,\ t \in [0,T].$$

Proof. Since $\left(\varphi_p\left(u^{\Delta}(t)\right)\right)^{\nabla} = -h(t) \leq 0$ and $\varphi_p(t)$ is a increasing function, thus $u^{\Delta}(t)$ is nonincreasing. So we obtain $u^{\Delta}(t) \leq u^{\Delta}(0) = 0$ and $u(t)$ is a nonincreasing function, that is $u(t) \geq u(T)$ for $t \in [0, T]$. It suffices to prove that $u(T) \geq 0$.

In view of

$$u(T) = \sum_{i=1}^{m-2} a_i u(\xi_i) \geq \sum_{i=1}^{m-2} a_i u(T),$$

and $\sum_{i=1}^{m-2} a_i < 1$, we have $u(T) \geq 0$. The proof is complete.

Now, let the Banach space $B = C_{ld}[0, T]$ (see [3]) be endowed with the norm $\|u\| = \sup_{t\in[0,T]} |u(t)|$, and choose the cone $P \subset B$ defined by

$$P = \left\{ \begin{array}{c} u \in B : u(t) \geq 0 \text{ for } t \in [0, T] \text{ and} \\ u^{\Delta\nabla}(t) \leq 0, u^{\Delta}(t) \leq 0 \text{ for } t \in (0, T),\ u^{\Delta}(0) = 0 \end{array} \right\}.$$

Clearly, $\|u\| = u(0)$ for $u \in P$. Define the operator $A : P \to B$ by

$$\begin{aligned} Au(t) \ = & -\int_0^t \varphi_q \left(\int_0^s a(\tau) f(\tau, u(\tau)) \nabla\tau \right) \Delta s \\ & + \frac{1}{d} \int_0^T \varphi_q \left(\int_0^s a(\tau) f(\tau, u(\tau)) \nabla\tau \right) \Delta s \\ & - \frac{1}{d} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q \left(\int_0^s a(\tau) f(\tau, u(\tau)) \nabla\tau \right) \Delta s. \end{aligned} \tag{2.4}$$

Obviously, $Au(t) \geq 0$ for $t \in [0, T]$.

From the definition of A, we claim that for each $u \in P$, $Au \in P$ and satisfies (1.6) and $Au(0)$ is the maximum value of $Au(t)$ on $[0, T]$.

In fact,

$$(Au)^{\Delta}(t) = -\varphi_q \left(\int_0^t a(\tau) f(\tau, u(\tau)) \nabla\tau \right) \geq 0,\ \ t \in (0, T),$$

is continuous and nonincreasing in $(0, T)$; moreover, $\varphi_q(x)$ is a monotone increasing continuously differentiable function and

$$\left(\int_0^t a(\tau) f(\tau, u(\tau)) \nabla\tau \right)^{\nabla} = a(t) f(t, u(t)) \geq 0,$$

then by the chain rule [6, Theorem 1.87, p31], we obtain

$$(Au)^{\Delta\nabla}(t) \leq 0,$$

so, $A : P \to P$.

Lemma 2.3. $A : P \to P$ *is completely continuous*.

Proof. First, we show that A maps bounded set into itself.

Assume $c > 0$ is a constant and $u \in \overline{P}_c = \{x \in P : \|x\| \le c\}$. Note that the continuity of $f(t,u)$ guarantees that there is a $C > 0$ such that $f(t,u) \le \varphi_p(C)$ for $t \in [0,T]$. So

$$\begin{aligned}
\|Au\| &= Au(0)\\
&= \frac{1}{d}\int_0^T \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\\
&\quad -\frac{1}{d}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\\
&\le \frac{1}{d}\int_0^T \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\\
&\le \frac{C}{d}\int_0^T \varphi_q\left(\int_0^s a(\tau))\nabla\tau\right)\Delta s.
\end{aligned}$$

That is, $A\overline{P}_c$ is uniformly bounded.

In addition, notice that

$$\begin{aligned}
|Au(t_1) - Au(t_2)| &= \left|\int_{t_1}^{t_2} \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\right|\\
&\le C\left|\int_{t_1}^{t_2} \varphi_q\left(\int_0^s a(s)\Delta s\right)\Delta\tau\right|\\
&\le C\,|t_1 - t_2|\,\varphi_q\left(\int_0^T a(s)\Delta s\right).
\end{aligned}$$

So, by applying Arzela-Ascoli theorem on time scales [2] we obtain that $A\overline{P}_c$ is relatively compact.

In view of Lebesgue's dominated convergence theorem on time scales [21], it is easy to prove that A is continuous. Hence, A is completely continuous. The proof is complete.

Lemma 2.4. *Let* $u \in P$*, then* $u(t) \ge \frac{T-t}{T}\|u\|$ *for* $t \in [0,T]$.

Proof. Since $u^{\Delta\nabla}(t) \le 0$, it follows that $u^{\Delta}(t)$ is nonincreasing. Thus, for $0 < t < T$,

$$u(t) - u(0) = \int_0^t u^{\Delta}(s)\Delta s \ge t u^{\Delta}(t)$$

and

$$u(T)-u(t)=\int_t^T u^{\Delta}(s)\Delta s\le (T-t)u^{\Delta}(t),$$

from which we have

$$u(t)\ge \frac{tu(T)+(T-t)u(0)}{T}\ge \frac{T-t}{T}u(0)=\frac{T-t}{T}\left\|u\right\|.$$

The proof is complete. ■

In the rest of this section, we provide some background material from the theory of cones in Banach spaces, and we then state several fixed point theorems which we needed later.

Let E be a Banach space and P be a cone in E. A map $\psi : P \to [0,+\infty)$ is said to be a nonnegative, continuous and increasing functional provided ψ is nonnegative, continuous and satisfies $\psi(x)\le\psi(y)$ for all $x, y \in P$ and $x \le y$.

Given a nonnegative continuous functional ψ on a cone P of a real Banach space E, we define, for each $d > 0$, the set

$$P(\psi, d)=\{x\in P:\psi(x)<d\}.$$

Lemma 2.5. [9, 13] *Let P be a cone in a Banach space E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$. If*

$$A : P\cap(\overline{\Omega}_2\backslash\Omega_1)\to P$$

is a completely continuous operator such that either

(*i*) $\left\|Ax\right\|\le\left\|x\right\|, \forall x\in P\cap\partial\Omega_1$ *and* $\left\|Ax\right\|\ge\left\|x\right\|, \forall x\in P\cap\partial\Omega_2$,

or

(*ii*) $\left\|Ax\right\|\ge\left\|x\right\|, \forall x\in P\cap\partial\Omega_1$ *and* $\left\|Ax\right\|\le\left\|x\right\|, \forall x\in P\cap\partial\Omega_2$. *Then A has a fixed point in $P\cap(\overline{\Omega_2}\backslash\Omega_1)$.*

Lemma 2.6. [5]. *Let P be a cone in a real Banach space E. Let α and γ be increasing, nonnegative continuous functional on P, and let θ be a nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some $c > 0$ and $H > 0$,*

$$\gamma(x)\le\theta(x)\le\alpha(x) \text{ and } \left\|x\right\|\le H\gamma(x)$$

for all $x \in \overline{P(\gamma, c)}$. Suppose there exist a completely continuous operator $A : \overline{P(\gamma, c)} \to P$ and $0 < a < b < c$ such that

$$\theta(\lambda x)\le\lambda\theta(x) \text{ for } 0\le\lambda\le 1 \text{ and } x\in\partial P(\theta, b)$$

and

(i) $\gamma(Ax) > c$ *for all* $x \in \partial P(\gamma, c)$;

(ii) $\theta(Ax) < b$ *for all* $x \in \partial P(\theta, b)$;

(iii) $P(\alpha, a) \neq \emptyset$ *and* $\alpha(Ax) > a$ *for* $x \in \partial P(\alpha, a)$.

Then, A *has at least two fixed points,* x_1 *and* x_2 *belonging to* $\overline{P(\gamma, c)}$ *satisfying*

$$a < \alpha(x_1) \text{ with } \theta(x_1) < b, \text{ and } b < \theta(x_2) \text{ with } \gamma(x_2) < c.$$

The following lemma is similar to Lemma 2.6.

Lemma 2.7. [18] *Let* P *be a cone in a real Banach space* E. *Let* α *and* γ *be increasing, nonnegative continuous functional on* P, *and let* θ *be a nonnegative continuous functional on* P *with* $\theta(0) = 0$ *such that, for some* $c > 0$ *and* $H > 0$,

$$\gamma(x) \leq \theta(x) \leq \alpha(x) \text{ and } \|x\| \leq H\gamma(x)$$

for all $x \in \overline{P(\gamma, c)}$. *Suppose there exist a completely continuous operator* $A : \overline{P(\gamma, c)} \to P$ *and* $0 < a < b < c$ *such that*

$$\theta(\lambda x) \leq \lambda\theta(x) \text{ for } 0 \leq \lambda \leq 1 \text{ and } x \in \partial P(\theta, b)$$

and

(i) $\gamma(Ax) < c$ *for all* $x \in \partial P(\gamma, c)$;

(ii) $\theta(Ax) > b$ *for all* $x \in \partial P(\theta, b)$;

(iii) $P(\alpha, a) \neq \emptyset$ *and* $\alpha(Ax) < a$ *for* $x \in \partial P(\alpha, a)$.

Then, A *has at least two fixed points,* x_1 *and* x_2 *belonging to* $\overline{P(\gamma, c)}$ *satisfying*

$$a < \alpha(x_1) \text{ with } \theta(x_1) < b, \text{ and } b < \theta(x_2) \text{ with } \gamma(x_2) < c.$$

Let $0 < a < b$ *be given and let* α *be a nonnegative continuous concave functional on the cone* P. *Define the convex sets* $P_a, P(\alpha, a, b)$ *by*

$$\begin{aligned} P_a &= \{x \in P : \|x\| < a\}, \\ P(\alpha, a, b) &= \{x \in P : a \leq \alpha(x), \|x\| \leq b\}. \end{aligned}$$

Finally we state the Leggett-Williams fixed point theorem [15].

Lemma 2.8. *Let* P *be a cone in a real Banach space* E, $A : \overline{P}_c \to \overline{P}_c$ *be completely continuous and* α *be a nonnegative continuous concave functional on* P *with* $\alpha(x) \leq \|x\|$ *for all* $x \in \overline{P}_c$. *Suppose there exists* $0 < d < a < b \leq c$ *such that*

(i) $\{x \in P(\alpha, a, b) : \alpha(x) > a\} \neq \emptyset$ *and* $\alpha(Ax) > a$ *for* $x \in P(\alpha, a, b)$;

(*ii*) $\|Ax\| < d$ *for* $\|x\| \leq d$;

(*iii*) $\alpha(Ax) > a$ *for* $x \in P(\alpha, a, c)$ *with* $\|Ax\| > b$.

Then A *has at least three fixed points* x_1, x_2, x_3 *satisfying*

$$\|x_1\| < d, \; a < \alpha(x_2), \; \|x_3\| > d \text{ and } \alpha(x_3) < a.$$

3. Single or Twin Solutions

In this section, let c be a constant of $(0, T)$ and define

$$f_0(t) = \lim_{u \to 0^+} \frac{f(t,u)}{\varphi_p(u)} \text{ and } f_\infty(t) = \lim_{u \to \infty} \frac{f(t,u)}{\varphi_p(u)}.$$

For the notational convenience, throughout this paper we denote

$$\begin{aligned} M &= \frac{T-l}{T} \int_0^l \varphi_q \left(\int_0^s a(\tau) \nabla \tau \right) \Delta s, \\ N &= \frac{1}{d} \int_0^T \varphi_q \left(\int_0^s a(\tau) \nabla \tau \right) \Delta s, \end{aligned}$$

and

$$L = \frac{T-c}{T} \int_0^c \varphi_q \left(\int_0^s a(\tau) \nabla \tau \right) \Delta s.$$

Theorem 3.1. *Suppose that there exist different positive numbers* a' *and* b' *such that*

$$f(t,u) \leq \varphi_p \left(\frac{a'}{N} \right), \; (t,u) \in [0,T] \times [0,a'], \tag{3.1}$$

$$f(t,u) \geq \varphi_p \left(\frac{b'}{L} \right), \; (t,u) \in [0,c] \times \left[\frac{T}{T-c} b', b' \right]. \tag{3.2}$$

Then (1.5) *and* (1.6) *has at least one positive solution* u^* *such that* $\|u^*\|$ *lies between* a' *and* b'.

Proof. Without loss of generality, we assume that $a' < b'$. Let

$$\Omega_{a'} = \left\{ u \in B : \|u\| < a' \right\} \text{ and } \Omega_{b'} = \left\{ u \in B : \|u\| < b' \right\}.$$

We claim that

(i) $\|Au\| \leq \|u\|$ for $u \in P \cap \partial\Omega_{a'}$; (ii) $\|Au\| \geq \|u\|$ for $u \in P \cap \partial\Omega_{b'}$.

To justify (i), let $u \in P \cap \partial\Omega_{a'}$, then $\|u\| = a'$ and $0 \le u(t) \le a'$ for $t \in [0, T]$. So, in view of (2.4) and (3.1), we have

$$\begin{aligned}
\|Au\| &= Au(0) \\
&= \frac{1}{d}\int_0^T \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s \\
&\quad -\frac{1}{d}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s \\
&\le \frac{1}{d}\int_0^T \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s \\
&\le \frac{a'}{dN}\int_0^T \varphi_q\left(\int_0^s a(\tau)\nabla\tau\right)\Delta s \\
&= a' = \|u\|\,.
\end{aligned}$$

To prove (ii), let $u \in P \cap \partial\Omega_{b'}$, then $\|u\| = b'$ and

$$\min_{t\in[0,c]} u(t) = u(c) \ge \frac{T-c}{T}\|u\| = \frac{T-c}{T}b'.$$

So

$$\frac{T-c}{T}b' \le u(t) \le b' \text{ for } t \in [0,c].$$

Hence, by Lemma 2.4, (2.4) and (3.1), we get

$$\begin{aligned}
Au(c) &\ge \frac{T-c}{T}\|Au\| = \frac{T-c}{T}Au(0) \\
&= \frac{T-c}{dT}\left[\int_0^T \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\right. \\
&\quad \left. -\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\right] \\
&\ge \frac{T-c}{dT}\left[\int_0^T \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\right. \\
&\quad \left. -\sum_{i=1}^{m-2} a_i \int_0^{T} \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\right] \\
&= \frac{T-c}{T}\int_0^T \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s \\
&\ge \frac{T-c}{T}\int_0^c \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s
\end{aligned}$$

$$\geq \frac{(T-c)b'}{TL}\int_0^c \varphi_q\left(\int_0^s a(\tau)\nabla\tau\right)\Delta s$$
$$= b' = \|u\|.$$

Therefore, $\|Au\| \geq Au(c) \geq \|u\|$.

Having obtained (i) and (ii), it follows from Lemma 2.5 that A has a fixed point $u^* \in P\cap\left(\overline{\Omega}_{b'}\backslash\Omega_{a'}\right)$, that is, u^* is a positive solution of (1.5) and (1.6) and $a' \leq \|u^*\| \leq b'$. The proof is complete.

Corollary 3.1. *Suppose that one of the following assumptions holds:*

(C_1) $f_0(t) < \varphi_p(\frac{1}{N})$, $t\in[0,T]$ *and* $f_\infty(t) > \varphi_p\left(\frac{T}{(T-c)L}\right)$ *for* $t\in[0,c]$;

(C_2) $f_0(t) > \varphi_p\left(\frac{T}{(T-c)L}\right)$, $t\in[0,c]$ *and* $f_\infty(t) < \varphi_p\left(\frac{1}{N}\right)$ *for* $t\in[0,T]$.

Then (1.5) *and* (1.6) *has at least one positive solution.*

Proof. First assume that (C_1) holds, then there exist sufficiently small $a'>0$ and sufficiently large $b'>0$ such that

$$\frac{f(t,u)}{\varphi_p(u)} \leq \varphi_p\left(\frac{1}{N}\right),\ (t,u)\in[0,T]\times[0,a'],$$

and

$$\frac{f(t,u)}{\varphi_p(u)} \geq \varphi_p\left(\frac{T}{(T-c)L}\right),\ (t,u)\in[0,c]\times\left[\frac{(T-c)b'}{T},+\infty\right).$$

Hence, we have

$$f(t,u) \leq \varphi_p\left(\frac{1}{N}\right)\varphi_p(u) \leq \varphi_p\left(\frac{a'}{N}\right),\ (t,u)\in[0,T]\times[0,a'],$$

$$f(t,u) \geq \varphi_p\left(\frac{T}{(T-c)L}\right)\varphi_p(u) \geq \varphi_p\left(\frac{b'}{L}\right),\ (t,u)\in[0,c]\times\left[\frac{T-c}{T}b',b'\right].$$

By Theorem 3.1 we know that (1.5) and (1.6) has at least one positive solution.

Next assume that (C_2) holds, then there exist $0<a'<b'$ such that

$$\frac{f(t,u)}{\varphi_p(u)} \geq \varphi_p\left(\frac{T}{(T-c)L}\right),\ (t,u)\in[0,c]\times[0,a'], \tag{3.3}$$

$$\frac{f(t,u)}{\varphi_p(u)} \leq \varphi_p\left(\frac{1}{N}\right),\ (t,u)\in[0,c]\times\left[b',+\infty\right). \tag{3.4}$$

By (3.3), we have

$$f(t,u) \geq \varphi_p\left(\frac{T}{(T-c)L}\right)\varphi_p(u) \geq \varphi_p\left(\frac{T}{(T-c)L}\right)\varphi_p\left(\frac{T-c}{T}a'\right) = \varphi_p\left(\frac{a'}{L}\right)$$

for $(t,u)\in[0,c]\times[\frac{T-c}{T}a',a']$. Hence, the condition (3.2) holds.

Now, we deal with (3.4). There are two cases to be considered.

Suppose that $f(t,u)$ is bounded. Then $f(t,u) \leq \varphi_p(K)$ for all $(t,u) \in [0,T] \times [0,+\infty)$ for some constant $K > 0$. In view of (3.4), there is $r > 0$ satisfying $\varphi_p(r) \geq \max\{\varphi_p(b'), K\varphi_p(N)\}$, such that

$$f(t,u) \leq K \leq \varphi_p(r/N), \ (t,u) \in [0,T] \times [0,r],$$

hence, condition (3.1) holds.

Suppose that $f(t,u)$ is unbounded. Then there exist $t_0 \in [0,T]$ and $r_1 \geq b'$ such that $f(t,u) \leq f(t_0,r_1)$ for $(t,u) \in [0,T] \times [0,r_1]$. By (3.4), we have $f(t,u) \leq f(t_0,r_1) \leq \varphi_p(r_1/N)$ for $(t,u) \in [0,T] \times [0,r_1]$. Therefore, (3.1) holds. Now by Theorem 3.1 we get the required conclusion. The proof is complete.

Remark 3.1. It is easy to see that Corollary 3.1 include the case that f is superlinear, i.e., $f_0(t) = 0$ and $f_\infty(t) = \infty, t \in [0,T]$; and the case that f is sublinear, i.e., $f_0(t) = \infty$ and $f_\infty(t) = 0, t \in [0,T]$.

Theorem 3.2. *Suppose that there exists $a' > 0$ such that (3.1) hold, and that*

$$f_0(t) > \varphi_p\left(\frac{T}{(T-c)L}\right), t \in [0,c],$$

and

$$f_\infty(t) > \varphi_p\left(\frac{T}{(T-c)L}\right), t \in [0,c].$$

Then (1.5) and (1.6) has at least two positive solutions u_1, u_2 such that

$$0 < \|u_1\| < a' < \|u_2\|.$$

Proof. From the proof of Corollary 3.1, we may take $0 < a_1 < a' < a_2$ such that

$$f(t,u) \geq \varphi_p\left(\frac{a_1}{L}\right), \ (t,u) \in [0,c] \times \left[\frac{T-c}{T}a_1, a_1\right],$$

and

$$f(t,u) \geq \varphi_p\left(\frac{a_2}{L}\right), \ (t,u) \in [0,c] \times \left[\frac{T-c}{T}a_2, a_2\right].$$

Hence, by Theorem 3.1, (1.5) and (1.6) have two positive solutions u_1 and u_2 such that $0 < \|u_1\| < a' < \|u_2\|$. The proof is complete.

Theorem 3.3. *Suppose that there exists $b' > 0$ such that (3.2) hold, and that*

$$f_0(t) < \varphi_p\left(\frac{1}{N}\right), \ t \in [0,T],$$

and

$$f_\infty(t) < \varphi_p\left(\frac{1}{N}\right),\ t \in [0, T].$$

Then (1.5) *and* (1.6) *has at least two positive solutions* u_1, u_2 *such that*

$$0 < \|u_1\| < b' < \|u_2\|.$$

The proof of Theorem 3.3 is similar to that of Theorem 3.2, we omit it here.

Remark 3.2. From Theorems 3.1, 3.2 and 3.3, it is easy to see that, when the assumption like (3.1), (3.2) and (C_1) or (C_2) are imposed appropriately on f, we can obtain the existence of an arbitrary number of positive solution of (1.5) and (1.6).

4. Further Results on Twin Solutions

In the following part of this paper, let $l = \max\{t \in \mathbb{T} : 0 \le t \le T/2\}$ and fix $c \in \mathbb{T}$ such that $c < l < T$. Define γ, θ, and α are nonnegative, increasing and continuous functionals on P with

$$\gamma(u) = \min_{t\in[c,l]} u(t) = u(l),$$

$$\theta(u) = \max_{t\in[l,T]} u(t) = u(l),$$

and

$$\alpha(u) = \max_{t\in[c,T]} u(t) = u(c).$$

We observe that, for each $u \in P$,

$$\gamma(u) = \theta(u) \le \alpha(u). \tag{4.1}$$

In addition, for each $u \in P, \gamma(u) = u(l) \ge \frac{T-l}{T}\|u\|$. Thus

$$\|u\| \le \frac{T}{T-l}\gamma(u), u \in P. \tag{4.2}$$

Finally, we also note that

$$\theta(\lambda u) = \lambda\theta(u),\ 0 \le \lambda \le 1 \text{ and } u \in \partial P(\theta, b').$$

We now present the results in this section.

Theorem 4.1. *Suppose that there are positive numbers* $a' < b' < c'$ *such that*

$$0 < a' < \frac{L}{N}b' < \frac{(T-l)L}{TN}c'.$$

Assume $f(t, u)$ *satisfies the following conditions:*

(*i*) $f(t,u) > \varphi_p\left(\frac{c'}{M}\right)$, $(t,u) \in [0,l] \times \left[c', \frac{T}{T-l}c'\right]$,

(*ii*) $f(t,u) < \varphi_p\left(\frac{b'}{N}\right)$, $(t,u) \in [0,T] \times \left[0, \frac{T}{T-l}b'\right]$,

(*iii*) $f(t,u) > \varphi_p\left(\frac{a'}{L}\right)$, $(t,u) \in [0,c] \times \left[a', \frac{T}{T-c}a'\right]$.

Then (1.5) *and* (1.6) *has at least two positive solutions* u_1 *and* u_2 *such that*

$a' < \max_{t\in[c,T]} u_1(t)$ *with* $\max_{t\in[l,T]} u_1(t) < b'$ *and* $b' < \max_{t\in[l,T]} u_2(t)$ *with* $\min_{t\in[c,l]} u_2(t) < c'$.

Proof. By the definition of operator A and its properties, it suffices to show that the conditions of Lemma 2.6 hold with respect to A.

We first show that if $u \in \partial P(\gamma, c')$, then $\gamma(Au) > c'$.

Indeed, if $u \in \partial P(\gamma, c')$, then

$$\gamma(u) = \min_{t\in[c,l]} u(t) = u(l) = c'.$$

Since $u \in P$, $\|u\| \leq \frac{T}{T-l}\gamma(u) = \frac{T}{T-l}c'$, we have

$$c' \leq u(t) \leq \frac{T}{T-l}c', t \in [0,l].$$

As a consequence of (i),

$$f(t,u(t)) > \varphi_p\left(\frac{c'}{M}\right),\ t \in [0,l].$$

Also, $Au \in P$ implies that

$$\begin{aligned}
\gamma(Au) &= Au\,(l) \geq \frac{T-l}{T}Au(0) \\
&= \frac{T-l}{Td}\int_0^T \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s \\
&\quad -\frac{T-l}{Td}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s \\
&\geq \frac{T-l}{Td}\int_0^T \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s \\
&\quad -\frac{T-l}{Td}\sum_{i=1}^{m-2} a_i \int_0^{T} \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s \\
&= \frac{T-l}{Td}\left(1 - \sum_{i=1}^{m-2} a_i\right)\int_0^T \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s
\end{aligned}$$

$$\geq \frac{T-l}{T}\int_0^l \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s$$
$$> \frac{(T-l)c'}{TM}\int_0^l \varphi_q\left(\int_0^s a(\tau)\nabla\tau\right)\Delta s = c'.$$

Next, we verify that $\theta(Au) < b'$ for $u \in \partial P(\theta, b')$.

Let us choose $u \in \partial P(\theta, b')$, then $\theta(u) = \max_{t\in[l,T]} u(t) = u\,(l) = b'$. This implies $0 \leq u(t) \leq b'$, $t \in [l, T]$. Since $u \in P$, we also have $b' \leq u(t) \leq \|u\| \leq \frac{T}{T-l}u(l) = \frac{T}{T-l}b'$ for $t \in [0, l]$. So

$$0 \leq u(t) \leq \frac{T}{T-l}b',\ t \in [0, T].$$

Using (ii),

$$f(t, u(t)) < \varphi_p\left(\frac{b'}{N}\right), t \in [0, T].$$

Also, $Au \in P$ implies that

$$\begin{aligned}
\theta(Au) &= Au(l) \leq Au(0)\\
&= \frac{1}{d}\int_0^T \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\\
&\quad -\frac{1}{d}\sum_{i=1}^{m-2} a_i\int_0^{\xi_i}\varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\\
&\leq \frac{1}{d}\int_0^T \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\\
&< \frac{b'}{dN}\int_0^T \varphi_q\left(\int_0^s a(\tau)\nabla\tau\right)\Delta s\\
&= b'.
\end{aligned}$$

Finally, we prove that $P(\alpha, a') \neq \emptyset$ and $\alpha(Au) > a'$ for all $u \in \partial P(\alpha, a')$.

In fact, the constant function $\frac{a'}{2} \in P(\alpha, a')$. Moreover, for $u \in \partial P(\alpha, a')$, we have

$$\alpha(u) = \max_{t\in[c,T]} u(t) = u(c) = a'.$$

This implies

$$a' \leq u(t) \leq \frac{T}{T-c}a',\ t \in [0, c].$$

Using assumption (iii),

$$f(t, u(t)) > \varphi_p\left(\frac{a'}{L}\right),\ t \in [0, c].$$

As before $Au \in P$, we obtain

$$\begin{aligned}
\alpha(Au) &= (Au)(c) \geq \frac{T-c}{T} Au(0) \\
&= \frac{T-c}{Td}\left[\int_0^T \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s \right.\\
&\quad \left. - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\right] \\
&\geq \frac{T-c}{Td}\left[\int_0^T \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s \right.\\
&\quad \left. - \sum_{i=1}^{m-2} a_i \int_0^{T} \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\right] \\
&= \frac{T-c}{T}\int_0^T \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s \\
&\geq \frac{T-c}{T}\int_0^c \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s \\
&> \frac{a'}{L}\frac{T-c}{T}\int_0^c \varphi_q\left(\int_0^s a(\tau)\nabla\tau\right)\Delta s = a'.
\end{aligned}$$

Thus, by Lemma 2.6, there exists at least two fixed points of A which are positive solutions u_1 and u_2, belonging to $\overline{P(\gamma, c')}$, of the BVP (1.5) and (1.6) such that

$$a' < \alpha(u_1) \text{ with } \theta(u_1) < b', \text{ and } b' < \theta(u_2) \text{ with } \gamma(u_2) < c'.$$

The proof is complete.

Theorem 4.2. *Assume that there are positive numbers $a' < b' < c'$ such that*

$$0 < a' < \frac{T-c}{T} b' < \frac{(T-c)N}{TM} c'.$$

Suppose $f(t,u)$ satisfies the following conditions:

(*i*) $f(t,u) < \varphi_p(\frac{c'}{N})$ *for* $(t,u) \in [0,T] \times [0, \frac{T}{T-l}c']$,

(*ii*) $f(t,u) > \varphi_p(\frac{b'}{M})$ *for* $(t,u) \in [0,l] \times [b', \frac{T}{T-l}b']$,

(*iii*) $f(t,u) < \varphi_p(\frac{a'}{N})$ *for* $(t,u) \in [0,c] \times [0, \frac{T}{T-c}a']$.

Then (1.5) *and* (1.6) *has at least two positive solutions* u_1 *and* u_2 *such that*

$$a' < \max_{t\in[c,T]} u_1(t) \text{ with } \max_{t\in[l,T]} u_1(t) < b' \text{ and } b' < \max_{t\in[l,T]} u_2(t) \text{ with } \max_{t\in[c,l]} u_2(t) < c'.$$

Using Lemma 2.7, the proof is similar to that of Theorem 4.1 and we omit it here.

Corollary 4.1. *Assume that f satisfies conditions*

(i) $f_0(t) > \varphi_p\left(\frac{1}{M}\right)$, $t \in [0, l\,]$ *and* $f_\infty(t) > \varphi_p\left(\frac{1}{L}\right)$, $t \in [0, c\,]$;

(ii) *there exists* $a' > 0$ *such that*

$$f(t,u) < \varphi_p\left(\frac{a'}{N}\right), (t,u) \in [0,T] \times [0,a'].$$

Then (1.5) *and* (1.6) *has at least two positive solutions.*

Corollary 4.2. *Assume that f satisfies conditions*

(i) $f_0(t) < \varphi_p\left(\frac{T-l}{TN}\right)$, $t \in [0,T]$ *and* $f_\infty(t) < \varphi_p\left(\frac{T-c}{TN}\right)$, $t \in [0,c]$;

(ii) *there exists* $b' > 0$ *such that*

$$f(t,u) > \varphi_p\left(\frac{b'}{N'}\right), (t,u) \in [0,l] \times \left[b', \frac{T}{T-l}b'\right].$$

Then (1.5) *and* (1.6) *has at least two positive solutions.*

By applying Theorem 4.1 and Theorem 4.2 respectively, the proof of Corollary 4.1 and Corollary 4.2 are easy and we omit them.

5. Triple Solutions

Let the nonnegative continuous concave functional $\Psi : P \to [0, \infty)$ be defined by

$$\Psi(u) = \min_{t\in[0,l]} u(t) = u(l), \ u \in P.$$

Note that for $u \in P, \Psi(u) \le \|u\|$.

Theorem 5.1. *Suppose that there exist constants $0 < d' < a'$ such that*

(i) $f(t,u) < \varphi_p(\frac{d'}{N})$, $(t,u) \in [0,T] \times [0,d']$;

(ii) $f(t,u) \ge \varphi_p(\frac{a'}{M})$, $(t,u) \in [0,l] \times [a', \frac{T}{T-l}a']$;

(iii) *one of the following conditions holds:*

(D_1) $\lim_{u\to\infty} \max_{t\in[0,T]} \frac{f(t,u)}{\varphi_p(u)} < \varphi_p(\frac{1}{N})$;

(D_2) *there exists a number* $c' > \frac{T}{T-l}a'$ *such that* $f(t,u) < \varphi_p(\frac{c'}{N})$ *for* $(t,u) \in [0,T] \times [0,c']$.

Then (1.5) *and* (1.6) *has at least three positive solutions.*

Proof. By the definition of operator A and its properties, it suffices to show that the conditions of Lemma 2.8 hold with respect to A.

We first show that if (D_1) holds, then there exists a number $l' > \frac{T}{T-l}a'$ such that $A : \overline{P}_{l'} \to P_{l'}$.

Suppose that

$$\lim_{u\to\infty} \max_{t\in[0,T]} \frac{f(t,u)}{\varphi_p(u)} < \varphi_p\left(\frac{1}{N}\right)$$

holds, then there are $\tau > 0$ and $\delta < \frac{1}{N}$ such that if $u > \tau$, then

$$\max_{t\in[0,T]} \frac{f(t,u)}{\varphi_p(u)} < \varphi_p(\delta).$$

That is to say,

$$f(t,u) \le \varphi_p(\delta u), (t,u) \in [0,T] \times [\tau,\infty).$$

Set $\lambda = \max\{f(t,u) : (t,u) \in [0,T] \times [0,\tau]\}$, then

$$f(t,u) \le \lambda + \varphi_p(\delta u), (t,u) \in [0,T] \times [0,\infty). \tag{5.1}$$

Taking

$$l' > \max\left\{\frac{T}{T-l}a', \varphi_q\left(\frac{\lambda\varphi_p(N)}{1-\varphi_p(\delta N)}\right)\right\}. \tag{5.2}$$

If $u \in \overline{P}_{l'}$, then by (2.4), (5.1) and (5.2), we obtain

$$\begin{aligned}
\|Au\| &= Au(0)\\
&= \frac{1}{d}\int_0^T \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s -\\
&\quad \frac{1}{d}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\\
&\le \frac{1}{d}\int_0^T \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\\
&\le \frac{1}{d}\int_0^T \varphi_q\left(\int_0^s a(\tau)\left(\lambda + \varphi_p(\delta u)\right)\nabla\tau\right)\Delta s\\
&\le \frac{1}{d}\varphi_q\left(\lambda + \varphi_p(\delta l')\right)\int_0^T \varphi_q\left(\int_0^s a(\tau)\nabla\tau\right)\Delta s\\
&= \varphi_q\left(\lambda + \varphi_p(\delta l')\right)N\\
&< l'.
\end{aligned}$$

Next we verify that if there is a positive number r' such that $f(t,u) < \varphi_p(r'/N)$ for $(t,u) \in [0,T] \times [0,r']$, then $A : \overline{P}_{r'} \to P_{r'}$.

Indeed, if $u \in \overline{P}_{r'}$, then

$$\begin{aligned}\|Au\| &= Au(0)\\ &= \frac{1}{d}\int_0^T \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\\ &\quad -\frac{1}{d}\sum_{i=1}^{m-2} a_i\int_0^{\xi_i} \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\\ &\le \frac{1}{d}\int_0^T \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\\ &< \frac{r'}{dN}\int_0^T \varphi_q\left(\int_0^s a(\tau)\nabla\tau\right)\Delta s\\ &= r',\end{aligned}$$

thus, $Au \in P_{r'}$.

Hence, we have shown that either (D_1) or (D_2) holds, then there exists a number c' with $c' > \frac{T}{T-l}a'$ such that $A : \overline{P}_{c'} \to P_{c'}$. It is also note from (i) that $A : \overline{P}_{d'} \to P_{d'}$.

Now, we show that $\left\{u \in P(\Psi, a', \frac{T}{T-l}a') : \Psi(u) > a'\right\} \neq \emptyset$ and $\Psi(Au) > a'$ for all $u \in P(\Psi, a', \frac{T}{T-l}a')$.

In fact,

$$u = \frac{(2T-l)a'}{T-l} \in \left\{u \in P(\Psi, a', \frac{T}{T-l}a') : \Psi(u) > a'\right\}.$$

For $u \in P(\Psi, a', \frac{T}{T-l}a')$, we have

$$a' \le \min_{t\in[0,l]} u(t) = u(l) \le u(t) \le \frac{T}{T-l}a',$$

for all $t \in [0,l]$. Then, in view of (ii), we know that

$$\begin{aligned}\Psi(Au) &= \min_{t\in[0,l]} Au(t) = Au\,(l) \ge \frac{T-l}{T}Au(0)\\ &= \frac{T-l}{Td}\int_0^T \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\\ &\quad -\frac{T-l}{Td}\sum_{i=1}^{m-2} a_i\int_0^{\xi_i} \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\tau\\ &\ge \frac{T-l}{Td}\int_0^T \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\end{aligned}$$

$$
\begin{aligned}
&-\frac{T-l}{Td}\sum_{i=1}^{m-2}a_i\int_0^T\varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\\
\geq\ &\frac{T-l}{T}\int_0^l\varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\\
>\ &\frac{(T-l)a'}{TM}\int_0^l\varphi_q\left(\int_0^s a(\tau)\nabla\tau\right)\Delta s=a'.
\end{aligned}
$$

Finally, we assert that if $u\in P(\Psi,a',c')$ and $\|Au\|>\frac{T}{T-l}a'$, then $\Psi(Au)>a'$. Suppose $u\in P(\Psi,a',c')$ and $\|Au\|>\frac{T}{T-l}a'$, then

$$
\begin{aligned}
\Psi(Au)\ &=\ \min_{t\in[0,l]}Au(t)=Au\left(l\right)\geq\frac{T-l}{T}Au(0)\\
&=\ \frac{T-l}{T}\|Au\|>a'.
\end{aligned}
$$

To sum up, the hypotheses of Lemma 2.8 are satisfied, hence BVP (1.5) and (1.6) has at least three positive solutions u_1,u_2,u_3 such that

$$\|u_1\|<d',a'<\min_{t\in[0,l]}u_2(t)\text{ and }\|u_3\|>d'\text{ with }\min_{t\in[0,l]}u_3(t)<a'.$$

The proof is complete.

From Theorem 5.1, we see that, when assumption like (i), (ii), and (iii) are imposed appropriately on f, we can establish the existence of an arbitrary number of positive solutions of (1.5), (1.6).

Theorem 5.2. *Suppose that there exist constants*

$$0<d_1'<a_1'<\frac{T}{T-l}a_1'<d_2'<a_2'<\frac{T}{T-l}a_2'<d_3'<\ldots<d_n',n\in\mathbb{N},$$

such that the following conditions are satisfied:

(*i*) $f(t,u)<\varphi_p\left(\frac{d_i'}{N}\right)$, $(t,u)\in[0,T]\times[0,d_i']$;

(*ii*) $f(t,u)\geq\varphi_p\left(\frac{a_i'}{M}\right)$, $(t,u)\in[0,l]\times[a_i',\frac{T}{T-l}a_i']$;

Then (1.5) and (1.6) has at least $2n-1$ *positive solutions.*

Proof. When $n=1$, it is immediate from condition (i) that $A:\overline{P}_{d_1'}\to P_{d_1'}\subset\overline{P}_{d_1'}$, which means that A has at least one fixed point $u_1\in\overline{P}_{d_1'}$ by the Schauder fixed point theorem. When $n=2$, it is clear that Theorem 5.1 holds (with $c_1=d_2'$). Then we can obtain at least three positive solutions u_1, u_2, and u_3 satisfying

$$\|u_1\|<d_1',\min_{t\in[0,l]}u_2(t)>a_1'\text{ and }\|u_3\|>d_1'\text{ with }\min_{t\in[0,l]}u_3(t)<a_1'.$$

Following this way, we finish the proof by induction. The proof is complete.

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