TAIWANESE JOURNAL OF MATHEMATICS
Vol. 12, No. 3, pp. 649-669, June 2008
This paper is available online at http://www.tjm.nsysu.edu.tw/

SENSITIVITY ANALYSIS OF VECTOR EQUILIBRIUM PROBLEMS

Kenji Kimura and Jen-Chih Yao*

Abstract. In this paper, we introduce a type of parametric vector equilibrium problem on topological vector spaces. This problem includes parametric vector variational inequality problem and ε-vector equilibrium problem and so on, as special cases. Existence and some continuity properties for the of solution mapping of the parametric vector equilibrium problem are established.

1. Introduction

Let $\mathfrak{X}$ be a real Hausdorff topological vector space, and X a nonempty subset of $\mathfrak{X}$. For any subset K of a topological space, we let $\operatorname{int} K$, $\operatorname{cl} K$ and K^{c} denote the topological interior, closure and complement of K, respectively. A subset C of a topological vector space Z is called a cone if $\tau x \in C$ for all $z \in C$ and $\tau \geq 0$. The cone C is called solid if $\operatorname{int} C \neq \emptyset$. C is called pointed if $C \cap (-C) = \{\theta_Z\}$ and C is called proper if $C \neq Z$, where θ_Z denotes the zero vector of Z. Let Z be a real topological vector space with a solid pointed convex cone $C \subset Z$. Let Γ and Λ be two (index sets) nonempty subsets of two Hausdorff topological spaces, respectively. Suppose that $f : \Gamma \times X \times X \to Z$ is a parameterized vector-valued function, and that $A : \Lambda \to 2^X \setminus \{\emptyset\}$ is a constraint mapping. We consider the following parametric vector equilibrium problem (PVEP, in short): for given $p \in \Gamma$ and $\lambda \in \Lambda$,

$$\text{(PVEP)} \qquad \begin{array}{l} \text{find } x \in A(\lambda) \text{ such that} \\ f(p, x, y) \notin -\operatorname{int} C \text{ for all } y \in A(\lambda). \end{array}$$

Received September 28, 2006, accepted November 20, 2006.
Communicated by Wen-Wei Lin.
2000 *Mathematics Subject Classification*: 46A40, 47H14, 49J, 54C60, 54C08,90C, 65K.
Key words and phrases: Vector equilibrium problem, Upper semicontinuity, Lower semicontinuity, Sensitivity.
This research was partially supported by the grant NSC 94-2115-M-110-004.
*Corresponding author.

The solution mapping $\Omega(p, \lambda)$ of PVEP is a set-valued mapping from $\Gamma \times \Lambda$ to 2^X defined by

$$\Omega(p,\lambda) = \{x \in A(\lambda) : f(p,x,y) \notin -\operatorname{int} C, \text{ for all } y \in A(\lambda)\}. \tag{1}$$

If A is a constant mapping, say $A(\lambda) = K$ for all $\lambda \in \Lambda$, and $f(p,x,y) = g(x,y)$ for all $p \in \Gamma$, $x, y \in K$, then PVEP reduces to the following vector equilibrium problem (VEP, in short):

$$\text{(VEP)} \qquad \begin{aligned} &\text{find } x \in K \text{ such that} \\ &g(x,y) \notin -\operatorname{int} C \text{ for all } y \in K. \end{aligned}$$

For VEP and its generalizations, we refer to [1-3, 14] and the references therein.

Liou et al. [10] introduced a weak PVVI as follows: for a given $x \in E$,

$$\text{(PVVI)} \qquad \begin{aligned} &\text{find } y^* \in \mathcal{A}(x) \text{ such that} \\ &\nabla_y \varphi(x,y^*)(y^* - v) \notin -\operatorname{int} C \text{ for all } v \in \mathcal{A}(x), \end{aligned}$$

where $\varphi = (\varphi_1, \ldots, \varphi_p) : E \times K \to \mathbb{R}^p$, $\varphi(x,\cdot)$ is differentiable in $\mathcal{A}(x)$ for a given $x \in E$. It is clear that weak PVVI is a special case of PVEP.

For given $\varepsilon \in \operatorname{int} C$, the ε-vector equilibrium problem is to find $x \in K$ such that

$$(\varepsilon\text{-VEP}) \qquad g(x,y) + \varepsilon \notin -\operatorname{int} C, \text{ for all } y \in K.$$

The solution mapping $S(\varepsilon)$ is defined by

$$S(\varepsilon) = \{x \in K : g(x,y) + \varepsilon \notin -\operatorname{int} C, \text{ for all } y \in K\}. \tag{2}$$

This problem studied in [8] is also a special case of PVEP.

The main purpose of this paper is to investigate some continuity properties of Ω. To this end, we will give some preliminaries which will be used for the rest of this paper in Section 2. We will establish some existence results of PVEP in Section 3. Sections 4-6 give some results on upper semicontinuity, lower semicontinuity and continuity of the solution mapping Ω of PVEP, respectively. In Section 7 we give some applications of the results established in Sections 3-6. Finally we investigate the differentiablity of the solution mapping Ω of PVEP in Section 5.

2. Preliminaries

We recall the cone-convexity of vector-valued functions by Tanaka [12]. Let X be a vector space, and Z also a vector space with a partial ordering defined by a pointed convex cone C. Suppose that K is a convex subset of X and that f is a vector-valued function from K to Z. The mapping f is said to be C-convex on K if for each $x_1, x_2 \in K$ and $\lambda \in [0,1]$, we have

$$\lambda f(x_1) + (1-\lambda)f(x_2) \in f(\lambda x_1 + (1-\lambda)x_2) + C.$$

As a special case, if $Z = \mathbb{R}$ (the set of real numbers) and $C = \mathbb{R}_+$ (the set of nonnegative real numbers), then C-convexity is the same as ordinary convexity.

Definition 2.1. [C-quasiconvexity, 5, 11, 12]. Let X be a vector space, and Z also a vector space with a partial ordering defined by a pointed convex cone C. Suppose that K is a convex subset of X and that f is a vector-valued function from K to Z. Then, f is said to be *C-quasiconvex on K* if it satisfies one of the following two equivalent conditions:

(i) for each $x_1, x_2 \in K$ and $\lambda \in [0,1]$,

$$f(\lambda x_1 + (1-\lambda)x_2) \in z - C, \text{ for all } z \in C(f(x_1), f(x_2)),$$

where $C(f(x_1), f(x_2))$ is the set of upper bounds of $f(x_1)$ and $f(x_2)$, i.e.,

$$C(f(x_1), f(x_2)) := \{z \in Z : z \in f(x_1) + C \text{ and } z \in f(x_2) + C\}.$$

(ii) for each $z \in Z$,

$$A(z) := \{x \in K : f(x) \in z - C\}$$

is convex or empty.

The statement in Definition 2.1 (i) is defined by Luc [11] and the statement (ii) is by Ferro [5].

Definition 2.2. (C-proper quasiconvexity, [12]). Let X be a vector space, and Z also a vector space with a partial ordering defined by a pointed convex cone C. Suppose that K is a convex subset of X and that f is a vector-valued function from K to Z. Then, f is said to be *C-properly quasiconvex on K* if for every $x_1, x_2 \in K$ and $\lambda \in [0,1]$ we have either

$$f(\lambda x_1 + (1-\lambda)x_2) \in f(x_1) - C,$$

or

$$f(\lambda x_1 + (1-\lambda)x_2) \in f(x_2) - C;$$

f is said to be *strictly C-properly quasiconvex on K* if for every $x_1, x_2 \in K$ and $\lambda \in (0,1)$ we have either

$$f(\lambda x_1 + (1-\lambda)x_2) \in f(x_1) - \operatorname{int} C,$$

or

$$f(\lambda x_1 + (1-\lambda)x_2) \in f(x_2) - \operatorname{int} C.$$

Remark 1. In real-valued function, ordinary quasiconvex function is also properly quasiconvex. For vector-valued function, we have the following example.

Example 1. Let g be a real-valued function from $\mathbb{R}^2$ to $\mathbb{R}$ defined by

$$g(u,v) = u^2 + v^2.$$

Let $h : (u,v) \mapsto (u,v,g(u,v))$ and ordering cone $C = \{(z_1,z_2,z_3) : z_1 > 0\} \cup \{\theta_Z\}$. Then h is strictly C-properly quasiconvex.

Recall that a set-valued mapping $T : X \to 2^Y$ where X and Y are two vector spaces is said to be convex if for any $x_1,\ x_2 \in X$ and $\alpha \in [0,1]$, we have

$$\alpha T(x_1) + (1-\alpha)T(x_2) \subseteq T(\alpha x_1 + (1-\alpha)x_2).$$

The proof of the following proposition is straightforward and here will be omitted.

Proposition 2.1. *Let P and X be two nonempty convex subsets of two vector spaces, respectively, and Z a topological vector space with a solid pointed convex cone C. Suppose that $g : P \times X \times X \to Z$ is $(-C)$-properly quasiconvex on $P \times X$ for each $y \in X$ and that $S : P \to 2^X$ is defined by*

$$S(p) = \{x \in X : g(p,x,y) \notin -\operatorname{int} C, \textit{ for all } y \in X\}$$

is nonempty for each $p \in P$. Then S is a convex set-valued mapping.

Definition 2.3. (C-continuity, [11, 13]). Let X be a topological space, and Z a topological vector space with a partial ordering defined by a solid pointed convex cone C. Suppose that f is a vector-valued function from X to Z. Then, f is said to be *C-continuous at $x \in X$* if it satisfies one of the following two equivalent conditions:

(i) For any neighbourhood $V_{f(x)} \subset Z$ of $f(x)$, there exists a neighbourhood $U_x \subset X$ of x such that $f(u) \in V_{f(x)} + C$ for all $u \in U_x$.

(ii) For any $k \in \operatorname{int} C$, there exists a neighbourhood $U_x \subset X$ of x such that $f(u) \in f(x) - k + \operatorname{int} C$ for all $u \in U_x$.

Moreover a vector-valued function f is said to be *C-continuous on X* if f is C-continuous at every x on X.

Remark 2. Whenever $Z = \mathbb{R}$ and $C = \mathbb{R}_+$, C-continuity and $(-C)$-continuity are the same as ordinary lower and upper semicontinuity, respectively. In [13, Definition 2.1, pp. 314-315], C-continuous function is called C-lower semicontinuous function, and $(-C)$-continuous function is called C-upper semicontinuous function.

Remark 3. By the balancedness of the topology of topological vector spaces, C of (i) can be replaced by $\operatorname{int} C$. See Proposition 3.2.

Definition 2.4. (see [4]). Let X and Y be two topological spaces, $T : X \to 2^Y$ a set-valued mapping.

(i) T is said to be upper semicontinuous (u.s.c. for short) at $x \in X$ if for each open set V containing $T(x)$, there is an open set U containing x such that for each $z \in U$, $T(z) \subset V$; T is said to be u.s.c. on X if it is u.s.c. at all $x \in X$, and also $T(x)$ is compact for all $x \in X$.

(ii) T is said to be *lower semicontinuous* (l.s.c. for short) at $x \in X$ if for each open set V with $T(x) \cap V \neq \emptyset$, there is an open set U containing x such that for each $z \in U$, $T(z) \cap V \neq \emptyset$; T is said to be l.s.c. on X if it is l.s.c. at all $x \in X$.

(iii) T is said to be *continuous* at $x \in X$ if $T(x)$ is both u.s.c. and l.s.c.; T is said to be continuous on X if it is both u.s.c. and l.s.c. at each $x \in X$.

(iv) T is said to be closed if the graph of T, denoted by $\mathrm{Gr}(T)$, is closed where

$$\mathrm{Gr}(T) = \{(x, y) \in X \times Y : y \in T(x)\}.$$

Remark 4. If $T(x)$ is singleton and u.s.c. at x then T is l.s.c. at x, i.e., continuous at x. If T is singleton in a neighborhood of x, then u.s.c. and l.s.c. of T at x are equivalent.

Definition 2.5. (KKM-map). Let X be a topological vector space, and K a nonempty subset of X. Suppose that F is a multifunction from K to 2^X. Then, F is said to be a *KKM-map*, if

$$\operatorname{co}\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i)$$

for each finite subset $\{x_1, \ldots, x_n\}$ of X.

Remark 5. Obviously, if F is a KKM-map, then $x \in F(x)$ for each $x \in K$.

Lemma 2.1. (Fan-KKM; see [6]). *Theorem 1.1. Let X be a Hausdorff topological vector space, and K a nonempty subset of X; and let G be a multifunction* *from K to 2^X. Suppose that G is a KKM-map and that $G(x)$ is a closed subset of X for each $x \in K$. If $G(\hat{x})$ is compact for at least one $\hat{x} \in K$, then $\bigcap_{x \in K} G(x) \neq \emptyset$.*

3. Existence Results for PVEP

First, we give some propositions that will e used in proving existence results of PVEP.

Proposition 3.1. *Let Z be a real topological vector space. Let K and E be two nonempty sets of Z. Suppose that K is open. Suppose that E is convex and* $\operatorname{int} E \neq \emptyset$. Then

$$K + \operatorname{cl} E = K + E = K + \operatorname{int} E.$$

Proof. Clearly $K + \operatorname{int} E \subset K + E \subset K + \operatorname{cl} E$. We next show $K + \operatorname{int} E \supset K + \operatorname{cl} E$. Let $z \in K + \operatorname{cl} E$. Then there exist $k \in K$ and $e \in \operatorname{cl} E$ such that $z = k + e$. Since Z is a topological vector space and K is an open set of Z, there is an balanced neighborhood $\mathcal{V}$ of θ_Z with $\mathcal{V} + k \subset K$. On the other hand, since E is convex and $e \in \operatorname{cl} E$, for any neighborhood $\mathcal{U}$ of θ_Z, $(\mathcal{U} + e) \cap \operatorname{int} E \neq \emptyset$. Hence there exists $v \in \mathcal{V}$ such that $v + e \in \operatorname{int} E$. Since $\mathcal{V}$ is balanced, $-v \in \mathcal{V}$. Therefore $z = k + e = (k - v) + (e + v)$ with $(k - v) \in K$ and $(e + v) \in \operatorname{int} E$. Hence $z \in K + \operatorname{int} E$. ∎

Proposition 3.2. *Let Z be a real topological vector space with solid pointed convex cone C. Then*

$$\operatorname{cl} C + \operatorname{int} C = \operatorname{int} C.$$

Proof. By Proposition 3.1, $\operatorname{cl} C + \operatorname{int} C = \operatorname{int} C + \operatorname{int} C$. Since C is convex cone, $\operatorname{int} C + \operatorname{int} C = \operatorname{int} C$. ∎

Proposition 3.3. *Let Z be a real topological vector space and C a solid pointed convex cone in Z. Then*

$$(\operatorname{cl} C)^{\mathrm{c}} - \operatorname{cl} C = (\operatorname{int} C)^{\mathrm{c}} - \operatorname{int} C = (\operatorname{cl} C)^{\mathrm{c}}.$$

Proof. Since $(\operatorname{cl} C)^{\mathrm{c}}$ is open and $\operatorname{cl} C$ is convex with $\operatorname{int} C \neq \emptyset$, by Proposition 3.1,

$$(\operatorname{cl} C)^{\mathrm{c}} - \operatorname{cl} C = (\operatorname{cl} C)^{\mathrm{c}} - \operatorname{int} C.$$

Since $(\operatorname{cl} C)^{\mathrm{c}} \subset (\operatorname{int} C)^{\mathrm{c}}$, we have

$$(\operatorname{cl} C)^{\mathrm{c}} - \operatorname{int} C \subset (\operatorname{int} C)^{\mathrm{c}} - \operatorname{int} C.$$

Next we have

$$(\operatorname{int} C)^{\mathrm{c}} - \operatorname{int} C \subset (\operatorname{cl} C)^{\mathrm{c}}.$$

Indeed, suppose that there exist $z \in (\operatorname{int} C)^{\mathrm{c}}$, $c \in \operatorname{int} C$ and $\hat{c} \in \operatorname{cl} C$ such that $z - c' = \hat{c}$. Then by Proposition 3.2, $z = \hat{c} + c' \in \operatorname{int} C$. This is contradiction.

Finally,

$$(\operatorname{cl} C)^{\mathrm{c}} = (\operatorname{cl} C)^{\mathrm{c}} - \theta_Z \subset (\operatorname{cl} C)^{\mathrm{c}} - \operatorname{cl} C.$$

Thus

$$(\operatorname{cl} C)^{\mathrm{c}} - \operatorname{cl} C \subset (\operatorname{int} C)^{\mathrm{c}} - \operatorname{int} C \subset (\operatorname{cl} C)^{\mathrm{c}} \subset (\operatorname{cl} C)^{\mathrm{c}} - \operatorname{cl} C,$$

i.e.,

$$(\operatorname{cl} C)^{\mathrm{c}} - \operatorname{cl} C = (\operatorname{int} C)^{\mathrm{c}} - \operatorname{int} C = (\operatorname{cl} C)^{\mathrm{c}}.$$ ■

We now establish some existence results for PVEP as follows.

Theorem 3.1. *Let X be a nonempty subset of a real Hausdorff topological vector space $\mathfrak{X}$, and Z a real topological vector space with a solid pointed convex cone $C \subset Z$. Let Γ and Λ be two nonempty sets, respectively. Suppose that A is a set-valued map from Λ to $2^X \setminus \{\emptyset\}$ and that f is a vector-valued function from $\Gamma \times X \times X$ to Z with $f(p, x, x) \notin -\operatorname{int} C$ for all $p \in \Gamma$ and $x \in X$. Also for each $p \in \Gamma$ and $\lambda \in \Lambda$, we assume the following conditions:*

(*i*) *$A(\lambda)$ is closed and convex for each $\lambda \in \Lambda$;*

(*ii*) *$\{x \in A(\lambda) : f(p, x, y) \in -\operatorname{int} C\}$ is open for each $y \in A(\lambda)$;*

(*iii*) *$\{y \in A(\lambda) : f(p, x, y) \in -\operatorname{int} C\}$ is convex for each $y \in A(\lambda)$;*

(*iv*) *for each $\lambda \in \Lambda$ there exist a set $\mathcal{B}_\lambda \subset X$ and $\hat{y} \in A(\lambda) \cap \mathcal{B}_\lambda$ such that $\mathcal{B}_\lambda$ is compact and*

$$f(p, x, \hat{y}) \in -\operatorname{int} C \text{ for all } x \in (A(\lambda) \setminus \mathcal{B}_\lambda).$$

Then the problem PVEP has at least one solution for each $p \in \Gamma$ and $\lambda \in \Lambda$.

Proof. Let $G : A(\lambda) \to 2^{A(\lambda)}$ be defined by for each $y \in A(\lambda)$

$$G(y) := \{x \in A(\lambda) : f(p, x, y) \notin -\operatorname{int} C\},$$

where $p \in \Gamma$ and $\lambda \in \Lambda$ are fixed. Then by $f(p, x, x) \notin -\operatorname{int} C$ and condition (ii), $G(y)$ is nonempty and closed, for each $y \in A(\lambda)$.

Next, by conditions (iv) and closedness of $G(y)$ for each $y \in A(\lambda)$, for corresponding $\hat{y} \in A(\lambda) \cap \mathcal{B}_\lambda$, $G(\hat{y})$ is compact.

Finally, we show that G is a KKM-map. Suppose to the contrary that there exists $\mu_i \in [0,1],\ x_i \in A(\lambda)\ i = 1, \ldots, n$ such that

$$\sum_{i=1}^{n} \mu_i x_i = x \notin \bigcup_{i=1}^{n} G(x_i).$$

Since $x_i \in A(\lambda)\ i = 1, \ldots, n$, by the convexity of $A(\lambda)$ we have $x \in A(\lambda)$. Hence $f(p, x, x_i) \in -\operatorname{int} C\ i = 1, \ldots, n$. This implies that $f(p, x, \sum_{i=1}^{n} \mu_i x_i) = f(p, x, x) \in -\operatorname{int} C$ because of condition (iii). This contradicts the fact that $f(p, x, x) \notin -\operatorname{int} C$ for each $p \in \Gamma$.

Thus we can apply Lemma 2.1, to get

$$\Omega(p, \lambda) = \bigcap_{y \in A(\lambda)} G(y) \neq \emptyset,$$

for each $p \in \Gamma$ and $\lambda \in \Lambda$. ■

The following result is an easy consequence of Theorem 3.1.

Theorem 3.2. *Let X be a nonempty compact subset of a real Hausdorff topological vector space $\mathfrak{X}$, and Z a real topological vector space with a solid pointed convex cone $C \subset Z$. Let Γ and Λ be two nonempty sets, respectively. Suppose that A is a set-valued map from Λ to $2^X \setminus \{\emptyset\}$ and that f is a vector-valud function from $\Gamma \times X \times X$ to Z with $f(p, x, x) \notin -\operatorname{int} C$ for all $p \in \Gamma$ and $x \in X$. Also we assume the following conditions:*

(i) *$A(\lambda)$ is compact convex for each $\lambda \in \Lambda$;*

(ii) *$f(p, \cdot, y)$ is $(-C)$-continuous on $A(\lambda)$ for each $p \in \Gamma, y \in A(\lambda)$;*

(iii) *$f(p, x, \cdot)$ is C-quasiconvex on $A(\lambda)$ for each $p \in \Gamma, x \in A(\lambda)$.*

Then the problem PVEP has at least one solution for each $p \in \Gamma$ and $\lambda \in \Lambda$.

Theorem 3.3. *Let X be a nonempty subset of a real Hausdorff topological vector space $\mathfrak{X}$, and Z a real topological vector space with a solid pointed convex cone $C \subset Z$. Let Γ and Λ be two nonempty sets, respectively. Suppose that A is a set-valued map from Λ to $2^X \setminus \{\emptyset\}$ and that f is a vector-valued function from $\Gamma \times X \times X$ to Z with $f(p, x, x) \notin -\operatorname{int} C$ for all $p \in \Gamma$ and $x \in X$. Also we assume the following conditions:*

(i) $\operatorname{cl} A(\lambda)$ *is compact convex for each $\lambda \in \Lambda$;*

(ii) *$f(p, \cdot, y)$ is $(-C)$-continuous on* $\operatorname{cl} A(\lambda)$ *for each $p \in \Gamma, y \in \operatorname{cl} A(\lambda)$;*

(iii) *$f(p, x, \cdot)$ is C-quasiconvex on* $\operatorname{cl} A(\lambda)$ *for each $p \in \Gamma, x \in \operatorname{cl} A(\lambda)$;*

(*iv*) $f(p, \cdot, \cdot)$ *is* C*-continuous on* $X \times X$ *for each* $p \in \Gamma$*;*

(*v*) *for some* $\hat{p} \in \Gamma$ *and* $\Gamma' \subset \Gamma$*,* $f(p, x, y) \in f(\hat{p}, x, y) + \operatorname{int} C$ *for each* $p \in \Gamma'$ *and* $(x, y) \in X \times X$*.*

Then the problem PVEP has at least one solution for each $p \in \Gamma'$ *and* $\lambda \in \Lambda$*.*

Proof. For $\hat{p} \in \Gamma$ of condition (v), let $G : A(\lambda) \to 2^{A(\lambda)}$ be defined by for each $y \in A(\lambda)$

$$G(y) := \{x \in \operatorname{cl} A(\lambda) : f(\hat{p}, x, y) \notin -\operatorname{int} C\},$$

where $\lambda \in \Lambda$ is fixed. Then by Theorem 3.2,

$$\mathcal{S} = \bigcap_{y \in \operatorname{cl} A(\lambda)} G(y) \neq \emptyset,$$

for each $\lambda \in \Lambda$. Next let $p \in \Gamma'$, $\lambda \in \Lambda$ and $x \in \mathcal{S}$. Then by condition (v), $f(p, x, y) \notin -\operatorname{cl} C$ for all $y \in \operatorname{cl} A(\lambda)$. Since $(-\operatorname{cl} C)^{\mathrm{c}}$ is a neighborhood of $f(p, x, y)$ for all $y \in \operatorname{cl} A(\lambda)$ and $(-\operatorname{cl} C)^{\mathrm{c}} + C = (-\operatorname{cl} C)^{\mathrm{c}}$, by condition (iv), for each $y \in \operatorname{cl} A(\lambda)$ there is a neighborhood $\mathcal{U}_{(x,y)}$ of (x, y) such that

$$f(p, u, v) \in (-\operatorname{cl} C)^{\mathrm{c}}, \text{ for all } (u, v) \in \mathcal{U}_{(x,y)}.$$

Let $\mathcal{U}_{(x,y)} = \mathcal{W}_y \times \mathcal{V}_y$, where $\mathcal{W}_y$ and $\mathcal{V}_y$ denote neighborhoods of x and y, respectively. Since $\operatorname{cl} A(\lambda)$ is compact, we can choose a finite subset $\{y_1, \ldots, y_n\}$ of $\operatorname{cl} A(\lambda)$ such that

$$\bigcup_{i=1}^{n} \mathcal{V}_{y_i} \supset \operatorname{cl} A(\lambda).$$

Let $\mathcal{W} = \bigcap\limits_{i=1}^{n} \mathcal{W}_{y_i}$. Then for any $w \in \mathcal{W}$, we have

$$f(p, w, y) \in (-\operatorname{cl} C)^{\mathrm{c}}, \text{ for all } y \in \operatorname{cl} A(\lambda).$$

Hence

$$f(p, w, y) \notin -\operatorname{int} C, \text{ for all } y \in \operatorname{cl} A(\lambda).$$

Since $\mathcal{W}$ is a neighborhood of x and $x \in \operatorname{cl} A(\lambda)$, $\mathcal{W} \cap A(\lambda) \neq \emptyset$. Thus for all $x' \in \mathcal{W} \cap A(\lambda)$

$$f(p, x', y) \notin -\operatorname{int} C, \text{ for all } y \in \operatorname{cl} A(\lambda).$$

Therefore

$$f(p, x', y) \notin -\operatorname{int} C, \text{ for all } y \in A(\lambda).$$

Consequently, $x' \in \Omega(p, \lambda)$, i.e., $\Omega(p, \lambda) \neq \emptyset$. The proof is now complete. ∎

Example 2. Let $f : \Gamma \times X \times X \to Z$, where $\Gamma,\ X,\ Z$ are $\mathbb{R}$. Let Γ' and Λ be $\mathbb{R}_{++} = \{t \in \mathbb{R} : 0 < t\}$, $\hat{p} = 0$ and $A(\lambda) = \{x \in \mathbb{Q} : 0 < x < 2 + \lambda\}$ and $C = \mathbb{R}_+$. Suppose that

$$f(p, x, y) = (y^2 - 2)^2 - (x^2 - 2)^2 + p.$$

Then PVEP has at least one solution for each $p \in \Gamma'$ and $\lambda \in \Lambda$.

4. Upper Semicontinuity of Ω

In this section, we investigate conditions under which the solution mapping of PVEP is upper semicontinuous.

Lemma 4.1. *Let X and Y be Topological space and Topological vector space, respectively. Let T be a set-valued mapping from X to 2^Y. Suppose that $T(x)$ is compact at $x \in X$. Then T is u.s.c. at $x \in X$ if and only if for any nets $\{x_\mu\} \subset X$ with $x_\mu \to x$ and $\{y_\mu\} \subset Y$ with $y_\mu \in T(x_\mu)$, there exists a subnet $\{y_\nu\} \subset \{y_\mu\}$ such that $y_\nu \to y$ for some $y \in T(x)$.*

Proof. Suppose that T is u.s.c. at $x \in X$ and that $\{x_\mu\} \subset X$ with $x_\mu \to x$ and $\{y_\mu\} \in Y$ with $y_\mu \in T(x_\mu)$. Let $\mathcal{V} \in \mathfrak{V}$, where $\mathfrak{V}$ stands for a basis of neighborhoods of θ_Y. Then $\bigcup_{u \in T(x)} \{u + \mathcal{V}\}$ is an open covering of $T(x)$. Since $T(x)$ is compact, there exists a finite subset $\{u_1, \dots, u_n\} \subset T(x)$ such that $\bigcup_{i=1}^{n} \{u_i + \mathcal{V}\} \supset T(x)$. Since T is u.s.c. at $x \in X$, there exists a neighborhood $\mathcal{U}$ of x such that

$$T(x') \subset \bigcup_{i=1}^{n} \{u_i + \mathcal{V}\}, \text{ for all } x' \in \mathcal{U}.$$

Because of $x_\mu \to x$ there exists μ' such that $\{x_\mu\} \subset \mathcal{U}$ for all $\mu \geq \mu'$. Hence for each $\mu \geq \mu'$, $y_\mu \in u_i + \mathcal{V}$ for some $u_i \in \{u_1, \dots, u_n\}$. Therefore there exist a subnet $\{y_\nu\} \subset \{y_\mu\}$ and $u_i \in \{u_1, \dots, u_n\}$ such that

$$\{y_\nu\} \subset u_i + \mathcal{V}.$$

Let this $u_i =: u^{\mathcal{V}}$, corresponding to $\mathcal{V} \in \mathfrak{V}$, i.e., for each $\mathcal{V} \in \mathfrak{V}$ there exist a subnet $\{y_\nu\} \subset \{y_\mu\}$ and $u^{\mathcal{V}} \in T(x)$ such that

$$\{y_\nu\} \subset u^{\mathcal{V}} + \mathcal{V}. \tag{3}$$

Let $\mathcal{V}_2 \geq \mathcal{V}_1$, for each $\mathcal{V}_1, \mathcal{V}_2 \in \mathfrak{V}$ if $\mathcal{V}_2 \subset \mathcal{V}_1$. Then $\mathfrak{V}$ is a directed set and $\{u^{\mathcal{V}}\}$ is a net of $T(x)$. Since $T(x)$ is compact, there exist $\mathfrak{V}' \subset \mathfrak{V}$ and $y \in T(x)$ such that

$u^{\mathcal{V}'} \to y$, where $\mathcal{V}' \in \mathfrak{V}'$. Let V be a neighborhood of y. Since Y is a topological vector space, there exists $\bar{\mathcal{V}} \in \mathfrak{V}$ such that $\bar{\mathcal{V}} + \bar{\mathcal{V}} \in V - y$. Because $u^{\mathcal{V}'} \to y$, there exists $\mathcal{V}'' \in \mathfrak{V}'$ such that

$$u^{\mathcal{V}'} \in y + \bar{\mathcal{V}}, \text{ for all } \mathcal{V}' \geq \mathcal{V}''.$$

Hence for any $\Upsilon \in \mathfrak{V}'$ with $\Upsilon \geq \bar{\mathcal{V}} \cap \mathcal{V}''$,

$$\{u^{\Upsilon}\} + \mathcal{V}' \subset (y + \bar{\mathcal{V}}) + \bar{\mathcal{V}} \subset V.$$

Therefore by (3), there exists a subnet $\{y_\nu\} \subset \{y_\mu\}$ such that

$$\{y_\nu\} \subset V.$$

Thus there exists a subnet $\{y_\nu\} \subset \{y_\mu\}$ such that $y_\nu \to y$ for some $y \in T(x)$.

Suppose that T is not u.s.c. at $x \in X$. Then there exists an open set $\mathcal{V}$ containing $T(x)$ such that for any neighborhood $\mathcal{U}_\mu$ of x, there is a point $x_\mu \in \mathcal{U}_\mu$ with $T(x_\mu) \cap \mathcal{V}^{\mathrm{c}} \neq \emptyset$. Hence there exist $\{x_\nu\} \subset X$ converging to x and $y_\nu \in T(x_\nu) \cap \mathcal{V}^{\mathrm{c}}$. Since $y_\nu \notin T(x)$ for all ν, $\{y_\nu\}$ doesn't have subnet converging to some point of $T(x)$. ∎

From Lemma 4.1, we can establish the following upper semicontinuity property of the solution mapping Ω for PVEP.

Theorem 4.1. *Let X be a nonempty subset of a real Hausdorff topological vector space $\mathfrak{X}$, and Z a real topological vector space with a solid pointed convex cone $C \subset Z$. Let Γ and Λ be two nonempty closed subsets of two Hausdorff spaces, respectively. Suppose that A is a set-valued map from Λ to $2^X \setminus \{\emptyset\}$ and that f is a vector-valued function from $\Gamma \times X \times X$ to Z with $f(p, x, x) \notin -\operatorname{int} C$ for all $p \in \Gamma$ and $x \in X$. Also we assume the following conditions:*

(*i*) *A is continuous with compact values;*

(*ii*) *f is $(-C)$-continuous on $\Gamma \times X \times X$;*

(*iii*) *$\Omega(p, \lambda) \neq \emptyset$ for each $p \in \Gamma$ and $\lambda \in \Lambda$.*

Then Ω is u.s.c. on $\Gamma \times \Lambda$.

Proof. Let $p \in \Gamma$ and $\lambda \in \Lambda$. By condition (ii), $\Omega(p, \lambda)$ is closed for each $(p, \lambda) \in \Gamma \times \Lambda$. By condition (i), $A(\lambda)$ is compact for each $\lambda \in \Lambda$. Hence $\Omega(p, \lambda)$ is compact. Therefore we can apply Lemma 4.1. Let $(p_\lambda, \lambda_\alpha) \to (p, \lambda)$ and $x_\alpha \in \Omega(p_\lambda, \lambda_\alpha)$. Then for each α

$$f(p_\alpha, x_\alpha, y_\alpha) \notin -\operatorname{int} C, \text{ for all } y_\alpha \in A(\lambda_\alpha). \tag{4}$$

Since A is u.s.c., without loss of generality , we can assume $x_\alpha \to x$ for some $x \in A(\lambda)$.

Suppose to the contrary that $x \notin \Omega(p,\lambda)$. Then there exists $y \in A(\lambda)$ such that

$$f(p,x,y) \in -\operatorname{int} C.$$

By condition (ii), there exists a neighborhood $\mathfrak{U}$ of (p,x,y) such that

$$f(p',x',y') \in -\operatorname{int} C, \text{ for all } (p',x',y') \in \mathfrak{U}.$$

Let $\mathfrak{U} := \mathcal{P} \times \mathcal{X} \times \mathcal{Y}$. Since A is l.s.c. and $y \in \mathcal{Y} \cap A(\lambda)$, i.e., $\mathcal{Y} \cap A(\lambda) \neq \emptyset$, there exists $\mathcal{L}$ which is a neighborhood of λ such that

$$\mathcal{Y} \cap A(\lambda') \neq \emptyset, \text{ for all } \lambda' \in \mathcal{L}.$$

Since $(p_\alpha, \lambda_\alpha, x_\alpha) \to (p,\lambda,x)$ there exists α' such that

$$(p_\alpha, \lambda_\alpha, x_\alpha) \in \mathcal{P} \times \mathcal{L} \times \mathcal{X}, \text{ for all } \alpha \geq \alpha'.$$

Hence for each $\alpha \geq \alpha'$ there exists $y_\alpha \in \mathcal{Y} \cap A(\lambda_\alpha)$ such that

$$f(p_\alpha, x_\alpha, y_\alpha) \in -\operatorname{int} C.$$

This contradicts to (4). Hence $x \in \Omega(p,\lambda)$. Thus Ω is u.s.c. at (p,λ). Since $(p,\lambda) \in \Gamma \times \Lambda$ is arbitrary, Ω is u.s.c. on $\Gamma \times \Lambda$. ■

For fixed $\lambda \in \Lambda$, we can get the following result.

Theorem 4.2. *Let X be a nonempty subset of a real Hausdorff topological vector space $\mathfrak{X}$, and Z a real topological vector space with a solid pointed convex cone $C \subset Z$. Let Γ be a nonempty subset of Hausdorff space and Λ an arbitrary set. Suppose that A is a set-valued map from Λ to $2^X \setminus \{\emptyset\}$ and that f is a vector-valued function from $\Gamma \times X \times X$ to Z with $f(p,x,x) \notin -\operatorname{int} C$ for all $p \in \Gamma$ and $x \in X$. Also we assume the following conditions:*

(i) *$A(\lambda)$ is compact for each $\lambda \in \Lambda$;*

(ii) *$f(\cdot,\cdot,y)$ is $(-C)$-continuous on $\Gamma \times X$ for each $y \in A(\lambda)$;*

(iii) *$\Omega(p,\lambda) \neq \emptyset$ for each $p \in \Gamma$ and $\lambda \in \Lambda$.*

Then $\Omega(\cdot,\lambda)$ is u.s.c. on Γ for each fixed $\lambda \in \Lambda$.

5. Lower Semicontinuity of Ω

Next, we investigate conditions under which the solutions mapping of PVEP is lower semicontinuous.

Theorem 5.1. *Let X be a nonempty subset of a real Hausdorff topological vector space $\mathfrak{X}$, and Z a real topological vector space with a solid pointed convex cone $C \subset Z$. Let Γ and Λ be two nonempty subsets of two Hausdorff spaces, respectively. Suppose that A is a set-valued map from Λ to $2^X \setminus \{\emptyset\}$ and that f is a vector-valud function from $\Gamma \times X \times X$ to Z with $f(p, x, x) \notin -\operatorname{int} C$ for all $p \in \Gamma$ and $x \in X$. Also we assume that the following conditions:*

(i) *A is continuous at λ_0 and $A(\lambda_0)$ is compact convex;*

(ii) *f is C-continuous on $P \times X \times X$ for a certain neighborhood P of p_0;*

(iii) *$f(p_0, \cdot, y)$ is strictly $(-C)$-properly quasiconvex on X for fixed $p_0 \in \Gamma$ and each $y \in X$;*

(iv) *for fixed $p_0 \in \Gamma$ and $\lambda_0 \in \Lambda$, $\Omega(p_0, \lambda_0)$ contains at least two points.*

Then Ω is l.s.c. at (p_0, λ_0).

Proof. Let $\mathcal{V} \subset \mathfrak{X}$ such that $\mathcal{V} \cap \Omega(p_0, \lambda_0) \neq \emptyset$. Then there exists $x \in \mathcal{V} \cap \Omega(p_0, \lambda_0)$. Since $\Omega(p_0, \lambda_0)$ has at least two elements, we can choose $\bar{x} \in \Omega(p_0, \lambda_0) \setminus \{x\}$. Because of the convexity of $A(\lambda_0)$ and condition (iii), $\Omega(p_0, \lambda_0)$ is convex. Then there is an $\mu \in (0, 1)$ such that $x' := (1 - \mu)x + \mu\bar{x}$ belongs to $\mathcal{V}$. Because of condition (iii), for each $y \in A(\lambda_0)$,

$$f(p_0, x', y) \in f(p_0, x, y) + \operatorname{int} C,$$

or

$$f(p_0, x', y) \in f(p_0, \bar{x}, y) + \operatorname{int} C.$$

Thus by Proposition 3.3, we have

$$f(p_0, x', y) \notin -\operatorname{cl} C \text{ for all } y \in A(\lambda),$$

because $\{x, \bar{x}\} \subset \Omega(p_0, \lambda_0)$. Hence for each $y \in A(\lambda_0)$ there exists a neighborhood $\mathcal{W}^y$ of $f(p_0, x', y)$ such that $\mathcal{W}^y \cap (-\operatorname{cl} C) = \emptyset$. Because of condition (ii), for each $y \in A(\lambda_0)$ there exists a neighborhood $\mathfrak{U}$ of (p_0, x', y) such that

$$f(\tilde{p}, \tilde{x}', \tilde{y}) \in \mathcal{W}^y + C, \text{ for all } (\tilde{p}, \tilde{x}', \tilde{y}) \in \mathfrak{U}.$$

Let $\mathfrak{U} := \mathcal{P}_y \times \mathcal{X}_y \times \mathcal{Y}_y$ where $\mathcal{P}_y, \mathcal{X}_y$, and $\mathcal{Y}_y$ are neighborhoods of p_0, x', and y, respectively. Since $\bigcup_{y \in A(\lambda_0)} \mathcal{Y}_y \supset A(\lambda_0)$ and $A(\lambda_0)$ is compact, then there is a finite subset $\{y_1, \ldots, y_n\} \subset A(\lambda_0)$ such that

$$\bigcup_{i=1}^{n} \mathcal{Y}_{y_i} \supset A(\lambda_0).$$

Let $\mathfrak{P} := \bigcap\limits_{i=1}^{n} \mathcal{P}_{y_i}$, $\mathfrak{Q} := \bigcap\limits_{i=1}^{n} \mathcal{X}_{y_i}$, and $\mathfrak{Y} := \bigcup_{i=1}^{n} \mathcal{Y}_{y_i}$. Then for each $\tilde{p} \in \mathfrak{P}$ and $\tilde{x}' \in \mathfrak{Q}$

$$f(\tilde{p}, \tilde{x}', \tilde{y}) \in \bigcup_{i=1}^{n} (\mathcal{W}^{y_i} + C) \text{ for all } \tilde{y} \in \mathfrak{Y}.$$

Now $\bigcup\limits_{i=1}^{n} \mathcal{W}^{y_i} \cap (-\mathrm{cl}\, C) = \emptyset$, i.e., $\bigcup\limits_{i=1}^{n} \mathcal{W}^{y_i} \subset (-\mathrm{cl}\, C)^{\mathrm{c}}$. Therefore,

$$\bigcup_{i=1}^{n} (\mathcal{W}^{y_i} + C) = \bigcup_{i=1}^{n} \mathcal{W}^{y_i} + C \subset (-\mathrm{cl}\, C)^{\mathrm{c}}$$

by Proposition 3.3. Hence for each $\tilde{p} \in \mathfrak{P}$ and $\tilde{x}' \in \mathfrak{Q}$

(5) $$f(\tilde{p}, \tilde{x}', \tilde{y}) \notin -\mathrm{int}\, C \text{ for all } \tilde{y} \in \mathfrak{Y}.$$

Since $\mathfrak{Y} \supset A(\lambda_0)$ is an open neighborhood of $A(\lambda_0)$ and A is u.s.c. at λ_0, there is a neighborhood $\mathcal{L}_1$ of λ_0 such that

$$A(\lambda') \subset \mathfrak{Y} \text{ for all } \lambda' \in \mathcal{L}_1.$$

Since both of $\mathcal{V}$ and $\mathfrak{Q}$ are neighborhoods of x', $\mathcal{V} \cap \mathfrak{Q} \neq \emptyset$. Let $\bar{\mathcal{V}} := \mathcal{V} \cap \mathfrak{Q}$. Since $x' \in \bar{\mathcal{V}} \cap A(\lambda_0)$, $\bar{\mathcal{V}} \cap A(\lambda_0) \neq \emptyset$. As A is l.s.c. at λ_0, there is a neighborhood $\mathcal{L}_2$ of λ such that

$$A(\lambda'') \cap \bar{\mathcal{V}} \neq \emptyset \text{ for all } \lambda'' \in \mathcal{L}_2.$$

Let $\mathfrak{L} := \mathcal{L}_1 \cap \mathcal{L}_2$. Then $A(\tilde{\lambda}) \subset \mathfrak{Y}$ for all $\tilde{\lambda} \in \mathcal{L}$ and $\mathfrak{Q} \supset \bar{\mathcal{V}} \cap A(\tilde{\lambda}) \neq \emptyset$. Therefore (5) implies for each $(\tilde{p}, \tilde{\lambda}) \in \mathfrak{P} \times \mathfrak{L}$ which is an open set containing (p_0, λ_0), there is $x^* \in A(\tilde{\lambda}) \cap \bar{\mathcal{V}}$ such that

$$f(\tilde{p}, x^*, u) \notin -\mathrm{int}\, C \text{ for all } u \in A(\tilde{\lambda}),$$

i.e. $x^* \in \Omega(\tilde{p}, \tilde{\lambda}) \cap \mathcal{V} \neq \emptyset$. Hence Ω is l.s.c. at (p_0, λ_0). ∎

Corollary 5.1. *Let X be a nonempty subset of a real Hausdorff topological vector space $\mathfrak{X}$, and Z a real topological vector space with a solid pointed convex cone $C \subset Z$. Let Γ and Λ be two nonempty subsets of two Hausdorff spaces, respectively. Suppose that A is a set-valued map from Λ to $2^X \setminus \{\emptyset\}$ and that f is a vector-valued function from $\Gamma \times X \times X$ to Z with $f(p, x, x) \notin -\mathrm{int}\, C$ for all $p \in \Gamma$ and $x \in X$. Also we assume that the following conditions:*

(*i*) *A has convex values and is continuous;*

(*ii*) *f is C-continuous on $\Gamma \times X \times X$;*

(iii) *$f(p,\cdot,y)$ is strictly $(-C)$-properly quasiconvex on X for each $p \in \Gamma$ and $y \in X$;*

(iv) *$\Omega(p,\lambda)$ contains at least two points for each $p \in \Gamma$ and $\lambda \in \Lambda$.*

Then Ω is l.s.c. on $\Gamma \times \Lambda$.

Remark 6. Lower semicontinuity of solution mapping Ω cannot be guaranteed even if f is real-valued and A is constant, by assuming only the continuity of f. See [7, p. 151]Fukushima. Stronger condition is needed, e.g., Ω is single-valued or condition (iii) and (iv) hold.

Example 3. Let $X = \mathbb{R}^2$ and let $\Gamma = [-1,2]$, $\Lambda = \{0\}$, $A(0) = K = \{(x_1, x_2) : x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0\}$, $C = \mathbb{R}_+$. Suppose that $f : \Gamma \times X \times X \to \mathbb{R}$ is defined by

$$f(p,x,y) = \big(-(1+p)y_1 - y_2\big) - \big(-(1+p)x_1 - x_2\big).$$

Then A is constant, f is continuous on $\Gamma \times X \times X$, and

$$\Omega(p,0) = \begin{cases} \{(0,1)\} & p \in [-1,0), \\ K & p = 0, \\ \{(1,0)\} & p \in (0,2]. \end{cases}$$

It is easy to see that Ω is not l.s.c. at $(0,0)$.

The next result removes the assumption that $\Omega(p,\lambda)$ contains at least two points in Corollary 5.1.

Theorem 5.2. *Let X be a nonempty subset of a real Hausdorff topological vector space $\mathfrak{X}$, and Z a real topological vector space with a solid pointed convex cone $C \subset Z$. Let Γ and Λ be two nonempty subsets of two Hausdorff spaces, respectively. Suppose that A is a set-valued map from Λ to $2^X \setminus \{\emptyset\}$ and that f is a vector-valued function from $\Gamma \times X \times X$ to Z with $f(p,x,x) \notin -\operatorname{int} C$ for all $p \in \Gamma$ and $x \in X$. Also we assume the following conditions:*

(i) *A is continuous and $A(\lambda)$ is compact convex for each $\lambda \in \Lambda$;*

(ii) *f is C-continuous on $\Gamma \times X \times X$;*

(iii) *$f(p,\cdot,y)$ is strictly $(-C)$-properly quasiconvex on X for each $p \in \Gamma$ and $y \in X$;*

(iv) *there exist $\Gamma' \subset \Gamma$ and $\hat{p} \in \Gamma$ such that for any $p \in \Gamma'$,*

$$f(p,x,y) \in f(\hat{p},x,y) + \operatorname{int} C, \ \textit{for all } x,y \in X;$$

(*v*) $\Omega(p,\lambda)\neq\emptyset$ *for each* $p\in\Gamma$ *and* $\lambda\in\Lambda$.

Then Ω *is l.s.c. on* $\Gamma'\times\Lambda$.

Proof. Let $p\in\Gamma',\lambda\in\Lambda$, $\mathcal{V}$ be an open satisfying $\mathcal{V}\cap\Omega(p,\lambda)\neq\emptyset$. Let $x\in\mathcal{V}\cap\Omega(p,\lambda)$. Then by condition (iv), there exists $\bar{p}\in\Gamma$ such that

$$f(p,x,y)\in f(\hat{p},x,y)+\operatorname{int}C,\ \text{ for all } x,y\in X.$$

Suppose that $\bar{x}\in\Omega(\hat{p},\lambda)$ which is nonempty by condition (iv) and that $x'\in\mathcal{V}\cap\{u\in A(\lambda):u=\mu x+(1-\mu)\bar{x},\ 0<\alpha<1\}$. Due to condition (iv), $\bar{x}\in\Omega(p,\lambda)$. Because of the way in selecting x', by condition (iii), we have

$$f(p,x',y)\in f(p,x,y)+\operatorname{int}C,$$

or

$$f(p,x',y)\in f(\hat{p},x,y)+\operatorname{int}C.$$

Therefore $f(p,x',y)\notin-\operatorname{cl}C$, for all $y\in A(\lambda)$. Then for each $y\in A(\lambda)$, there exists a neighborhood $\mathcal{W}_y$ of $f(p,x',y)$ such that $\mathcal{W}_y\cap(-\operatorname{cl}C)=\emptyset$. By condition (ii), for each $y\in A(\lambda)$, there exists a neighborhood $\mathfrak{U}_y$ of (p,x',y) such that

$$f(\tilde{p},\tilde{x}',\tilde{y})\in\mathcal{W}_y+C,\ \text{ for all } (\tilde{p},\tilde{x}',\tilde{y})\in\mathfrak{U}_y.$$

Let $\mathfrak{U}_y:=\mathcal{P}_y\times\mathcal{X}_y\times\mathcal{Y}_y$, where $\mathcal{P}_y,\mathcal{X}_y$ and $\mathcal{Y}_y$ are neighborhoods of p,x' and y, respectively. Since $A(\lambda)$ is compact, there exists a finite subset $\{y_1,\dots,y_n\}$ of $A(\lambda)$ such that $\bigcup_{i=1}^{n}\mathcal{Y}_{y_i}\supset A(\lambda)$. Let $\mathfrak{P}=\bigcap_{i=1}^{n}\mathcal{P}_{y_i}$, $\mathfrak{Q}=(\bigcap_{i=1}^{n}\mathcal{X}_{y_i})\cap\mathcal{V}$ and $\mathfrak{Y}=\bigcup_{i=1}^{n}\mathcal{Y}_{y_i}$. Then for each $(\tilde{p},\tilde{x},\tilde{y})\in\mathfrak{P}\times\mathfrak{Q}\times\mathfrak{Y}$

$$f(\tilde{p},\tilde{x},\tilde{y})\notin-\operatorname{int}C.$$

In addition, $\mathfrak{Y}$ is a neighborhood of $A(\lambda)$ and $\mathfrak{Q}$ is an open set with $\mathfrak{Q}\cap A(\lambda)\neq\emptyset$. Since A is continuous, by the same argument as that in the proof of Theorem 5.1 we can show that there is a neighborhood $\mathcal{L}$ of λ such that

$$A(\tilde{\lambda})\subset\mathfrak{Y}\ \text{and}\ A(\tilde{\lambda})\cap\mathfrak{Q}\neq\emptyset,\ \text{ for all } \tilde{\lambda}\in\mathcal{L}.$$

Hence for each $(\tilde{p},\tilde{\lambda})\in\mathfrak{P}\times\mathcal{L}$ which is an open set containing (p,λ), there exists $\bar{x}\in\mathfrak{Q}\cap A(\tilde{\lambda})$ such that

$$f(\tilde{p},\bar{x},\tilde{y})\notin-\operatorname{int}C,\ \text{ for all } \tilde{y}\in A(\tilde{\lambda}),$$

i.e., $\tilde{x}\in\Omega(\tilde{p},\tilde{\lambda})\cap\mathcal{V}\neq\emptyset$. Hence Ω is l.s.c. at (p,λ). Since (p,λ) is arbitrary, Ω is l.s.c. on $\Gamma'\times\Lambda$. ∎

By similar argument as that in the proof of Theorem 5.1, we can prove the following result.

Theorem 5.3. *Let X be a nonempty subset of a real Hausdorff topological vector space $\mathfrak{X}$, and Z a real topological vector space with a solid pointed convex cone $C \subset Z$. Let Γ be a nonempty subset of a Hausdorff space and Λ an arbitrarily set. Suppose that A is a set-valued map from Λ to $2^X \setminus \{\emptyset\}$ and that f is a vector-valued function from $\Gamma \times X \times X$ to Z with $f(p, x, x) \notin -\mathrm{int}\, C$ for all $p \in \Gamma$ and $x \in X$. Also we assume that the following conditions:*

- (i) *$A(\lambda)$ is compact convex for each $\lambda \in \Lambda$;*
- (ii) *$f(\cdot, x, \cdot)$ is C-continuous on $\Gamma \times X$ for each $x \in X$;*
- (iii) *$f(p, \cdot, y)$ is strictly $(-C)$-properly quasiconvex on X for each $p \in \Gamma$ and $y \in X$;*
- (iv) *$\Omega(p, \lambda)$ contains at least two points for each $p \in \Gamma$ and $\lambda \in \Lambda$.*

Then $\Omega(\cdot, \lambda)$ is l.s.c. on Γ for each $\lambda \in \Lambda$.

By the same argument as that in the proof of Theorem 5.2, we can prove the following result.

Theorem 5.4. *Let X be a nonempty subset of a real Hausdorff topological vector space $\mathfrak{X}$, and Z a real topological vector space with a solid pointed convex cone $C \subset Z$. Let Γ be a nonempty subset of a Hausdorff space and Λ an arbitrarily set. Suppose that A is a set-valued map from Λ to $2^X \setminus \{\emptyset\}$ and that f is a vector-valued function from $\Gamma \times X \times X$ to Z with $f(p, x, x) \notin -\mathrm{int}\, C$ for all $p \in \Gamma$ and $x \in X$. Also we assume that the following conditions:*

- (i) *$A(\lambda)$ is compact convex for each $\lambda \in \Lambda$;*
- (ii) *$f(\cdot, x, \cdot)$ is C-continuous on $\Gamma \times X$ for each $x \in X$;*
- (iii) *$f(p, \cdot, y)$ is strictly $(-C)$-properly quasiconvex on X for each $p \in \Gamma$ and $y \in X$;*
- (iv) *there exist $\Gamma' \subset \Gamma$ and $\hat{p} \in \Gamma$ such that for any $p \in \Gamma$,*

$$f(p, x, y) \in f(\hat{p}, x, y) + \mathrm{int}\, C, \ \textit{for all } x, y \in X;$$

- (v) *$\Omega(p, \lambda) \neq \emptyset$ for each $p \in \Gamma$ and $\lambda \in \Lambda$.*

Then $\Omega(\cdot, \lambda)$ is l.s.c. on Γ' for each $\lambda \in \Lambda$.

6. Continuity of Ω

By combining results in Sections 4 and 5, we can derive the following results concerning the continuity of the solution mapping of PVEP.

Theorem 6.1. *Let X be a nonempty subset of a real Hausdorff topological vector space $\mathfrak{X}$, and Z a real topological vector space with a solid pointed convex cone $C \subset Z$. Let Γ and Λ be two nonempty subsets of two Hausdorff spaces, respectively. Suppose that A is a set-valued map from Λ to $2^X \setminus \{\emptyset\}$ and that f is a vector-valued function from $\Gamma \times X \times X$ to Z with $f(p,x,x) \notin -\operatorname{int} C$ for all $p \in \Gamma$ and $x \in X$. Also we assume the following conditions:*

(i) *A is continuous and $A(\lambda)$ is compact convex for each λ;*

(ii) *f is C-continuous and $(-C)$-continuous on $\Gamma \times X \times X$;*

(iii) *$f(p,\cdot,y)$ is strictly $(-C)$-properly quasiconvex on X for each $p \in \Gamma$ and $y \in X$;*

(iv) *$\Omega(p,\lambda) \neq \emptyset$ for each $p \in \Gamma$ and $\lambda \in \Lambda$.*

Then Ω is continuous. on $\Gamma \times \Lambda$.

Proof. By Theorem 4.1, Ω is u.s.c. on $\Gamma \times \Lambda$. Let $p \in \Gamma$ and $\lambda \in \Lambda$. If $\Omega(p,\lambda)$ is a singleton, then Ω is continuous at (p,λ) because Ω is u.s.c. on $\Gamma \times \Lambda$. If $\Omega(p,\lambda)$ has at least two points, then by Theorem 5.1, Ω is continuous at (p,λ). Hence Ω is continuous on $\Gamma \times \Lambda$. ∎

Remark 7. We observe that the condition that f is both C-continuous and $(-C)$-continuous doesn't imply that f is continuous. See [11, pp.22-23, Theorem 5.3 and Remark 5.4].

Example 4. Let $X = l^1$, $Z = l^\infty$, and $f : X \to Z$ defined by

$$f(x) = (y(x_1), \ldots, y(x_i), \ldots),$$

where

$$y(x_i) = \begin{cases} -1 & x_i < -\dfrac{1}{i}, \\ i \cdot x_i & -\dfrac{1}{i} \leq x_i \leq \dfrac{1}{i}, \quad (i \in \mathbb{N}); \\ 1 & \dfrac{1}{i} < x_i, \end{cases}$$

and let $C = \bigcap_{i=2}^{\infty} C_i$, where $C_i := \{z \in Z : z_1 \geq \frac{1}{i} z_i,\ z = (z_1, \ldots, z_i, \ldots)\}$. Then f is C-continuous at θ_X for each fixed y and $(-C)$-continuous at θ_X, but not continuous at θ_X.

Theorem 6.2. *Let X be a nonempty compact subset of a real Hausdorff topological vector space $\mathfrak{X}$, and Z a real topological vector space with a solid pointed convex cone $C \subset Z$. Let Γ be nonempty subsets of Hausdorff space and Λ*

an arbitrary set. Suppose that A is a set-valued map from Λ to $2^X \setminus \{\emptyset\}$ and that f is a vector-valued function from $\Gamma \times X \times X$ to Z with $f(p, x, x) \notin -\operatorname{int} C$ for all $p \in \Gamma$ and $x \in X$. Also we assume the following conditions:

(i) *$A(\lambda)$ is compact convex for each $\lambda \in \Lambda$;*

(ii) *f is $(-C)$-continuous on $\Gamma \times X$ for each $y \in A(\lambda)$;*

(iii) *$f(\cdot, x, \cdot)$ is C-continuous on $\Gamma \times X$ for each $x \in X$;*

(iv) *$f(p, \cdot, y)$ is strictly $(-C)$-properly quasiconvex on X for each $p \in \Gamma$ and $y \in X$;*

(v) *$\Omega(p, \lambda) \neq \emptyset$ for each $p \in \Gamma$ and $\lambda \in \Lambda$.*

Then $\Omega(\cdot, \lambda)$ is continuous on Γ for each $\lambda \in \Lambda$.

Proof. By Theorem 4.2, $\Omega(\cdot, \lambda)$ is u.s.c. on Γ for each $\lambda \in \Lambda$. Let $p \in \Gamma$ and $\lambda \in \Lambda$. If $\Omega(p, \lambda)$ is a singleton, then Ω is continuous at (p, λ) because Ω is u.s.c. on $\Gamma(\cdot, \lambda)$ for each $\lambda \in \Lambda$. If $\Omega(p, \lambda)$ has at least two points, then by Theorem 5.3, $\Omega(\cdot, \lambda)$ is continuous at (p, λ) for each $\lambda \in \Lambda$. Hence $\Omega(\cdot, \lambda)$ is continuous on $\Gamma \times \Lambda$. ∎

Theorem 6.3. *Let X be a nonempty subset of a real Hausdorff topological vector space $\mathfrak{X}$, and Z a real topological vector space with a solid pointed convex cone $C \subset Z$. Let Γ and Λ be two nonempty subsets of two Hausdorff spaces, respectively. Suppose that A is a set-valued map from Λ to $2^X \setminus \{\emptyset\}$ and that f is a vector-valued function from $\Gamma \times X \times X$ to Z with $f(p, x, x) \notin -\operatorname{int} C$ for all $p \in \Gamma$ and $x \in X$. Also we assume that the following conditions:*

(i) *A has compact convex values and is continuous;*

(ii) *f is C-continuous and $(-C)$-continuous on $\Gamma \times X \times X$;*

(iii) *$f(p, x, \cdot)$ is C-quasiconvex on $A(\lambda)$ for each $p \in \Gamma, x \in A(\lambda)$;*

(iv) *$f(p, \cdot, y)$ is strictly $(-C)$-properly quasiconvex on X for each $p \in \Gamma$ and $y \in X$.*

Then Ω is continuous on $\Gamma \times \Lambda$.

Proof. By Theorem 4.1, $\Omega(p, \lambda)$ is nonempty for each $p \in \Gamma$ and $\lambda \in \Lambda$. Hence by Theorem 6.1, Ω is continuous on $\Gamma \times \Lambda$. ∎

7. Applications

In this section we will give some applications of the results established in Sections 3-6.

Theorem 7.1. *Let X be a real Hausdorff topological vector space. Let Z be a real topological vector space with a solid pointed convex cone $C \subset Z$. Suppose that K is a nonempty subset of X, that g is a vector-valued function from $X \times X$ to Z with $g(x, x) = \theta_Z$ for all $x \in X$. Also we assume that the following conditions:*

(i) $\mathrm{cl}\, K$ *is compact convex;*

(ii) $g(x, \cdot)$ *is C-quasiconvex on X for each $x \in X$;*

(iii) $g(\cdot, y)$ *is $(-C)$-continuous on X for each $y \in X$;*

(iv) g *is C-continuous on $X \times X$.*

Then the problem ε-VEP has at least one solution, i.e., $S(\varepsilon)$ is nonempty for each $\varepsilon \in \mathrm{int} C$.

Proof. Putting, $\Gamma = \{\theta_Z\} \cup \mathrm{int}\, C$, $\Gamma' = \mathrm{int}\, C$, $\Lambda = \{0\}$, $A(\lambda) = K$, $f(p, x, y) = g(x, y) + p, p \in \mathrm{int}\, C$, and $\hat{p} = \theta_Z$ in Theorem 3.3, we get the conclusion. ■

Theorem 7.2. *Let X be a real Hausdorff topological vector space. Let Z be a real topological vector space with a solid pointed convex cone $C \subset Z$. Suppose that K is a nonempty subset of X, that f is a vector-valued function from $X \times X$ to Z with $f(x, x) = \theta_Z$ for all $x \in X$. Also we assume that the following conditions:*

(i) *K is compact;*

(ii) $g(\cdot, y)$ *is $(-C)$-continuous on X for each $y \in X$;*

(iii) $S(\varepsilon)$ *is nonempty for each $\varepsilon \in \mathrm{int} C$.*

Then S is u.s.c. on $\mathrm{int}\, C$.

Proof. Let $\Gamma = \mathrm{int}\, C$, $\Lambda = \{0\}$, $A(0) = K$ and $f(\varepsilon, x, y) = g(x, y) + \varepsilon$. Then $f(\cdot, \cdot, y)$ is $-C$-continuous on $\Gamma \times X$. The result then follows from Theorem 4.2. ■

Theorem 7.3. *Let X be a real Hausdorff topological vector space. Let Z be a real topological vector space with a solid pointed convex cone $C \subset Z$. Suppose that K is a nonempty subset of X, that g is a vector-valued function from $X \times X$ to Z with $g(x, x) = \theta_Z$ for all $x \in X$. Also we assume that the following conditions:*

(i) *K is compact convex;*

(ii) $g(x, \cdot)$ *is C-continuous on X for each $x \in X$;*

(iii) $g(\cdot, y)$ *is strictly $(-C)$-properly quasiconvex on K for each $y \in K$;*

(iv) $S(\varepsilon)$ *is nonempty for each $\varepsilon \in \mathrm{int}\, C$.*

Then S is l.s.c. on $\mathrm{int}\, C$.

Proof. Let $\Gamma' = \mathrm{int}\, C$, $\Lambda = \{0\}$, $A(0) = K$, $f(\varepsilon, x, y) = g(x, y) + \varepsilon$, $\Gamma = \{\theta_Z\} \cup \mathrm{int}\, C$ and $\theta_Z = \hat{p}$. The conclusion follows from Theorem 5.4 ■

REFERENCES

1. Q. H. Ansari, S. Schaible and J. C. Yao, The system of generalized vector equilibrium problems with applications, *J. Global Optim.*, **22** (2002), 3-16.

2. Q. H. Ansari and Y. C. Yao, An existence result for generalized vector equilibrium problem, *Appl. Math. Lett.* **12** (1999), 53-56.

3. Q. H. Ansari, S. Schaible and J. C. Yao, System of vector equlibrium problems and its applications, *J. Optim. Theory Appl.*, **107** (2000), 547-557.

4. C. Berge, *Topological Space*, Oliver & Boyd, Edinburgh and London, 1963.

5. F. Ferro, A Minimax Theorem for Vector-Valued Functions, *J. Optim. Theory Appl.*, **60(1)** (1989), 19-31.

6. K. Fan, A Generalization of Tychonoff's Fixed Point Theorem, *Math. Ann.*, **142** (1961), 305-310.

7. M. Fukushima, *Bases of Nonlinear Analysis*, Asakura Publishing Co., Ltd, Tokyo, 2001.

8. K. Kimura, Y. C. Liou and J. C. Yao, Semicontinuity of the Solution Mapping of $\epsilon - vector$ equilibrium problem, *Pacific Journal of Optimization*, **3(2)** (2007), 345-359.

9. K. Kimura and T. Tanaka, Existence theorem of cone saddle-point applying a nonlinear scalarization, *Taiwanese J. Math.*, **10(2)** (2006), 563-571.

10. Y. C. Liou, X. Q. Yang and J. C. Yao, Mathematical programs with vector optimization constraints, *J. Optim. Theory Appl.*, **126(2)** (2005), 345-355.

11. D. T. Luc, Connectedness of the efficient point sets in quasiconcave vector maximization, *J. Math. Anal. Appl.*, **122** (1987), 346-354.

12. T. Tanaka, Cone-quasiconvexity of vector-valued functions, *Sci.Rep. Hirosaki Univ.*, **42** (1995), 157-163.

13. T. Tanaka, Generalized semicontinuity and existence theorems for cone saddle points, *Appl. Math. Optim.*, **36** (1997), 313-322.

14. L. C. Zeng, S. Y. Wu and J. C. Yao, Generalized KKM Theorem with applications to generalized minimax inequalities and generalized equlibrium problems, *Taiwanese J. Math.*, **10** (2006), 1497-1514.

Kenji Kimura and Jen-Chih Yao*
Department of Applied Mathematics,
National Sun Yat-sen University,
Kaohsiung 80424,
Taiwan, R.O.C.
E-mail: kimura@math.nsysu.edu.tw
yaojc@math.nsysu.edu.tw