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CONCAVITY OF CERTAIN MATRIX TRACE FUNCTIONS

Fumio Hiai

Abstract. We demonstrate how Epstein's method using theory of Pick functions improves the existing results and also proves new ones on the joint concavity of trace functions of the form $\mathrm{Tr}\,(F(A_1,\dots,A_k))$, where $F(A_1,\dots,A_k)$ is a matrix-valued function of positive semidefinite matrices $A_1,\dots,A_k$.

INTRODUCTION

We are concerned with the joint concavity of a trace function $\mathrm{Tr}\,(F(A_1,\dots,A_k))$, where $F(A_1,\dots,A_k)$ is a certain matrix-valued function of positive semidefinite matrices $A_1,\dots,A_k$. It sometimes happens that even though the function $F(A_1,\dots,A_k)$ is not at all operator concave in the order of positive semidefiniteness, its trace function $\mathrm{Tr}\,(F(A_1,\dots,A_k))$ is jointly concave, i.e.,

$$\begin{aligned}&\mathrm{Tr}\,(F(\lambda A_1+(1-\lambda)B_1,\dots,\lambda A_k+(1-\lambda)B_k))\\&\qquad\ge\lambda\mathrm{Tr}\,(F(A_1,\dots,A_k))+(1-\lambda)\mathrm{Tr}\,(F(B_1,\dots,B_k))\end{aligned}$$

for positive semidefinite matrices A_j, B_j and $0<\lambda<1$. For instance, the (joint) concavity of $\mathrm{Tr}\,(F(\cdot))$ is known when the function $F(\cdot)$ is any of the following:

(i) $F(A)=(CA^pC^*)^{1/p}$, where $0<p\le 1$ (see [6]),

(ii) $F(A_1,\dots,A_k)=\big(\sum_{j=1}^k A_j^p\big)^{1/p}$, where $0<p\le 1$ (see [4]),

(iii) $F(A,B)=A^{\alpha/2}CB^{\beta}C^*A^{\alpha/2}$, where $\alpha,\beta>0$ and $\alpha+\beta\le 1$ (see [8, 6]),

(iv) $F(A_1,\dots,A_k)=\exp\big(L+\sum_{j=1}^k p_j\log A_j\big)$, where $p_1,\dots,p_k>0$, $\sum_{j=1}^k p_j\le 1$ and L is a Hermitian matrix (see [8, 6]).

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In particular, the joint concavity of $\mathrm{Tr}\,(F(A,B)) = \mathrm{Tr}\,(A^\alpha C B^\beta C^*)$ in the case (iii) is known as the Lieb concavity in [8] (also see [1]). In [6], Epstein developed a powerful method using the integral representation of Pick functions to prove the concavity of the trace function of the above (i) (as well as those of the single-variable cases of (iii) and (iv)). In this paper, the same method will be systematically exemplified to improve the known joint concavity results for (i)–(iv) and also to prove new ones for some trace functions involving Hadamard products and operator means.

The general form of trace functions treated in this paper is

$$\mathrm{Tr}\left(\{F(A_1^p,\dots,A_k^p)\}^{1/p}\right) \quad \text{with} \quad 0<p\le 1 \tag{0.1}$$

or

$$\mathrm{Tr}\left(\exp\{F(\log A_1,\dots,\log A_k)\}\right), \tag{0.2}$$

and the function $F(\cdot)$ is allowed to contain several linear maps on matrix spaces. The joint concavity of such trace functions is strongly related to the fact that the functions x^p $(0<p\le 1)$ and $\log x$ in $x>0$ are operator monotone. When x^p and $\log x$ are analytically continued to the domain $\mathbb{C}\setminus(-\infty,0]$, their images of the upper-half plane $\mathbb{C}^+ = \{z : \mathrm{Im}\, z > 0\}$ are rather simple; in fact, the image of z^p is the sector $\{re^{i\theta} : r>0,\, 0<\theta<p\pi\}$ and that of $\log z$ is the strip $\{x+iy : 0<y<\pi\}$. These facts are essential in Epstein's method, so it does not seem easy to deal with trace functions involving more general operator monotone functions beyond the forms (0.1) and (0.2).

It may be worthwhile to mention the following remark about the joint "convexity" problem in the case where $F(A_1,\dots,A_k) = A_1+\cdots+A_k$. In this case, one may expect that the trace function (0.1) would become jointly convex when the condition on p is converted to $p>1$. However, it was shown in [2, 4] that the function $(A,B)\mapsto \mathrm{Tr}\left((A^p+B^p)^{1/p}\right)$ when $p>2$ is not jointly convex (not even separately) while it is jointly convex when $p=2$. Its joint convexity when $1<p<2$ is a conjecture of Carlen and Lieb [4] and it is still open. So the convexity problem of $\mathrm{Tr}\left((A_1^p+\cdots+A_k^p)^{1/p}\right)$ when $p>1$ is more subtle than its concavity part when $0<p<1$.

This paper is organized as follows. Section 1 is a preparation mostly taken from [6]. The above mentioned joint concavity results are improved in Section 2. Next, other types of trace functions are proved to be jointly concave; in Section 3 we consider trace functions involving tensor products and Hadamard products, and in Section 4 those involving operator means.

1. Preliminaries

Let $\mathbf{M}_n$ be the algebra of $n \times n$ complex matrices and $\mathbf{M}_n^+$ the set of all positive semidefinite $A \in \mathbf{M}_n$. We write $A > 0$ when $A \in \mathbf{M}_n$ is positive definite, that is, $A \in \mathbf{M}_n^+$ and A is invertible; also $A < 0$ when $-A > 0$. Let $\mathbf{M}_n^{++}$ denote the set of all $A > 0$ in $\mathbf{M}_n$. The identity matrix in $\mathbf{M}_n$ is denoted by I (sometimes I_n to be precise). Let $\operatorname{Im} X$ be the imaginary part of $X \in \mathbf{M}_n$, i.e., $\operatorname{Im} X := (X - X^*)/2i$. The trace of $X \in \mathbf{M}_n$ is denoted by $\operatorname{Tr} X$, and $\sigma(X)$ stands for the set of all eigenvalues of X.

We often use the following notations

$$\mathbb{C}^+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}, \quad \mathbb{C}^- := \{z \in \mathbb{C} : \operatorname{Im} z < 0\},$$

$$\mathcal{I}_n^+ := \{X \in \mathbf{M}_n : \operatorname{Im} X > 0\}, \quad \mathcal{I}_n^- := \{X \in \mathbf{M}_n : \operatorname{Im} X < 0\},$$

and for $0 < p \le 1$,

$$\Gamma_{p\pi} := \{re^{i\theta} : r > 0,\, 0 < \theta < p\pi\}, \quad \Gamma_{-p\pi} := \{re^{i\theta} : r > 0,\, 0 > \theta > -p\pi\}.$$

The assertions in the next lemma were given in [6]; in fact, if $X = A + iB$ with selfadjoint A and $B > 0$, then the inverse of X is

$$X^{-1} = B^{-1/2}(B^{-1/2}AB^{-1/2} + iI)^{-1}B^{-1/2}.$$

Lemma 1.1. *If $X \in \mathcal{I}_n^+$, then X is invertible and moreover $\sigma(X) \subset \mathbb{C}^+$. For invertible $X \in \mathbf{M}_n$, $X \in \mathcal{I}_n^+$ if and only if $X^{-1} \in \mathcal{I}_n^-$.*

Next, let us recall basic facts on analytic functional calculus and Pick functions, which will be frequently used in the discussions below. Let f be an analytic function in an open set Ω in $\mathbb{C}$. For every $X \in \mathbf{M}_n$ such that $\sigma(X) \subset \Omega$, the analytic functional calculus $f(X)$ is defined by

$$f(X) := \frac{1}{2\pi i}\int_\Gamma f(z)(zI - X)^{-1}\,dz,$$

where Γ is a piecewise smooth curve in Ω surrounding $\sigma(X)$. The spectral mapping theorem says that $\sigma(f(X)) = f(\sigma(X))$. The analytic functional calculus satisfies the composition rule $g(f(X)) = (g \circ f)(X)$ whenever f is as above and g is analytic in an open set containing $f(\sigma(X))$. Moreover, the map $X \mapsto f(X)$ has the Fréchet derivative

$$Df(X)(Y) = \frac{1}{2\pi i}\int_\Gamma f(z)(zI - X)^{-1}Y(zI - X)^{-1}\,dz \qquad (Y \in \mathbf{M}_n),$$

so $f(X(z))$ is analytic whenever $X(z)$ is an analytic function satisfying $\sigma(X(z)) \subset \Omega$. (See [10] for details on these facts.)

An analytic function φ in $\mathbb{C}^+ \cup \mathbb{C}^-$ is called a Pick function if φ maps $\mathbb{C}^+$ into itself and φ in $\mathbb{C}^-$ is the reflection of φ in $\mathbb{C}^+$, i.e., $\varphi(\overline{z}) = \overline{\varphi(z)}$ for all $z \in \mathbb{C}^+$. According to Löwner's theory (see [3, V.4]), any Pick function φ admits an integral expression

$$\varphi(z) = a + bz + \int_{-\infty}^{\infty} \frac{1+tz}{t-z}\, d\nu(t), \tag{1.1}$$

where $a \in \mathbb{R}$, $b \geq 0$ and ν is a finite measure on $\mathbb{R}$. Furthermore, a, b and ν are uniquely determined by φ, and if φ is analytically continued across an interval (α, β) in $\mathbb{R}$ (where $-\infty \leq \alpha < \beta \leq \infty$), then the measure ν is supported in $\mathbb{R} \setminus (\alpha, \beta)$.

For $p > 0$, the function x^p ($x > 0$) has the analytic continuation z^p in $\mathbb{C} \setminus (-\infty, 0]$ defined by

$$z^p := r^p e^{ip\theta} \qquad (z = re^{i\theta},\ r > 0,\ -\pi < \theta < \pi).$$

When $0 < p < 1$, it has the well-known integral expression (see [3, p. 116] for example)

$$z^p = \frac{\sin p\pi}{\pi} \int_0^{\infty} \frac{t^{p-1} z}{t+z}\, dt.$$

For every $X \in \mathcal{I}_n^+$ (resp. $X \in \mathcal{I}_n^-$), since $\sigma(X) \subset \mathbb{C}^+$ (resp. $\sigma(X) \subset \mathbb{C}^-$), one can define X^p via analytic functional calculus and it coincides with

$$X^p = \frac{\sin p\pi}{\pi} \int_0^{\infty} t^{p-1} X(tI + X)^{-1}\, dt. \tag{1.2}$$

In fact, this matrix-valued integral is absolutely convergent whenever $0 < p < 1$.

On the other hand, the analytic continuation $\log z$ in $\mathbb{C} \setminus (-\infty, 0]$ defined by

$$\log z = \log r + i\theta \qquad (z = re^{i\theta},\ r > 0,\ -\pi < \theta < \pi),$$

has the integral expression

$$\log z = \int_0^{\infty} \Big(\frac{1}{t+1} - \frac{1}{t+z} \Big)\, dt.$$

For every $X \in \mathcal{I}_n^+ \cup \mathcal{I}_n^-$, one can define $\log X$ via analytic functional calculus and it admits the integral expression

$$\log X = \int_0^{\infty} \Big(\frac{1}{t+1} I - (tI + X)^{-1} \Big)\, dt. \tag{1.3}$$

In fact, this integral is absolutely convergent because $\|(t+1)^{-1}I-(tI+X)^{-1}\| = O(t^{-2})$ as $t \to \infty$.

The following two lemmas are taken from [6]. One can easily show the first lemma from the expressions (1.2) and $(e^{-ip\pi}X^p)^{-1} = (-X^{-1})^p$ for $X \in \mathcal{I}_n^+$; the latter is seen from the analytic functional calculus of $(e^{-ip\pi}z^p)^{-1} = (-z^{-1})^p$ for $z \in \mathbb{C}^+$. The second lemma is seen from (1.3) and $\log(-X) = \log X - i\pi I_n$ for $X \in \mathcal{I}_n^+$.

Lemma 1.2. *Let $0 < p \le 1$. If $X \in \mathcal{I}_n^+$, then $X^p \in \mathcal{I}_n^+$ and $e^{-ip\pi}X^p \in \mathcal{I}_n^-$. Also, If $X \in \mathcal{I}_n^-$, then $X^p \in \mathcal{I}_n^-$ and $e^{ip\pi}X^p \in \mathcal{I}_n^+$.*

Lemma 1.3. *If $X \in \mathcal{I}_n^+$, then $\log X \in \mathcal{I}_n^+$ and $0 < \mathrm{Im}\,(\log X) < \pi I$. Also, if $X \in \mathcal{I}_n^-$, then $\log X \in \mathcal{I}_n^-$ and $0 > \mathrm{Im}\,(\log X) > -\pi I$.*

2. Improvements of Existing Results

In this section, we prove three theorems on the joint concavity of trace functions, thus improving the known results for the cases (i)–(iv) listed in the introduction.

The first theorem (also Corollary 2.2) is a generalization of both cases (i) and (ii). The proof of the theorem is an adaptation of Epstein's method in [6] based on theory of Pick functions to our generalized setting. But we give it in detail because the same method will be repeatedly used in the paper.

Theorem 2.1. *Let $m, n_1, \dots, n_k \in \mathbb{N}$, and for $j = 1, \dots, k$ let Φ_j be a positive linear map from $\mathbf{M}_{n_j}$ to $\mathbf{M}_m$. If $0 < p \le 1$, then the function*

$$(A_1, \dots, A_k) \in \prod_{j=1}^{k} \mathbf{M}_{n_j}^+ \mapsto \mathrm{Tr}\left(\Big\{\sum_{j=1}^{k} \Phi_j(A_j^p)\Big\}^{1/p}\right)$$

is jointly concave.

Proof. By approximation, we may assume that Φ_j's are strictly positive, that is, $A > 0$ implies $\Phi_j(A) > 0$. (We may take $\Phi_j(A) + \varepsilon \mathrm{Tr}\,(A) I_m$ for $\varepsilon > 0$.) To show the theorem, it suffices to prove that if $A_j \in \mathbf{M}_{n_j}^{++}$ and $H_j \in \mathbf{M}_{n_j}$ is Hermitian for $j = 1, \dots, k$, then

$$\frac{d^2}{dx^2}\mathrm{Tr}\left(\Big\{\sum_{j=1}^{k} \Phi_j\big((A_j + xH_j)^p\big)\Big\}^{1/p}\right) \le 0$$

for all $x > 0$ small enough. Indeed, once this has been proved, it is immediate to see that if $A_j, B_j \in \mathbf{M}_{n_j}^{++}$ for $j = 1, \ldots, k$, then

$$\frac{d^2}{dx^2}\mathrm{Tr}\left(\Big\{\sum_{j=1}^k \Phi_j\big((xA_j + (1-x)B_j)^p\big)\Big\}^{1/p}\right) \le 0$$

for all $0 < x < 1$, which yields the result.

Now let us show that the function

$$\varphi(z) := \mathrm{Tr}\left(\Big\{\sum_{j=1}^k \Phi_j\big((zA_j + H_j)^p\big)\Big\}^{1/p}\right) \qquad (z \in \mathbb{C}^+ \cup \mathbb{C}^-),$$

is a well-defined Pick function. Set $X_j(z) := zA_j + H_j$ for $z \in \mathbb{C}$; then clearly $X_j(z) \in \mathcal{I}_{n_j}^+$ (resp. $X_j(z) \in \mathcal{I}_{n_j}^-$) if $z \in \mathbb{C}^+$ (resp. $z \in \mathbb{C}^-$). For any $z \in \mathbb{C}^+$, since Lemma 1.2 implies that $X_j(z)^p \in \mathcal{I}_{n_j}^+$ and $e^{-ip\pi}X_j(z)^p \in \mathcal{I}_{n_j}^-$, we get

$$\mathrm{Im}\left(\sum_{j=1}^k \Phi_j(X_j(z)^p)\right) = \sum_{j=1}^k \Phi_j\big(\mathrm{Im}\,(X_j(z)^p)\big) > 0$$

and

$$\mathrm{Im}\left(e^{-ip\pi}\sum_{j=1}^k \Phi_j(X_j(z)^p)\right) = \sum_{j=1}^k \Phi_j\big(\mathrm{Im}\,(e^{-ip\pi}X_j(z)^p)\big) < 0\,.$$

These imply by Lemma 1.1 that

$$\sigma\left(\sum_{j=1}^k \Phi_j(X_j(z)^p)\right) \subset \Gamma_{p\pi}\,.$$

Therefore, $\big\{\sum_{j=1}^k \Phi_j(X_j(z)^p)\big\}^{1/p}$ can be defined by analytic functional calculus so that its eigenvalues are in $\mathbb{C}^+$. In this way, we infer that $\varphi(z) = \mathrm{Tr}\,\big(\big\{\sum_{j=1}^k \Phi_j(X_j(z)^p)\big\}^{1/p}\big)$ is a well-defined analytic function in $\mathbb{C}^+$ so that $\varphi(\mathbb{C}^+) \subset \mathbb{C}^+$. Here the analyticity of φ follows from the Fréchet differentiability of analytic functional calculus as remarked in Section 1. Similarly, φ is analytic in $\mathbb{C}^-$ and $\varphi(\mathbb{C}^-) \subset \mathbb{C}^-$. Moreover, since

$$\left(\Big\{\sum_{j=1}^k \Phi_j(X_j(z)^p)\Big\}^{1/p}\right)^* = \Big\{\sum_{j=1}^k \Phi_j\big((X_j(z)^p)^*\big)\Big\}^{1/p} = \Big\{\sum_{j=1}^k \Phi_j(X_j(\overline{z})^p)\Big\}^{1/p},$$

we get $\varphi(\overline{z}) = \overline{\varphi(z)}$ for $z \in \mathbb{C}^+$, and so φ is a Pick function.

Consequently, φ has the integral representation (1.1) with $a \in \mathbb{R}$, $b \geq 0$ and a finite measure ν on $\mathbb{R}$. But since

$$\varphi(z) = z\mathrm{Tr}\left(\Big\{\sum_{j=1}^{k} \Phi_j\big((A_j + z^{-1}H_j)^p\big)\Big\}^{1/p}\right),$$

it is clear that φ is analytic in $\mathbb{C} \setminus (-\infty, R]$ for some $R > 0$ sufficiently large, and so the measure ν must be supported in $(-\infty, R]$. Hence, for $x > 0$ small enough, we have

$$\mathrm{Tr}\left(\Big\{\sum_{j=1}^{k} \Phi_j\big((A_j + xH_j)^p\big)\Big\}^{1/p}\right) = x\varphi(x^{-1}) = ax + b + \int_{-\infty}^{R} \frac{x(x+t)}{xt-1}\, d\nu(t).$$

Since

$$\frac{d}{dx}\Big(\frac{x(x+t)}{xt-1}\Big) = \frac{x^2t - 2x - t}{(xt-1)^2}, \quad \frac{d^2}{dx^2}\Big(\frac{x(x+t)}{xt-1}\Big) = \frac{2(t^2+1)}{(xt-1)^3}, \tag{2.1}$$

we have

$$\frac{d^2}{dx^2}\mathrm{Tr}\left(\Big\{\sum_{j=1}^{k} \Phi_j\big((A_j + xH_j)^p\big)\Big\}^{1/p}\right) = -2\int_{-\infty}^{R} \frac{t^2+1}{(1-xt)^3}\, d\nu(t),$$

which is ≤ 0 for small $x > 0$, as desired. Since two functions of $t \in (-\infty, R]$ in (2.1) have a uniform bound whenever the parameter x is restricted to $\alpha \leq x \leq \beta$ with $0 < \alpha < \beta < 1/R$, one can use the dominated convergence theorem twice to justify the interchange of the order of integral and differential in the above. ■

In the above proof, we showed that $\sigma(X_0) \subset \mathbb{C}^+$ and hence $\mathrm{Tr}\, X_0 \in \mathbb{C}^+$ for $X_0 := \big\{\sum_{j=1}^{k} \Phi_j(X_j(z)^p)\big\}^{1/p}$; however, $X_0 \in \mathcal{I}_m^+$ does not follow so that our arguments are not valid when Tr is replaced by a general positive linear functional.

Corollary 2.2. *Let C_j be an $m \times n_j$ matrix for $j = 1, \ldots, k$. If $0 < p \leq 1$, then the function*

$$(A_1, \ldots, A_k) \in \prod_{j=1}^{k} \mathbf{M}_{n_j}^+ \mapsto \mathrm{Tr}\left(\Big\{\sum_{i=1}^{k} C_j A_j^p C_j^*\Big\}^{1/p}\right)$$

is jointly concave.

The second theorem strengthens the Lieb concavity twofold; it involves the p- and $1/p$-powers as well as positive linear maps.

Theorem 2.3. *Let $m, n_1, n_2 \in \mathbb{N}$, and let $\Phi : \mathbf{M}_{n_1} \to \mathbf{M}_m$ and $\Psi : \mathbf{M}_{n_2} \to \mathbf{M}_m$ be positive linear maps. If $0 < p \le 1$ and $\alpha, \beta > 0$ with $\alpha + \beta \le 1$, then the function*

$$(A, B) \in \mathbf{M}_{n_1}^+ \times \mathbf{M}_{n_2}^+ \mapsto \mathrm{Tr}\left(\left\{\Psi(B^{p\beta})^{1/2}\Phi(A^{p\alpha})\Psi(B^{p\beta})^{1/2}\right\}^{1/p}\right)$$

is jointly concave.

Proof. First, we show that the assertion in the case $\alpha + \beta < 1$ follows from that in the case $\alpha + \beta = 1$. When $\alpha + \beta < 1$, let $\alpha' := 1 - \beta$ so $0 < \alpha < \alpha' < 1$. Let $A_j, B_j \in \mathbf{M}_{n_j}^+$ $(j = 1, 2)$ and $0 < \lambda < 1$. Then, since x^γ $(x \ge 0)$, where $0 < \gamma < 1$, is operator concave as well as operator monotone, we get

$$(\lambda A_1 + (1-\lambda)A_2)^{p\alpha} \ge (\lambda A_1^{\alpha/\alpha'} + (1-\lambda)A_2^{\alpha/\alpha'})^{p\alpha'},$$

so that

$$\Phi\bigl((\lambda A_1 + (1-\lambda)A_2)^{p\alpha}\bigr) \ge \Phi\bigl((\lambda A_1^{\alpha/\alpha'} + (1-\lambda)A_2^{\alpha/\alpha'})^{p\alpha'}\bigr).$$

Therefore, we have

$$\begin{aligned}
&\mathrm{Tr}\Bigl(\bigl\{\Psi((\lambda B_1 + (1-\lambda)B_2)^{p\beta})^{1/2}\Phi((\lambda A_1 + (1-\lambda)A_2)^{p\alpha}) \\
&\quad \times\Psi((\lambda B_1 + (1-\lambda)B_2)^{p\beta})^{1/2}\bigr\}^{1/p}\Bigr) \\
&\ge \mathrm{Tr}\Bigl(\bigl\{\Psi((\lambda B_1 + (1-\lambda)B_2)^{p\beta})^{1/2}\Phi((\lambda A_1^{\alpha/\alpha'} + (1-\lambda)A_2^{\alpha/\alpha'})^{p\alpha'}) \\
&\quad \times\Psi((\lambda B_1 + (1-\lambda)B_2)^{p\beta})^{1/2}\bigr\}^{1/p}\Bigr) \\
&\ge \lambda\mathrm{Tr}\Bigl(\bigl\{\Psi(B_1^{p\beta})^{1/2}\Phi((A_1^{\alpha/\alpha'})^{p\alpha'})\Psi(B_1^{p\beta})^{1/2}\bigr\}^{1/p}\Bigr) \\
&\quad +(1-\lambda)\mathrm{Tr}\Bigl(\bigl\{\Psi(B_2^{p\beta})^{1/2}\Phi((A_2^{\alpha/\alpha'})^{p\alpha'})\Psi(B_2^{p\beta})^{1/2}\bigr\}^{1/p}\Bigr) \\
&= \lambda\mathrm{Tr}\Bigl(\bigl\{\Psi(B_1^{p\beta})^{1/2}\Phi(A_1^{p\alpha})\Psi(B_1^{p\beta})^{1/2}\bigr\}^{1/p}\Bigr) \\
&\quad +(1-\lambda)\mathrm{Tr}\Bigl(\bigl\{\Psi(B_2^{p\beta})^{1/2}\Phi(A_2^{p\alpha})\Psi(B_2^{p\beta})^{1/2}\bigr\}^{1/p}\Bigr).
\end{aligned}$$

The latter inequality in the above is due to the assumption for the case $\alpha' + \beta = 1$. In this way, we may and do assume that $\alpha + \beta = 1$.

We may assume as in the proof of Theorem 2.1 that Φ and Ψ are strictly positive. It suffices to prove that if $A, H \in \mathbf{M}_{n_1}$ and $B, K \in \mathbf{M}_{n_2}$ are such that $A, B > 0$ and H, K are Hermitian, then

$$\frac{d^2}{dx^2}\mathrm{Tr}\left(\left\{\Psi\bigl((B + xK)^{p\beta}\bigr)^{1/2}\Phi\bigl((A + xH)^{p\alpha}\bigr)\Psi\bigl((B + xK)^{p\beta}\bigr)^{1/2}\right\}^{1/p}\right) \le 0$$

for all $x > 0$ small enough. For $z \in \mathbb{C}$, set $X(z) := zA + H$ and $Y(z) := zB + K$. For any $z \in \mathbb{C}^+$, since $X(z) \in \mathcal{I}_{n_1}^+$, $Y(z) \in \mathcal{I}_{n_2}^+$ and Lemma 1.2 implies

$$\operatorname{Im}\Phi(X(z)^{p\alpha}) = \Phi(\operatorname{Im}X(z)^{p\alpha}) > 0, \quad \operatorname{Im}\Psi(Y(z)^{p\beta}) = \Psi(\operatorname{Im}Y(z)^{p\beta}) > 0,$$

we get $\Phi(X(z)^{p\alpha}), \Psi(Y(z)^{p\beta}) \in \mathcal{I}_m^+$, and so $\Psi(Y(z)^{p\beta})^{1/2} \in \mathcal{I}_m^+$ is well-defined. Now we define

$$F(z) := \Psi(Y(z)^{p\beta})^{1/2}\Phi(X(z)^{p\alpha})\Psi(Y(z)^{p\beta})^{1/2},$$

and prove that

$$\sigma(F(z)) \subset \Gamma_{p\pi} \quad \text{if} \quad z \in \mathbb{C}^+. \tag{2.2}$$

To obtain (2.2) it suffices to show the following properties:

(a) When $z = re^{i\theta}$ with a fixed $0 < \theta < \pi$, $\sigma(F(z)) \subset \Gamma_{p\pi}$ for sufficiently large $r > 0$.

(b) $\sigma(F(z)) \cap [0, \infty) = \emptyset$ for all $z \in \mathbb{C}^+$.

(c) $\sigma(F(z)) \cap \{re^{ip\pi} : r \geq 0\} = \emptyset$ for all $z \in \mathbb{C}^+$.

In fact, if (2.2) does not hold for some $z_0 = r_0e^{i\theta_0} \in \mathbb{C}^+$, then according to (a) and the continuity of the eigenvalues of $F(z)$ we must have $\sigma(F(z)) \cup \partial\Gamma_{p\pi} \neq \emptyset$ for some $z \in \{re^{i\theta_0} : r > r_0\}$, which says that (b) or (c) must be violated.

Proof of (a). We have

$$F(z) = z^p\Psi\big((B + z^{-1}K)^{p\beta}\big)^{1/2}\Phi\big((A + z^{-1}H)^{p\alpha}\big)\Psi\big((B + z^{-1}K)^{p\beta}\big)^{1/2}. \tag{2.3}$$

When $z = re^{i\theta_0}$ with $0 < \theta_0 < \pi$ fixed and $r \to \infty$, note that

$$\sigma\Big(\Psi\big((B + z^{-1}K)^{p\beta}\big)^{1/2}\Phi\big((A + z^{-1}H)^{p\alpha}\big)\Psi\big((B + z^{-1}K)^{p\beta}\big)^{1/2}\Big)$$

converges to $\sigma\big(\Psi(B^{p\beta})^{1/2}\Phi(A^{p\alpha})\Psi(B^{p\beta})^{1/2}\big) \subset (0, \infty)$. Hence (a) follows.

Proof of (b). For any $0 \leq r < \infty$, we have

$$F(z) - rI_m = \Psi(Y(z)^{p\beta})^{1/2}\big(\Phi(X(z)^{p\alpha}) - r\Psi(Y(z)^{p\beta})^{-1}\big)\Psi(Y(z)^{p\beta})^{1/2}.$$

Since $\Phi(X(z)^{p\alpha}), \Psi(Y(z)^{p\beta}) \in \mathcal{I}_m^+$ as already mentioned,

$$\Phi(X(z)^{p\alpha}) - r\Psi(Y(z)^{p\beta})^{-1} \in \mathcal{I}_m^+$$

so that $F(z) - rI_m$ is invertible by Lemma 1.1.

Proof of (c). For any $0 \le r < \infty$, we have

$$F(z) - re^{ip\pi}I_m = e^{ip\alpha\pi}\Psi(Y(z)^{p\beta})^{1/2}\big(\Phi(e^{-ip\alpha\pi}X(z)^{p\alpha}) \\ -r\Psi(e^{-ip\beta\pi}Y(z)^{p\beta})^{-1}\big)\Psi(Y(z)^{p\beta})^{1/2}$$

thanks to $\alpha + \beta = 1$. Since $\Phi(e^{-ip\alpha\pi}X(z)^{p\alpha}) - r\Psi(e^{-ip\beta\pi}Y(z)^{p\beta})^{-1} \in \mathcal{I}_m^-$ by Lemma 1.2, $F(z) - re^{ip\pi}I_m$ is invertible.

We have shown (2.2) and similarly

$$\sigma(F(z)) \subset \Gamma_{-p\pi} \quad \text{if} \quad z \in \mathbb{C}^-.$$

Then $F(z)^{1/p}$ can be defined for $z \in \mathbb{C}^+ \cup \mathbb{C}^-$ by analytic functional calculus so that $\sigma(F(z)^{1/p}) \subset \mathbb{C}^+$ for $z \in \mathbb{C}^+$ and $\sigma(F(z)^{1/p}) \subset \mathbb{C}^-$ for $z \in \mathbb{C}^-$. Since $(F(z)^{1/p})^* = F(\overline{z})^{1/p}$, we can define a Pick function $\varphi(z) := \mathrm{Tr}\,(F(z)^{1/p})$ for $z \in \mathbb{C}^+ \cup \mathbb{C}^-$, which is analytic in $\mathbb{C} \setminus (-\infty, R]$ for some $R > 0$ as clearly seen from (2.3). Since, thanks to $\alpha + \beta = 1$,

$$\mathrm{Tr}\left(\left\{\Psi\big((B + xK)^{p\beta}\big)^{1/2}\Phi\big((A + xH)^{p\alpha}\big)\Psi\big((B + xK)^{p\beta}\big)^{1/2}\right\}^{1/p}\right) = x\varphi(x^{-1})$$

for small $x > 0$, we can proceed in the same way as in the proof of Theorem 2.1.■

It should be noted that (2.2) is a consequence of [6, Lemma 2]; however, we prefer a direct proof because the proof of [6, Lemma 2] is not easily accessible.

Corollary 2.4. *If $0 < p \le 1$ and $\alpha, \beta > 0$ with $\alpha + \beta \le 1$ and if C is an $m \times n$ matrix, then the function*

$$(A, B) \in \mathbf{M}_n^+ \times \mathbf{M}_m^+ \mapsto \mathrm{Tr}\left((B^{p\beta/2}CA^{p\alpha}C^*B^{p\beta/2})^{1/p}\right)$$

is jointly concave.

Even the particular case $p = 1$ of Theorem 2.3 yields some operator concavity for tensor products and Hadamard products as we will mention in the next section (see Corollaries 3.4 and 3.5).

In the third theorem we show the joint concavity of exponential-logarithmic trace functions extending the case (iv).

Theorem 2.5. *Let $m, n_1, \dots, n_k \in \mathbb{N}$, and for $j = 1, \dots, k$ let Φ_j be a positive linear map from $\mathbf{M}_{n_j}$ to $\mathbf{M}_m$. Let $L \in \mathbf{M}_m$ be any Hermitian matrix. If $\sum_{j=1}^k \Phi_j(I_{n_j}) \le I_m$, then the function*

$$(A_1, \dots, A_k) \in \prod_{j=1}^k \mathbf{M}_{n_j}^{++} \mapsto \mathrm{Tr}\left(\exp\Big\{L + \sum_{j=1}^k \Phi_j(\log A_j)\Big\}\right)$$

is jointly concave.

Proof. We prove that if $A_j \in \mathbf{M}_{n_j}^{++}$ and $H_j \in \mathbf{M}_{n_j}$ is Hermitian for $j = 1, \dots, k$, then

$$\frac{d^2}{dx^2}\mathrm{Tr}\left(\exp\Bigl\{L + \sum_{j=1}^{k} \Phi_j\bigl(\log(A_j + xH_j)\bigr)\Bigr\}\right) \le 0$$

for any sufficiently small $x > 0$. To do so, we define

$$F(z) := L + (\log z)\Bigl(I_m - \sum_{j=1}^{k} \Phi_j(I_{n_j})\Bigr) + \sum_{j=1}^{k} \Phi_j\bigl(\log(zA_j + H_j)\bigr),$$

and show that the function

$$\varphi(z) := \mathrm{Tr}\,(\exp F(z)) \qquad (z \in \mathbb{C}^+ \cup \mathbb{C}^-)$$

is a Pick function. When $z \in \mathbb{C}^+$, since $0 < \mathrm{Im}\,(\log z) < \pi$ and Lemma 1.3 implies

$$0 < \mathrm{Im}\,(\log(zA_j + H_j)) < \pi I_{n_j} \qquad (j = 1, \dots, k),$$

we get

$$\mathrm{Im}\,F(z) = (\mathrm{Im}\,(\log z))\Bigl(I_m - \sum_{j=1}^{k} \Phi_j(I_{n_j})\Bigr) + \sum_{j=1}^{k} \Phi_j\bigl(\mathrm{Im}\,(\log(zA_j + H_j))\bigr) > 0$$

and

$$\mathrm{Im}\,F(z) < \pi\Bigl(I_m - \sum_{j=1}^{k} \Phi_j(I_{n_j})\Bigr) + \sum_{j=1}^{k} \pi\Phi_j(I_{n_j}) = \pi I_m.$$

Therefore, it follows that

$$\sigma(F(z)) \subset \{x + iy : x \in \mathbb{R},\, 0 < y < \pi\}$$

so that $\sigma(\exp F(z)) \subset \mathbb{C}^+$, implying $\varphi(z) \in \mathbb{C}^+$. Thus, $\varphi(z) \in \mathbb{C}^+$ if $z \in \mathbb{C}^+$, and similarly $\varphi(z) \in \mathbb{C}^-$ if $z \in \mathbb{C}^-$. Moreover, φ is analytic in $\mathbb{C}^+ \cup \mathbb{C}^-$, and $\varphi(\overline{z}) = \overline{\varphi(z)}$ for $z \in \mathbb{C}^+$ because $F(z)^* = F(\overline{z})$, so φ is a Pick function. Since

$$\begin{aligned}
&\mathrm{Tr}\left(\exp\Bigl\{L + \sum_{j=1}^{k} \Phi_j\bigl(\log(A_j + xH_j)\bigr)\Bigr\}\right)\\
&\quad = \mathrm{Tr}\left(\exp\Bigl\{L + (\log x)\sum_{j=1}^{k} \Phi_j(I_{n_j}) + \sum_{j=1}^{k} \Phi_j\bigl(\log(x^{-1}A_j + H_j)\bigr)\Bigr\}\right)\\
&\quad = \mathrm{Tr}\left(\exp\bigl\{(\log x)I_m + F(x^{-1})\bigr\}\right) = x\varphi(x^{-1})
\end{aligned}$$

for small $x > 0$, the remaining proof is the same as before. ∎

Corollary 2.6. *Let C_j be an $m \times n_j$ matrix for $j = 1, \dots, k$, and $L \in \mathbf{M}_m$ be Hermitian. If $\sum_{j=1}^k C_j C_j^* \le I_m$, then the function*

$$(A_1, \dots, A_k) \in \prod_{j=1}^k \mathbf{M}_{n_j}^{++} \mapsto \mathrm{Tr}\left(\exp\Big\{L + \sum_{j=1}^k C_j(\log A_j)C_j^*\Big\}\right)$$

is jointly concave.

The assumption $\sum_{j=1}^k C_i C_j^* \le I_m$ in the above corollary as well as $\sum_{j=1}^k \Phi_j(I_{n_j}) \le I_m$ in Theorem 2.5 is essential. In fact, in the scalar case, if $p > 1$ then $a > 0 \mapsto \exp(p^{1/2}(\log a)p^{1/2}) = \exp(p \log a) = a^p$ is convex.

3. Trace Functions Involving Tensor Products and Hadamard Products

Let $X \otimes Y$ be the tensor product of $X \in \mathbf{M}_{m_1}$ and $Y \in \mathbf{M}_{m_2}$. We write $X \circ Y$ for the Hadamard product of $X, Y \in \mathbf{M}_m$, that is, $X \circ Y$ is the entrywise product of X and Y. It is well-known that the k-fold Hadamard product $X_1 \circ X_2 \circ \cdots \circ X_k$ of $X_1, \dots, X_k \in \mathbf{M}_m$ is a compression of $X_1 \otimes X_2 \otimes \cdots \otimes X_k$; so one can write

$$X_1 \circ X_2 \circ \cdots \circ X_k = E(X_1 \otimes X_2 \otimes \cdots \otimes X_k)E \tag{3.1}$$

with some orthogonal projection E in $\bigotimes_1^k \mathbf{M}_m \cong \mathbf{M}_{m^k}$ (more precisely, the above right-hand side should be restricted to the range of E).

In this section, we prove the joint concavity of trace functions involving tensor products and Hadamard products. We first give a lemma.

Lemma 3.1. *Let $X_1, \dots, X_k \in \mathbf{M}_m$ be mutually doubly commuting, i.e., $X_j X_{j'} = X_{j'} X_j$ and $X_j X_{j'}^* = X_{j'}^* X_j$ for all $j \ne j'$. Let $p_1, \dots, p_k > 0$ with $\sum_{j=1}^k p_j \le 1$. If $X_j \in \mathcal{I}_m^+$ and $e^{-ip_j\pi} X_j \in \mathcal{I}_m^-$ for $j = 1, \dots, k$, then $X_1 \cdots X_k \in \mathcal{I}_m^+$ and $e^{-i(p_1+\cdots+p_k)} X_1 \cdots X_k \in \mathcal{I}_m^-$.*

Proof. Once the case $k = 2$ has been proved, a simple induction argument works to get the general case. So we may concentrate to the case $k = 2$. Let $X_j = A_j + iB_j$ with Hermitian A_j and B_j. The assumption $X_j \in \mathcal{I}_m^+$ means $B_j > 0$. Since

$$e^{-ip_j\pi} X_j = \big\{(\cos p_j\pi)A_j + (\sin p_j\pi)B_j\big\} + i\big\{(\sin p_j\pi)A_j - (\cos p_j\pi)B_j\big\}$$

belongs to $\mathcal{I}_m^-$, we also get

$$(\sin p_j\pi)A_j - (\cos p_j\pi)B_j > 0,$$

or $A_j > (\cot p_j\pi)B_j$ thanks to $\sin p_j\pi > 0$. Since X_1 and X_2 are doubly commuting, $\{A_1, B_1\}$ and $\{A_2, B_2\}$ are commuting so that

$$X_1X_2 = (A_1A_2 - B_1B_2) + i(A_1B_2 + B_1A_2)$$

with Hermitian $A_1A_2 - B_1B_2$ and $A_1B_2 + B_1A_2$. We have

$$\begin{aligned} A_1B_2 + B_1A_2 &\geq B_2^{1/2}A_1B_2^{1/2} + B_1^{1/2}A_2B_1^{1/2} \\ &> (\cot p_1\pi)B_2^{1/2}B_1B_2^{1/2} + (\cot p_2\pi)B_1^{1/2}B_2B_1^{1/2} \\ &= (\cot p_1\pi + \cot p_2\pi)B_2^{1/2}B_1B_2^{1/2} > 0 \end{aligned}$$

because of

$$\cot p_1\pi + \cot p_2\pi = \frac{\sin(p_1+p_2)\pi}{\sin p_1\pi \sin p_2\pi} > 0.$$

So $X_1X_2 \in \mathcal{I}_m^+$ is obtained.

To show that $e^{-i(p_1+p_2)\pi}X_1X_2 \in \mathcal{I}_m^-$, set $\tilde{X}_j := e^{ip_j\pi}X_j^*$; then $\tilde{X}_j \in \mathcal{I}_m^+$ follows from the assumption $e^{-ip_j\pi}X_j \in \mathcal{I}_m^-$ and also $e^{-ip_j\pi}\tilde{X}_j \in \mathcal{I}_m^-$ from $X_j \in \mathcal{I}_m^+$. Hence the first assertion applied to $\tilde{X}_1, \tilde{X}_2$ implies that $\tilde{X}_1\tilde{X}_2 \in \mathcal{I}_m^+$. This means $e^{i(p_1+p_2)\pi}(X_1X_2)^* \in \mathcal{I}_m^+$ or, equivalently, $e^{-i(p_1+p_2)\pi}X_1X_2 \in \mathcal{I}_m^-$. ■

Theorem 3.2. *For $j = 1, \dots, k$, let $m_j, n_j \in \mathbb{N}$ and $\Phi_j : \mathbf{M}_{n_j} \to \mathbf{M}_{m_j}$ be a positive linear map. Moreover, let $l \in \mathbb{N}$ and $\Psi : \bigotimes_{j=1}^k \mathbf{M}_{m_j}$ ($\cong \mathbf{M}_m$, where $m := \sum_{j=1}^k m_j$) $\to \mathbf{M}_l$ be a positive linear map. If $0 < p \leq 1$ and $\alpha_1, \dots, \alpha_k > 0$ with $\sum_{j=1}^k \alpha_j \leq 1$, then the function*

$$(A_1, \dots, A_k) \in \prod_{j=1}^k \mathbf{M}_{n_j}^+ \mapsto \mathrm{Tr}\left(\left\{\Psi\big(\Phi_1(A_1^{p\alpha_1}) \otimes \cdots \otimes \Phi_k(A_k^{p\alpha_k})\big)\right\}^{1/p}\right)$$

is jointly concave.

Proof. First, note that the assertion in the case $\sum_{j=1}^k \alpha_j < 1$ follows from that in the case $\sum_{j=1}^k \alpha_j = 1$. This can be seen as in the proof of Theorem 2.3, so we omit the details. We may further assume that all Φ_j and Ψ are strictly positive. What we have to prove is that if $A_j \in \mathbf{M}_{n_j}^{++}$ and $H_j \in \mathbf{M}_{n_j}$ is Hermitian for $j = 1, \dots, k$, then

$$\frac{d^2}{dx^2}\mathrm{Tr}\left(\left\{\Psi\big(\Phi_1\big((A_1+xH_1)^{p\alpha_1}\big) \otimes \cdots \otimes \Phi_k\big((A_k+xH_k)^{p\alpha_k}\big)\big)\right\}^{1/p}\right) \leq 0 \tag{3.2}$$

for all $x > 0$ small enough. Set $Y_j(z) := zA_j + H_j$ for $z \in \mathbb{C}$. For any $z \in \mathbb{C}^+$, since $Y_j(z) \in \mathcal{I}_{n_j}^+$, we can define $Y_j(z)^{p\alpha_j}$ and set

$$X_j := I_{m_1} \otimes \cdots \otimes I_{m_{j-1}} \otimes \Phi_j(Y_j(z)^{p\alpha_j}) \otimes I_{m_{j+1}} \otimes \cdots \otimes I_{m_k} \quad (j = 1, \dots, k).$$

Obviously, $X_1, \dots, X_k$ are mutually doubly commuting. By Lemma 1.2, we get

$$\begin{aligned} \operatorname{Im} X_j &= I_{m_1} \otimes \cdots \otimes \Phi_j\bigl(\operatorname{Im}(Y_j(z)^{p\alpha_j})\bigr) \otimes \cdots \otimes I_{m_k} > 0, \\ \operatorname{Im}(e^{-ip\alpha_j\pi} X_j) &= I_{m_1} \otimes \cdots \otimes \Phi_j\bigl(\operatorname{Im}(e^{-ip\alpha_j\pi} Y_j(z)^{p\alpha_j})\bigr) \otimes \cdots \otimes I_{m_k} < 0. \end{aligned}$$

Hence by Lemma 3.1 we have $X_1 \cdots X_k \in \mathcal{I}_m^+$ and $e^{-ip\pi} X_1 \cdots X_k \in \mathcal{I}_m^-$; namely,

$$\begin{aligned} \Phi_1(Y_1(z)^{p\alpha_1}) \otimes \cdots \otimes \Phi_k(Y_k(z)^{p\alpha_k}) &\in \mathcal{I}_m^+, \\ e^{-ip\pi} \Phi_1(Y_1(z)^{p\alpha_1}) \otimes \cdots \otimes \Phi_k(Y_k(z)^{p\alpha_k}) &\in \mathcal{I}_m^- . \end{aligned}$$

These imply by Lemma 1.1 that

$$\sigma\Bigl(\Psi\bigl(\Phi_1(Y_1(z)^{p\alpha_1}) \otimes \cdots \otimes \Phi_k(Y_k(z)^{p\alpha_k})\bigr)\Bigr) \subset \Gamma_{p\pi} \quad \text{if} \quad z \in \mathbb{C}^+,$$

while similarly

$$\sigma\Bigl(\Psi\bigl(\Phi_1(Y_1(z)^{p\alpha_1}) \otimes \cdots \otimes \Phi_k(Y_k(z)^{p\alpha_k})\bigr)\Bigr) \subset \Gamma_{-p\pi} \quad \text{if} \quad z \in \mathbb{C}^-.$$

Thus, one can define

$$z \in \mathbb{C}^+ \cup \mathbb{C}^- \mapsto \Bigl\{\Psi\bigl(\Phi_1(Y_1(z)^{p\alpha_1}) \otimes \cdots \otimes \Phi_k(Y_k(z)^{p\alpha_k})\bigr)\Bigr\}^{1/p}$$

via analytic functional calculus so that the function

$$\varphi(z) := \operatorname{Tr}\Bigl(\Bigl\{\Psi\bigl(\Phi_1(Y_1(z)^{p\alpha_1}) \otimes \cdots \otimes \Phi_k(Y_k(z)^{p\alpha_k})\bigr)\Bigr\}^{1/p}\Bigr)$$

maps $\mathbb{C}^+$ (resp. $\mathbb{C}^-$) into itself. Furthermore, one can see as before that φ is a Pick function and it is analytic in $\mathbb{C} \setminus (-\infty, R]$ for some $R > 0$. The assumption $\sum_{j=1}^k \alpha_j = 1$ guarantees that $\operatorname{Tr}\bigl(\{\cdots\}^{1/p}\bigr)$ in (3.2) is equal to $x\varphi(x^{-1})$ whenever $x > 0$ is sufficiently small, and so the remaining proof of (3.2) is the same as in the proof of Theorem 2.1. ∎

The next theorem is just an application of Theorem 3.2 to $\Psi(E \cdot E)$ instead of Ψ, where E is as in (3.1).

Theorem 3.3. *Let $\Phi_j : \mathbf{M}_{n_j} \to \mathbf{M}_m$ be a positive linear map for $j = 1, \dots, k$, and $\Psi : \mathbf{M}_m \to \mathbf{M}_l$ be a positive linear map. If $0 < p \le 1$ and $\alpha_1, \dots, \alpha_k > 0$ with $\sum_{j=1}^k \alpha_j \le 1$, then the function*

$$(A_1, \dots, A_k) \in \prod_{j=1}^k \mathbf{M}_{n_j}^+ \mapsto \operatorname{Tr}\Bigl(\bigl\{\Psi\bigl(\Phi_1(A_1^{p\alpha_1}) \circ \cdots \circ \Phi_k(A_k^{p\alpha_k})\bigr)\bigr\}^{1/p}\Bigr)$$

is jointly concave. In particular,

$$(A_1, \dots, A_k) \in \prod_{j=1}^{k} \mathbf{M}_{n_j}^{+} \mapsto \mathrm{Tr}\,\Big(\big\{\Phi_1(A_1^{p\alpha_1}) \circ \cdots \circ \Phi_k(A_k^{p\alpha_k})\big\}^{1/p}\Big)$$

is jointly concave.

When all Φ_j are completely positive, the tensor product map $\Phi_1 \otimes \cdots \otimes \Phi_k$ is completely positive again (see [9]), and since the expression inside the trace in Theorem 3.2 is equal to

$$\Big\{\Psi(\Phi_1 \otimes \cdots \otimes \Phi_k)\big((A_1^{\alpha_1} \otimes \cdots \otimes A_k^{\alpha_k})^p\big)\Big\}^{1/p},$$

the assertion can be also seen from Theorem 2.1 (for $k = 1$) and the joint operator concavity of

$$(A_1, \dots, A_k) \in \prod_{j=1}^{k} \mathbf{M}_{n_j}^{+} \mapsto A_1^{\alpha_1} \otimes \cdots \otimes A_k^{\alpha_k} \tag{3.3}$$

due to Ando [1].

In the rest of this section, we complement some "operator concavity" results obtained from Theorem 2.3.

Corollary 3.4. *Let $\Phi : \mathbf{M}_{n_1} \to \mathbf{M}_{m_1}$ and $\Psi : \mathbf{M}_{n_2} \to \mathbf{M}_{m_2}$ be positive linear maps. For every $\alpha, \beta > 0$ with $\alpha + \beta \le 1$, the map*

$$(A, B) \in \mathbf{M}_{n_1}^{+} \times \mathbf{M}_{n_2}^{+} \mapsto \Phi(A^{\alpha}) \otimes \Psi(B^{\beta})$$

is jointly operator concave in the order of positive semidefiniteness.

Proof. First, assume $m_1 = m_2 = m$. Note that the transpose $X \mapsto X^t$ is a positive linear map on $\mathbf{M}_m$. For any $X \in \mathbf{M}_m$, if we take $p = 1$, $X^*\Phi(\cdot)X$ for Φ and $(\Psi(\cdot))^t$ for Ψ in Theorem 2.3, then we have the joint concavity of

$$(A, B) \in \mathbf{M}_{n_1}^{+} \times \mathbf{M}_{n_2}^{+} \mapsto \mathrm{Tr}\,\big(X^*\Phi(A^{\alpha})X(\Psi(B^{\beta}))^t\big). \tag{3.4}$$

Consider $\mathbf{M}_m$ as a Hilbert space with respect to the inner product $\langle X, Y\rangle := \mathrm{Tr}\,(Y^*X)$. Then $\mathbf{M}_m \otimes \mathbf{M}_m$ is faithfully represented on the Hilbert space $\mathbf{M}_m$ by the representation $\pi(A \otimes B)X := AXB^t$ for $A, B, X \in \mathbf{M}_m$. Since the right-hand side of (3.4) is

$$\langle \pi(\Phi(A^{\alpha}) \otimes \Psi(B^{\beta}))X, X\rangle,$$

it follows that $(A, B) \mapsto \Phi(A^{\alpha}) \otimes \Psi(B^{\beta})$ is jointly operator concave.

When $m_1 \neq m_2$, choose $m \geq m_1, m_2$ and set $\tilde{\Phi}(A) := \Phi(A) \oplus 0_{m-m_1}$ and $\tilde{\Psi}(B) := \Psi(B) \oplus 0_{m-m_2}$. Then $\Phi(A^\alpha) \otimes \Psi(B^\beta)$ is identified with the $m_1 m_2 \times m_1 m_2$ corner of $\tilde{\Phi}(A^\alpha) \otimes \tilde{\Psi}(B^\beta)$. So the result follows from the above case. ∎

The following is obvious from the above corollary because the Hadamard product is a compression of the tensor product.

Corollary 3.5. *Let $\Phi : \mathbf{M}_{n_1} \to \mathbf{M}_m$ and $\Psi : \mathbf{M}_{n_2} \to \mathbf{M}_m$ be positive linear maps. For every $\alpha, \beta > 0$ with $\alpha + \beta \leq 1$, the map*

$$(A, B) \in \mathbf{M}_{n_1}^+ \times \mathbf{M}_{n_2}^+ \mapsto \Phi(A^\alpha) \circ \Psi(B^\beta)$$

is jointly operator concave.

It may be conjectured that the assertion of Corollary 3.4 holds for more than two components, that is, the map

$$(A_1, \ldots, A_k) \in \prod_{j=1}^k \mathbf{M}_{n_j}^+ \mapsto \Phi_1(A_j^{\alpha_1}) \otimes \cdots \otimes \Phi_k(A_k^{\alpha_k})$$

is jointly operator concave for positive linear maps $\Phi_j : \mathbf{M}_{n_j} \to \mathbf{M}_{m_j}$ and $\alpha_j > 0$ with $\sum_{j=1}^k \alpha_j \leq 1$. A positive linear map $\Phi : \mathbf{M}_n \to \mathbf{M}_n$ is said to be decomposable if there exist completely positive linear maps $\Phi^{(1)}$ and $\Phi^{(2)}$ such that $\Phi(X) = \Phi^{(1)}(X) + \Phi^{(2)}(X^t)$. Not all positive linear maps on $\mathbf{M}_n$ are decomposable if $n \geq 3$ (see [5, Appendix B]). It is easily seen from the operator concavity of (3.3) that the above conjecture is true when all Φ_j are decomposable positive linear maps.

4. Trace Functions Involving Operator Means

Let m be an operator mean in the sense of Kubo and Ando [7]. It admits the integral representation

$$A \, m \, B = aA + bB + \int_0^\infty \frac{1+t}{t} \{(tA) : B\} \, d\nu(t),$$

where $a, b \geq 0$ and ν is a finite measure on $(0, \infty)$. In the above representation, the parallel sum $A : B$ of $A, B \in \mathbf{M}_n^{++}$ is defined by

$$A : B := (A^{-1} + B^{-1})^{-1},$$

and $A : B := \lim_{\varepsilon \to +0} (A + \varepsilon I) : (B + \varepsilon I)$ for general $A, B \in \mathbf{M}_n^+$.

In this section, we prove the joint concavity of trace functions involving the operator mean m. To do so, we need to define the operator mean $X \, m \, Y$ of

$X, Y \in \mathcal{I}_n^+$ in such a way that we get $X\,m\,Y \in \mathcal{I}_n^+$. When $X, Y \in \mathcal{I}_n^+$, since $X^{-1} + Y^{-1} \in \mathcal{I}_n^-$ by Lemma 1.1, the parallel sum

$$X\,:\,Y := (X^{-1} + Y^{-1})^{-1}$$

is defined and belongs to $\mathcal{I}_n^+$. So one can define the operator mean $X\,m\,Y$ belonging to $\mathcal{I}_n^+$ as

$$X\,m\,Y := aX + bY + \int_0^\infty \frac{1+t}{t}\{(tX) : Y\}\,d\nu(t), \tag{4.1}$$

whenever the above integral is absolutely convergent. To check the absolute convergence of the integral, we estimate as follows. Since

$$\begin{aligned} tX : Y &= tX^{1/2}(I + tX^{1/2}Y^{-1}X^{1/2})^{-1}X^{1/2} \\ &= Y^{1/2}(I + t^{-1}Y^{1/2}X^{-1}Y^{1/2})^{-1}Y^{1/2}, \end{aligned}$$

we get

$$\frac{1+t}{t}\|tX : Y\| \le \frac{(1+t)\|X^{1/2}\|^2}{1 - t\|X^{1/2}Y^{-1}X^{1/2}\|} \quad \text{for small} \quad t > 0 \tag{4.2}$$

and

$$\frac{1+t}{t}\|tX : Y\| \le \frac{(1+t)\|Y^{1/2}\|^2}{t - \|Y^{1/2}X^{-1}Y^{1/2}\|} \quad \text{for large} \quad t > 0. \tag{4.3}$$

Hence $t^{-1}(1+t)\|tX : Y\|$ is bounded for all $0 < t < \infty$, and the operator mean $X\,m\,Y \in \mathcal{I}_n^+$ is well-defined. On the other hand, when $X, Y \in \mathcal{I}_n^-$, one can define $X\,m\,Y \in \mathcal{I}_n^-$ by the same formula (4.1). It is obvious that $(X\,m\,Y)^* = X^*\,m\,Y^*$ if $X, Y \in \mathcal{I}_n^+$.

Lemma 4.1. *If $X, Y \in \mathcal{I}_n^+$ and $e^{-ip\pi}X, e^{-ip\pi}Y \in \mathcal{I}_n^-$, then $X\,m\,Y \in \mathcal{I}_n^+$ and $e^{-ip\pi}(X\,m\,Y) \in \mathcal{I}_n^-$.*

Proof. The above discussion implies that $(e^{-ip\pi}X)\,m\,(e^{-ip\pi}Y) \in \mathcal{I}_n^-$ as well as $X\,m\,Y \in \mathcal{I}_n^+$. Since

$$(e^{-ip\pi}X) : (e^{-ip\pi}Y) = e^{-ip\pi}(X : Y),$$

it follows immediately that $(e^{-ip\pi}X)\,m\,(e^{-ip\pi}Y) = e^{-ip\pi}(X\,m\,Y)$. Hence we obtain the assertion. ∎

Lemma 4.2. *If $X(z), Y(z) : \Omega \to \mathcal{I}_n^+$ (or $\mathcal{I}_n^-$) are analytic functions in an open set Ω in $\mathbb{C}$, then the function $z \in \Omega \mapsto X(z)\,m\,Y(z)$ is analytic in Ω.*

Proof. For any $z_0 \in \Omega$, it follows from the estimates (4.2) and (4.3) that the convergence

$$\int_{[\alpha,\beta]} \frac{1+t}{t}\{(tX(z)) : Y(z)\}\, d\nu(t) \to \int_{(0,\infty)} \frac{1+t}{t}\{(tX(z)) : Y(z)\}\, d\nu(t)$$

as $\alpha \to +0$ and $\beta \to \infty$ is uniform for z in some neighborhood of z_0. So we may show that for each $0 < \alpha < \beta < \infty$ the function

$$z \in \Omega \mapsto F(z) := \int_{[\alpha,\beta]} \frac{1+t}{t}\{(tX(z)) : Y(z)\}\, d\nu(t)$$

is analytic. When $z_0 \in \Omega$ and $z_0 + u \in \Omega$ as $u \to 0$, since

$$\begin{aligned}
&(tX(z_0+u))^{-1} \\
&\quad = t^{-1}\bigl(X(z_0) + uX'(z_0) + o(u)\bigr)^{-1} \\
&\quad = t^{-1}X(z_0)^{-1/2}\bigl\{I + X(z_0)^{-1/2}(uX'(z_0) + o(u))X(z_0)^{-1/2}\bigr\}^{-1}X(z_0)^{-1/2} \\
&\quad = t^{-1}X(z_0)^{-1} - t^{-1}uX(z_0)^{-1}X'(z_0)X(z_0)^{-1} + o(u)
\end{aligned}$$

and similarly

$$Y(z_0+u)^{-1} = Y(z_0)^{-1} - uY(z_0)^{-1}Y'(z_0)Y(z_0)^{-1} + o(u),$$

we have

$$\begin{aligned}
&(tX(z_0+u))^{-1} + Y(z_0+u)^{-1} \\
&\quad = (tX(z_0))^{-1} + Y(z_0)^{-1} \\
&\qquad - u\bigl\{t^{-1}X(z_0)^{-1}X'(z_0)X(z_0)^{-1} + Y(z_0)^{-1}Y'(z_0)Y(z_0)^{-1}\bigr\} + o(u).
\end{aligned}$$

Therefore, we estimate

$$\begin{aligned}
&(tX(z_0+u)) : Y(z_0+u) \\
&\quad = \bigl\{(tX(z_0)) : Y(z_0)\bigr\} + u\bigl\{(tX(z_0)) : Y(z_0)\bigr\}^{-1} \\
&\qquad \times\bigl\{t^{-1}X(z_0)^{-1}X'(z_0)X(z_0)^{-1} + Y(z_0)^{-1}Y'(z_0)Y(z_0)^{-1}\bigr\} \\
&\qquad \times\bigl\{(tX(z_0)) : Y(z_0)\bigr\}^{-1} + o(u),
\end{aligned}$$

where $o(u)$ as $u \to 0$ is uniform for all t restricted to $[\alpha, \beta]$. This estimate implies that $F(z)$ is differentiable at z_0, completing the proof. ■

Theorem 4.3. *Let m be any operator mean. If $0 < p \le 1$, then the function*

$$(A, B) \in \mathbf{M}_n^+ \times \mathbf{M}_n^+ \mapsto \mathrm{Tr}\,\bigl((A^p\, m\, B^p)^{1/p}\bigr)$$

is jointly concave.

Proof. It suffices to show that if $A, B > 0$ and H, K are Hermitian, then

$$\frac{d^2}{dx^2}\mathrm{Tr}\left((A+xH)^p\, m\,(B+xK)^p\right)^{1/p} \le 0$$

for sufficiently small $x > 0$. Set $X(z) := zA + H$ and $Y(z) := zB + K$ for $z \in \mathbb{C}$. When $z \in \mathbb{C}^+$, since $X(z)^p$ and $Y(z)^p$ satisfy the assumptions in Lemma 4.1, we have

$$\sigma\big(X(z)^p\, m\, Y(z)^p\big) \subset \Gamma_{p\pi}.$$

Hence $\{X(z)^p\, m\, Y(z)^p\}^{1/p}$ can be defined via analytic functional calculus and we have

$$\sigma\big(\{X(z)^p\, m\, Y(z)^p\}^{1/p}\big) \subset \mathbb{C}^+ \quad \text{if} \quad z \in \mathbb{C}^+.$$

When $z \in \mathbb{C}^-$, we can similarly define $\{X(z)^p\, m\, Y(z)^p\}^{1/p}$ so that

$$\sigma\big(\{X(z)^p\, m\, Y(z)^p\}^{1/p}\big) \subset \mathbb{C}^- \quad \text{if} \quad z \in \mathbb{C}^-.$$

In this way, the function

$$\varphi(z) := \mathrm{Tr}\left(\{X(z)^p\, m\, Y(z)^p\}^{1/p}\right)$$

maps $\mathbb{C}^+$ (resp. $\mathbb{C}^-$) into itself. It follows from Lemma 4.2 that $X(z)^p\, m\, Y(z)^p$ is analytic in $\mathbb{C}^+ \cup \mathbb{C}^-$ and hence so is $\{X(z)^p\, m\, Y(z)^p\}^{1/p}$. Furthermore, since

$$\left(\{X(z)^p\, m\, Y(z)^p\}^{1/p}\right)^* = \left\{(X(z)^p\, m\, Y(z)^p)^*\right\}^{1/p} = \{X(\overline{z})^p\, m\, Y(\overline{z})^p\}^{1/p},$$

we get $\varphi(\overline{z}) = \overline{\varphi(z)}$ for $z \in \mathbb{C}^+$, and so φ is a Pick function. Finally, it is easily seen from Lemma 4.2 that φ is analytic in $\mathbb{C} \setminus (-\infty, R]$ for some $R > 0$, and the remaining proof is the same as before. ■

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Graduate School of Information Sciences, Tohoku University
Aoba-ku, Sendai 980-8577, Japan
E-mail: hiai@math.is.tohoku.ac.jp