TAIWANESE JOURNAL OF MATHEMATICS
Vol. 7, No. 2, pp. 293-308, June 2003

THE GENERALIZED CESÀRO OPERATOR ON THE UNIT POLYDISK

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Abstract. Let $D_n = \{(z_1, \dots, z_n) \in \mathbf{C}^n : |z_j| < 1,\ j = 1, \dots, n\}$ be the unit polydisk in $\mathbf{C}^n$. The aim of this paper is to prove the boundedness of the generalized Cesàro operators $\mathcal{C}^{\vec{\gamma}}$ on $H^p(D_n)$ (Hardy) and $\mathcal{A}^{p,q}_{\vec{\mu}}(D_n)$ (the generalized Bergman) spaces, for $0 < p, q < \infty$ and $\vec{\gamma} = (\gamma_1, \dots, \gamma_n)$ with *Re* $(\gamma_j) > 1$, $j = 1, \dots, n$. Here $\vec{\mu} = (\mu_1, \dots, \mu_n)$ and each μ_j is a positive Borel measure on the interval $[0, 1)$. Also we present a class of invariant spaces under the action of this operator.

1. Introduction and Preliminaries

The classical Cesàro operator $\mathcal{C}$ is defined by

$$\mathcal{C}(f)(z) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_k \right) z^n,$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an analytic function on the unit disk $D = \{z \in \mathbf{C} : |z| < 1\}$ in the complex plane $\mathbf{C}$. This operator has been studied extensively by many mathematicians in the past decade. One of the major interests in this operator is its behavior on function spaces. It is known that the operator $\mathcal{C}$ is bounded on the Hardy spaces $H^p(D)$ for $0 < p < \infty$. Basic facts on Hardy spaces can be found, for example, in [6]. For $1 < p < \infty$, the boundedness of the operator $\mathcal{C}$ on $H^p(D)$ is a consequence of a classical result of Hardy [9], and further information can be found in [12]. The case $p = \infty$ was considered in [5]. The boundedness on $H^1(D)$ was

Received April 1, 2002.
Communicated by S. B. Hsu.
2000 *Mathematics Subject Classification*: 47B38, 46E15.
Key words and phrases: Analytic functions, Cesàro operator, polydisk, Hardy spaces, Bergman spaces, invariant spaces.
Research partially supported by a grant from National Science Foundation and by a William Fulbright Research Grant.

given by Siskakis [14] by a particularly elegant method. A different proof of the result can be found in [8]. Following ideas in [10], Miao [11] extended the result to $H^p(D)$ with $0 < p < 1$. It has been shown that the operator $\mathcal{C}$ is also bounded on the Bergman spaces (see *e.g.,* [15]) as well as on the weighted Bergman spaces (see *e.g.,* [1] and [3]). $\mathcal{D}_{2,a}$, for $a \in (0,1)$. $\mathcal{D}_{2,0}$. Indeed, $\mathcal{C}(1)(z) \notin \mathcal{D}_{2,0}$.

For each complex γ with $Re\ (\gamma) > -1$ and k nonnegative integer let A_k^γ be defined as the kth coefficient in the expression

$$\frac{1}{(1-x)^{\gamma+1}} = \sum_{k=0}^{\infty} A_k^\gamma x^k,$$

so that $A_k^\gamma = \frac{(\gamma+1)\cdots(\gamma+k)}{k!}$.

For an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on D, the generalized Cesàro operator is defined by

$$\mathcal{C}^\gamma(f)(z) = \sum_{n=0}^{\infty}\left(\frac{1}{A_n^{\gamma+1}}\sum_{k=0}^{n} A_{n-k}^{\gamma} a_k\right) z^n. \tag{1}$$

The integral form of $\mathcal{C}^\gamma$ is (see [16])

$$\mathcal{C}^\gamma(f)(z) = \frac{\gamma+1}{z^{\gamma+1}}\int_0^z f(\zeta)\frac{(z-\zeta)^\gamma}{(1-\zeta)^{\gamma+1}}d\zeta,$$

or, taking simply as a path the segment joining 0 and z,

$$\mathcal{C}^\gamma(f)(z) = (\gamma+1)\int_0^1 f(tz)\frac{(1-t)^\gamma}{(1-tz)^{\gamma+1}}dt.$$

These operators were first introduced in [16] and have been subsequently studied in [2] and [20]. The adjoint operator of $\mathcal{C}^\gamma$ was considered in [2], [7], [12], [13], [16], [19] and [20]. Note that when $\gamma = 0$, we obtain the classical Cesàro operator $\mathcal{C}^0 = \mathcal{C}$. Stempak proved that $\mathcal{C}^\gamma$ is bounded on $H^p(D)$ for $0 < p \le 2$. For $0 < p \le 1$, his method is similar to that of Miao; for $p = 2$, it is based on the boundedness of an appropriate sequence transformation, and an interpolation then yields the result for $1 < p < 2$. After that, Andersen [2] and Xiao [20] proved the boundedness of $\mathcal{C}^\gamma$, on $H^p(D)$ for $p > 2$ using different methods.

Motivated by [16], in [17] we defined a family of Cesàro operators $\mathcal{C}^{\vec{\gamma}}$, for the polydisk D_n. Let $\vec{\gamma} = (\gamma_1, ..., \gamma_n) \in \mathbf{C}^n$, $Re\ (\gamma_j) > -1, j = 1, ..., n$. The *generalized Cesáro operator* $\mathcal{C}^{\vec{\gamma}}$ is defined by

$$\mathcal{C}^{\vec{\gamma}}(f)(z) = \sum_{|\alpha|=0}^{\infty}\left(\frac{\sum_{\beta\le\alpha} a_{\alpha-\beta}\prod_{j=1}^{n} A_{\beta_j}^{\gamma_j}}{\prod_{j=1}^{n} A_{\alpha_j}^{\gamma_j+1}}\right) z^\alpha,$$

whenever $f(z) = \sum_{|\alpha|=0}^{\infty} a_\alpha z^\alpha$ is an analytic function on D_n (α and β are multi-indices from $(\mathbf{Z}_+)^n$). A simple calculation with power series then gives

$$\text{(2)} \quad \mathcal{C}^{\vec{\gamma}}(f)(z) = \prod_{j=1}^{n}(\gamma_j + 1) \int_0^1 \cdots \int_0^1 \frac{f(\tau_1 z_1, ..., \tau_n z_n)}{\prod_{j=1}^{n}(1 - \tau_j z_j)^{\gamma_j+1}} \prod_{j=1}^{n}(1 - \tau_j)^{\gamma_j} d\tau,$$

where $d\tau = d\tau_1 \cdots d\tau_n$.

In what follows, for $z, w \in \mathbf{C}^n$ we write $z \cdot w = (z_1 w_1, ..., z_n w_n)$; $e^{i\theta}$ is an abbreviation for $(e^{i\theta_1}, ..., e^{i\theta_n})$; $d\tau = d\tau_1 \cdots d\tau_n$; $d\theta = d\theta_1 \cdots d\theta_n$ and r, τ, s are vectors in $\mathbf{C}^n$. If we write $0 \le r < 1$, where $r = (r_1, ..., r_n)$ it means $0 \le r_j < 1$ for $j = 1, ..., n$.

In [17] we proved the following theorem:

Theorem A. *Let $0 < p \le 1$, $\vec{\gamma} = (\gamma_1, ..., \gamma_n)$ such that $Re\,(\gamma_j) > -1$, $j = 1, ..., n$ and $0 \le r < 1$. Then there is a constant C independent of f and r such that*

$$\int_{[0,2\pi]^n} |\mathcal{C}^{\vec{\gamma}}(f)(r \cdot e^{i\theta})|^p d\theta \le C \int_{[0,2\pi]^n} |f(r \cdot e^{i\theta})|^p d\theta,$$

for all $H(D_n)$.

We proved Theorem A, following Miao's arguments from [11], which are modifications of the corresponding arguments used in the case of the unit disk.

In [3], quite independently the authors introduced and considered the case $\vec{\gamma} = \vec{0}$. They proved a result similar to Theorem A for the operator $\mathcal{C}^{\vec{0}}$ in the case of $0 < p < \infty$. In the case $1 < p < \infty$, their method is based on the following result (Theorem 1.8 in [3]):

Theorem B. *Let $p \in [1, \infty)$, $\alpha_j > -1$, $j = 1, ..., n$ and m be a fixed positive integer and let $\mathbf{k} = (k_1, ..., k_n) \in (\mathbf{Z}_+)^n$. Let f be a holomorphic function defined on the polydisc D_n in $\mathbf{C}^n$. Then for $\vec{\alpha} = (\alpha_1, ..., \alpha_n)$, $f \in \mathcal{A}^p(dV_{\vec{\alpha}})$ if and only if*

$$\left[\prod_{j=1}^{n}(1 - |z_j|^2)^{k_j}\right] \frac{\partial^{|\mathbf{k}|} f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}(z) \in \mathcal{L}^p(dV_{\vec{\alpha}}), \qquad \forall\ \mathbf{k} \textit{ with } |\mathbf{k}| = m.$$

Moreover,

$$||f||_{\mathcal{A}^p(dV_{\vec{\alpha}})} \asymp \left(\sum_{|\mathbf{k}|=0}^{m-1} \left| \frac{\partial^{|\mathbf{k}|} f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}(0) \right| + \sum_{|\mathbf{k}|=m} \left\| \left[\prod_{j=1}^{n}(1 - |z_j|^2)^{k_j}\right] \frac{\partial^m f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} \right\|_{\mathcal{L}^p(dV_{\vec{\alpha}})} \right).$$

We first would like to mention that in [18], the second author proved similar results as Theorem B with general weights. Readers can consult the paper [18] and references therein to study results of weighted Bergman speces. We also want to point out that there is a small gap in the proof of the case $1 < p < \infty$ of Theorem A in [3]. Following Andersen's ideas in [2], we would like to provide a complete proof for the case $1 < p < \infty$. To make this paper self-contained, we will also give a proof for the case $0 < p \le 1$ in section 2. Let us state our first main result as the following theorem.

Theorem 1. *Let $0 < p < \infty$, $\gamma = (\gamma_1, ..., \gamma_n)$ such that Re $(\gamma_j) > -1$, $j = 1, ..., n$ and $0 \le r < 1$. Then there is a constant C independent of f and r such that*

$$\int_{[0,2\pi]^n} |C^{\vec{\gamma}}(f)(r \cdot e^{i\theta})|^p d\theta \le C \int_{[0,2\pi]^n} |f(r \cdot e^{i\theta})|^p d\theta,$$

for all $H(D_n)$.

In order to prove Theorem 1 we need several auxiliary results which are incorporated in the following lemmas.

Lemma 1. ([4]) *Let $0 < p < \infty$ and $0 \le r < 1$. Then there is a constant C independent of f and r such that*

$$\int_{[0,2\pi]^n} \sup_{0 \le \tau < 1} |f(\tau \cdot r \cdot e^{i\theta})|^p d\theta \le C \int_{[0,2\pi]^n} |f(r \cdot e^{i\theta})|^p d\theta$$

for all $f \in H(D_n)$.

Lemma 2. ([6, p.65]) *For each $1 < a < \infty$ there is a positive constant $C = C(a)$ such that*

$$\int_{-\pi}^{\pi} |1 - \rho e^{i\theta}|^{-a} d\theta \le C(1-\rho)^{1-a}, \ \ \textit{if} \ \ 0 \le \rho < 1.$$

The following lemma is a well-known generalization of a theorem in [9].

Lemma 3. *Let $0 < p < \infty$, $1 < a < \infty$ and $0 \le r < 1$. Then there is a constant C independent of f and r such that*

$$\int_{[0,1)^n} \left(\int_{[0,2\pi]^n} |f(\tau \cdot r \cdot e^{i\theta})|^{pa} d\theta \right)^{1/a} \prod_{j=1}^{n} (1-\tau_j)^{-1/a} d\tau \le C \int_{[0,2\pi]^n} |f(r \cdot e^{i\theta})|^p d\theta,$$

for all $f \in H(D_n)$.

For real y and $\sigma > -1$, set

$$H^{\sigma}(y) = \frac{1}{1+|y|} \begin{cases} 1+|y|^{\sigma}, & \text{if } \sigma < 0 \\ \log(2+1/|y|), & \text{if } \sigma = 0 \\ 1, & \text{if } \sigma > 0. \end{cases}$$

Lemma 4. ([2]) *For $\sigma > -1$, there is a constant $C = C(\sigma)$ such that*

$$\int_0^1 \frac{x^{\sigma+1}dx}{[x^2+\varphi^2][x^2+\theta^2]^{(\sigma+1)/2}} \cdot \; C\frac{H^{\sigma}(\varphi/\theta)}{|\theta|}$$

for all real φ and $\theta \neq 0$.

For any measurable function $g(e^{i\theta})$, define $E_s g(e^{i\theta}) = E_{s_1,\dots,s_n} g(e^{i\theta})$ by

$$E_s g(e^{i\theta}) = \begin{cases} g(e^{i(s+1)\theta}), & \text{if } |s_j\theta_j| \cdot \; \pi \text{ for all } j \in \{1,\dots,n\}, \\ 0, & \text{otherwise.} \end{cases}$$

The following lemma is a generalization of Lemma 2.2 in [2].

Lemma 5. *Let $\sigma_j > -1$, $j = 1,..,n$, $1 < p < \infty$ and*

$$A_{\vec{\sigma},p} = 2^{n/p} \int_{\mathbf{R}^n} \prod_{j=1}^{n} \frac{H^{\sigma_j}(s_j)}{|s_j+1|^{1/p}} ds.$$

Then $A_{\vec{\sigma},p} < \infty$ and

$$\int_{[-\pi,\pi]^n} \left(\int_{\mathbf{R}^n} \prod_{j=1}^{n} H^{\sigma_j}(s_j) E_s g(e^{i\theta}) ds \right)^p d\theta \cdot \; A^p_{\vec{\sigma},p} \int_{[-\pi,\pi]^n} g^p(e^{i\theta}) d\theta,$$

for all measurable $g \geq 0$.

Proof. The first assertion can be easily proved. Let $H^{\sigma}(s) = \prod_{j=1}^{n} H^{\sigma_j}(s_j)$. By Minkowski's inequality we obtain

$$\left(\int_{[-\pi,\pi]^n} \left(\int_{\mathbf{R}^n} H^{\sigma}(s) E_s g(e^{i\theta}) ds \right)^p d\theta \right)^{1/p}$$

$$\cdot \; \int_{\mathbf{R}^n} H^{\sigma}(s) \left(\int_{[-\pi,\pi]^n} [E_s g(e^{i\theta})]^p d\theta \right)^{1/p} ds.$$

On the other hand, since for real b, $\min\{|b+1|, |(b+1)/b|\} \cdot \; 2$, for $s_j \neq -1$, $j = 1,\dots,n$, we obtain

$$
\begin{aligned}
&\int_{[-\pi,\pi]^n}[E_s g(e^{i\theta})]^p d\theta \\
&= \int_{\bigcap_{j=1}^{n}\{\theta_j\,:\,|s_j\theta_j|\le\,\pi\}\cap\{\theta_j\,:\,|\theta_j|\le\,\pi\}} g^p(e^{i(s+1)\theta})d\theta \\
(4)\qquad &= \prod_{j=1}^{n}\frac{1}{|s_j+1|}\int_{\bigcap_{j=1}^{n}\{\varphi_j\,:\,|s_j\varphi_j|\le\,|s_j+1|\pi\}\cap\{\varphi_j\,:\,|\varphi_j|\le\,|s_j+1|\pi\}} g^p(e^{i\varphi})d\varphi \\
&\le \prod_{j=1}^{n}\frac{1}{|s_j+1|}\int_{\bigcap_{j=1}^{n}\{|\varphi_j|\le\,2\pi\}} g^p(e^{i\varphi})d\varphi \\
&= \prod_{j=1}^{n}\frac{2}{|s_j+1|}\int_{[-\pi,\pi]^n} g^p(e^{i\varphi})d\varphi.
\end{aligned}
$$

From (3) and (4) the result follows.

2. Proof of Theorem 1

In this section we give a proof of Theorem 1. Throughout the following proof C will denotes a constant which may change from line to line.

Proof of Theorem 1. In what follows, for the sake of simplicity, we assume that γ_j, $j = 1, ..., n$, are real numbers such that $\gamma_j > -1$.

Case $0 < p \le 1$. Let $f \in H(D_n)$ and $t_k = 1 - 2^{-k}$, $k \in \mathbf{N} \cup \{0\}$. Let

$$
I = M_p^p(\mathcal{C}^{\vec{\gamma}}(f), r) = \int_{[0,2\pi]^n} |\mathcal{C}^{\vec{\gamma}}(f)(r\cdot e^{i\theta})|^p d\theta.
$$

By Lemma 1 and some simple calculations, we obtain

$$
\begin{aligned}
I &\le C\int_{[0,2\pi]^n}\left(\int_{[0,1)^n}\frac{|f(\tau\cdot r\cdot e^{i\theta})|}{|\prod_{j=1}^{n}(1-\tau_j r_j e^{i\theta_j})^{\gamma_j+1}|}\prod_{j=1}^{n}(1-\tau_j)^{\gamma_j}d\tau\right)^p d\theta \\
&\le C\sum_{k_1,\dots,k_n=1}^{\infty}\int_{[0,2\pi]^n} \\
(5)\qquad &\left(\int_{t_{k_1-1}}^{t_{k_1}}\cdots\int_{t_{k_n-1}}^{t_{k_n}}\frac{|f(\tau\cdot r\cdot e^{i\theta})|}{|\prod_{j=1}^{n}(1-tau_j r_j e^{i\theta_j})^{\gamma_j+1}|}\prod_{j=1}^{n}(1-\tau_j)^{\gamma_j}d\tau\right)^p d\theta \\
&\le C\sum_{k_1,\dots,k_n=1}^{\infty}\frac{1}{2^{p\sum_{j=1}^{n}k_j(\gamma_j+1)}}\int_{[0,2\pi]^n}\sup_{t_{\mathbf{k}-1}<\tau<t_{\mathbf{k}}} \\
&\left(\frac{|f(\tau\cdot r\cdot e^{i\theta})|}{|\prod_{j=1}^{n}(1-\tau_j r_j e^{i\theta_j})^{\gamma_j+1}|}\right)^p d\theta
\end{aligned}
$$

$$
\begin{aligned}
&\cdot\ C \sum_{k_1,\ldots,k_n=1}^{\infty} \frac{1}{2^{p\sum_{j=1}^{n} k_j(\gamma_j+1)}} \int_{[0,2\pi]^n} \sup_{0\cdot\ \tau<t_{\mathbf{k}}} \\
&\left(\frac{|f(\tau\cdot r\cdot e^{i\theta})|}{|\prod_{j=1}^{n}(1-\tau_j r_j e^{i\theta_j})^{\gamma_j+1}|}\right)^p d\theta \\
&\cdot\ C \sum_{k_1,\ldots,k_n=1}^{\infty} \frac{1}{2^{p\sum_{j=1}^{n} k_j(\gamma_j+1)}} \int_{[0,2\pi]^n} \left(\frac{|f(t_{\mathbf{k}}\cdot r\cdot e^{i\theta})|}{|\prod_{j=1}^{n}(1-t_{k_j} r_j e^{i\theta_j})^{\gamma_j+1}|}\right)^p d\theta \\
&\cdot\ C \sum_{k_1,\ldots,k_n=1}^{\infty} \int_{t_{k_1}}^{t_{k_1+1}} \cdots \int_{t_{k_n}}^{t_{k_n+1}} \int_{[0,2\pi]^n} \left(\frac{|f(\tau\cdot r\cdot e^{i\theta})|}{|\prod_{j=1}^{n}(1-\tau_j r_j e^{i\theta_j})^{\gamma_j+1}|}\right)^p d\theta \\
&\times \prod_{j=1}^{n}(1-\tau_j)^{p(\gamma_j+1)-1} d\tau \\
&\cdot\ C \int_{[0,1)^n} \int_{[0,2\pi]^n} \left(\frac{|f(\tau\cdot r\cdot e^{i\theta})|}{|\prod_{j=1}^{n}(1-\tau_j r_j e^{i\theta_j})^{\gamma_j+1}|}\right)^p d\theta \prod_{j=1}^{n}(1-\tau_j)^{p(\gamma_j+1)-1} d\tau.
\end{aligned}
$$

Here, $t_{\mathbf{k}}$ denotes $(t_{k_1},\ldots,t_{k_n})$.

Choose $a>1$ such that $\max_{j=1,..,n}\{1-p(\gamma_j+1)\}<1/a$. Then by Hölder's inequality with exponents a and $b=a/(a-1)$, and using Lemma 2 we obtain

$$
\begin{aligned}
(6)\qquad &\int_{[0,2\pi]^n} \left(\frac{|f(\tau\cdot r\cdot e^{i\theta})|}{|\prod_{j=1}^{n}(1-\tau_j r_j e^{i\theta_j})^{\gamma_j+1}|}\right)^p d\theta \\
&\cdot \left(\int_{[0,2\pi]^n} |f(\tau\cdot r\cdot e^{i\theta})|^{pa} d\theta\right)^{1/a} \left(\int_{[0,2\pi]^n} \frac{d\theta}{|\prod_{j=1}^{n}(1-\tau_j r_j e^{i\theta_j})^{\gamma_j+1}|^{pb}}\right)^{1/b} \\
&\cdot\ C\left(\int_{[0,2\pi]^n} |f(\tau\cdot r\cdot e^{i\theta})|^{pa} d\theta\right)^{1/a} \prod_{j=1}^{n}(1-\tau_j r_j)^{-(\gamma_j+1)p+1-1/a} \\
&\cdot\ C\left(\int_{[0,2\pi]^n} |f(\tau\cdot r\cdot e^{i\theta})|^{pa} d\theta\right)^{1/a} \prod_{j=1}^{n}(1-\tau_j)^{-(\gamma_j+1)p+1-1/a}.
\end{aligned}
$$

From (5) and (6) we obtain

$$
\begin{aligned}
M_p^p(\mathcal{C}^{\vec{\gamma}}(f),r)\cdot\ &C\int_{[0,1)^n} \left(\int_{[0,2\pi]^n} |f(\tau\cdot r\cdot e^{i\theta})|^{pa} d\theta\right)^{1/a} \prod_{j=1}^{n}(1-\tau_j)^{-1/a} d\tau \\
\cdot\ &C\int_{[0,2\pi]^n} |f(r\cdot e^{i\theta})|^p d\theta,
\end{aligned}
$$

as desired.

Case $p > 1$. Let $f \in H(D_n)$ and $0 < r < 1$, set $f_r(e^{i\varphi}) = f(r \cdot e^{i\varphi})$. Then for $0 < \tau < 1$, $f(\tau \cdot r \cdot e^{i\theta})$ is given by the following integral

$$f(\tau \cdot r \cdot e^{i\theta}) = \frac{1}{(2\pi)^n} \int_{[-\pi,\pi]^n} f_r(e^{i\varphi}) \prod_{j=1}^{n} P(\tau_j, \varphi_j - \theta_j) d\varphi \tag{7}$$

where $P(\rho, \phi)$ is the Poisson's kernel i.e.

$$P(\rho, \phi) = \frac{1 - \rho^2}{1 - 2\rho \cos\, \phi + \rho^2}.$$

Combining (2) and (7) and using Fubini's theorem, we obtain

$$\mathcal{C}^{\vec{\gamma}}(f)(r \cdot e^{i\theta}) = \prod_{j=1}^{n} \frac{\gamma_j + 1}{2\pi} \int_{[-\pi,\pi]^n} K_r^{\vec{\gamma}}(\theta, \varphi) f_r(e^{i(\theta+\varphi)}) d\varphi,$$

where

$$K_r^{\vec{\gamma}}(\theta, \varphi) = \prod_{j=1}^{n} \int_0^1 \frac{(1 + \tau_j)(1 - \tau_j)^{\gamma_j + 1}}{(1 - 2\tau_j \cos\, \varphi_j + \tau_j^2)(1 - r_j \tau_j e^{i\theta_j})^{\gamma_j + 1}} d\tau_j.$$

Using an estimate in [2,p.621], we have that there is a constant $C = C(\vec{\gamma})$ such that

$$|K_r^{\vec{\gamma}}(\theta, \varphi)| \cdot \; C \prod_{j=1}^{n} \int_0^1 \frac{x^{\gamma_j + 1} dx}{[x^2 + \varphi_j^2][x^2 + \theta_j^2]^{(\gamma_j + 1)/2}}$$

for $|\theta_j| \cdot \; \pi$, $|\phi_j| \cdot \; \pi$, $j = 1, ..., n$. Thus, by Lemma 4, we obtain

$$|K_r^{\vec{\gamma}}(\theta, \varphi)| \cdot \; C \prod_{j=1}^{n} \frac{H^{\gamma_j}(\varphi_j / \theta_j)}{|\theta_j|}$$

for $0 < |\theta_j| \cdot \; \pi$, $|\phi_j| \cdot \; \pi$, $0 < r < 1$. Hence

$$\begin{aligned} |\mathcal{C}^{\vec{\gamma}}(f)(r \cdot e^{i\theta})| \cdot \; & C \int_{[-\pi,\pi]^n} \prod_{j=1}^{n} \frac{H^{\gamma_j}(\varphi_j / \theta_j)}{|\theta_j|} |f_r(e^{i(\theta+\varphi)})| d\varphi \\ &= C \int_{\mathbf{R}^n} \prod_{j=1}^{n} H^{\gamma_j}(s_j) E_s |f_r|(e^{i\theta}) ds. \end{aligned}$$

From this estimates, using Lemma 5 and the 2π periodicity of the subintegral function, the result follows.

The Hardy space $H^p(D_n)$ $(0 < p < \infty)$ is defined on D_n by

$$H^p(D_n) = \{\, f \mid f \in H(D_n) \quad \text{and} \quad ||f||_{H^p(D_n)} < \infty \,\},$$

where

$$||f||^p_{H^p(D_n)} = \sup_{0 \le r<1} \int_{[0,2\pi]^n} |f(r \cdot e^{i\theta})|^p d\theta.$$

We obtain the following corollaries from Theorem 1.

Corollary 1. *The generalized Cesàro operator is bounded on* $H^p(D_n)$ *for* $p > 0$.

Given $0 < p, q < \infty$, and positive Borel measures μ_j, $j = 1, ..., n$ on the interval $(0, 1)$, the weighted space $\mathcal{A}^{p,q}_{\vec{\mu}}(D_n)$ consists of those functions f analytic on D_n for which

$$||f||_{\mathcal{A}^{p,q}_{\vec{\mu}}(D_n)} = \left[\int_{[0,1)^n} \left(\int_{[0,2\pi]^n} |f(r \cdot e^{i\theta})|^p d\theta \right)^{q/p} \prod_{j=1}^{n} d\mu_j(r_j) \right]^{1/q} < \infty.$$

Of particular interest are the absolutely continuous measures of the form $d\mu_j(r_j) = (1 - r_j)^a r_j^b dr_j$; the spaces obtained include the Bergman spaces.

Corollary 2. *The generalized Cesàro operator is bounded on* $\mathcal{A}^{p,q}_{\vec{\mu}}(D_n)$ *for* $p, q > 0$. *Moreover, there is a constant* C *independent of* f, *such that*

$$||\mathcal{C}^{\vec{\gamma}}(f)||_{\mathcal{A}^{p,q}_{\vec{\mu}}(D_n)} \le C||f||_{\mathcal{A}^{p,q}_{\vec{\mu}}(D_n)}.$$

Remark 1. If $p = \infty$ then the operator $\mathcal{C}^{\vec{\gamma}}$ is not bounded. Indeed, taking $g(z) \equiv 1$ we obtain

$$\mathcal{C}^{\vec{\gamma}}(g)(z) = \prod_{j=1}^{n} \sum_{k=1}^{\infty} \frac{\gamma_j + 1}{k + \gamma_j + 1} z_j^k \ge \prod_{j=1}^{n} \frac{1 + \gamma_j}{2 + \gamma_j} \sum_{k=1}^{\infty} \frac{z_j^k}{k}, \quad \text{for} \quad 0 < z_j < 1,$$

which is not in $H^\infty(D_n)$.

3. Some Invariant Spaces of the Generalized Cesáro Operators

The a-Bloch space $\mathcal{B}^a(D_n)$ is the space of all analytic functions f on D_n such that

$$b_a(f) = \max_{j=1,\dots,n} \sup_{z \in D_n} (1 - |z_j|^2)^a \left| \frac{\partial f}{\partial z_j}(z) \right| < \infty.$$

It is clear that $\mathcal{B}^a$ is a normed space, modulo constant functions and $\mathcal{B}^{a_1} \subset \mathcal{B}^{a_2}$ for $a_1 < a_2$. Hardy and Littlewood (see [10]) proved that $\mathcal{B}^a(D) = \Lambda_{1-a}(D)$ for $0 < a < 1$. Here $\Lambda_\beta(D)$ is the Lipschitz space of order β which can be used to characterize the dual space of Hardy space $H^p(D)$ for $0 < p < 1$ (see [6]). Therefore, $\mathcal{B}^a$ are important in the theory of Hardy spaces. This is the main reason for us to bring a-Bloch space into the picture.

With $\mathcal{S}_{\vec{\alpha}}$ we denote the space which consists of all analytic functions f on D_n such that

$$N(f)_{\mathcal{S}_{\vec{\alpha}}} = \sup_{z \in D_n} |f(z)| \prod_{j=1}^{n} (1 - |z_j|)^{\alpha_j} < \infty,$$

where $\vec{\alpha} = (\alpha_1, ..., \alpha_n)$, $\alpha_j > 0$, $j = 1, ..., n$.

It is well-known that when $n = 1$ and $a > 1$, the following are equivalent:

$$b_a(f) < \infty \Leftrightarrow \sup_{z \in D} (1 - |z|^2)^a \left| z \frac{\partial f}{\partial z}(z) \right| < \infty \Leftrightarrow N(f)_{\mathcal{S}_{a-1}} < \infty.$$

In order to prove the main theorem in this section, we need the following auxiliary result:

Lemma 6. *Let* $\gamma > -1$ *and* $\alpha > 0$. *Then*

$$\int_0^1 \frac{(1-t)^\gamma}{(1-t|z|)^{\alpha+\gamma+1}} dt \cdot \frac{\alpha+\gamma+1}{\alpha(\gamma+1)} \frac{1}{(1-|z|)^\alpha}.$$

Proof. We have

$$\begin{aligned}
\int_0^1 \frac{(1-t)^\gamma}{(1-t|z|)^{\alpha+\gamma+1}} dt &= \int_0^{|z|} \frac{(1-t)^\gamma}{(1-t|z|)^{\alpha+\gamma+1}} dt + \int_{|z|}^1 \frac{(1-t)^\gamma}{(1-t|z|)^{\alpha+\gamma+1}} dt \\
&\cdot \int_0^{|z|} \frac{1}{(1-t)^{\alpha+1}} dt + \int_{|z|}^1 \frac{(1-t)^\gamma}{(1-|z|)^{\alpha+\gamma+1}} dt \\
&= \frac{1}{\alpha} \frac{1}{(1-|z|)^\alpha} - \frac{1}{\alpha} + \frac{1}{\gamma+1} \frac{1}{(1-|z|)^\alpha} \\
&\cdot \frac{\alpha+\gamma+1}{\alpha(\gamma+1)} \frac{1}{(1-|z|)^\alpha},
\end{aligned}$$

as desired.

Remark 2. For *Re* $(\gamma) > 0$, it is easy to obtain the following identity:

$$\begin{aligned}
z\mathcal{C}^\gamma(f)'(z) &= \gamma(\gamma+1) \int_0^1 f(tz) \frac{(1-t)^{\gamma-1}}{(1-tz)^{\gamma+1}} dt \\
&\quad -(\gamma+1)^2 \int_0^1 f(tz) \frac{(1-t)^\gamma}{(1-tz)^{\gamma+1}} dt = I + II.
\end{aligned}$$

By this identity we can see that the generalized Cesàro operator $\mathcal{C}^\gamma$ is bounded from $\mathcal{B}^a(D)$ to $\mathcal{B}^a(D)$ for $1 < a < \infty$. Indeed, assuming that $f \in \mathcal{B}^a(D)$, we have

$$
\begin{aligned}
|I| \cdot \quad & |\gamma(\gamma+1)| \int_0^1 \frac{|f(tz)(1-t)^{\gamma-1}|}{|1-tz|^{\gamma+1}} dt \\
\cdot \quad & |\gamma(\gamma+1)| \int_0^1 \frac{|f(tz)|(1-t|z|)^{a-1}(1-t)^{\gamma-1}}{(1-t|z|)^{a+\gamma}} dt \\
\cdot \quad & |\gamma(\gamma+1)| N(f)_{\mathcal{S}_{a-1}} \int_0^1 \frac{(1-t)^{\gamma-1}}{(1-t|z|)^{a+\gamma}} dt \\
\cdot \quad & \left| \frac{(\gamma+1)(a+\gamma)}{\alpha} \right| N(f)_{\mathcal{S}_{a-1}} \frac{1}{(1-|z|)^a}
\end{aligned}
$$

It follows that

$$(1-|z|^2)^a |I| \cdot \quad \left| 2^a \frac{(\gamma+1)(a+\gamma)}{\alpha} \right| N(f)_{\mathcal{S}_{a-1}}.$$

Similarly, one has

$$(1-|z|^2)^a |II| \cdot \quad \left| \frac{2^a(\gamma+1)(a+\gamma+1)}{\alpha} \right| N(f)_{\mathcal{S}_{a-1}}.$$

Combining the estimates for I and II, we immediately obtain the following:

$$\sup_{z \in D} (1-|z|^2)^a |z\mathcal{C}^\gamma(f)'(z)| \cdot \quad C_{a,\gamma} N(f)_{\mathcal{S}_{a-1}} < \infty,$$

as desired.

For the case $0 < a < 1$, we choose $f \equiv 1$, then

$$\mathcal{C}^\gamma(f)(z) = \sum_{k=0}^{\infty} \frac{\gamma+1}{k+\gamma+1} z^k \geq C \sum_{k=1}^{\infty} \frac{z^k}{k}$$

for some $C > 0$ and for all $0 < z < 1$. However, $\sum_{k=1}^{\infty} \frac{z^k}{k} = -\log(1-z) \notin H^\infty(D)$. Therefore, $\mathcal{C}^\gamma(f) \notin \mathcal{B}^a(D)$ since $\mathcal{B}^a(D) = \Lambda_{1-a}(D) \subset H^\infty(D)$.

As for the case $a = 1$, we may consider $f(z) = \frac{1}{z} \log \frac{1}{1-z}$. It is easy to see that $f \in \mathcal{B}^1(D)$. However, $\mathcal{C}^\gamma(f) \notin \mathcal{B}^1(D)$.

It can be shown that for $0 < a < \infty$,

$$f \in \mathcal{B}^a(B_n) \Leftrightarrow \sup_{z \in B_n} |\mathcal{R}(f)(z)|(1-|z|^2)^a < \infty$$

where $\mathcal{R}(f)(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)$ is the radial derivative of f. Therefore, we may use this property to obtain some similar results for sliced Cesàro operator. However, when we turn to polydisk, the situation is quite different. The following lemma is a natural generalization of the result for unit disk D to polydisk D_n.

Lemma 7. *Let $a > 1$. Then $\mathcal{B}^a(D_n) \subset \mathcal{S}_{\vec{a}-1}(D_n)$, where $\vec{a} - 1 = (a - 1, ..., a - 1)$.*

Proof. Without loss of generality, we may assume $n = 2$. It is clear that

$$\begin{aligned}
|f(z_1, z_2) - f(0,0)| &= \left| \int_0^1 \frac{\partial f}{\partial z_2}(z_1, tz_2) z_2 dt + \int_0^1 \frac{\partial f}{\partial z_1}(tz_1, 0) z_1 dt \right| \\
&\cdot \int_0^1 \left| \frac{\partial f}{\partial z_2}(z_1, tz_2) \right| \frac{(1 - t|z_2|)^a}{(1 - t|z_2|)^a} |z_2| dt \\
&+ \int_0^1 \left| \frac{\partial f}{\partial z_1}(tz_1, 0) \right| \frac{(1 - t|z_1|)^a}{(1 - t|z_1|)^a} |z_1| dt \\
&\cdot \frac{||f||_{\mathcal{B}_a}}{a - 1} \left(\frac{1}{(1 - |z_1|)^{a-1}} + \frac{1}{(1 - |z_2|)^{a-1}} \right).
\end{aligned}$$

From (8), it follows

$$(1 - |z_1|)^{a-1} (1 - |z_2|)^{a-1} |f(z_1, z_2)| \cdot \; |f(0,0)| + \frac{2}{a - 1} ||f||_{\mathcal{B}_a},$$

as desired.

Since $\mathcal{S}_{\vec{a}-1} \subset \mathcal{S}_{\vec{\alpha}-1}$, when $a \cdot \; \min\{\alpha_1, ..., \alpha_n\}$, we may obtain the result easily.

Corollary 3. *Let $a > 1$ and $\alpha_j > 1, j = 1, ..., n$. Then $\mathcal{B}^a(D_n) \subset \mathcal{S}_{\vec{\alpha}-1}(D_n)$, when $a \cdot \; \min\{\alpha_1, ..., \alpha_n\}$.*

Remark 3. The inclusion in Lemma 7 is proper. Indeed, let

$$f(z_1, ..., z_n) = \prod_{k=1}^{n} \frac{c_k}{(1 - z_k)^{a-1}},$$

then

$$\prod_{k=1}^{n} (1 - |z_k|)^{a-1} |f(z)| \cdot \; \prod_{k=1}^{n} |c_k|,$$

hence $f \in \mathcal{S}_{\vec{a}-1}(D_n)$.

On the other hand, it is easy to see that

$$\sup_{z \in D_n} (1 - |z_k|)^a \left| \frac{\partial f}{\partial z_k}(z) \right| = \infty$$

for all $k \in \{1, ..., n\}$.

The main result in this section is the following theorem:

Theorem 2. *The space $\mathcal{S}_{\vec{\alpha}}(D_n)$, $\alpha > 1$ is invariant for the generalized Cesàro operators on the polydisk D_n. Moreover there is a constant C independent of f such that*

$$N(\mathcal{C}^{\vec{\gamma}}(f))_{\mathcal{S}_{\vec{\alpha}}} \cdot \ \ C N(f)_{\mathcal{S}_{\vec{\alpha}}}.$$

Proof. Without loss of generality, we may assume that γ_j, $j = 1, ..., n$, are real numbers such that $\gamma_j > -1$. Let $f \in \mathcal{S}_{\vec{\alpha}}$. Then

$$\begin{aligned}
|\mathcal{C}^{\vec{\gamma}} f(z)| \cdot \ & \prod_{j=1}^{n} |\gamma_j + 1| \int_0^1 \cdots \int_0^1 \frac{|f(\tau \cdot z)|}{\prod_{j=1}^{n} |1 - \tau_j z_j|^{\gamma_j + 1}} \prod_{j=1}^{n} (1 - \tau_j)^{\gamma_j} d\tau \\
\cdot \ & \prod_{j=1}^{n} |\gamma_j + 1| \int_0^1 \cdots \int_0^1 \frac{|f(\tau \cdot z)| \prod_{j=1}^{n} (1 - \tau_j |z_j|)^{\alpha_j}}{\prod_{j=1}^{n} (1 - \tau_j |z_j|)^{\alpha_j + \gamma_j + 1}} \prod_{j=1}^{n} (1 - \tau_j)^{\gamma_j} d\tau \\
\cdot \ & \prod_{j=1}^{n} |\gamma_j + 1| \, N(f)_{\mathcal{S}_{\vec{\alpha}}} \int_0^1 \cdots \int_0^1 \frac{\prod_{j=1}^{n} (1 - \tau_j)^{\gamma_j}}{\prod_{j=1}^{n} (1 - \tau_j |z_j|)^{\alpha_j + \gamma_j + 1}} d\tau \\
= \ & \prod_{j=1}^{n} |\gamma_j + 1| \, N(f)_{\mathcal{S}_{\vec{\alpha}}} \prod_{j=1}^{n} \int_0^1 \frac{(1 - \tau_j)^{\gamma_j}}{(1 - \tau_j |z_j|)^{\alpha_j + \gamma_j + 1}} d\tau_j \\
\cdot \ & \prod_{j=1}^{n} \frac{|\alpha_j + \gamma_j + 1|}{|\alpha_j|} N(f)_{\mathcal{S}_{\vec{\alpha}}} \prod_{j=1}^{n} \frac{1}{(1 - |z_j|)^{\alpha_j}} \qquad \text{(by Lemma 6)}
\end{aligned}$$

from which the result follows with $C = N(f)_{\mathcal{S}_{\vec{\alpha}}} \prod_{j=1}^{n} \frac{|\alpha_j + \gamma_j + 1|}{|\alpha_j|}$.

Corollary 4. *Let $a > 1$. Then $\mathcal{C}^{\vec{\gamma}}$ is bounded operator from $\mathcal{B}^{a}(D_n)$ to $\mathcal{S}_{\vec{a}-1}$ (D_n).*

Denote $\mathcal{B}_{a,a-1}$ (D_n) the collection of all analytic functions f on D_n such that

$$P(f)_{\mathcal{B}_{a,a-1}} = \max_{k=1,\dots,n} \sup_{z \in D_n} \prod_{j=1}^{n} (1 - |z_j|)^{a-1} (1 - |z_k|) \left| \frac{\partial f}{\partial z_k}(z) \right| < \infty.$$

By Lemma 7 and Theorem 2, we obtain the following result.

Lemma 8. *Let $a > 1$. Then $\mathcal{B}_{a,a-1}(D_n) = \mathcal{S}_{\vec{a}-1}(D_n)$.*

Proof. Without loss of generality, we may assume $n = 2$. As in Lemma 7 we have

$$\begin{aligned}
&|f(z_1, z_2) - f(0,0)| \\
&= \left| \int_0^1 \frac{\partial f}{\partial z_2}(z_1, tz_2) z_2 dt + \int_0^1 \frac{\partial f}{\partial z_1}(tz_1, 0) z_1 dt \right| \\
&\cdot \int_0^1 \left| \frac{\partial f}{\partial z_2}(z_1, tz_2) \right| \frac{(1 - t|z_2|)^a (1 - |z_1|)^{a-1}}{(1 - t|z_2|)^a (1 - |z_1|)^{a-1}} |z_2| dt \\
&+ \int_0^1 \left| \frac{\partial f}{\partial z_1}(tz_1, 0) \right| \frac{(1 - t|z_1|)^a (1 - |0|)^{a-1}}{(1 - t|z_1|)^a (1 - |0|)^{a-1}} |z_1| dt \\
&\cdot \frac{P(f)_{\mathcal{B}_{a,a-1}}}{a-1} \left(\frac{1}{(1 - |z_1|)^{a-1} (1 - |z_2|)^{a-1}} + \frac{1}{(1 - |z_2|)^{a-1}} \right) \\
&\cdot \frac{2P(f)_{\mathcal{B}_{a,a-1}}}{(a-1)(1 - |z_1|)^{a-1} (1 - |z_2|)^{a-1}}.
\end{aligned}$$

From which it follows that $\mathcal{B}_{a,a-1}(D_n) \subset \mathcal{S}_{\vec{a}-1}(D_n)$.

On the other hand, by Cauchy's integral formula we have

$$\frac{\partial f}{\partial z_k}(z) = \frac{1}{(2\pi)^n} \int_{\prod_{j=1}^n \partial B(z_j, (1-|z_j|)/2)} \frac{f(\zeta) d\zeta}{(\zeta_k - z_k) \prod_{j=1}^n (\zeta_j - z_j)}. \tag{9}$$

Since $\zeta_j \in \partial B(z_j, (1 - |z_j|)/2)$, one has

$$\frac{(1 - |z_j|)}{2} < (1 - |\zeta_j|) < \frac{3(1 - |z_j|)}{2}. \tag{10}$$

From (9) and (10) we obtain

$$\begin{aligned}
&\left| \frac{\partial f}{\partial z_k}(z) \right| \\
&\cdot \frac{C}{(1 - |z_k|) \prod_{j=1}^n (1 - |z_j|)} \int_{\prod_{j=1}^n \partial B(z_j, (1-|z_j|)/2)} |f(\zeta)| \prod_{j=1}^n |d\zeta_j| \\
&\cdot \frac{C}{(1 - |z_k|) \prod_{j=1}^n (1 - |z_j|)} \int_{\prod_{j=1}^n \partial B(z_j, (1-|z_j|)/2)} \frac{N(f)_{\mathcal{S}_{\vec{a}-1}}}{\prod_{j=1}^n (1 - |\zeta_j|)^{a-1}} \prod_{j=1}^n |d\zeta_j| \\
&\cdot \frac{C N(f)_{\mathcal{S}_{\vec{a}-1}}}{(1 - |z_k|) \prod_{j=1}^n (1 - |z_j|)^{a-1}}.
\end{aligned}$$

It follows that $\mathcal{S}_{\vec{a}-1}(D_n) \subset \mathcal{B}_{a,a-1}(D_n)$. Moreover, we conclude that

$$P(f)_{\mathcal{B}_{a,a-1}} \cdot \ C N(f)_{\mathcal{S}_{\vec{a}-1}}.$$

Remark 4. Applying Lemma 8 we can prove that for $a > 1$, there is a constant C independent of f such that

$$P(\mathcal{C}^{\vec{\gamma}}(f))_{\mathcal{B}_{a,a-1}} \cdot \; C\,N(f)_{\mathcal{S}_{\vec{a}-1}}.$$

Indeed, without loss of generality, we may assume $n = 2$ and $\gamma_1, \gamma_2 \in (-1, \infty)$. For $a > 1$, we have

$$\begin{aligned}
&\left|\frac{\partial \mathcal{C}^{\gamma}(f)}{\partial z_1}(z)\right| \\
&= C\left|\int_0^1\int_0^1 \left(\frac{t_1}{(1-t_1z_1)^{\gamma_1+1}}\frac{\partial f}{\partial z_1} + \frac{(\gamma_1+1)t_1 f}{(1-t_1z_1)^{\gamma_1+2}}\right)\frac{(1-t_1)^{\gamma_1}(1-t_2)^{\gamma_2}}{(1-t_2z_2)^{\gamma_2+1}}dt\right| \\
&\cdot\; C\int_0^1\int_0^1 \left(\left|\frac{\partial f}{\partial z_1}\right| \frac{(1-t_1|z_1|)^{a}(1-t_2|z_2|)^{a-1}}{(1-t_1|z_1|)^{a+\gamma_1+1}(1-t_2|z_2|)^{a-1}} + \right. \\
&\left.\frac{|\gamma_1+1|\;|f|(1-t_1|z_1|)^{a-1}(1-t_2|z_2|)^{a-1}}{(1-t_1|z_1|)^{a+\gamma_1+1}(1-t_2|z_2|)^{a-1}}\right)\frac{(1-t_1)^{\gamma_1}(1-t_2)^{\gamma_2}}{(1-t_2|z_2|)^{\gamma_2+1}}dt \qquad (11) \\
&\cdot\; C\,N(f)_{\mathcal{S}_{\vec{a}-1}}\int_0^1\int_0^1 \left(\frac{(1-t_1)^{\gamma_1}(1-t_2)^{\gamma_2}}{(1-t_1|z_1|)^{a+\gamma_1+1}(1-t_2|z_2|)^{a+\gamma_2}}\right)dt \\
&\cdot\; C\,N(f)_{\mathcal{S}_{\vec{a}-1}}\left(\frac{1}{(1-|z_1|)^{a}(1-|z_2|)^{a-1}}\right) \quad \text{(by Lemma 6)} \\
&\cdot\; \frac{C\,N(f)_{\mathcal{S}_{\vec{a}-1}}}{(1-|z_1|)^{a}(1-|z_2|)^{a-1}}.
\end{aligned}$$

Similarly we obtain

$$(1-|z_1|)^{a-1}(1-|z_2|)^{a}\left|\frac{\partial \mathcal{C}^{\gamma}(f)}{\partial z_2}(z)\right| \cdot \; C\,N(f)_{\mathcal{S}_{\vec{a}-1}}. \qquad (12)$$

From (11) and (12) the result follows immediately.

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